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To cite this version:
Alain Haraux, Mitsuharu Otani. Analyticity and regularity for a class of second order evolution equations. 2007. <hal-00174022>

HAL Id: hal-00174022
https://hal.archives-ouvertes.fr/hal-00174022
Submitted on 21 Sep 2007

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ANALYTICITY AND REGULARITY FOR A CLASS OF SECOND ORDER EVOLUTION EQUATIONS

Alain HARAUX
Laboratoire Jacques-Louis Lions, U.M.R C.N.R.S. 7598, Université Pierre et Marie Curie, Boite courrier 187, 75252 Paris Cedex 05, France
haraux@ann.jussieu.fr

Mitsuharu Ōtani
Department of Applied physics, Waseda University, 3-4-1 Okubo, Shinjuku-ku, Tokyo, Japan.
otani@waseda.jp

Résumé. On étudie la conservation de la régularité et l’effet régularisant pour l’équation $u'' + Au + cA^\alpha u' = 0$ où $A$ est un opérateur auto-adjoint positif sur un espace de Hilbert réel $H$ et $\alpha \in (0, 1]$; $c > 0$. Si $\alpha \geq \frac{1}{2}$ l’équation engendre un semi-groupe analytique sur $D(A^{1/2}) \times H$, et si $\alpha \in (0, \frac{1}{2})$ une propriété de régularisation plus faible mais optimale est prouvée. Enfin des propriétés de conservation de la régularité associée à d’autres normes sont obtenues, avec comme exemple typique d’application l’équation des ondes $u_{tt} - \Delta u - c\Delta u_t = 0$ avec condition de Dirichlet homogène dans un domaine borné pour laquelle la régularité $C_0(\Omega) \times C_0(\Omega)$ est conservée pour $t > 0$, un saisissant contraste avec le cas conservatif $u_{tt} - \Delta u = 0$ dans lequel la régularité $C_0(\Omega)$ peut-être perdue même pour un état initial $(u_0, 0)$ avec $u_0 \in C_0(\Omega) \cap C^1(\Omega)$.

Abstract. The regularity conservation as well as the smoothing effect are studied for the equation $u'' + Au + cA^\alpha u' = 0$ where $A$ is a positive selfadjoint operator on a real Hilbert space $H$ and $\alpha \in (0, 1]$; $c > 0$. When $\alpha \geq \frac{1}{2}$ the equation generates an analytic semigroup on $D(A^{1/2}) \times H$, and if $\alpha \in (0, \frac{1}{2})$ a weaker optimal smoothing property is established. Some conservation properties in other norms are established, a typical example being the strongly dissipative wave equation $u_{tt} - \Delta u - c\Delta u_t = 0$ with Dirichlet boundary conditions in a bounded domain for which the space $C_0(\Omega) \times C_0(\Omega)$ is conserved for $t > 0$, in sharp contrast with the conservative case $u_{tt} - \Delta u = 0$ for which $C_0(\Omega)$-regularity can be lost even starting from an initial state $(u_0, 0)$ with $u_0 \in C_0(\Omega) \cap C^1(\Omega)$.

Keywords: Regularity, analytic semi-group, smoothing effect

AMS classification numbers: 35B65, 35D10, 35L05, 35L90
1. INTRODUCTION AND NOTATION

This paper is mainly devoted to a detailed study of the regularity conservation as well as the smoothing effect for the equation

\[(1.1) \quad u'' + Au + cA^\alpha u' = 0\]

where \(A\) is a positive selfadjoint operator on a real Hilbert space \(H\) and \(\alpha \geq 0, \ c > 0\). Throughout the text we assume \(A \geq \eta I, \ \eta > 0\).

In particular the operator \(A^s\) is well defined and bounded \(H \to H\) for all \(s \leq 0\). We identify \(H\) with its topological dual and we therefore have

\[D(A) \subset V = D(A^{1/2}) \subset H = H' \subset V' \subset D(A)^'.\]

More generally we define a monotone nonincreasing one-parameter family of Hilbert spaces by the formula

\[H^s = \begin{cases} 
D(A^{s/2}) & \text{if } s \geq 0 \\
(D(A^{-s/2}))' & \text{if } s < 0.
\end{cases}\]

We are interested in the smoothing effect and the conservation of regularity for the evolution equation (1.1). The plan of the paper is as follows: Section 2 is devoted to well-posedness of (1.1) considered as a first order system in \(V \times H\). Section 3 deals with compactness properties of the resolvent and the semi-group associated to (1.1). Sections 4 and 5 are devoted to the study of time and spatial smoothing properties and especially to a simple direct proof of analyticity for \(\alpha \geq 1/2\). A more general result, motivated by a conjecture of Chen & Russell [2] can be found in [3] but their proof is quite involved, relying on a stationary type of argument involving complex resolvent. Here we give a pure real and dynamical argument based only on elementary tools such as inner products and integration with respect to \(t\). In Section 6 we investigate some regularity conservation properties which are specific to (1.1) with \(c > 0\) since the conservative problem corresponding to \(c = 0\) does not satisfy those properties anymore, cf. eg. [5]. Finally Section 5 contains the basic examples of application.

