On the p-adic Leopoldt Transform of a power series
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Let \( p \) be an odd prime number. Let \( X \) be the projective limit for the norm maps of the \( p \)-Sylow subgroups of the ideal class groups of \( \mathbb{Q}(\zeta_p^{n+1}) \), \( n \geq 0 \). Let \( \Delta = \text{Gal}(\mathbb{Q}(\zeta_p)/\mathbb{Q}) \) and let \( \theta \) be an even and non-trivial character of \( \Delta \). Then \( X \) is a \( \mathbb{Z}_p[[T]] \)-module and the characteristic ideal of the isotypic component \( X(\omega\theta^{-1}) \) is generated by a power series \( f(T, \theta) \in \mathbb{Z}_p[[T]] \) such that (see for example [2]):

\[
\forall n \geq 1, \ n \equiv 0 \pmod{p-1}, \ f((1 + p)^{1-n} - 1, \theta) = L(1-n, \theta),
\]

where \( L(s, \theta) \) is the usual Dirichlet \( L \)-series. Therefore, it is natural and interesting to study the properties of the power series \( f(T, \theta) \).

We denote by \( \overline{f(T, \theta)} \in \mathbb{F}_p[[T]] \) the reduction of \( f(T, \theta) \) modulo \( p \). Then B. Ferrero and L. Washington have proved (3):

\[
\overline{f(T, \theta)} \neq 0.
\]

Note that, in fact, we have (4):

\[
\overline{f(T, \theta)} \not\in \mathbb{F}_p[[T^p]].
\]

W. Sinnott has proved the following (8):

\[
\overline{f(T, \theta)} \not\in \mathbb{F}_p(T).
\]

But, note that \( \forall a \in \mathbb{Z}_p^*, \mathbb{F}_p[[T]] = \mathbb{F}_p[((1 + T)^a - 1)] \). Therefore it is natural to introduce the notion of a pseudo-polynomial which is an element \( F(T) \) in \( \mathbb{F}_p[[T]] \) such that there exist an integer \( r \geq 1, c_1, \ldots, c_r \in \mathbb{F}_p, a_1, \ldots, a_r \in \mathbb{Z}_p \), such that \( F(T) = \sum_{i=1}^r c_i(1 + T)^{a_i} \). An element of \( \mathbb{F}_p[[T]] \) will be called a pseudo-rational function if it is the quotient of two pseudo-polynomials. In this paper, we prove that \( \overline{f(T, \theta)} \) is not a pseudo-rational function (part 1 of Theorem 1.3). This latter result suggests the following question: is \( \overline{f(T, \theta)} \) algebraic over \( \mathbb{F}_p(T) \)? We suspect that this is not the case but we

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have no evidence for it. Note that, by the result of Ferrero and Washington, we can write:
\[ f(T, \theta) = T^{\lambda(\theta)} U(T), \]
where \( \lambda(\theta) \in \mathbb{N} \) and \( U(T) \in \mathbb{F}_p[[T]]^* \). S. Rosenberg has proved that (\[6\]):
\[ \lambda(\theta) \leq (4p(p - 1))^{\phi(p-1)}, \]
where \( \phi \) is Euler’s totient function. In this paper, we improve Rosenberg’s bound (part 2) of Theorem 4.5):
\[ \lambda(\theta) < \left(\frac{p - 1}{2}\right)^{\phi(p-1)}. \]
This implies that the lambda invariant of the field \( \mathbb{Q}(\zeta_p) \) is less than \( 2(\frac{p - 1}{2})^{\phi(p-1)+1} \)
(see Corollary \[10\] for the precise statement for an abelian number field).

