Deciding Unambiguity and Sequentiality from a Finitely Ambiguous Max-Plus Automaton

Ines Klimann, Sylvain Lombardy, Jean Mairesse, Christophe Prieur

Deciding Unambiguity and Sequentiality starting from a Finitely Ambiguous Max-Plus Automaton

Ines Klimann, Sylvain Lombardy, Jean Mairesse, and Christophe Prieur*

July 4, 2004

Abstract

Finite automata with weights in the max-plus semiring are considered. The main result is: it is decidable in an effective way whether a series that is recognized by a finitely ambiguous max-plus automaton is unambiguous, or is sequential. A collection of examples is given to illustrate the hierarchy of max-plus series with respect to ambiguity.

1 Introduction

A *max-plus automaton* is a finite automaton with multiplicities in the max-plus semiring $\mathbb{R}_{\max} = (\mathbb{R} \cup \{-\infty\}, \max, +)$. Roughly speaking, it is an automaton with two tapes: an input tape labelled by a finite alphabet Σ, and an output tape weighted in $\mathbb{R}_{\max}$. The weight of a word in Σ^* is the maximum over all successful paths of the sum of the weights along the path.

Max-plus automata, and their min-plus counterparts, are studied under various names in the literature: distance automata, finance automata, cost automata. They have also appeared in various contexts: to study logical problems in formal language theory (star height, finite power property) [13, 23], to model the dynamic of some Discrete Event Systems (DES) [10, 12], or in the context of automatic speech recognition [18].

Two automata are *equivalent* if they recognize the same series, *i.e.* if they have the same input/output behavior. The problem of equivalence of two max-plus automata is undecidable [15]. The same problem for finitely ambiguous max-plus automata is decidable [14, 25].

The *sequentiality* problem is defined as follows: given a max-plus automaton, is there an equivalent max-plus automaton which is *sequential* (*i.e.* deterministic in input). Let us give some motivations on why the sequentiality problem is important. In the case of a sequential automaton, the time complexity of

*LIAFA, CNRS (UMR 7089) - Université Paris 7, 2, place Jussieu - 75251 Paris Cedex 5 - France. email: {klimann,lombardy,mairesse,prieur}@liafa.jussieu.fr

computing the output is roughly linear in the length of the input. This time efficiency is central in speech processing, see [18]. Consider now a DES modelled by a max-plus automaton. If the automaton is unambiguous, or *a fortiori* sequential, then one can compute the optimal, as well as the average behavior, of the DES, see [10, 11].
Sequentiality is decidable for unambiguous max-plus automata [18]. In the present paper, we prove that sequentiality is decidable for finitely ambiguous max-plus automata. To the best of our knowledge, it is not known if the finite ambiguity of a max-plus series (defined *via* an infinitely ambiguous automaton) is a decidable problem. In particular, the status of the sequentiality problem is still open for a general max-plus automaton (even if the multiplicities are restricted to be in $\mathbb{Z}_{\max}$, $\mathbb{N}_{\max}$ or $\mathbb{Z}^{-}_{\max}$). To be complete, it is necessary to mention that in [18, §3.5], it is claimed that any max-plus automaton admits an effectively computable equivalent unambiguous one. If that was true, it would imply the decidability of the sequentiality for general max-plus automata. However, the statement is erroneous and counter-examples are provided in §3 of the present paper[1].

The sequentiality problem can be asked for automata over any semiring $\mathbb{K}$. For transducers, *i.e.* when $\mathbb{K}$ is the set of rational subsets of a free monoid (with union and concatenation as the two laws), the problem is completely solved in the functional case (when, for every input, the output is a language of cardinality at most one) [3, 7, 8]. For a general transducer, the problem is wide open. Observe that the semiring $\{a^{\geqslant n}, n \in \mathbb{N}\} = \{a^n a^*, n \in \mathbb{N}\}$ is isomorphic to $\mathbb{N}_{\min}$: $a^n a^* + a^m a^* = a^{\min(n,m)} a^*$ and $a^n a^* \cdot a^m a^* = a^{n+m} a^*$. Similarly, the semiring $\{a^{\leqslant n}, n \in \mathbb{N}\}$ is isomorphic to $\mathbb{N}_{\max}$ (where $a^{\leqslant n} = \{\varepsilon, a, \dots, a^n\}$). Hence automata over $\mathbb{N}_{\max}$ or $\mathbb{N}_{\min}$ translate into transducers, but *not* functional ones. Also the translation does not work for automata over $\mathbb{R}_{\max}$. Hence, the vast literature on transducers is of limited use in our context.

In the present paper, we work with $\mathbb{R}_{\max}$. Decidability and complexity should be interpreted under the assumption that two real numbers can be added or compared in constant time.

2 Preliminaries

2.1 Max-plus semiring and series

The free monoid over a finite set (alphabet) Σ is denoted by Σ^* and the empty word is denoted by ε. The structure $\mathbb{R}_{\max} = (\mathbb{R} \cup \{-\infty\}, \max, +)$ is a semiring, which is called the *max-plus semiring*. It is convenient to use the notations $\oplus = \max$ and $\otimes = +$. The neutral elements of $\oplus$ and $\otimes$ are denoted respectively by $\mathbb{0} = -\infty$ and $\mathbb{1} = 0$. The subsemirings $\mathbb{N}_{\max}$, $\mathbb{Z}_{\max}$, ..., are defined in the natural way. The *min-plus semiring* $\mathbb{R}_{\min}$ is obtained by replacing max by min and $-\infty$ by $+\infty$ in the definition of $\mathbb{R}_{\max}$. The results of this paper can be easily adapted to the min-plus setting. Observe that the subsemiring

[1]The version of [18] available on the author's website has been correctly modified.

$\mathbb{B} = (\{\mathbb{0}, \mathbb{1}\}, \oplus, \otimes)$ is isomorphic to the Boolean semiring. For matrices A, B of appropriate sizes with entries in $\mathbb{R}_{\max}$, we set $(A \oplus B)_{ij} = A_{ij} \oplus B_{ij}$, $(A \otimes B)_{ij} = \bigoplus_k A_{ik} \otimes B_{kj}$, and for $a \in \mathbb{R}_{\max}$, $(a \otimes A)_{ij} = a \otimes A_{ij}$. We usually omit the $\otimes$ sign, writing for instance AB instead of $A \otimes B$.

Consider the set $\mathbb{R}_{\max}\langle\langle\Sigma^*\rangle\rangle$ of *(formal power) series (over* Σ^* *with coefficients in* $\mathbb{R}_{\max}$*)*, that is the set of maps from Σ^* to $\mathbb{R}_{\max}$. We denote by $\langle S, u\rangle$ the coefficient of the word u in the series S. The *support* of a series S is the set $\operatorname{Supp} S = \{u \in \Sigma^* \mid \langle S, u\rangle \neq \mathbb{0}\}$. It is convenient to use the notation $S = \bigoplus_{u\in\Sigma^*} \langle S, u\rangle u = \bigoplus_{u\in \operatorname{Supp}\,(S)} \langle S, u\rangle u$. Equipped with the addition ($\oplus$) and the Cauchy product ($\otimes$), the set $\mathbb{R}_{\max}\langle\langle\Sigma^*\rangle\rangle$ forms a semiring. The image of $\lambda \in \mathbb{R}_{\max}$ by the canonical injection into $\mathbb{R}_{\max}\langle\langle\Sigma^*\rangle\rangle$ is still denoted by λ. In particular, the neutral elements of $\mathbb{R}_{\max}\langle\langle\Sigma^*\rangle\rangle$ are $\mathbb{0}$ and $\mathbb{1}$. The characteristic series of a language L is the series $\mathbb{1}_L$ such that $\langle \mathbb{1}_L, w\rangle = \mathbb{1}$ if $w \in L$, and $\langle \mathbb{1}_L, w\rangle = \mathbb{0}$ otherwise.

2.2 Max-plus automaton

Let Q and Σ be two finite sets. A *max-plus automaton* of set of states (dimension) Q over the alphabet Σ, is a triple $\mathcal{A} = (\alpha, \mu, \beta)$, where $\alpha \in \mathbb{R}_{\max}^{1\times Q}$, $\beta \in \mathbb{R}_{\max}^{Q\times 1}$, and where $\mu : \Sigma^* \to \mathbb{R}_{\max}^{Q\times Q}$ is a morphism of monoids. The morphism μ is uniquely determined by the family of matrices $\{\mu(a), a \in \Sigma\}$, and for $w = a_1 \cdots a_n$, we have $\mu(w) = \mu(a_1) \otimes \cdots \otimes \mu(a_n)$. The series *recognized* (or *realized*) *by* $\mathcal{A}$ is by definition $S(\mathcal{A}) = \bigoplus_{u\in\Sigma^*} (\alpha\mu(u)\beta)u$. This is just a specialization to the max-plus semiring of the classical notion of an automaton with multiplicities over a semiring [4, 9, 17]. By the Kleene-Schützenberger Theorem [21], the set of series recognized by a max-plus automaton is equal to the set of rational series over $\mathbb{R}_{\max}$. We denote it by Rat.

A state $i \in Q$ is *initial*, resp. *final*, if $\alpha_i \neq \mathbb{0}$, resp. $\beta_i \neq \mathbb{0}$. As usual a max-plus automaton is represented graphically by a labelled weighted digraph with ingoing and outgoing arcs for initial and final states, see *e.g.* Figure 6 (the input or output weights equal to $\mathbb{1}$ are omitted). The terminology of graph theory is used accordingly (*e.g.* (simple) path or circuit of an automaton, union of automata, ...). A path which is both starting with an ingoing arc and ending with an outgoing arc is called a *successful path*. The *label of a path* is the concatenation of the labels of the successive arcs (so called *transitions*), the *weight of a path* is the product ($\otimes$) of the weights of the successive arcs (including the ingoing and the outgoing arc, need it be). We denote by *weight*(π) the weight of the path π. We use the following notations for paths in an automaton $\mathcal{A} = (\alpha, \mu, \beta)$:

$$p \to q, \quad \to p \to q, \quad p \to q \to, \quad p \xrightarrow{u|x} q, \quad \Big[p \xrightarrow{u|x} q\Big]_{\mathcal{A}}, \text{ if } \mu(u)_{pq} = x \text{ in } \mathcal{A}\,.$$

The first example is a path (of any length) from p to q, the second also includes an ingoing arc, the third an outgoing arc, in the fourth the weight and the label are added and in the fifth the underlying automaton is recalled.

An automaton is *trim* if any state belongs to at least one successful path.