2. THE INITIAL VALUE PROBLEM

In this section we consider the equation

\[(2.1) \quad u'' + Au + Bu' = 0\]

where \(B = B^* \geq 0\) on \(H\) and in addition for some constants \(C > c > 0\)

\[cA^\alpha \leq B \leq CA^\alpha.\]

**Theorem 2.1.** Let \((u^0, u^1) \in V \times H\) be given. For any \(T > 0\) there is a unique \(u \in C([0, T], V) \cap C^1([0, T], H)\) such that, setting \(\beta = \max\{1, \alpha\}\) we have

\[
\begin{cases} 
  u' \in L^2(0, T; H^\alpha) \quad & u'' \in L^2(0, T; H^{-\beta}) \\
  u'' = -Au - Bu' \quad & \text{in } L^2(0, T; H^{-\beta}) \\
  u(0) = u^0 \quad & u'(0) = u^1.
\end{cases}
\]
Moreover the function
\[ E(t) := \frac{1}{2} \{ |u(t)|_V^2 + |u'(t)|_H^2 \} \]
is absolutely continuous with
\[ E'(t) = -< Bu', u' >_H - \alpha, H^\alpha \]
almost everywhere on \((0, T)\).

Proof. For the existence part we introduce \( J_\lambda = (I + \lambda A)^{-1} \) and \( K_\lambda = J_\lambda^n \) for some integer \( n \geq \max\{1, \alpha\} \). Then we solve
\[ u'' + A u + K_\lambda B K_\lambda u' = 0; \quad u_\lambda(0) = u^0, \quad u'_\lambda(0) = u^1. \]
The identity
\[ \int_0^T (BK_\lambda u', K_\lambda u') dt + \frac{1}{2} \{ ||u'_\lambda(T)||_V^2 + ||u_\lambda(T)||_V^2 \} = \frac{1}{2} \{ ||u_0||_H^2 + ||u_0||_V^2 \} \]
allows to pass to the limit as \( \lambda \to 0 \) along a suitable subsequence.

For the uniqueness part as well as the energy identity we start with a solution \( u \in L^2(0, T; V) \cap H^1(0, T; H) \) of
\[ u'' + A u = f \in L^2(0, T; H^{-\beta}) \]
We show that \( v_\lambda = K_\lambda u \) satisfies
\[ \frac{d}{dt} \frac{1}{2} \{ ||u'_\lambda(t)||_V^2 + ||v_\lambda(t)||^2 \} = (K_\lambda f, K_\lambda u') = < f, K_\lambda^2 u' > \]
Then we integrate and let \( \lambda \to 0 \). Finally we choose \( f = -Bu' \). Uniqueness follows then by linearity from the energy identity applied with \( u^0 = u^1 = 0 \).

3. Compactness of the resolvent

In this section we assume \( B = c A^\alpha, \ c > 0 \). Setting \( v = u' \), \( U = (u, v) \in V \times H = \mathcal{H} \), and denoting by \( \tilde{B} \) the unique extension to \( \mathcal{L}(V, H^{1-2\alpha}) \) of \( B \in \mathcal{L}(D(B), H) \), we find that the equation (2.1) becomes
\[ U'' + A U = 0 \]
where
\[ D(A) = \{(u, v) \in V \times V, \ Au + \tilde{B} v \in H \} \]
and
\[ A(u, v) = (-v, Au + \tilde{B} v) \]
so that
\[ ||A(u, v)||_H \sim ||v||_V + ||Au + \tilde{B} v||_H \]

**Theorem 3.1.** a) If \( \alpha < 1 \), then \( D(A) \subset H^\gamma \times V \) with \( \gamma = \min\{2, 3 - 2\alpha\} \).
As a consequence if the imbedding \( V \to H \) is compact, then so is \((I + A)^{-1} : H \to \mathcal{H}) .

b) If \( \alpha \geq 1 \), then we have
\[ \forall z \in V, \ U(z) =: (cz, -A^{1-\alpha} z) \in D(A) \]
with
\[ ||U(z)||_H + ||AU(z)||_H \leq K ||z||_V. \]
In particular if \( \dim H = \infty \), \((I + A)^{-1} : \mathcal{H} \to \mathcal{H}\) is not compact. Hence in this case the semi-group \( S(t) \) generated on \( \mathcal{H} \) by \( A \) is never compact.