Note that this bound is certainly far from the truth, because according to a heuristic argument due to Ferrero and Washington (see \[3\]) and to Grennberg’s conjecture:
\[ \lambda(\mathbb{Q}(\zeta_p)) = \sum_{\theta \in \hat{\Delta}, \theta \neq 1 \text{ and even}} \lambda(\theta) \leq \frac{\text{Log}(p)}{\text{Log(Log}(p))}. \]

The author is indebted to Warren Sinnott for communicating some of his unpublished works (note that Lemma \[4.2\] is due to Warren Sinnott). The author also thanks Filippo Nuccio for pointing out the work of J. Kraft and L. Washington (\[4\]).

1 Notations

Let \( p \) be an odd prime number and let \( K \) be a finite extension of \( \mathbb{Q}_p \). Let \( O_K \) be the valuation ring of \( K \) and let \( \pi \) be a prime of \( K \). We set \( \mathbb{F}_q = O_K/\pi O_K \), it is a finite field having \( q \) elements and its characteristic is \( p \). Let \( T \) be an indeterminate over \( K \), we set \( \Lambda = O_K[[T]] \). Observe that \( \Lambda/\pi \Lambda \simeq \mathbb{F}_q[[T]] \). Let \( F(T) \in \Lambda \setminus \{0\} \), then we can write in an unique way (\[1\], Theorem 7.3):
\[ F(T) = \pi^{\mu(F)} P(T) U(T), \]
where $U(T)$ in an unit of $\Lambda$, $\mu(F) \in \mathbb{N}$, $P(T) \in O_K[T]$ is a monic polynomial such that $P(T) \equiv \lambda(T)(\mod \pi)$ for some integer $\lambda(F) \in \mathbb{N}$. If $F(T) = 0$, we set $\mu(F) = \lambda(F) = \infty$. An element $F(T) \in \Lambda$ is called a pseudo-polynomial (see also [6], Definition 2) if there exist some integer $r \geq 1$, $c_1, \ldots, c_r \in O_K$, $a_1, \ldots, a_r \in \mathbb{Z}_p$, such that:

$$F(T) = \sum_{i=1}^{r} c_i (1 + T)^{a_i}.$$ 

We denote the ring of pseudo-polynomials in $\Lambda$ by $A$. Let $\delta \in \mathbb{Z}/(p - 1)\mathbb{Z}$ and $F(T) \in \Lambda$, we set:

$$\gamma_\delta(F(T)) = \frac{1}{p - 1} \sum_{\eta \in \mu_{p-1}} \eta^\delta F((1 + T)^\eta - 1).$$

Then $\gamma_\delta : \Lambda \to \Lambda$ is a $O_K$-linear map and:
- for $\delta, \delta' \in \mathbb{Z}/(p - 1)\mathbb{Z}$, $\gamma_\delta \gamma_{\delta'} = 0$ if $\delta \neq \delta'$ and $\gamma_\delta^2 = \gamma_\delta$,
- $\sum_{\delta \in \mathbb{Z}/(p - 1)\mathbb{Z}} \gamma_\delta = \text{Id}_\Lambda$.

For $F(T) \in \Lambda$, we set:

$$D(F(T)) = (1 + T) \frac{d}{dT} F(T),$$

$$U(F(T)) = F(T) - \frac{1}{p} \sum_{\zeta \in \mu_p} F(\zeta(1 + T) - 1) \in \Lambda.$$ 

Then $D, U : \Lambda \to \Lambda$ are $O_K$-linear maps. Observe that:
- $U^2 = U$,
- $DU = UD$,
- $\forall \delta \in \mathbb{Z}/(p - 1)\mathbb{Z}$, $\gamma_\delta U = U \gamma_\delta$,
- $\forall \delta \in \mathbb{Z}/(p - 1)\mathbb{Z}$, $D \gamma_\delta = \gamma_{\delta + 1} D$.

If $F(T) \in \Lambda$, we denote its reduction modulo $\pi$ by $\overline{F(T)} \in \mathbb{F}_q[[T]]$. If $f : \Lambda \to \Lambda$ is a $O_K$-linear map, we denote its reduction modulo $\pi$ by $\overline{f} : \mathbb{F}_q[[T]] \to \mathbb{F}_q[[T]]$. For all $n \geq 0$, we set $\omega_n(T) = (1 + T)^{p^n} - 1$.