Let I be a finite set. The *tensor product automaton* of $(\mathcal{A}_i = (\alpha^{\llcorner i}, \mu^{\llcorner i}, \beta^{\llcorner i}))_{i \in I}$, denoted by $\odot_{i \in I} \mathcal{A}_i$, is defined as follows. It is the max-plus automaton (A, M, B) of dimension $Q = \prod_i Q_i$, where Q_i is the dimension of $\mathcal{A}_i$, and such that

$$\forall p, q \in Q, \quad A_p = \bigotimes_{i \in I} \alpha^{\llcorner i}_{p_i}, \quad \forall a \in \Sigma, \; M(a)_{p,q} = \bigotimes_{i \in I} \mu^{\llcorner i}(a)_{p_i, q_i}, \quad B_p = \bigotimes_{i \in I} \beta^{\llcorner i}_{p_i} .$$

2.3 Heap model

A *heap* or *Tetris model* [24], consists of a finite set of slots $\mathcal{R}$, and a finite set of rectangular pieces Σ. Each piece $a \in \Sigma$ is of height 1 and occupies a determined subset $\mathcal{R}(a)$ of the slots. To a word $u = u_1 \cdots u_k \in \Sigma^*$ is associated the *heap* obtained by piling up in order the pieces $u_1, \ldots, u_k$, starting with a horizontal ground and according to the Tetris game mechanism (pieces are subject to gravity and fall down vertically until they meet either a previously piled up piece or the ground). Consider the morphism generated by the matrices $M(a) \in \mathbb{R}_{\max}^{\mathcal{R} \times \mathcal{R}}$, $a \in \Sigma$, defined by

$$M(a)_{ij} = \begin{cases} 1 & \text{if } i, j \in \mathcal{R}(a), \\ 0 & \text{if } i = j \notin \mathcal{R}(a), \\ -\infty & \text{otherwise} . \end{cases}$$

Let $x(u)_i$ be the height of the heap u on slot $i \in \mathcal{R}$. We have ([5, 11, 12]): $x(u)_i = \mathbb{1} M(u) \delta_i$, where $\mathbb{1} = (\mathbb{1}, \ldots, \mathbb{1}) \in \mathbb{R}_{\max}^{1 \times \mathcal{R}}$ and $\delta_i \in \mathbb{R}_{\max}^{\mathcal{R} \times 1}$ is defined by $(\delta_i)_j = \mathbb{1}$ if $j = i$ and $\mathbb{0}$ otherwise. In other words, the application $x(\cdot)_i : \Sigma^* \to \mathbb{R}_{\max}$ is recognized by the max-plus automaton $(\mathbb{1}, M, \delta_i)$. We call $(\mathbb{1}, M, \delta)$, $\delta = \bigoplus_{i \in I} \delta_i, I \subseteq \mathcal{R}$, a *heap automaton* (associated with the heap model). Among max-plus automata, heap automata are particularly convenient and playful, due to the underlying geometric interpretation. Here, they are used as a source of examples and counter-examples, *e.g.* Figures 3, 4 and 7.

We represent a heap automaton graphically as in Figure 1.

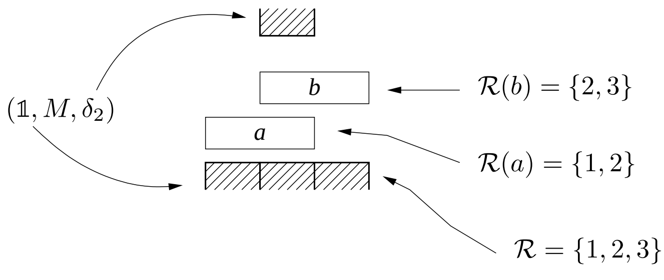


Figure 1: A heap automaton

2.4 Ambiguity and Sequentiality

Consider a max-plus automaton $\mathcal{A} = (\alpha, \mu, \beta)$ of dimension Q over Σ. The automaton is *sequential* if there is a unique initial state and if for all $i \in Q$, and for all $a \in \Sigma$, there is at most one $j \in Q$ such that $\mu(a)_{ij} \neq \mathbb{0}$. In the case of a Boolean automaton, we also say *deterministic* for sequential. The automaton $\mathcal{A}$ is *unambiguous* if for any word $u \in \Sigma^*$, there is at most one successful path of label u. The automaton is *finitely ambiguous* if there exists some $k \in \mathbb{N}$ such that for any word $u \in \Sigma^*$, there are at most k successful paths of label u. The minimal such k is called the *degree of ambiguity* of the automaton. Clearly, 'sequential' implies 'unambiguous' which implies 'finitely ambiguous'. The automaton is *infinitely ambiguous* if it is not finitely ambiguous.

Consider a series $S \in \mathrm{Rat}$. The series is *sequential* (resp. *unambiguous*, *finitely ambiguous*) if there exists a sequential (resp. unambiguous, finitely ambiguous) max-plus automaton recognizing it. The series is *infinitely ambiguous* if there exists no finitely ambiguous max-plus automaton recognizing it. The *degree of ambiguity* of a finitely ambiguous series is the minimal degree of ambiguity of an automaton recognizing it. The sets of sequential, unambiguous, and finitely ambiguous series are denoted respectively by Seq, NAmb, and FAmb. Define $\mathrm{FSeq} = \{S \mid \exists k, \exists S_1, \ldots, S_k \in \mathrm{Seq},\ S = S_1 \oplus \cdots \oplus S_k\}$.

Consider a total order on Σ^*. Given a series $S \neq \mathbb{0}$, define the *normalized series* $\varphi(S)$ by $\varphi(S) = \bigoplus_{u \in \Sigma^*} (\langle S, u \rangle - \langle S, u_0 \rangle) u$, where u_0 is the smallest word of $\mathrm{Supp}\, S$. The *(left) quotient* of a series S by a word w is the series $w^{-1}S$ defined by $w^{-1}S = \bigoplus_{u \in \Sigma^*} \langle S, wu \rangle u$.

A series S is rational if and only if the semi-module of series $\langle w^{-1}S, w \in \Sigma^* \rangle$ is finitely generated, *i.e.* if there exists $S_1, \ldots, S_k$, such that:

$$\forall w \in \Sigma^*,\ \exists \lambda_1, \ldots, \lambda_k \in \mathbb{R}_{\max},\ w^{-1}S = \bigoplus_i \lambda_i S_i.$$

A series S is sequential if and only if the set of series $\{\varphi(w^{-1}S), w \in \Sigma^*\}$ is finite.

Proposition 1 *A trim automaton $\mathcal{A}$ of dimension Q is infinitely ambiguous if and only if there exist $p, q \in Q, p \neq q$, and $v \in \Sigma^*$, such that $p \xrightarrow{v} p$, $p \xrightarrow{v} q$, $q \xrightarrow{v} q$. This can be checked in polynomial time.*

For a proof, see [27] and the references therein. Observe that the (in)finite ambiguity is independent of the underlying semiring. Next result is due to Mohri [18] and is an adaptation of a classical result of Choffrut on functional transducers, see [3, 7, 8] (for the decidability) and [2, 26] (for the polynomial complexity).

Theorem 1 *Let $\mathcal{A}$ be an unambiguous max-plus automaton. There exists a polynomial time algorithm to decide whether $S(\mathcal{A})$ is a sequential series.*

If $\mathcal{A}$ is unambiguous and $S(\mathcal{A})$ is sequential, a sequential automaton recognizing the series can be effectively constructed from $\mathcal{A}$ using an adaptation of the subset construction of Boolean automata [1, 6, 18].

It is useful to detail Theorem 1. We need to introduce several definitions. Given two words $u, v \in \Sigma^*$, let $u \wedge v$ be the longest common prefix of u and

v, and define $d(u,v) = |u| + |v| - 2|u \wedge v|$. It is easy to check that $d(.,.)$ is a distance on Σ^*. A series S is *M-Lipschitz* ($M \in \mathbb{R}_+$) if:

$$\forall u, v \in \text{Supp } S,\ \ |\langle S, u\rangle - \langle S, v\rangle| \leqslant M d(u,v) \ ;$$

and S is *Lipschitz* if it is M-Lipschitz for some M. The set of Lipschitz series is denoted by Lip. Consider a trim max-plus automaton $\mathcal{A}$ of dimension Q. Two states $p, q \in Q$ are *twins* if:

$$\left[\xrightarrow{x_0} i \xrightarrow{u_1|x_1} p \xrightarrow{u_2|x_2} p, \quad \xrightarrow{y_0} j \xrightarrow{u_1|y_1} q \xrightarrow{u_2|y_2} q\right] \implies [x_2 = y_2]\ .$$

If all the states are twins, the automaton $\mathcal{A}$ is said to satisfy the *twin property*. We denote the set of all such automata by Twin. The following implications hold:

$$\Big[\mathcal{A} \in \text{Twin}\Big] \implies \Big[S(\mathcal{A}) \in \text{Seq}\Big] \implies \Big[S(\mathcal{A}) \in \text{Lip}\Big]\ . \qquad (1)$$

Furthermore,

$$\Big[\mathcal{A} \in \text{NAmb}, \quad S(\mathcal{A}) \in \text{Lip}\Big] \implies \Big[\mathcal{A} \in \text{Twin}\Big]\ . \qquad (2)$$

The twin property can be checked in polynomial time, hence Theorem 1 follows from the above implications.

3 Hierarchy of Series

The examples in this section illustrate the classes of series on which we work.

$$\text{Seq} \subsetneq \underset{(\S 3.1)}{(\text{NAmb} \cap \text{FSeq})} \begin{array}{c} \subsetneq \\ \subsetneq \end{array} \begin{array}{c} \underset{(\S 3.2)}{\text{FSeq}} \\ \\ \underset{(\S 3.3)}{\text{NAmb}} \end{array} \begin{array}{c} \subsetneq \\ \subsetneq \end{array} \underset{(\S 3.4)}{\text{FAmb}} \subsetneq \underset{(\S 3.5)}{\text{Rat}} \subsetneq \underset{(\S 3.6)}{\text{Series}}$$

3.1 A Series in $\overline{\textbf{Seq}} \cap \textbf{NAmb} \cap \textbf{FSeq}$

An example over a one-letter alphabet is provided in Figure 2. The recognized series is

$$\langle S, a^n\rangle = \begin{cases} 0 & \text{if } n \text{ is odd,} \\ n & \text{if } n \text{ is even.} \end{cases}$$

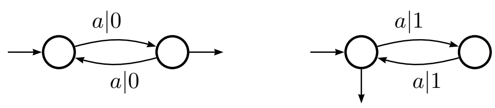


Figure 2: $\overline{\text{Seq}} \cap \text{NAmb} \cap \text{FSeq}$

The series is not Lipschitz, since $|\langle S, a^{n+1}\rangle - \langle S, a^n\rangle| \geqslant n$, and consequently the series cannot be sequential (see (1)). It is clear that it is an unambiguous series (the only successful path of label a^n is the right or left one depending on the parity of n) and a sum of sequential series. In fact, any max-plus rational series over a one-letter alphabet is unambiguous and a sum of sequential series [16, 19].

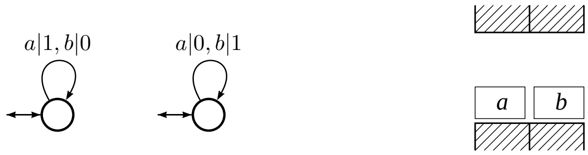


Figure 3: $\mathrm{FSeq} \cap \overline{\mathrm{NAmb}}$

3.2 A Series in $\mathbf{FSeq} \cap \overline{\mathbf{NAmb}}$

The series $\langle S, u\rangle = |u|_a \oplus |u|_b$ over the alphabet $\{a, b\}$ is a sum of two sequential series: the heap automaton of Figure 3 recognizes this series.