Proof. a) If \( U \in D(A) \), then \( v \in V \) and \( Au + \tilde{B}v \in H \), hence
\[
u \in D(A) + A^{-1}(\tilde{B}V) = H^2 + A^{\alpha-1}(H^1) = H^\gamma.
\]
If \( V \to H \) is compact and \( \alpha < 1 \), then \( H^\gamma \to V \) is compact, therefore since
\[
(I + A)^{-1} \in L(H_2, H^\gamma \times \mathcal{V})
\]
the result is now obvious.

b) Assume now \( \alpha \geq 1 \). Then clearly \( \forall z \in V \), \( U(z) = (cz, -A^{1-\alpha}z) \in D(A) \) because both \( cz \) and \( -A^{1-\alpha}z \) are in \( V \) and in addition
\[
A(cz) + cA^\alpha(-A^{1-\alpha}z) = 0 \in H.
\]
Moreover we have
\[
\|U(z)\|_H \leq \|cz\|_V + \|A^{1-\alpha}z\|_H \leq c\|z\|_V + C'\|z\|_V \leq C''\|z\|_V.
\]
and
\[
\|AU(z)\|_H \leq \|A^{1-\alpha}z\|_V + \|A(cz) + cA^\alpha(-A^{1-\alpha}z)\|_H = \|A^{1-\alpha}z\|_V \leq C''\|z\|_V.
\]
Hence
\[
\|U(z)\|_H + \|AU(z)\|_H \leq K\|z\|_V
\]
with \( K := C'' + C''' \). Finally when \( z \in V \) varies in the unit ball
\[
\{\|z\|_V \leq 1\}
\]
the first projection of \( U(z) \) covers the entire ball of radius \( c \) in \( V \), therefore \((I + A)^{-1} : \mathcal{H} \to \mathcal{H}\) is not compact \( \square \)

4. Analytic Type Time Smoothing Effect.

In this section we give a new and short proof of a result previously obtained by Chen & Triggiani [3]. However our proof seems to be limited to the case \( B = cA^\alpha \), \( c > 0 \) or at least to require that \( B \) commutes with \( A \).

**Theorem 4.1.** For any \( \alpha \geq 1/2 \), the semi-group \( S(t) \) generated on \( \mathcal{H} \) by \( A \) is analytic, more precisely
\[
\forall t > 0, \quad S(t)U^0 \in D(A).
\]
and
\[
(4.1) \quad \forall t > 0, \quad \|AS(t)U^0\|_H \leq \frac{C}{t} \|U^0\|_H.
\]

Proof. We set
\[
E := \frac{1}{2} (|u'(0)|^2 + |A^{1/2}u(0)|^2) = \frac{1}{2} \|U^0\|^2_H.
\]
Multiplying the equation by \( u' \) we have immediately
\[
(4.2) \quad \forall t \geq 0, \quad \int_0^t (Bu', u') ds \leq E.
\]
In particular, with \( C = 1/c \)
\[
(4.3) \quad \forall t \geq 0, \quad \int_0^t |A^{1/2}u'(s)|^2 ds \leq CE.
\]
Taking the inner product in \(H\) of (2.1) by \(A^{1-\alpha}u(s)\) and integrating, we find
\[
\int_0^t |A^{1-\frac{\alpha}{2}}u(s)|^2 ds = - \int_0^t (u'' + Bu', A^{1-\alpha}u) ds.
\]
Now we have
\[
\int_0^t (Bu', A^{1-\alpha}u) ds = c \int_0^t (A^\alpha u', A^{1-\alpha}u) ds = c \int_0^t (Au, u') ds
\]
\[
= \frac{c}{2} (|A^{1/2}u(t)|^2 - |A^{1/2}u(0)|^2) \geq -cE.
\]
Next, integrating by parts we find
\[
\int_0^t (u''(s), A^{1-\alpha}u) ds = [u'(s), A^{1-\alpha}u]_0^t - \int_0^t (u', A^{1-\alpha}u') ds.
\]
Since \(\alpha \geq 1/2\), we have \(1 - \alpha \leq 1/2\) and \(1 - \alpha \leq \alpha\), hence
\[- \int_0^t (u'', A^{1-\alpha}u) ds \leq C_1 E\]
therefore
\[(4. 4) \quad \int_0^t |A^{1-\frac{\alpha}{2}}u(s)|^2 ds \leq (c + C_1)E = C_2 E
\]
and hence
\[
\int_0^t (|A^{1-\frac{\alpha}{2}}u(s)|^2 + |A^{\frac{1}{2}}u'(s)|^2) ds \leq C_3 E.
\]
Since \(1 - \alpha \leq \frac{\alpha}{2}\), we deduce
\[(4. 5) \quad \int_0^t (|A^{1-\frac{\alpha}{2}}u(s)|^2 + |A^{\frac{1}{2}}u'(s)|^2) ds \leq C_4 E.
\]
We introduce now
\[w(t) = A^{1-\frac{\alpha}{2}}u(t)\]
The basic estimate (4.2) applied to \(w\) instead of \(u\) gives
\[
\int_s^t |A^{\frac{1}{2}}u'(\sigma)|^2 d\sigma = \int_s^t |A^{\frac{1}{2}}u'(\sigma)|^2 d\sigma \leq C_5 (|A^{1-\frac{\alpha}{2}}u(s)|^2 + |A^{\frac{1}{2}}u'(s)|^2).
\]
By integrating on \((0, t)\) and using (4.5) we obtain
\[
\int_0^t \int_s^t |A^{\frac{1}{2}}u'(\sigma)|^2 d\sigma ds \leq C_5 C_4 E
\]
which by Fubini’s theorem reduces to
\[
\int_0^t \sigma |A^{\frac{1}{2}}u'(\sigma)|^2 d\sigma \leq C_5 C_4 E.
\]
Hence for some \(K > 0\)
\[(4. 6) \quad \forall t > 0, \quad \int_0^t s|A^{\frac{1}{2}}u'(s)|^2 ds \leq KE.
\]
We shall in fact establish for some \(M > 0\)
\[(4. 7) \quad \forall t > 0, \quad \int_0^t s(|A^{\frac{1}{2}}u'(s)|^2 + |u''(s)|^2) ds \leq ME.
\]
To this end, first we observe that as a consequence of (4.6)