Let $B$ be a commutative and unitary ring. We denote the set of invertible elements of $B$ by $B^*$. We fix $\kappa$ a topological generator of $1 + p\mathbb{Z}_p$. Let $x \in \mathbb{Z}_p$ and let $n \geq 1$, we denote the unique integer $k \in \{0, \ldots, p^n - 1\}$ such that $x \equiv k \pmod{p^n}$ by $[x]_n$. Let $\omega : \mathbb{Z}_p^* \to \mu_{p - 1}$ be the Teichmüller character, i.e. $\forall a \in \mathbb{Z}_p^*$, $\omega(a) \equiv a \pmod{p}$. Let $x, y \in \mathbb{Z}_p$, we write:
- $x \sim y$ if there exists $\eta \in \mu_{p-1}$ such that $y = \eta x$.
- $x \equiv y \pmod{\mathbb{Q}^*}$ if there exists $z \in \mathbb{Q}^*$ such that $y = zx$.

The function $\text{Log}_p$ will denote the usual $p$-adic logarithm. $v_p$ will denote the usual $p$-adic valuation on $\mathbb{C}_p$ such that $v_p(p) = 1$.

Let $\rho$ be a Dirichlet character of conductor $f_\rho$. Recall that the Bernoulli numbers $B_{n,\rho}$ are defined by the following identity:

$$\sum_{a=1}^{f_\rho} \rho(a)e^{aZ} - 1 = \sum_{n \geq 0} \frac{B_{n,\rho} n! Z^n}{n!},$$

where $e^Z = \sum_{n \geq 0} Z^n/n!$. If $\rho = 1$, for $n \geq 2$, $B_{n,1}$ is the $n$th Bernoulli number.

Let $x \in \mathbb{R}$. We denote the biggest integer less than or equal to $x$ by $\lfloor x \rfloor$.

The function $\text{Log}$ will denote the usual logarithm.

## 2 Preliminaries

Let $\delta \in \mathbb{Z}/(p - 1)\mathbb{Z}$. In this section, we will recall the construction of the $p$-adic Leopoldt transform $\Gamma_\delta$ (see [4], Theorem 6.2) which is an $O_K$-linear map from $\Lambda$ to $\Lambda$.

First, observe that $(\pi^n, \omega_n(T)) = \pi^n \Lambda + \omega_n(T)\Lambda$, $n \geq 1$, is a basis of neighbourhood of zero in $\Lambda$:

**Lemma 2.1**

1) $\forall n \geq 1$, $(\pi, T)^{2n} \subset (\pi^n, T^n) \subset (\pi, T)^n$.
2) $\forall n \geq 1$, $\omega_n(T) \in (p^{[n/2]}/, T^{[n/2]+1})$.
3) Let $N \geq 1$, set $n = \lfloor \log(N)/\log(p) \rfloor$. We have:

$$T^N \in (p^{[n/2]}, \omega_{[n/2]+1}(T)).$$

**Proof** Note that assertion 1) is obvious. Assertion 2) comes from the fact:

$$\forall k \in \{1, \cdots, p^n\}, v_p\left(\frac{p^n}{k!(p^n-k)!}\right) = n - v_p(k).$$

To prove assertion 3), it is enough to prove the following:
\(\forall n \geq 0\), there exist \(\delta_0^{(n)}(T), \ldots, \delta_n^{(n)}(T) \in \mathbb{Z}[T]\) such that:

\[
T^{p^n} = \sum_{i+j=n} \omega_i(T)p^j \delta_j^{(n)}(T).
\]