Assume that S is unambiguous. The series S is 1-Lipschitz. So it has to be sequential, see (2) and (1). Consequently, there exist series $S_1, \ldots S_k$ such that:

$$\forall u \in \Sigma^*,\ \exists i,\ \exists \lambda_u \in \mathbb{R}_{\max} \quad u^{-1}S = \lambda_u \otimes S_i.$$

By the pigeon-hole principle, there must exist $i \in \{1, \ldots k\}$ and two integers $m < n$ such that

$$\exists \lambda_n, \lambda_m \quad (a^n)^{-1}S = \lambda_n \otimes S_i, \qquad (a^m)^{-1}S = \lambda_m \otimes S_i.$$

Consequently, we have

$$\langle (a^n)^{-1}S, b^{m+1}\rangle - \langle (a^n)^{-1}S, \varepsilon\rangle = \langle (a^m)^{-1}S, b^{m+1}\rangle - \langle (a^m)^{-1}S, \varepsilon\rangle.$$

However

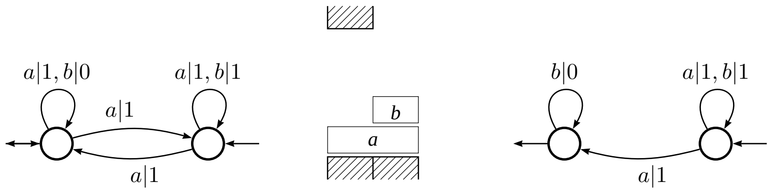
$$\langle (a^n)^{-1}S, b^{m+1}\rangle - \langle (a^n)^{-1}S, \varepsilon\rangle = \langle S, a^n b^{m+1}\rangle - \langle S, a^n\rangle = n - n = 0$$
$$\langle (a^m)^{-1}S, b^{m+1}\rangle - \langle (a^m)^{-1}S, \varepsilon\rangle = \langle S, a^m b^{m+1}\rangle - \langle S, a^m\rangle = m + 1 - m = 1.$$

This is a contradiction, consequently S is not sequential and thus cannot be an unambiguous series.

3.3 Series in $\mathbf{NAmb} \cap \overline{\mathbf{FSeq}}$

a) The first example is the series S given by the heap automaton of Figure 4 (a), or equivalently by the automaton of Figure 4 (b).

Consider the series $\tilde{S}$ defined by $\langle \tilde{S}, w\rangle = \langle S, w\rangle - |w|$. An automaton recognizing $\tilde{S}$ can clearly be obtained from an automaton recognizing S by removing 1 from each output weight. Hence S and $\tilde{S}$ are both sum of sequential series or none of them is.



(a) (b)

Figure 4: NAmb $\cap$ $\overline{\text{FSeq}}$

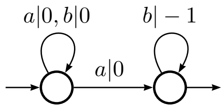


Figure 5: NAmb $\cap$ $\overline{\text{FSeq}}$

The series $\tilde{S}$ is recognized by the automaton of Figure 5. Suppose that $\tilde{S} = S_1 \oplus S_2 \oplus \cdots \oplus S_k$, where $k \in \mathbb{N}$ and the S_i are sequential series.

Since the S_i are sequential series, they are Lipschitz. Let N be the maximal Lipschitz coefficient of the S_i. Let $(N_i)_{i \geqslant 0}$ be a sequence of integers such that

$$N_0 > N, \quad N(N_{k-1} + 1) < N_k - N_{k-1} \text{ for all } k \geqslant 1.$$

The coefficient of ab^{N_k} in $\tilde{S}$ is $-N_k$, and it comes, for instance, from S_1. The coefficient of $ab^{N_k}ab^{N_{k-1}}$ is $-N_{k-1}$. We have:

$$d(ab^{N_k}, ab^{N_k}ab^{N_{k-1}}) = N_{k-1}+1 \text{ and } |\langle \tilde{S}, ab^{N_k}\rangle - \langle \tilde{S}, ab^{N_k}ab^{N_{k-1}}\rangle| = N_k - N_{k-1}.$$

The coefficient of $ab^{N_k}ab^{N_{k-1}}$ in $\tilde{S}$ does not come from S_1, since

$$\begin{aligned} |\langle S_1, ab^{N_k}\rangle - \langle S_1, ab^{N_k}ab^{N_{k-1}}\rangle| &\leqslant N(N_{k-1} + 1) \\ &< N_k - N_{k-1} = |\langle S_1, ab^{N_k}\rangle - \langle \tilde{S}, ab^{N_k}ab^{N_{k-1}}\rangle|. \end{aligned}$$

In the same way, we prove that any two words of the set

$$\{ab^{N_k}, ab^{N_k}ab^{N_{k-1}}, \ldots, ab^{N_k}ab^{N_{k-1}} \cdots ab^{N_0}\}$$

cannot be recognized by the same S_i. But this set has cardinality $k+1$ and thus there is a contradiction.

b) The second example is the series given by the automaton of Figure 6. The series recognized by this automaton is:

$$\langle S, a^{m_1}b^{n_1} \cdots a^{m_p}b^{n_p}\rangle = \sum_{m_i \text{even}} m_i,$$

where $m_1 \in \mathbb{N}$, $m_{k+1} \in \mathbb{N} - \{0\}$, $n_k \in \mathbb{N} - \{0\}$ for $1 \leqslant k \leqslant p-1$, and $n_p \in \mathbb{N}$. The automaton is clearly unambiguous. Furthermore, it is not a finite sum of sequential series. To simplify notations, let us prove that S is not the sum of two sequential series. Suppose that $S = S_1 \oplus S_2$, with $S_1, S_2 \in \mathrm{Seq}$.

The series S_i, $i \in \{1,2\}$, are sequential, so they are Lipschitz by (1). Let N be such that S_i, $i \in \{1,2\}$, are N-Lipschitz. Let us consider words of the form $a^r b^n a^s$, with $n > 0$. We discuss on the parity of r and s. The coefficient of the word $a^{2p+1}b^n a^{2q+1}$ in S, which is equal to 0, comes from one of the S_i. For instance

$$\langle S, a^{2p+1}b^n a^{2q+1}\rangle = 0 = \langle S_1, a^{2p+1}b^n a^{2q+1}\rangle. \tag{3}$$

Set $q > N$. Since S_1 is N-Lipschitz and $d(a^{2p+1}b^n a^{2q+1}, a^{2p+1}b^n a^{2q}) = 1$, we have

$$\langle S, a^{2p+1}b^n a^{2q}\rangle = 2q = \langle S_2, a^{2p+1}b^n a^{2q}\rangle. \tag{4}$$

Fix q and n. Since S_1 and S_2 are Lipschitz, there exists an integer M such that:

$$\forall u, v \in \mathrm{Supp}\ S_i,\ d(u,v) \leqslant 2n + 4q + 2 \Rightarrow |\langle S_i, u\rangle - \langle S_i, v\rangle| \leqslant M. \tag{5}$$

We have:

$$d(a^{2p}b^n a^{2q}, a^{2p+1}b^n a^{2q+1}) = 2n{+}4q{+}2 \quad \text{and} \quad d(a^{2p}b^n a^{2q}, a^{2p+1}b^n a^{2q}) = 2n{+}4q{+}1.$$

So, by Equation (5), we know that:

- If $a^{2p}b^n a^{2q} \in \mathrm{Supp}\ S_1$, then

  $$2p + 2q = |\langle S_1, a^{2p}b^n a^{2q}\rangle - \langle S_1, a^{2p+1}b^n a^{2q+1}\rangle| \leqslant M,$$

  which is wrong for p large enough.

- If $a^{2p}b^n a^{2q} \in \mathrm{Supp}\ S_2$, then

  $$2p = |\langle S_2, a^{2p}b^n a^{2q}\rangle - \langle S_2, a^{2p+1}b^n a^{2q}\rangle| \leqslant M,$$

  which is also wrong for p large enough.

Consequently, S is not the sum of two sequential series. To extend the result to the sum of m sequential series, one has to consider words of the form $a^{r_1}b^{n_1}a^{r_2}\cdots a^{r_{m-1}}b^{n_{m-1}}a^{r_m}$.

3.4 Series in $\overline{\mathbf{NAmb}} \cap \overline{\mathbf{FSeq}} \cap \mathbf{FAmb}$

a) Consider the heap automaton given in Figure 7 (a). The corresponding series is at most two-ambiguous since it is also recognized by the two-ambiguous automaton of Figure 7 (b). It cannot be unambiguous: on $\{a,b\}^*$, since it coincides with the series of Figure 3 which is in $\overline{\mathrm{NAmb}}$. It cannot be a finite sum of sequential series: on $\{b,c\}^*$, it coincides with the series of Figure 4 which is in $\overline{\mathrm{FSeq}}$.

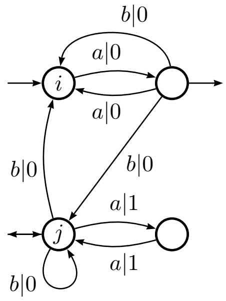


Figure 6: NAmb $\cap$ $\overline{\text{FSeq}}$

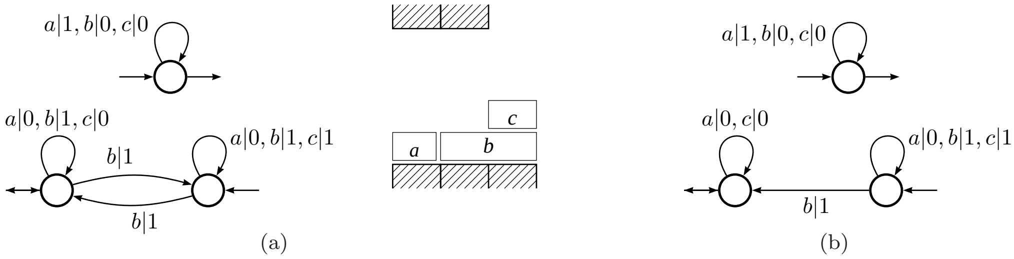


Figure 7: $\overline{\text{NAmb}} \cap \overline{\text{FSeq}} \cap$ FAmb

b) Another example is provided by the automaton $\mathcal{A}$ of Figure 8.

Denote by S the series recognized by this automaton, by S_1 the series recognized by the left part, say $\mathcal{A}_1$, of the automaton, and by S_2 the series recognized by the right part, say $\mathcal{A}_2$.

The automaton $\mathcal{A}_1$ is the one introduced in Section 3.3 and the automaton $\mathcal{A}_2$ is the same one after permutation of the a's and b's in the labels. Recall that $\mathcal{A}_1$ and $\mathcal{A}_2$ are unambiguous, so S is at most two-ambiguous.

Let us prove that S is not a finite sum of sequential series. Denote by L the language of words whose blocks of b's have odd length. Let u be a word of L: in $\mathcal{A}_2$, the b-blocks of u are always read in the upper part of the automaton, so $\langle S_2, u\rangle = 0$. Since the coefficient of u in $\mathcal{A}_1$ is at least 0, we have $S \odot \mathbb{1}_L = S_1 \odot \mathbb{1}_L$. Suppose that S is a finite sum of sequential series. Then so is $S \odot \mathbb{1}_L$ and $S_1 \odot \mathbb{1}_L$. And this is false since one can choose an odd n in the proof of Section 3.3 for the automaton of Figure 6.

Let us prove that S is not unambiguous. Let M be the rational language of words whose a-blocks and b-blocks have even lengths and let u be a word of M. In $\mathcal{A}_1$, the a-blocks have to be read in the lower part of $\mathcal{A}_1$ and so $\langle S_1, u\rangle = |u|_a$. In the same way: $\langle S_2, u\rangle = |u|_b$. So we have $S \odot \mathbb{1}_M = S' \odot \mathbb{1}_M$, where S' is the series recognized by the automaton of Figure 3. Consequently, if S is unambiguous, so is $S' \odot \mathbb{1}_M$. We now apply the arguments of Section 3.2

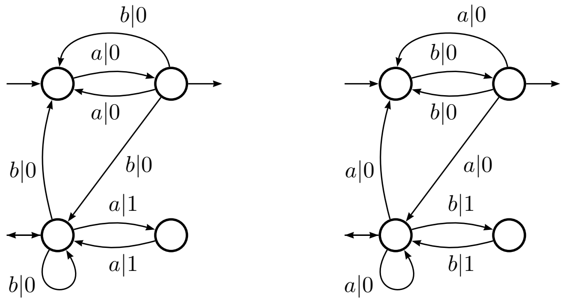
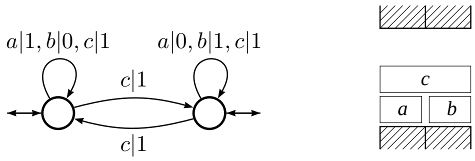


Figure 8: $\overline{\mathrm{NAmb}} \cap \overline{\mathrm{FSeq}} \cap \mathrm{FAmb}$



(a) (b)

Figure 9: $\overline{\mathrm{FAmb}} \cap \mathrm{Rat}$

to show that $S' \odot \mathbb{1}_M$ is not unambiguous.

c) Besides, Weber has given examples of series which are k-ambiguous and not $(k-1)$-ambiguous [25, Theorem 4.2].