\[ \inf_{t \leq s \leq 2t} s^2 |A^{\frac{1}{2}} u'(s)|^2 \leq 2t \inf_{t \leq s \leq 2t} s |A^{\frac{1}{2}} u'(s)|^2 \leq 2KE. \]

Now we choose \( \tau \in (t, 2t) \) for which \( u'(\tau) \in D(A^{1/2}) \) with

\[ \tau^2 |A^{\frac{1}{2}} u'(\tau)|^2 \leq 2KE \]

and we integrate on \((0, \tau)\) after taking the inner product by \( su''(s) \). We find

\[ \int_0^\tau s |u''|^2 ds = - \int_0^\tau [s(Au'', u'') + s(Bu', u'')] ds. \]

First we have

\[- \int_0^\tau s(Bu', u'') ds = - \frac{1}{2} \int_0^\tau s(Bu', u') ds = - \frac{1}{2} (Bu', u')_0 + \int_0^\tau \frac{1}{2} (Bu', u') ds \leq \frac{1}{2} E. \]

Then

\[- \int_0^\tau s(Au', u'') ds = - \int_0^\tau (sAu, (u')') ds = - (sAu, u')_0 + \int_0^\tau ((sAu)', u') ds \]

\[ \leq \tau |A^{\frac{1}{2}} u(\tau)| |A^{\frac{1}{2}} u'(\tau)| + \int_0^\tau (Au, u') ds + \int_0^\tau s |A^{\frac{1}{2}} u'|^2 ds \]

\[ \leq \tau |A^{\frac{1}{2}} u(\tau)| |A^{\frac{1}{2}} u'(\tau)| + \frac{1}{2} |A^{\frac{1}{2}} u(\tau)|^2 + \int_0^\tau s |A^{\frac{1}{2}} u'|^2 ds \leq K'E. \]

By adding we find

\[ \int_0^\tau s |u''|^2 ds \leq K''E \]

and together with (4.6) this provides (4.7) with \( t \) replaced by \( \tau \). Since \( t \leq \tau \) we obtain (4.7). Finally since the function

\[ t \to (|A^{\frac{1}{2}} u'(t)|^2 + |u''(t)|^2) \]

is nonincreasing, we have

(4. 8) \( \forall t > 0, \ 2 \int_0^t s(|A^{\frac{1}{2}} u'(s)|^2 + |u''(s)|^2) ds \geq t^2 (|A^{\frac{1}{2}} u'(t)|^2 + |u''(t)|^2). \)

By combining (4.7) and (4.8), we obtain (4.1) \( \square \)