Let’s prove this latter fact by recurrence on \(n\). Note that the result is clear if \(n = 0\). Let’s assume that it is true for \(n\) and let’s prove the assertion for \(n + 1\). Let \(r(T) \in \mathbb{Z}[T]\) such that:

\[
\frac{\omega_{n+1}(T)}{\omega_n(T)} + pr(T) = T^{p^n(p-1)}.
\]

Then:

\[
T^{p^{n+1}} = T^{p^n} \frac{\omega_{n+1}(T)}{\omega_n(T)} + pr(T)T^{p^n}.
\]

Note that there exists \(q(T) \in \mathbb{Z}[T]\) such that:

\[
\frac{\omega_{n+1}(T)}{\omega_n(T)} = \omega_n(T)^{p-1} + pq(T).
\]

Thus:

\[
T^{p^{n+1}} = \omega_{n+1}(T)\delta_0^{(n)}(T) + \sum_{i+j=n, j \geq 1} (\omega_n(T)^{p-1} + pq(T))\omega_i(T)p^j \delta_j^{(n)}(T) + \sum_{i+j=n} \omega_i(T)p^j \delta_j^{(n)}(T)r(T).
\]

Thus, there exist \(\delta_0^{(n+1)}(T), \ldots, \delta_{n+1}^{(n+1)}(T) \in \mathbb{Z}[T]\) such that:

\[
T^{p^{n+1}} = \sum_{i+j=n+1} \omega_i(T)p^j \delta_j^{(n+1)}(T). \checkmark
\]

The following Lemma will be useful in the sequel (for a similar result see \[6\], Lemma 5):

**Lemma 2.2** Let \(F(T) \in A\). Write \(F(T) = \sum_{i=1}^r \beta_i(1 + T)^{\alpha_i}\), \(\beta_1, \ldots, \beta_r \in O_K, \alpha_1, \ldots, \alpha_r \in \mathbb{Z}_p\), and \(\alpha_i \neq \alpha_j\) for \(i \neq j\). Let \(N = \text{Max}\{v_p(\alpha_i - \alpha_j), i \neq j\}\). Let \(n \geq 1\) be an integer. Then:

\[
F(T) \equiv 0 \pmod{\omega_{N+1}(T))} \iff \forall i = 1, \cdots r, \beta_i \equiv 0 \pmod{\pi^n}.
\]
Proof We have:

\[ F(T) \equiv \sum_{i=1}^{r} \beta_i (1 + T)^{[\alpha_i]N+1} \pmod{\omega_{N+1}(T)}. \]

Therefore \( F(T) \equiv 0 \pmod{\omega_{N+1}(T)} \) if and only if we have:

\[ \sum_{i=1}^{r} \beta_i (1 + T)^{[\alpha_i]N+1} \equiv 0 \pmod{\pi^n}. \]

But for \( i \neq j \), \([\alpha_i]_{N+1} \neq [\alpha_j]_{N+1} \). Therefore \( \sum_{i=1}^{r} \beta_i (1 + T)^{[\alpha_i]N+1} \equiv 0 \pmod{\pi^n} \) if and only if:

\[ \forall i = 1, \ldots, r, \beta_i \equiv 0 \pmod{\pi^n}. \]

Observe that \( U, D, \gamma_\delta \) are continuous \( O_K \)-linear maps by Lemma 2.1 and the following Lemma:

Lemma 2.3 Let \( F(T) \in \Lambda \) and let \( n \geq 0 \).

1) \( F(T) \equiv 0 \pmod{\omega_n(T)} \Rightarrow \gamma_\delta(F(T)) \equiv 0 \pmod{\omega_n(T)} \).

2) \( F(T) \equiv 0 \pmod{\omega_n(T)} \Rightarrow D(F(T)) \equiv 0 \pmod{(p^n, \omega_n(T))} \).

3) If \( n \geq 1 \), \( F(T) \equiv 0 \pmod{\omega_n(T)} \Rightarrow U(F(T)) \equiv 0 \pmod{\omega_n(T)} \).