3.5 Series in $\overline{\mathbf{FAmb}} \cap \mathbf{Rat}$

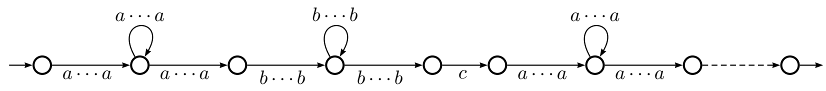
Consider the series S recognized by the automaton of Figure 9. Assume that S is finitely ambiguous. Using the result of Corollary 1 below, S is recognized by a finite union of unambiguous automata with the same support, say $\mathcal{A}_1, \ldots, \mathcal{A}_k$.

Denote by S_i the series recognized by $\mathcal{A}_i$, for $1 \leqslant i \leqslant k$, and by n the maximal dimension of an automaton $\mathcal{A}_i$. Observe that Supp $S = \Sigma^*$. Since all the S_i have the same support, we have Supp $S_i = \Sigma^*$.

Now, consider the word $w_0 = \big(a^n b^n c\big)^k$. For any i, there is a single successful path labelled by w_0 in $\mathcal{A}_i$. Note that a path of length n contains necessarily a circuit.

So, each automaton $\mathcal{A}_i$ contains a path of the form:

π_i :



For every $j \in \{1, \ldots, k\}$, we choose in the subpath labelled by the j-th factor a^n (resp. b^n) a circuit that is called the j-th a-loop (resp. the j-th b-loop).

The coefficient of a word in S is less than or equal to its lentgh, it is thus the same for its coefficients in the S_i. Consequently, the mean weights of the loops of π_i are less than or equal to 1. Denote by $av(\pi_i, a, j)$ the mean weight of the j-th a-loop in the path π_i, and define $av(\pi_i, b, j)$ similarly.

Set $j \in \{1, \ldots, k\}$. For $\lambda \in \mathbb{N} - \{0\}$, consider the word

$$w_\lambda = (a^n b^n c) \cdots (a^n b^n c) \underbrace{(a^{n+\lambda n!} b^{n+\lambda n!} c)}_{j\text{-th block}} (a^n b^n c) \cdots (a^n b^n c)$$

This word can be read on each path π_i by turning into the j-th a- and b-loops, whose lengths are less than or equal to n and so divide $n!$.

Let $i \in \{1, \ldots, k\}$ be such that $\langle S, w_0 \rangle = \langle S_i, w_0 \rangle$. We have $\langle S, w_\lambda \rangle - \langle S, w_0 \rangle = \lambda n!$, and so $\langle S_i, w_\lambda \rangle - \langle S_i, w_0 \rangle \leqslant \lambda n!$. But $\langle S_i, w_\lambda \rangle - \langle S_i, w_0 \rangle = (av(\pi_i, a, j) + av(\pi_i, b, j))\lambda n!$, consequently

$$av(\pi_i, a, j) + av(\pi_i, b, j) \leqslant 1. \tag{6}$$

Consider any u in $\{01, 10\}^k$. For all $p \in \mathbb{N} - \{0\}$, let us define the word

$$v_p(u) = (a^{n+\lambda_1 n!} b^{n+\mu_1 n!} c) \cdots (a^{n+\lambda_j n!} b^{n+\mu_j n!} c) \cdots (a^{n+\lambda_k n!} b^{n+\mu_k n!} c),$$

where $(\lambda_j, \mu_j) = (p, 0)$ if $(u_{2j-1}, u_{2j}) = (1, 0)$ (we say then that the dominant j-th loop is the j-th a-loop) and $(\lambda_j, \mu_j) = (0, p)$ otherwise (the dominant j-th loop is the j-th b-loop), for any $j \in \{1, \ldots, k\}$.

By the pigeon-hole principle, for some i, there are infinitely many words of the form $v_p(u)$ such that $\langle S, v_p(u) \rangle = \langle S_i, v_p(u) \rangle = k + nk + kpn!$. Such words are read on the path π_i. The dominant j-th loop in π_i has then necessarily mean weight 1, and by Equation (6), the non-dominant j-th loop in π_i has mean weight less than or equal to 0.

Consequently, we have built an injection from the language $\{01, 10\}^k$ into the set of paths $\{\pi_i\}$. But the language has cardinality 2^k and the set of paths has cardinality k. So we have a contradiction.

3.6 Rational and Non-Rational Series ($\overline{\text{Rat}}$)

A max-plus series is non-rational as soon as its support is a non-rational language. Here, we present a less trivial example of non-rational max-plus series.

In this paragraph, it is necessary to distinguish between $\mathbb{R}_{\min}$ and $\mathbb{R}_{\max}$: for R = Rat or NAmb, we use the respective notations $\mathbb{R}_{\min}R$, $\mathbb{R}_{\max}R$. If

$S \in \mathbb{R}_{\max}\langle\langle\Sigma^*\rangle\rangle$, we identify S with $\tilde{S} \in \mathbb{R}_{\min}\langle\langle\Sigma^*\rangle\rangle$ such that $\langle\tilde{S}, w\rangle = \langle S, w\rangle$ if $w \in \operatorname{Supp} S$ and $\langle\tilde{S}, w\rangle = +\infty$ if $\langle S, w\rangle = -\infty$.

Clearly, we have

$$\mathbb{R}_{\max}\mathrm{NAmb} = \mathbb{R}_{\min}\mathrm{NAmb} = \mathrm{NAmb}.$$

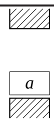
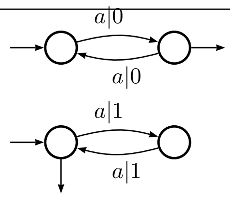
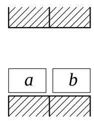
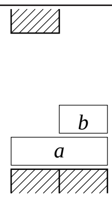
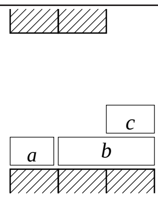
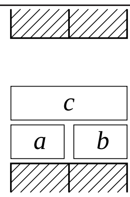
On the other hand, it is easy to find $S \in \mathbb{R}_{\min}\mathrm{FSeq} \cap \mathbb{R}_{\min}\overline{\mathrm{NAmb}}$ such that $S \notin \mathbb{R}_{\max}\mathrm{Rat}$.

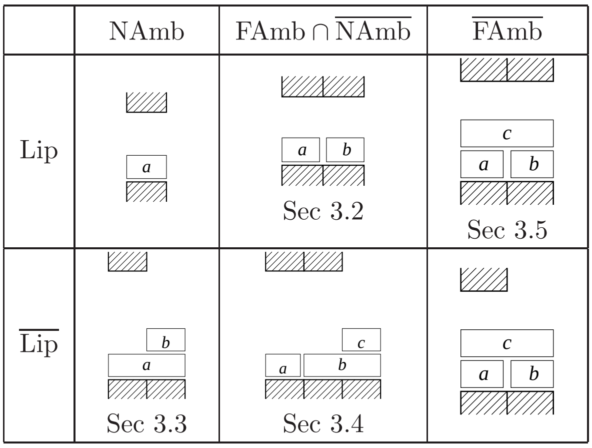
Consider for instance the series $S = \min(|w|_a, |w|_b)$ (recognized by the automaton of Figure 3 seen as a min-plus automaton). Let us prove that S does not belong to $\mathbb{R}_{\max}\mathrm{Rat}$. If it does: let $S_1, \dots S_n$ be a minimal generating family of $\langle u^{-1}S, u \in \Sigma^*\rangle$ (see §2.4), we have: $\forall u \in \Sigma^*$, $\exists \lambda_1^{(u)}, \dots \lambda_n^{(u)}$, $u^{-1}S = \bigoplus_i \lambda_i^{(u)} \otimes S_i$. The restrictions of the quotients of S to b^* are bounded, hence so are the restrictions of the S_i. Let k_i be such that: $\langle S_i, b^{k_i}\rangle = \max_k \langle S_i, b^k\rangle$. It follows that for any word u: $\max_k \langle u^{-1}S, b^k\rangle = \max_{k_i} \langle u^{-1}S, b^{k_i}\rangle$. Consider $k > \max_i k_i$. Then arises a contradiction:

$$\max_l \langle (a^k)^{-1}S, b^l\rangle = k > \max_{k_i} \langle (a^k)^{-1}S, b^{k_i}\rangle = \max_i k_i.$$

3.7 Ambiguity vs. sequentiality and Ambiguity vs. Lipschitz

Here are some examples of series that are in several classes described in Section 3:

	NAmb	$\mathrm{FAmb} \cap \overline{\mathrm{NAmb}}$	$\overline{\mathrm{FAmb}}$
Seq		impossible	impossible
$\mathrm{FSeq} \cap \overline{\mathrm{Seq}}$		 Sec 3.2	impossible
$\overline{\mathrm{FSeq}}$	 Sec 3.3	 Sec 3.4	 Sec 3.5



4 From Finitely Ambiguous to Union of Unambiguous

Weber [25] has proved that a finitely ambiguous $\mathbb{N}_{\max}$-automaton can be turned into an union of unambiguous ones. We present a completely different and simpler proof that holds in any semiring, in particular $\mathbb{R}_{\max}$.

In this section, we work on the structure of the automata. So we consider simply Boolean automata.

Below, given a set S, we identify the vectors of $\mathbb{B}^S$ with the subsets of S, *i.e.* $x \in \mathbb{B}^S$ is identified with $\{i \in S \mid x_i = \mathbb{1}\}$.