**Theorem 4.2.** For any \( \alpha < 1/2 \), the semi-group \( S(t) \) generated on \( \mathcal{H} \) by \( A \) satisfies

\[ \forall t > 0, \quad S(t)u^0 \in D(A) \]

and

(4. 9) \( \forall t > 0, \quad \|AS(t)u^0\|_\mathcal{H} \leq \frac{C}{t^{\beta}} \|u^0\|_\mathcal{H} \)

with \( \beta = \frac{1}{2\alpha} \). In addition if \( A \) is unbounded with \( A^{-1} \) compact, (4.9) is not satisfied for any \( \beta > \frac{1}{2\alpha} \). In particular in this case the semi-group \( S(t) \) is not analytic.
Proof. The beginning of proof of Theorem 4.1 applies until formula (4.3). Then taking the inner product in $H$ of (2.1) by $A^\alpha u(s)$ and integrating we find
\[
\int_0^t |A^{1 + \frac{\alpha}{2}} u(s)|^2 ds = \int_0^t (u'' + Bu, A^\alpha u) ds.
\]
Now we have since $\alpha \leq 1/2$
\[
\int_0^t (Bu, A^\alpha u) ds = c \int_0^t (A^\alpha u, A^\alpha u) ds = \frac{c}{2}(|A^{\alpha/2} u(t)|^2 - |A^{\alpha/2} u(0)|^2) \geq -C_1 E.
\]
Next, integrating by parts we find
\[
\int_0^t (u'', A^\alpha u) ds = [(u', A^\alpha u)]_0^t - \int_0^t (u', A^\alpha u') ds
\]
hence since $\alpha \leq 1/2$ and by using (4.3)
\[
-\int_0^t (u'', A^\alpha u) ds \leq C_2 E
\]
therefore
\[
(4.10) \quad \int_0^t |A^{1 + \frac{\alpha}{2}} u(s)|^2 ds \leq (C_1 + C_2) = C_3 E.
\]
Combining (4.10) with (4.3) we deduce
\[
(4.11) \quad \int_0^t (|A^{\frac{1}{2} + \frac{\alpha}{2}} u(s)|^2 + |A^{\frac{\alpha}{2}} u'(s)|^2) ds \leq C_4 E
\]
from which we deduce
\[
(4.12) \quad |A^{\frac{1}{2} + \frac{\alpha}{2}} u(t)| + |A^{\frac{\alpha}{2}} u'(t)| \leq \frac{C}{\sqrt{t}}(|A^{\frac{1}{2}} u(0)| + |A^0 u'(0)|).
\]
Since time-translation and multiplication by $A^\alpha$ commutes with the equation, replacing $t$ by $\frac{x}{2}$ and iterating (4.12) $n$ times we deduce easily
\[
(4.13) \quad |A^{\frac{1}{2} + \frac{n\alpha}{2}} u(t)| + |A^{\frac{n\alpha}{2}} u'(t)| \leq \frac{C(n)}{(\sqrt{t})^n}(|A^{\frac{1}{2}} u(0)| + |A^0 u'(0)|).
\]
Now (4.13) is valid for integer values of $n$ and by interpolation, we extend it easily for all real $n > 0$. Finally choosing $n = \frac{1}{\alpha}$ we obtain (4.9).

In order to prove the optimality result, we set $p = \frac{1}{2\alpha}$, we select $\gamma > 0$ and for any $\lambda > 0$ with $\gamma \lambda^p - \lambda^2 > 0$ we set
\[
\omega := \sqrt{\gamma \lambda^p - \lambda^2}; \quad y(t) = y_\lambda(t) = e^{-\lambda t} \cos \omega t.
\]
Then
\[
y'(t) = -\lambda e^{-\lambda t} \cos \omega t - \omega e^{-\lambda t} \sin \omega t
\]
\[
y''(t) = \lambda^2 e^{-\lambda t} \cos \omega t + 2\omega \lambda e^{-\lambda t} \sin \omega t - \omega^2 e^{-\lambda t} \cos \omega t
\]
and
\[
y''(t) + 2\lambda y'(t) = -(\lambda^2 + \omega^2) e^{-\lambda t} \cos \omega t
\]
so that $y$ is a solution of
\[
(4.14) \quad y'' + \gamma \lambda^p y + 2\lambda y' = 0
\]
which satisfies
\[
\lambda^p y^2(0) + y'^2(0) = \lambda^p + \lambda^2.
\]
We are interested in the behavior of the energy for large values of \( \lambda \) and small values of \( t \) when \( \gamma \geq \gamma_0 > 0 \). We select
\[
t := t(\lambda) = \pi \left[ \frac{\omega}{\lambda} \right] + 1/2
\]
where \( \left[ \frac{\omega}{\lambda} \right] \) denotes the integer part of \( \frac{\omega}{\lambda} \). As \( \lambda \to \infty \) we have
\[
t(\lambda) \sim \frac{\pi}{\lambda}
\]
and it follows
\[
\lambda^p y''^2(t(\lambda)) \sim \lambda^p \omega^2 e^{-2\pi}
\]
\[
t^p(\lambda) \lambda y''^2(t(\lambda)) \sim \pi^p \omega^2 e^{-2\pi} \sim \pi^p e^{-2\pi} \gamma \lambda^p
\]
In particular for any \( \varepsilon > 0 \) we have
\[
\lim_{\lambda \to \infty} (t(\lambda))^{p-\varepsilon} \frac{\lambda^p y''^2(t(\lambda)) + y''^2(t(\lambda))}{\lambda^p y^2(0) + y^2(0)} = \infty
\]
uniformly for \( \gamma \geq \gamma_0 > 0 \).

Finally let \( \mu \) be a large eigenvalue of \( A \) with associated eigenfunction \( \phi_\mu \) and set
\[
\gamma = \mu (c/2)^{-p}; \quad \lambda = \left( \frac{\mu}{\gamma} \right)^{1/p} \iff \gamma \lambda^p = \mu.
\]
Since \( y = y(\lambda) \) is a solution of
\[
y'' + \mu y + 2\gamma^{-1/p} \mu^{1/p} y' = y'' + \mu y + c \mu^\alpha y' = 0.
\]
It is now easy to see that \( u(t) := y(t) \phi_\mu \) is a solution of (2.1) which does not satisfy (4.4) for any \( \beta > \frac{1}{2\alpha} \). \( \square \)

Theorems 4.1 and 4.2 imply a stronger time-smoothing effect property. More precisely we have

**Theorem 4.3.** For any \( \alpha > 0 \), and for any solution \( u \) of
\[
u'' + Au + c A^\alpha u' = 0
\]
we have
\[
\forall \delta > 0, u \in C^\infty([\delta, \infty), V))
\]
In addition the operator
\[
(u(0), u'(0)) \in V \times H \to u^{(k)} \in L^\infty([\delta, \infty), V)
\]
is bounded for each fixed value of \( k \).

5. **Existence of a spatial smoothing effect**

Combining the result of Theorem 4.3 and the inclusion \( D(A) \subset H^\gamma \times V \) obtained in Theorem 3.1 for \( 0 < \alpha < 1 \), by an easy induction argument we obtain

**Theorem 5.1.** Assuming \( \alpha < 1 \), for any solution \( u \) of
\[
u'' + Au + c A^\alpha u' = 0
\]
we have
\[
\forall n \in N, \quad \forall \delta > 0, u \in C^\infty([\delta, \infty), D(A^n))
\]
In addition the operator
\[(u(0), u'(0)) \in V \times H \rightarrow u^{(k)} \in L^\infty([\delta, \infty), D(A^n))\]
is bounded for each fixed value of \(k\) and \(n\).

Remark. If on the other hand \(\alpha \geq 1\) there is no spatial smoothing effect anymore. For instance if we consider the special case
\[
\Omega = (0, \pi) \quad H = L^2(\Omega) \quad V = H^1_0(\Omega) \quad A = -\Delta, \quad B = 2A
\]
we have special solutions of the form
\[
u(t, x) = \sum a_ne^{(-n^2+\sqrt{n^2(n^2-1)})t}\sin nx
\]
with
\[u(0) = \sum a_n \sin nx\]
and
\[
\|u(0)\|_{D(A^s)}^2 = \sum n^{4s}a_n^2
\]
\[
\|u(t)\|_{D(A^s)}^2 = \sum n^{4s}a_n^2e^{(-n^2+\sqrt{n^2(n^2-1)})^2t}.
\]
Since
\[-n^2 + \sqrt{n^2(n^2-1)} = \frac{-n^2}{n^2 + \sqrt{n^2(n^2-1)}} \geq -1
\]
we find
\[
\|u(0)\|_{D(A^s)}^2 \leq \|u(t)\|_{D(A^s)}^2e^{2t}
\]
This formal estimate can be easily worked out to show that if \(u(t) \in D(A^s)\) for some \(t > 0\), we must have \(u(0) \in D(A^s)\). Hence there is no spatial smoothing effect whatsoever in such a situation.

6. Regularity: Conservation properties

Let \(L = L^s \geq 0\) on \(H\) and assume that there is a second Banach space \(X\) such that
\[
\bigcap_{n \geq 1} D(L^n) \subset X \subset H
\]
with dense imbeddings. The norm in \(X\) is denoted by \(\|\cdot\|\) and we assume that \(\exp(-tL)\) is a \(C^0\) semigroup of bounded operators on \(X\).