Let $\mathcal{A} = (\alpha, \mu, \beta)$ be a trim automaton. The *past* of a state p is the set of words that label a path from some initial state to p. The *future* of p is the set of words that label a path from p to some final state. We write:

$$\mathsf{Past}_{\mathcal{A}}(p) = \{w \in \Sigma^* \mid (\alpha\mu(w))_p = \mathbb{1}\}, \qquad \mathsf{Fut}_{\mathcal{A}}(p) = \{w \in \Sigma^* \mid (\mu(w)\beta)_p = \mathbb{1}\}.$$

Let $\mathcal{A} = (\alpha, \mu : \Sigma^* \to \mathbb{B}^{Q\times Q}, \beta)$ be an automaton. Let us recall the usual determinization procedure of $\mathcal{A}$ *via* the subset construction. Let R be the least subset of $\mathbb{B}^Q$ inductively defined by:

$$\alpha \in R, \quad X \in R \Rightarrow \forall a \in \Sigma, X\mu(a) \in R.$$

Let $\mathcal{D} = \mathcal{D}(\mathcal{A}) = (J, \nu : \Sigma^* \to \mathbb{B}^{R\times R}, U)$ be the *determinized automaton* of $\mathcal{A}$ defined by:

$$J = \{\alpha\}, \qquad U = \{P \in R \mid P\beta = \mathbb{1}\}, \qquad \nu(a)_{P,P'} = \mathbb{1} \Longleftrightarrow P' = P\mu(a).$$

Lemma 1 *i) Let $\mathcal{A}$ be an automaton and $\mathcal{D}$ its determinized automaton. Then for each state P of D,*

$$\mathsf{Past}_{\mathcal{D}}(P) \subseteq \bigcap_{p\in P} \mathsf{Past}_{\mathcal{A}}(p), \qquad \textit{and} \qquad \mathsf{Fut}_{\mathcal{D}}(P) = \bigcup_{p\in P} \mathsf{Fut}_{\mathcal{A}}(p).$$

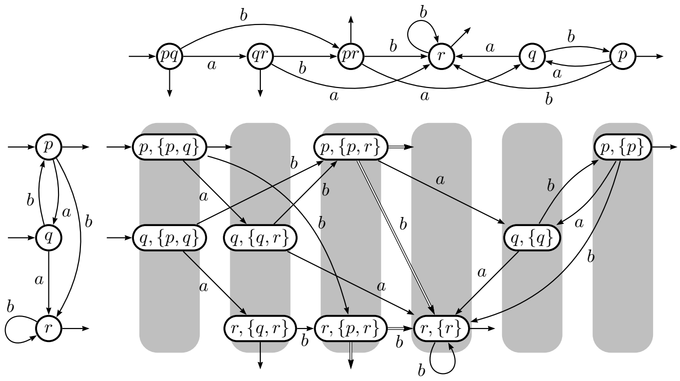


Figure 10: $\mathcal{A}$ (left), the determinized automaton (top) and the Schützenberger covering

ii) Let $\mathcal{A}$ and $\mathcal{B}$ be two automata and $\mathcal{A} \odot \mathcal{B}$ their tensor product (cf. §2.4), then, for all state (p, q) of $\mathcal{A} \odot \mathcal{B}$,

$$\mathsf{Past}_{\mathcal{A}\odot\mathcal{B}}(p,q) = \mathsf{Past}_{\mathcal{A}}(p) \cap \mathsf{Past}_{\mathcal{B}}(q), \qquad \mathsf{Fut}_{\mathcal{A}\odot\mathcal{B}}(p,q) = \mathsf{Fut}_{\mathcal{A}}(p) \cap \mathsf{Fut}_{\mathcal{B}}(q).$$

The constructions and results given in Propositions 2 and 3 are inspired by Schützenberger [22]. They have been explicitely stated by Sakarovitch in [20].

Let $\mathcal{A}$ be an automaton and $\mathcal{D}$ its determinized automaton. The trim part of the product $\mathcal{A} \odot \mathcal{D}$ is called the *Schützenberger covering $\mathcal{S}$ of $\mathcal{A}$*.

Proposition 2 *Let $\mathcal{A} = (\alpha, \mu, \beta)$ be a trim automaton, $\mathcal{D}$ its determinized automaton and $\mathcal{S}$ its Schützenberger covering.*
i) The states of $\mathcal{S}$ are exactly the pairs (p, P), where P is a state of $\mathcal{D}$ and $p \in P$. We call the set $\{(p, P) \mid p \in P\}$ of states of $\mathcal{S}$ a column *(in gray on Figure 10).*
ii) The canonical surjection ψ from the transitions of $\mathcal{S}$ onto the transitions of $\mathcal{A}$ induces a one-to-one mapping between the successful paths of $\mathcal{S}$ and $\mathcal{A}$.
iii) Let P be a state of $\mathcal{D}$. Then, for every p in P,

$$\mathsf{Past}_{\mathcal{S}}(p,P) = \mathsf{Past}_{\mathcal{D}}(P), \qquad \mathsf{Fut}_{\mathcal{S}}(p,P) = \mathsf{Fut}_{\mathcal{A}}(p).$$

Thus, all the states of a given column have the same past.

proof. *i)* A state (p, P) of $\mathcal{S}$ is initial if and only if p is initial in $\mathcal{A}$ (*i.e.* $p \in \alpha$) and P is initial in $\mathcal{D}$ (*i.e.* $P = \{\alpha\}$). Now, let (p, P) be a state of $\mathcal{S}$ such that $p \in P$ and (q, Q) a successor of (p, P) by a. Then, there exist two transitions:

$$\left[p \xrightarrow{a} q\right]_{\mathcal{A}} \quad \text{and} \quad \left[P \xrightarrow{a} Q\right]_{\mathcal{D}}.$$

By definition of $\mathcal{D}$, q belongs thus to Q.

Conversely, let P be a state of $\mathcal{D}$ and p an element of P. For every w in $\mathsf{Past}_{\mathcal{D}}(P)$, w belongs to $\mathsf{Past}_{\mathcal{A}}(p)$ (Lemma 1). Therefore there is a path in $\mathcal{S}$ from an initial state to (p, P).

ii) Let π be a successful path of $\mathcal{A}$, with label $w = w_1 w_2 \cdots w_n$. Let θ be the (unique) successful path with label w in $\mathcal{D}$:

$$\pi = \Big[\rightarrow p_0 \xrightarrow{w_1} p_1 \xrightarrow{w_2} \ldots \xrightarrow{w_n} p_n \rightarrow\Big]_{\mathcal{A}}, \quad \theta = \Big[\rightarrow P_0 \xrightarrow{w_1} P_1 \xrightarrow{w_2} \ldots \xrightarrow{w_n} P_n \rightarrow\Big]_{\mathcal{D}}.$$

There is a path in $\mathcal{S}$: $\pi' = \rightarrow (p_0, P_0) \xrightarrow{w_1} (p_1, P_1) \xrightarrow{w_2} \ldots \xrightarrow{w_n} (p_n, P_n) \rightarrow$.
The function $\pi \mapsto \pi'$ is obviously one-to-one.

iii) By results from Lemma 1:

$$\left.\begin{array}{c} \mathsf{Past}_{\mathcal{S}}(p, P) = \mathsf{Past}_{\mathcal{A}}(p) \cap \mathsf{Past}_{\mathcal{D}}(P) \\ \forall p \in P, \mathsf{Past}_{\mathcal{D}}(P) \subseteq \mathsf{Past}_{\mathcal{A}}(p) \end{array}\right\} \Rightarrow \forall p \in P, \mathsf{Past}_{\mathcal{S}}(p, P) = \mathsf{Past}_{\mathcal{D}}(P).$$

$$\left.\begin{array}{c} \mathsf{Fut}_{\mathcal{S}}(p, P) = \mathsf{Fut}_{\mathcal{A}}(p) \cap \mathsf{Fut}_{\mathcal{D}}(P) \\ \forall p \in P, \mathsf{Fut}_{\mathcal{A}}(p) \subseteq \mathsf{Fut}_{\mathcal{D}}(P) \end{array}\right\} \Rightarrow \forall p \in P, \mathsf{Fut}_{\mathcal{S}}(p, P) = \mathsf{Fut}_{\mathcal{A}}(p).$$

□

Definition 1 *In $\mathcal{S}$, different transitions with the same label, the same destination and whose origins belong to the same column are said to be* competing. *Likewise, different final states of the same column are* competing. *A* competing set *is a maximal set of competing transitions or competing final states.*

Let $\mathcal{U}$ be an automaton obtained from $\mathcal{S}$ by removing all transitions except one in every competing set and by turning all final states of a column, except one, into non-final states. The choice of the transition (or the final state) to keep in a competing set is arbitrary.

For instance, the covering $\mathcal{S}$ of Figure 10 has two competing sets (drawn with double lines); the first one contains two transitions with label b that arrive in $(r, \{r\})$, the second one contains the states $(p, \{p, r\})$ and $(r, \{p, r\})$ which are both final. The above selection principle gives rise to four possible automata, the automaton of Fig. 11 being one of them.

Proposition 3 *Let $\mathcal{S}$ and $\mathcal{U}$ be two automata defined as above. Then,*
i) $\forall P, \forall p \in P$, $\mathsf{Past}_{\mathcal{U}}(p, P) = \mathsf{Past}_{\mathcal{S}}(p, P)$.
ii) *Futures of states in a column of $\mathcal{U}$ are disjoint and*

$$\forall P, \forall p \in P, \bigcup_{p \in P} \mathsf{Fut}_{\mathcal{U}}(p, P) = \bigcup_{p \in P} \mathsf{Fut}_{\mathcal{S}}(p, P)\,.$$

Consequently, the automaton $\mathcal{U}$ is unambiguous and equivalent to $\mathcal{A}$.

proof. *i*) The proof is by induction on the length of words. If (p, P) is initial in $\mathcal{S}$, it is still initial in $\mathcal{U}$. Let wa be a word of $\mathsf{Past}_{\mathcal{S}}(p, P)$ and π a path labelled by this word from an initial state to (p, P). We consider the last transition of π:

$$\Big[(q, P') \xrightarrow{a} (p, P)\Big]_{\mathcal{S}}.$$

If this transition does not belong to a competing set, it still appears in $\mathcal{U}$ and, by induction, $w \in \mathsf{Past}_{\mathcal{U}}(q, P')$, thus $wa \in \mathsf{Past}_{\mathcal{U}}(p, P)$. If this transition belongs to a competing set, there exist $q' \in P'$ and a transition

$$\Big[(q', P') \xrightarrow{a} (p, P)\Big]_{\mathcal{S}}$$

which still appears in $\mathcal{U}$, and by induction, since $\mathsf{Past}_{\mathcal{S}}(q, P') = \mathsf{Past}_{\mathcal{S}}(q', P')$, $w \in \mathsf{Past}_{\mathcal{U}}(q', P')$, so $wa \in \mathsf{Past}_{\mathcal{U}}(p, P)$.

ii) We prove this by induction on the length of words. If there are several final states in a column of $\mathcal{S}$, exactly one remains in $\mathcal{U}$, so there is at most one state whose future contains the empty word. Now let (p, P) and (p', P') be two states in the same column such that the word au belongs to $\mathsf{Fut}_{\mathcal{A}}(p)$ and $\mathsf{Fut}_{\mathcal{A}}(p')$:

$$\Big[p \xrightarrow{a} q \xrightarrow{u} t \rightarrow\Big]_{\mathcal{A}}$$

$$\Big[p' \xrightarrow{a} q' \xrightarrow{u} t' \rightarrow\Big]_{\mathcal{A}}$$

Both transitions $p \xrightarrow{a} q$ and $p' \xrightarrow{a} q'$ correspond to the same transition in $\mathcal{D}$. Thus q and q' belong to the same column and, by induction, $q = q'$. Since there is no competing set in $\mathcal{U}$, $p = p'$.
Obviously $\mathsf{Fut}_{\mathcal{U}}(p, P) \subseteq \mathsf{Fut}_{\mathcal{S}}(p, P)$. If au is in the future of a state (p_0, P_0) of $\mathcal{S}$, there exist a state (p_1, P_1) and a transition $(p_0, P_0) \xrightarrow{a} (p_1, P_1)$, such that u is in $\mathsf{Fut}_{\mathcal{S}}(p_1, P_1)$. By induction, there exists p'_1 in P_1 such that u is in $\mathsf{Fut}_{\mathcal{U}}(p'_1, P_1)$, and there exists a transition $(p'_0, P_0) \xrightarrow{a} (p'_1, P_1)$, thus au is in $\mathsf{Fut}_{\mathcal{U}}(p'_0, P_0)$.

Let w be a word accepted by $\mathcal{A}$. For any factorization uv of w, there is exactly one column P of $\mathcal{U}$ such that, for every p in P, u is in $\mathsf{Past}_{\mathcal{U}}(p, P)$ and there is exactly one state (p, P) in this column such that v is in $\mathsf{Fut}_{\mathcal{U}}(p, P)$. This characterizes the only successful path with label w in $\mathcal{U}$. $\square$

We show now how the Schützenberger covering can be used to convert a finitely ambiguous automaton $\mathcal{A}$ into a finite union of unambiguous automata, each of them recognizing the same language as $\mathcal{A}$.