First we consider the problem
\[
u'' + L^2u + cLu' = 0
\]
Then we have

**Theorem 6.1.** Let \(c \geq 2\), and let \(L\) be coercive on \(X\). Introducing
\[D_X(L) = \{x \in D(L), Lx \in X\}\]
assume that the set \(\{x \in D_X(L), \|Lx\| \leq 1\}\) is closed in \(X\) and that \(\exp(-tL)\) is an analytic semigroup of bounded operators on \(X\). Let \(u\) be the unique solution of (6.2) satisfying
\[u(0) = u^0 \in D_X(L), \quad u'(0) = u^1 \in X\]
whose existence is insured by Theorem 2.1 with \( A = L^2 \) and \( B = cL \). Then we have
\[
\forall t > 0, \quad u(t) \in D_X(L), \quad u'(t) \in X
\]
with
\[
(6.3) \quad \sup_{t > 0} \{ \| u(t) \| + \| Lu(t) \| + \| u'(t) \| \} \leq C(\| u^0 \| + \| Lu^0 \| + \| u^1 \|)
\]

Proof. We start with
\[
(u^0, u^1) \in \left( \bigcap_{n \geq 1} D(L^n) \right)^2
\]
The main idea is to look for \( \alpha, \beta > 0 \) such that
\[
u'' + L^2 u + cLu' = (u' + \alpha Lu)' + \beta L(u' + \alpha Lu).
\]
This identity reduces to the system
\[
\alpha + \beta = c; \quad \alpha \beta = 1
\]
which has real solutions \((\alpha, \beta)\) as a consequence of the assumption \( c \geq 2 \).
Then since \( v = u' + \alpha Lu \) is a solution of
\[
v' + \beta Lv = 0
\]
we have (cf. e.g. [6]) with \( K = \| u^1 \| + \alpha \| Lu^0 \| \)
\[
\forall t > 0, \quad \| v(t) \|_{D_X(L^{1/2})} \leq C_1 t^{-1/2} \| u^1 + \alpha Lu^0 \| \leq C_1 t^{-1/2} K.
\]
Then we have
\[
u(t) = e^{-\alpha L_t} u^0 + \int_0^t e^{-\alpha L(t-s)} v(s) ds
\]
hence by [6]
\[
\forall t > 0, \quad \| u(t) \|_{D_X(L)} \leq C_2 \| u^0 \|_{D_X(L)} + C_3 \int_0^t C_1 (t-s)^{-1/2} s^{-1/2} K ds.
\]
This provides the bound on \( \| u(t) \| + \| Lu(t) \| \). Finally since
\[
\forall t > 0, \quad \| v(t) \| \leq K
\]
and
\[
u'(t) = v(t) - \alpha Lu(t)
\]
the estimate on \( \| u'(t) \| \) follows easily. Then the general case
\[
(u^0, u^1) \in D_X(L) \times X
\]
follows by a density argument.

We next consider the problem
\[
(6.4) \quad u'' + Lu + cLu' = 0.
\]
We assume that \( \exp(-tL) \) is a \( C^0 \) semigroup of bounded operators on \( X \). Then we have
Theorem 6.2. Under the condition (6.1), let \( u \) be the unique solution of (6.4) satisfying
\[
\begin{align*}
    u(0) &= u^0 \in X, \\
    u'(0) &= u^1 \in X
\end{align*}
\]
whose existence is insured by Theorem 2.1 with \( A = L \) and \( B = cL \). Then we have
\[
\forall t > 0, \quad u(t) \in X, \quad u'(t) \in X
\]
If in addition we assume
\[
\| \exp(-tL) \|_{\mathcal{L}(X)} \leq Me^{-\lambda t}
\]
for some \( M > 1, \lambda > 0 \) such that
\[
(6.6) \quad c^2 \lambda > M + 1
\]
then \((u, u')\) is bounded for \( t \geq 0 \) and
\[
(6.7) \quad \sup_{t>0} \{ \| u(t) \| + \| u'(t) \| \} \leq C(c, M, \lambda)(\| u^0 \| + \| u^1 \|)
\]
Proof. As before we consider first
\[
(u^0, u^1) \in \bigcap_{n \geq 1} D(L^n)^2
\]
The main idea is to introduce
\[
v := u' + \frac{1}{c} u
\]
so that
\[
v' + cLv = u'' + \frac{1}{c} u' + cLu' + Lu = \frac{1}{c} u' = \frac{1}{c} (v - \frac{1}{c} u)
\]
Let \( J \) be any closed subinterval of \([0, +\infty)\). By setting
\[
\| u \|_{J, \infty} := \sup_{t \in J} \| u(t) \|
\]
and similarly
\[
\| v \|_{J, \infty} := \sup_{t \in J} \| v(t) \|
\]
from
\[
v' + cL v - \frac{1}{c} v = -\frac{1}{c^2} u
\]
we obtain
\[
\forall t \in J, \quad \| v(t) \| \leq M \exp\left(\frac{1}{c} t\right) \| v^0 \| + \frac{M}{c^2} \exp\left(\frac{1}{c} t\right) |J| \| u \|_{J, \infty} \leq C(\| v^0 \| + |J| \| u \|_{J, \infty})
\]
for \( |J| \leq 1 \) and
\[
\forall t \in J, \quad \| u(t) \| \leq \| u^0 \| + c |J| \| v \|_{J, \infty}
\]
The result easily follows locally by selecting \( |J| \) small enough, for instance \( Cc|J| \leq \frac{1}{2} \) and using a density argument. Then a simple induction argument concludes the proof, since the condition on \( |J| \) is independent of the initial data and the equation is autonomous.
When (6.5)-(6.6) are assumed, the same calculation for an arbitrary $J$ now gives
\[\forall t \in J, \quad \|v(t)\| \leq M \exp[-(\lambda c - \frac{1}{c})t] \|v_0\| + \frac{M}{\lambda c^3 - c} \|u\|_{\infty}\]
and
\[\forall t \in J, \quad \|u(t)\| \leq \|u^0\| + c\|v\|_{\infty}\]
Hence if $\lambda c^2 > 1$ we find
\[\|v\|_{\infty} \leq M\|v_0\| + \frac{M}{\lambda c^3 - c} \|u\|_{\infty} \leq M\|v_0\| + \frac{M}{\lambda c^3 - c} (\|u^0\| + c\|v\|_{\infty})\]
which gives the desired bound for $v$ if
\[\frac{M}{\lambda c^3 - 1} < 1\]
a condition equivalent to (6.6). Then the bound on $u$ follows automatically. This bound is valid on each interval $J = [0, T]$ and the result (6.7) follows. The general case is obtained by density.