Proposition 4 *Let $\mathcal{S}$ be the Schützenberger covering of a finitely ambiguous automaton. Then, competing transitions of $\mathcal{S}$ do not belong to any circuit of $\mathcal{S}$. Thus a path of $\mathcal{S}$ contains at most one transition of each competing set.*

proof. Assume that a competing transition τ belongs to a circuit:

$$\rightarrow i \xrightarrow{u} (p, P) \xrightarrow[\tau]{a} (q, Q) \xrightarrow{w} (p, P) \xrightarrow[\tau]{a} (q, Q) \xrightarrow{v} t \rightarrow .$$

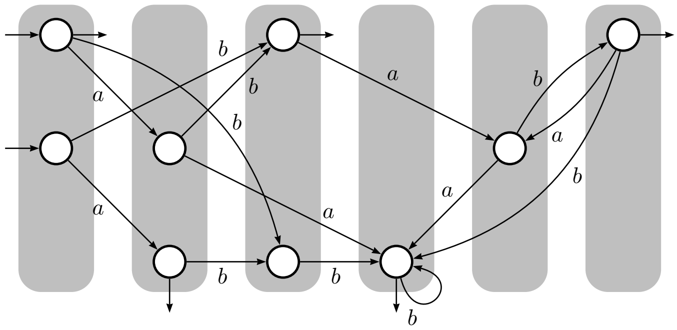


Figure 11: An unambiguous automaton equivalent to $\mathcal{S}$

Hence, $u(aw)^*$ is a subset of $\mathsf{Past}_{\mathcal{A}}(p)$. Let τ' be another transition that belongs to the same competing set: $(p', P) \xrightarrow[\tau']{a} (q, Q)$. From Lemma 1, $u(aw)^*$ is a subset of $\mathsf{Past}_{\mathcal{A}}(p')$. Thus, for every n, for every k in $\{0, \ldots, n\}$, there exists a path:

$$\to i \xrightarrow{u(aw)^k} (p', P) \xrightarrow[\tau']{a} \Big[(q, Q) \xrightarrow{w} (p, P) \xrightarrow[\tau]{a} (q, Q)\Big]^{n-k} \xrightarrow{v} t \to .$$

Therefore, there are at least $n + 1$ successful paths with label $u(wa)^n v$ in $\mathcal{S}$, which is in contradiction with the finite ambiguity of $\mathcal{S}$ and $\mathcal{A}$.

If there exists a path of $\mathcal{S}$ that contains two competing transitions τ and τ':

$$(p, P) \xrightarrow[\tau]{a} (q, Q) \xrightarrow{w} (p', P) \xrightarrow[\tau']{a} (q, Q),$$

then τ' belongs to a circuit, which is impossible. $\square$

Assume that $\mathcal{A}$ is finitely ambiguous. As a consequence of Proposition 4, for every path in $\mathcal{S}$ (and thus for every path in $\mathcal{A}$), one can compute an unambiguous automaton $\mathcal{U}$ that contains this path. Consider the following algorithm.

As they do not belong to any circuit, competing sets of $\mathcal{S}$ are partially ordered.

- Compute C, the set of maximal competing sets of $\mathcal{S}$ (there is no path from any element of C to another competing set).
- Let $\mathcal{S}_1$ and $\mathcal{S}_2$ be two copies of $\mathcal{S}$. For every competing set X in C, let x be an element of X;
- if x is a transition, remove every transition of $X\setminus\{x\}$ in $\mathcal{S}_1$ and remove x in $\mathcal{S}_2$;
- if x is a final state, make every state of $X\setminus\{x\}$ in $\mathcal{S}_1$ non-final and make x in $\mathcal{S}_2$ non-final.
- Apply inductively this algorithm to $\mathcal{S}_1$ and $\mathcal{S}_2$.

The result is a finite set of unambiguous automata. Each of them recognizes the language of $\mathcal{A}$ and every path of $\mathcal{S}$ appears in at least one of these automata. Notice that the cardinality of this set may be larger than the degree of ambiguity of $\mathcal{A}$. Denote by $\mathcal{F}$ the automaton obtained by taking the union of the automata in this set.

Assume now that $\mathcal{A}$ is any automaton with multiplicities over an idempotent semiring. Since there is a canonical mapping from the transitions (resp. initial states, resp. final states) of the Schützenberger covering $\mathcal{S}$ onto the transitions (resp. initial states, resp. final states) of $\mathcal{A}$, one can decorate every transition (resp. initial state, resp. final state) of $\mathcal{S}$ with the corresponding multiplicity in $\mathcal{A}$. This decoration can be carried out in the same way on the automaton $\mathcal{F}$.

Obviously, since there is a one-to-one mapping between the successful paths of $\mathcal{A}$ and those of $\mathcal{S}$, the series realized by $\mathcal{S}$ is equal to the one realized by $\mathcal{A}$.

Furthermore, as every path of $\mathcal{S}$ appears in $\mathcal{F}$, the automaton $\mathcal{F}$ realizes the same series as $\mathcal{A}$. Notice that a path of $\mathcal{S}$ may appear several times in $\mathcal{F}$, with no consequence since the semiring is idempotent.

The construction of $\mathcal{F}$ could be modified in order to get a one-to-one relation between paths of $\mathcal{A}$ and paths of $\mathcal{F}$, but then the automata in the union would not have the same support, which would be less convenient in the sequel.

Corollary 1 *A finitely ambiguous max-plus automaton can be effectively turned into an equivalent finite union of unambiguous max-plus automata, all with the same support.*

5 The Decidability Result

In this section, we show that a series, realized by a finite union of unambiguous automata having the same support, is unambiguous if and only if a certain property denoted by (**P**) holds. Associated with Theorem 1 and Corollary 1, this enables to prove Theorem 2, stated at the end of the paper.

Consider a finite family of max-plus automata $(\mathcal{A}_i)_{i\in I}$ with respective dimensions $(Q_i)_{i\in I}$. Set $\mathcal{A}_i = (\alpha^{\llcorner i}, \mu^{\llcorner i}, \beta^{\llcorner i})$. The corresponding product automaton $\mathcal{P}$ is an automaton with multiplicities in the product semiring $\mathbb{R}_{\max}^I$, defined as follows.
Set $Q = \prod_{i\in I} Q_i$ and consider $A, B \in (\mathbb{R}_{\max}^I)^Q, M : \Sigma^* \to (\mathbb{R}_{\max}^I)^{Q\times Q}$ with

$$\forall \mathbf{p}, \mathbf{q} \in Q, \;\; A_{\mathbf{p}} = (\alpha^{\llcorner i}_{p_i})_{i\in I},$$

$$\forall a \in \Sigma, \;\; M(a)_{\mathbf{p},\mathbf{q}} = \begin{cases} (\mu^{\llcorner i}(a)_{p_i,q_i})_{i\in I} & \text{if } \forall i, \;\; \mu^{\llcorner i}(a)_{p_i,q_i} \neq \mathbb{0} \\ (\mathbb{0}, \dots, \mathbb{0}) & \text{otherwise} \end{cases}$$

$$B_{\mathbf{p}} = (\beta^{\llcorner i}_{p_i})_{i\in I} \,.$$

A state $\mathbf{q} \in Q$ is *initial* if $\forall i, \;\; (A_{\mathbf{q}})_i \neq \mathbb{0}$. A state $\mathbf{q} \in Q$ is *final* if $\forall i, \;\; (B_{\mathbf{q}})_i \neq \mathbb{0}$. The trim part of (A, M, B) with respect to the above definition of initial and final states is the *product automaton* $\mathcal{P}$.

Clearly, if the automata $(\mathcal{A}_i)_{i\in I}$ are unambiguous and all have the same support, then the product automaton $\mathcal{P}$ is also unambiguous and satisfies

$$\forall u \in \Sigma^*, \forall i \in I,\ \langle S(\mathcal{P}), u\rangle_i = \alpha^{\llcorner i}\mu^{\llcorner i}(u)\beta^{\llcorner i}$$
$$\Rightarrow \bigoplus_{i\in I}\langle S(\mathcal{P}), u\rangle_i = \langle \bigoplus_{i\in I} S(\mathcal{A}_i), u\rangle = \bigoplus_{i\in I}\alpha^{\llcorner i}\mu^{\llcorner i}(u)\beta^{\llcorner i}.$$

Definition 2 *Let θ be a simple circuit of $\mathcal{P}$, whose weight is $(x^{\llcorner i})_{i\in I}$. The set of* victorious coordinates of θ, *denoted by $Vict(\theta)$, is the set of coordinates on which the weight of θ is maximal,* i.e. $Vict(\theta) = \big\{i \in I \mid x^{\llcorner i} = \max_{j\in I}\{x^{\llcorner j}\}\big\}$.

This definition is extended in a natural way to a strongly connected subgraph C of $\mathcal{P}$: the set of *victorious coordinates of* C is the intersection of the sets of victorious coordinates of the simple circuits of C. We also extend the definition to a path π of $\mathcal{P}$: the set of *victorious coordinates of* π is the intersection of the sets of victorious coordinates of the strongly connected subgraphs of $\mathcal{P}$ crossed by π.

Let us define the 'dominance' property (**P**):

For each successful path π of the product automaton $\mathcal{P}$, the set of victorious coordinates of π is not empty.

Obviously, the number of simple circuits is finite. Hence (**P**) is a decidable property.

Let $(\mathcal{A}_i = (\alpha^{\llcorner i} \in \mathbb{R}_{\max}^{Q_i}, \mu^{\llcorner i} : \Sigma^* \to \mathbb{R}_{\max}^{Q_i\times Q_i}, \beta^{\llcorner i} \in \mathbb{R}_{\max}^{Q_i}))_{i\in I}$ be a finite family of unambiguous trim automata, all with the same support, and let $\mathcal{P}$ be the product automaton with set of states $Q \subseteq \Pi_{i\in I} Q_i$. We assume that $\mathcal{P}$ satisfies the dominance property (**P**).

Let $N = |Q|$ and $M = \max(\max_{i,a,p,q}\mu^{\llcorner i}(a)_{p,q}, \max_{i,p}\beta^{\llcorner i}_p) - \min(\min_{i,a,p,q}\mu^{\llcorner i}(a)_{p,q}, \min_{i,p}\beta^{\llcorner i}_p)$, where the minima are taken over non-$\mathbb{0}$ terms. In words, M is the difference between the largest and the smallest non-initial weights appearing in the automata.

We use the following notations as shortcuts. For $\mathbf{x} = (x^{\llcorner i})_{i\in I} \in \mathbb{R}_{\max}^I$, set $\check{\mathbf{x}} = \min_{i\in I}\{x^{\llcorner i} \mid x^{\llcorner i} \neq -\infty\}$ and $\underline{\mathbf{x}} = \mathbf{x} - (\check{\mathbf{x}}, \dots, \check{\mathbf{x}})$.

Set $I = \{1, \dots, n\}$. We now define an automaton $\mathcal{U}$ that is shown to be unambiguous and to realize the series $\bigoplus_{i\in I} S(\mathcal{A}_i)$.

The states of $\mathcal{U}$ belong to $\mathbb{R}_{\max}^n \times Q$.

Initial states. All the initial states are defined as follows. If $\mathbf{q} = (q^{\llcorner 1}, \dots, q^{\llcorner n})$ is a tuple such that $q^{\llcorner i}$ is an initial state of $\mathcal{A}_i$, and if we set $\boldsymbol{\alpha} = (\alpha^{\llcorner 1}_{q^{\llcorner 1}}, \dots, \alpha^{\llcorner n}_{q^{\llcorner n}})$, then $(\underline{\boldsymbol{\alpha}}, \mathbf{q})$ is an initial state of $\mathcal{U}$ and the weight of the ingoing arc is $\check{\boldsymbol{\alpha}}$.