7. Main examples

Example 7.1. Let $\Omega$ be a bounded open subset of $\mathbb{R}^N$. We consider the problem

\[
\begin{cases}
u_{tt} - \Delta u - c\Delta u_t = 0 & \text{on } \mathbb{R}^+ \times \Omega \\
u = \Delta u = 0 & \text{on } \mathbb{R}^+ \times \partial\Omega \\
u(0) = u^0 \quad u'(0) = u^1
\end{cases}
\]

We set
\[H = L^2(\Omega) ; \quad V = H^1_0(\Omega)\]
\[X = C_0(\Omega) = \{u \in C(\overline{\Omega}), \quad u = 0 \text{ on } \partial\Omega\}\]

For any $(u^0, u^1) \in V \times H$, the unique mild solution $u$ of (7.1) belongs to $C^\infty((0, \infty), V)$. If in addition we assume $(u^0, u^1) \in X \times X$, then for all $t \geq 0$ \((u(t), u_t(t)) \in X \times X\). For $c$ large enough we have the stronger property $u \in W^{1,\infty}((0, \infty), X)$. The same result is also valid if $X$ is replaced by $X_p = L^p(\Omega)$ for any $p \in [2, \infty)$.

Example 7.2. Let $\Omega$ be a bounded open subset of $\mathbb{R}^N$. We consider the problem

\[
\begin{cases}
u_{tt} + \Delta^2 u - c\Delta u_t = 0 & \text{on } \mathbb{R}^+ \times \Omega \\
u = \Delta u = 0 & \text{on } \mathbb{R}^+ \times \partial\Omega \\
u(0) = u^0 \quad u'(0) = u^1
\end{cases}
\]

For any $(u^0, u^1) \in X \times X$ we set
\[H = L^2(\Omega) ; \quad V = \{u \in H^1_0(\Omega), \Delta u \in H\}\]
\[X = C_0(\Omega) = \{u \in C(\overline{\Omega}), \quad u = 0 \text{ on } \partial\Omega\}\]

For any $(u^0, u^1) \in V \times H$, the unique mild solution $u$ of (7.1) belongs to $C^\infty((0, \infty), V) \cap C^\infty(0, \infty) \times \Omega)$. If in addition we assume $c \geq 2$ and $(u^0, u^1) \in X \times X$, then $u \in L^\infty((0, \infty), X)$. Finally if $c \geq 2$, $(u^0, u^1) \in X \times X$
\[ \Delta^2 u^0 \in X, \text{ then } \Delta^2 u \in L^\infty((0, \infty), X); \quad u_t \in L^\infty((0, \infty), X). \] The same result is also valid if \( X \) is replaced by \( X_p = L^p(\Omega) \) for any \( p \in [2, \infty) \).

**Acknowledgement.** The main part of this work was done during a visit of the first author at Department of Applied Physics of Waseda University in March 2003.

**References**