States and transitions. If $(\mathbf{z}, \mathbf{p})$ is a state of $\mathcal{U}$, then for each transition in $\mathcal{P}$ of type: $\mathbf{p} \xrightarrow{a|\mathbf{x}} \mathbf{q}$ such that $x^{\llcorner i} \neq -\infty$ for all i, there is a transition in $\mathcal{U}$ leaving

$\mathbf{p}$, labelled by the letter a, and that we now describe. Set $\mathbf{t} = \mathbf{z} + \mathbf{x}$. Let V be the set of victorious coordinates of the maximal strongly connected subgraph of $\mathbf{q}$ in $\mathcal{P}$. Since $\mathcal{P}$ satisfies $(\mathbf{P})$, the set $V \cap \{t^{\llcorner k} \neq -\infty\}$ is non-empty. Let $j \in V$ be such that $t^{\llcorner j} = \min_{k \in V}\{t^{\llcorner k} \mid t^{\llcorner k} \neq -\infty\}$, and let $\mathbf{y} \in \mathbb{R}^n_{\max}$ be defined by:

$$\forall i, \quad y^{\llcorner i} = \begin{cases} -\infty & \text{if } t^{\llcorner i} < t^{\llcorner j} - NM, \\ t^{\llcorner i} & \text{otherwise.} \end{cases}$$

Now $(\underline{\mathbf{y}}, \mathbf{q})$ is a state of $\mathcal{U}$ and we have the following transition:

$$\Big[(\mathbf{z}, \mathbf{p}) \xrightarrow{a|\check{\mathbf{y}}} (\underline{\mathbf{y}}, \mathbf{q})\Big]_{\mathcal{U}} .$$

Final states. All the final states are defined as follows. If $(\mathbf{z}, \mathbf{q})$ is a state of $\mathcal{U}$, and if $q^{\llcorner i}$ is a final state of $\mathcal{A}_i$ for all i, then $(\mathbf{z}, \mathbf{q})$ is a final state of $\mathcal{U}$ and the weight of the outgoing arc is $\max_{i \in I}\{z^{\llcorner i} + \beta^{\llcorner i}_{q^{\llcorner i}}\}$.

Lemma 2 *The set of states of $\mathcal{U}$ is finite.*

proof. First, given a state $(\mathbf{z_1}, \mathbf{q})$ of $\mathcal{U}$, we show that there are finitely many states of the form $(\mathbf{z_2}, \mathbf{q})$ that can be reached from $(\mathbf{z_1}, \mathbf{q})$.

Observe that a path leading from $(\mathbf{z_1}, \mathbf{q})$ to $(\mathbf{z_2}, \mathbf{q})$ in $\mathcal{U}$ corresponds to a circuit leading from $\mathbf{q}$ to $\mathbf{q}$ in $\mathcal{P}$ that can be fully decomposed into simple circuits belonging to the strongly connected component of $\mathbf{q}$. Let V be the set of victorious coordinates of the strongly connected component of $\mathbf{q}$. By definition of victorious coordinates, for all $i \in V$ the value of $z_2^{\llcorner i} - z_1^{\llcorner i}$ is a constant, that we denote by x, and for all $i \notin V$ one has $z_2^{\llcorner i} \leqslant z_1^{\llcorner i} + x$.

Let $\mathcal{C}$ be the (finite) set of simple circuits of $\mathcal{P}$. For a circuit $\theta \in \mathcal{C}$, let the weight of the circuit in $\mathcal{P}$ be denoted by $(\mathit{weight}(\theta)^{\llcorner 1}, \ldots, \mathit{weight}(\theta)^{\llcorner n})$. Set also $\mathit{weight}(\theta) = \max_{i \leqslant n} \mathit{weight}(\theta)^{\llcorner i}$. Now define

$$\delta = \min_{\theta \in \mathcal{C}} \Big[\mathit{weight}(\theta) - \max_i\{\mathit{weight}(\theta)^{\llcorner i} \mid \mathit{weight}(\theta)^{\llcorner i} < \mathit{weight}(\theta)\}\Big] .$$

By definition, we have $\delta > 0$. By construction, for $i \notin V$, either $z_2^{\llcorner i} = z_1^{\llcorner i} + x$, or $z_2^{\llcorner i} \leqslant z_1^{\llcorner i} + x - \delta$. Furthermore, there is at least one index i and one index j such that $z_1^{\llcorner i} = 0$ and $z_2^{\llcorner j} = 0$. At last, for $j \notin V$, we have by construction $z_2^{\llcorner j} \geqslant \min_{i \in V} z_2^{\llcorner i} - NM$, or $z_2^{\llcorner j} = -\infty$. Alltogether, it shows that there are finitely many possible values for $\mathbf{z_2} = (z_2^{\llcorner 1}, \ldots, z_2^{\llcorner n})$.

Consequently, any acyclic path in $\mathcal{U}$ is of finite length. Since the number of initial states is finite, it follows easily from König Lemma that the number of states of $\mathcal{U}$ is finite. $\square$

Lemma 3 *The automaton $\mathcal{U}$ is unambiguous.*

proof. Define the surjective map

$$\Psi \ : \ \begin{array}{ccc} \mathcal{U} & \longrightarrow & \mathcal{P} \\ (\mathbf{z}, \mathbf{p}) & \longmapsto & \mathbf{p} \end{array} .$$

By construction of $\mathcal{U}$, the following properties hold.

$i)$ The map Ψ restricted to the initial states of $\mathcal{U}$ defines a bijection between the initial states of $\mathcal{U}$ and $\mathcal{P}$.

$ii)$ Consider $\left[\mathbf{p} \xrightarrow{a} \mathbf{q}\right]_{\mathcal{P}}$. Then $\forall(\mathbf{z},\mathbf{p}) \in \Psi^{-1}(\mathbf{p}), \exists!(\mathbf{z'},\mathbf{q}) \in \Psi^{-1}(\mathbf{q})$ such that $\left[(\mathbf{z},\mathbf{p}) \xrightarrow{a} (\mathbf{z'},\mathbf{q})\right]_{\mathcal{U}}$.

$iii)$ A state $(\mathbf{z},\mathbf{q})$ is a final state of $\mathcal{U}$ if and only if $\mathbf{q}$ is a final state of $\mathcal{P}$.

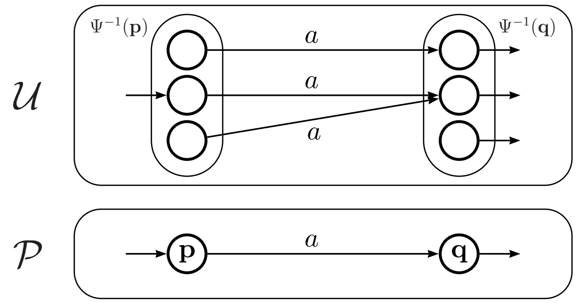


Figure 12: The properties of the map Ψ.

These three properties together imply that there is a bijection between successful paths in $\mathcal{P}$ and successful paths in $\mathcal{U}$. As $\mathcal{P}$ is unambiguous, so is $\mathcal{U}$. $\square$

Lemma 4 *The automaton $\mathcal{U}$ recognizes the series $\bigoplus_{i\in I} S(\mathcal{A}_i)$.*

proof. Let ℓ be an integer and $u = a_0a_1\cdots a_{\ell-1}$ be a word in the common support of the series $S(\mathcal{A}_i)$.

By Lemma 3, there exists exactly one successful path labelled by u in the automaton $\mathcal{U}$:

$$\pi = \left[\rightarrow (\mathbf{z_0},\mathbf{q_0}) \xrightarrow{a_0} (\mathbf{z_1},\mathbf{q_1}) \xrightarrow{a_1} \cdots \xrightarrow{a_{\ell-2}} (\mathbf{z_{\ell-1}},\mathbf{q_{\ell-1}}) \xrightarrow{a_{\ell-1}} (\mathbf{z_\ell},\mathbf{q_\ell}) \rightarrow\right]_{\mathcal{U}}$$

• Fix $i \in \{1,\ldots,n\}$. Assume that $z_\ell^{\llcorner i} = -\infty$.
Then i is not a victorious coordinate of π. Let j be a victorious coordinate, we show that $\langle S(\mathcal{A}_i),u\rangle < \langle S(\mathcal{A}_j),u\rangle$. Hence the coefficient of u in $\bigoplus_{i\in I} S(\mathcal{A}_i)$ is not realized by the coordinate i, which means that there is no damage in having $z_\ell^{\llcorner i} = -\infty$.

In the path π, there exists a minimal state q_h such that the coordinate $z_h^{\llcorner i}$ is equal to $-\infty$. That means that the difference between $z_h^{\llcorner i}$ and $z_h^{\llcorner j}$ would have been larger than NM. Let π' in $\mathcal{P}$ be the path that corresponds to π (by the proof of Lemma 3, there is a canonical bijection between successful paths of $\mathcal{U}$ and $\mathcal{P}$) and let q'_h be the state of π' that corresponds to q_h. Let π'_h be the end of

π' from q'_h onwards (including the final arrow). Let us prove that the difference of weights on π'_h between the coordinates i and j is smaller than NM, that is:

$$\mathit{weight}\,(\pi'_h)^{\llcorner i} - \mathit{weight}\,(\pi'_h)^{\llcorner j} \leqslant NM. \tag{7}$$

Actually, on every circuit, the weight with respect to i is smaller than or equal to the weight with respect to j (which is victorious), and, if we delete all the circuits in π'_h, we obtain an acyclic path that is necessarily shorter than $N-1$. On every transition, the difference between the weights of the coordinates i and j is at most M. Likewise, the difference between terminal functions is smaller than M. Hence we proved (7). It means that the weight of coordinate i cannot catch up with the one of coordinate j. In particular, we have: $\langle S(\mathcal{A}_i), u\rangle < \langle S(\mathcal{A}_j), u\rangle \leqslant \langle \bigoplus_{i\in I} S(\mathcal{A}_i), u\rangle$.

• Assume that $z_\ell^{\llcorner i} \neq -\infty$. Set $\boldsymbol{\alpha} = (\alpha^{\llcorner i}_{q_0^{\llcorner i}})_{i\in I}$ and $\boldsymbol{\beta} = (\beta^{\llcorner i}_{q_\ell^{\llcorner i}})_{i\in I}$. Let π' be the path in $\mathcal{P}$ that corresponds to π:

$$\pi' = \Big[\xrightarrow{\boldsymbol{\alpha}} \mathbf{q_0} \xrightarrow{a_0|\mathbf{x_0}} q_1 \xrightarrow{a_1|\mathbf{x_1}} \cdots \xrightarrow{a_{\ell-1}|\mathbf{x_{\ell-1}}} \mathbf{q}_\ell \xrightarrow{\boldsymbol{\beta}}\Big]_{\mathcal{P}}.$$

We have, by construction of the automaton $\mathcal{U}$:

$$\begin{aligned}\langle S(\mathcal{A}_i), u\rangle =& \alpha^{\llcorner i}_{q_0^{\llcorner i}} + \sum_{k=0}^{\ell-1} x_k^{\llcorner i} + \beta^{\llcorner i}_{q_\ell^{\llcorner i}}\\ =& \check{\boldsymbol{\alpha}} + z_0^{\llcorner i} + \sum_{k=0}^{\ell-1}(y_k + z_{k+1}^{\llcorner i} - z_k^{\llcorner i}) + \beta^{\llcorner i}_{q_\ell^{\llcorner i}}\\ =& \check{\boldsymbol{\alpha}} + \sum_{k=0}^{\ell-1} y_k + z_l^{\llcorner i} + \beta^{\llcorner i}_{q_\ell^{\llcorner i}}\end{aligned}$$

Therefore, $\langle S(\mathcal{A}_i), u\rangle = \langle \bigoplus_{j\in I} S(\mathcal{A}_j), u\rangle$ if and only if $z_\ell^{\llcorner i} + \beta^{\llcorner i}_{q_\ell^{\llcorner i}} = \max_j[z_\ell^{\llcorner j} + \beta^{\llcorner j}_{q_\ell^{\llcorner j}}]$. Now observe that by construction,

$$\langle \mathcal{U}, u\rangle = \check{\boldsymbol{\alpha}} + \sum_{k=0}^{\ell-1} y_k + \max_i [z_\ell^{\llcorner i} + \beta^{\llcorner i}_{q_\ell^{\llcorner i}}].$$

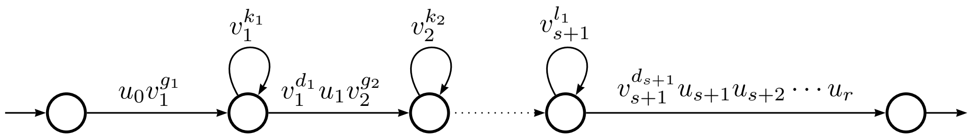
The equality $\langle \bigoplus_{i\in I} S(\mathcal{A}_i), u\rangle = \langle \mathcal{U}, u\rangle$ follows easily. □

We now have all the ingredients to prove the proposition below.

Proposition 5 *Consider a finite family $(\mathcal{A}_i)_{i\in I}$ of trim and unambiguous max-plus automata having the same support. Let $\mathcal{P}$ be the corresponding product automaton. The series $\bigoplus_{i\in I} S(\mathcal{A}_i)$ is unambiguous if and only if $\mathcal{P}$ satisfies the property (**P**). In this case, the automaton $\mathcal{U}$ defined above is finite, unambiguous, and realizes the series $\bigoplus_{i\in I} S(\mathcal{A}_i)$.*

proof. Lemmas 2, 3 and 4 show that (**P**) is a sufficient condition for $\bigoplus_{i\in I} S(\mathcal{A}_i)$ to be unambiguous. Let us prove that (**P**) is also a necessary condition.

By way of contradiction, assume that $S = \bigoplus_{i\in I} S(\mathcal{A}_i)$ is recognized by an unambiguous automaton $\mathcal{U}$ and that (**P**) does not hold. There exists a path π of $\mathcal{P}$ that can be decomposed into $\pi_0, \theta_1, \pi_1, \theta_2, \ldots, \pi_r$, where every θ_i is a circuit and $\bigcap_i \mathit{Vict}(\theta_i) = \varnothing$. Let u_i be the label of π_i and v_i the label of θ_i. Let s be the maximal integer such that $V = \bigcap_{i\leqslant s} \mathit{Vict}(\theta_i) \neq \varnothing$. Let $w_{k,l} = u_0 v_1^k u_1 \cdots v_s^k u_s v_{s+1}^l u_{s+1} u_{s+2} \cdots u_r$. For every k, l, $w_{k,l}$ is accepted by $\mathcal{P}$ and thus by $\mathcal{U}$ (with an unique successful path). Let k_0, l_0 be greater than the number of states d of $\mathcal{U}$. By the pigeon-hole principle, every path in $\mathcal{U}$ labelled by $v_i^{k_0}$ (for $i \in \{1, \ldots, s\}$) has a sub-circuit labelled by $v_i^{k_i}$ (with $k_i < d$). Likewise, the path labelled by $v_{s+1}^{l_0}$ has a sub-circuit labelled by $v_{s+1}^{l_1}$. It means that there exist $(g_i, k_i, d_i)_{i\in[1,s]}$ and (g_{s+1}, l_1, d_{s+1}) such that the successful path labelled by w_{k_0,l_0} in $\mathcal{U}$ has the following shape:



Let $K = \prod_{i\leqslant s} k_i$. Since $\mathcal{U}$ is unambiguous, for every pair of integers (α, β), the word $w_{k_0+\alpha K, l_0+\beta l_1}$ is accepted by a path that has the same shape; hence, there exist $x = \langle S, w_{k_0,l_0}\rangle$, ρ and λ such that, for every $(\alpha, \beta) \in \mathbb{N} \times \mathbb{N}$, $\langle S, w_{k_0+\alpha K, l_0+\beta l_1}\rangle = x + \alpha\rho + \beta\lambda$.

The word $w_{k_0+\alpha K, l_0+\beta l_1}$ labels in $\mathcal{P}$ a successful path that is the concatenation of π_0, $(k_0+\alpha K)$ times θ_0, π_1,…,π_s, $(l_0+\beta l_1)$ times θ_{s+1},…. Therefore, for every β, there exists N_β such that, for every $\alpha > N_\beta$, the successful coordinates of the path labelled by $w_{k_0+\alpha K, l_0+\beta l_1}$ belong to V and the weight is equal to $y + \alpha\rho_1 + \beta\lambda_1$, where y is a constant, ρ_1 is the sum of the maximal weights of the circuits θ_1 to θ_s, and $\lambda_1 = \max_{i\in V} \mathit{weight}(\theta_{s+i})^{\llcorner i}$.

Likewise, for every α, there exists M_α such that, for every $\beta > M_\alpha$, the successful coordinate of the path labelled by $w_{k_0+\alpha K, l_0+\beta l_1}$ is a victorious coordinate of θ_{s+1} and the weight of this path is equal to $z + \alpha\rho_2 + \beta\lambda_2$, where z is a constant, ρ_2 is the maximum over the victorious coordinates of θ_{s+1} of the sums of the weights of the circuits θ_1 to θ_s, and λ_2 is the maximal weight of θ_{s+1}.

To summerize, the following equalities hold:

$$\begin{aligned} \forall\alpha, \beta,\ \langle S, w_{k_0+\alpha K, l_0+\beta l_1}\rangle &= x + \alpha\rho + \beta\lambda \\ \forall\beta, \forall\alpha > N_\beta,\ \langle S, w_{k_0+\alpha K, l_0+\beta l_1}\rangle &= y + \alpha\rho_1 + \beta\lambda_1 \\ \forall\alpha, \forall\beta > M_\alpha,\ \langle S, w_{k_0+\alpha K, l_0+\beta l_1}\rangle &= z + \alpha\rho_2 + \beta\lambda_2 \end{aligned}$$

Therefore, $\rho_1 = \rho = \rho_2$ and $\lambda_1 = \lambda = \lambda_2$. Thus, there exists a coordinate that belongs to V and that is victorious on θ_{s+1}; this contradicts the maximality of s.

It would be possible to use an argument similar to the one in §3.6, to prove the above. □

The main result is now a corollary of Proposition 5:

Theorem 2 *One can decide in an effective way, whether the series recognized by a finitely ambiguous max-plus automaton is unambiguous, and whether it is sequential.*

More precisely, turn first the finitely ambiguous automaton into an equivalent finite union of unambiguous automata, all having the same support (Corollary 1). Then check the property (**P**) on the new family of automata. If (**P**) is satisfied the series is unambiguous; build the unambiguous automaton $\mathcal{U}$ (Proposition 5), then decide the sequentiality of $\mathcal{U}$ (Theorem 1).

References

[1] C. Allauzen and M. Mohri. Efficient algorithms for testing the twins property. *Journal of Automata, Languages and Combinatorics*, 8(2):117–144, 2003.

[2] M.-P. Béal, O. Carton, C. Prieur, and J. Sakarovitch. Squaring transducers: An efficient procedure for deciding functionality and sequentiality. *Theor. Comput. Sci.*, 292:45–63, 2003.

[3] J. Berstel. *Transductions and context-free languages*. B. G. Teubner, 1979.

[4] J. Berstel and C. Reutenauer. *Rational Series and their Languages*. Springer Verlag, 1988.

[5] M. Brilman and J.M. Vincent. Dynamics of synchronized parallel systems. *Stochastic Models*, 13(3):605–619, 1997.

[6] A.L. Buchsbaum, R. Giancarlo, and J.R. Westbrook. On the determinization of weighted finite automata. *SIAM J. Comput.*, 30(5):1502–1531, 2000.

[7] C. Choffrut. Une caractérisation des fonctions séquentielles et des fonctions sous-séquentielles en tant que relations rationnelles. *Theor. Comput. Sci.*, 5:325–337, 1977.

[8] C. Choffrut. *Contribution à l'étude de quelques familles remarquables de fonctions rationnelles*. Thèse d'état, Univ. Paris VII, 1978.

[9] S. Eilenberg. *Automata, languages and machines*, vol. A. Academic Press, 1974.

[10] S. Gaubert. Performance evaluation of (max,+) automata. *IEEE Trans. Aut. Cont.*, 40(12):2014–2025, 1995.

[11] S. Gaubert and J. Mairesse. Task resource models and (max,+) automata. In J. Gunawardena, editor, *Idempotency*, volume 11, pages 133–144. Cambridge University Press, 1998.

[12] S. Gaubert and J. Mairesse. Modeling and analysis of timed Petri nets using heaps of pieces. *IEEE Trans. Aut. Cont.*, 44(4):683–698, 1999.

[13] K. Hashigushi. Algorithms for determining relative star height and star height. *Inf. Comput.*, 78(2):124–169, 1988.

[14] K. Hashigushi, K. Ishiguro, and S. Jimbo. Decidability of the equivalence problem for finitely ambiguous finance automata. *Int. J. Algebra Comput.*, 12(3):445–461, 2002.

[15] D. Krob. The equality problem for rational series with multiplicities in the tropical semiring is undecidable. *Int. J. Algebra Comput.*, 4(3):405–425, 1994.

[16] D. Krob and A. Bonnier-Rigny. A complete system of identities for one-letter rational expressions with multiplicities in the tropical semiring. *Theor. Comput. Sci.*, 134:27–50, 1994.

[17] W. Kuich and A. Salomaa. *Semirings, Automata, Languages*, volume 5 of *EATCS*. Springer-Verlag, 1986.

[18] M. Mohri. Finite-state transducers in language and speech processing. *Comput. Linguist.*, 23(2):269–311, 1997.

[19] P. Moller. *Théorie algébrique des systèmes à événements discrets*. PhD thesis, École des Mines de Paris, 1988.

[20] J. Sakarovitch. A construction in finite automata that has remained hidden. *Theor. Comput. Sci.*, 204:205–231, 1998.

[21] M.-P. Schützenberger. On the definition of a family of automata. *Information and Control*, 4(2–3):245–270, 1961.

[22] M.-P. Schützenberger. Sur les relations rationnelles entre monoïdes libres. *Theor. Comput. Sci.*, 3:243–259, 1976.

[23] I. Simon. Recognizable sets with multiplicities in the tropical semiring. In *Mathematical Foundations of Computer Science, Proc. 13th Symp.*, number 324 in LNCS, pages 107–120, 1988.

[24] G.X. Viennot. Heaps of pieces, I: Basic definitions and combinatorial lemmas. In Labelle and Leroux, editors, *Combinatoire Énumérative*, number 1234 in Lect. Notes in Math., pages 321–350. Springer, 1986.

[25] A. Weber. Finite-valued distance automata. *Theor. Comput. Sci.*, 134:225–251, 1994.

[26] A. Weber and R. Klemm. Economy of description for single-valued transducers. *Information and Computation*, 118(2):327–340, 1995.

[27] A. Weber and H. Seidl. On the degree of ambiguity of finite automata. *Theor. Comput. Sci.*, 88(2):325–349, 1991.