Proof The assertions 1) and 2) are obvious. It remains to prove 3). Observe that, by [7], Proposition 7.2, we have:

\[ \forall G(T) \in \Lambda, G(T) \equiv 0 \pmod{\omega_n(T)} \Leftrightarrow \forall \zeta \in \mu_{p^n}, G(\zeta - 1) = 0. \]

Now, let \( F(T) \in \Lambda, F(T) \equiv 0 \pmod{\omega_n(T)} \). Since the map: \( \mu_{p^n} \to \mu_{p^n}, x \mapsto \zeta x \), is a bijection for all \( \zeta \in \mu_{p^n} \), we get:

\[ \forall \zeta \in \mu_{p^n}, U(F)(\zeta - 1) = 0. \]

Therefore:

\[ U(F(T)) \equiv 0 \pmod{\omega_n(T)}. \]

Let \( s \in \mathbb{Z}_p \). For \( n \geq 0 \), set:

\[ k_n(s, \delta) = [s]_{n+1} + \delta_n p^{n+1} \in \mathbb{N} \setminus \{0\}, \]

where \( \delta_n \in \{1, \ldots, p-1\} \) is such that \([s]_{n+1} + \delta_n \equiv \delta \pmod{p-1}\). Observe that:
- \( \forall n \geq 0, k_n(s, \delta) \equiv \delta \pmod{p-1} \) and \( k_n(s, \delta) \equiv s \pmod{p^{n+1}} \),
- \( \forall n \geq 0, k_{n+1}(s, \delta) > k_n(s, \delta) \),
- \( s = \lim_n k_n(s, \delta) \).

In particular:

\[
\forall n \geq 0, k_n^1(s, \delta) \equiv \delta \pmod{p-1} \quad \text{and} \quad k_n^1(s, \delta) \equiv s \pmod{p^n+1}.
\]

Now, let \( F(T) \in A \). Write \( F(T) = \sum_{i=1}^r \beta_i (1 + T)^{\alpha_i}, \beta_1, \ldots, \beta_r \in O_K, \alpha_1, \ldots, \alpha_r \in \mathbb{Z}_p \). We set:

\[
\Gamma_{\delta}(F(T)) = \sum_{\alpha_i \in \mathbb{Z}_p^*} \beta_i \omega^\delta(\alpha_i) (1 + T)^{\log_{p}(\alpha_i)}.
\]

Thus, we have a surjective \( O_K \)-linear map: \( \Gamma_{\delta} : A \to A \). Note that:

**Lemma 2.4** Let \( F(T) \in A \).

1) Let \( s \in \mathbb{Z}_p \). Then:

\[
\forall n \geq 0, \Gamma_{\delta}(F)(\kappa^s - 1) \equiv D^{k_n(s, \delta)}(F)(0) \pmod{p^{n+1}}.
\]

2) Let \( n \geq 1 \). Assume that \( F(T) \equiv 0 \pmod{\omega_n(T)} \). Then \( \Gamma_{\delta}(F(T)) \equiv 0 \pmod{\omega_n-1(T)} \).

**Proof** For \( a \in \mathbb{Z}_p^* \), write \( a = \omega(a) < a >, \) where \( < a > \in 1 + p\mathbb{Z}_p \). Let’s write:

\[
F(T) = \sum_{i=1}^r \beta_i (1 + T)^{\alpha_i}, \beta_1, \ldots, \beta_r \in O_K, \alpha_1, \ldots, \alpha_r \in \mathbb{Z}_p \).
\]

We have:

\[
D^{k_n(s, \delta)}(F(T)) = \sum_{i=1}^r \beta_i k_n(s, \delta) (1 + T)^{\alpha_i}.
\]

Thus:

\[
D^{k_n(s, \delta)}(F(T)) \equiv \sum_{\alpha_i \in \mathbb{Z}_p^*} \beta_i \omega^\delta(\alpha_i) < \alpha_i >^s (1 + T)^{\alpha_i} \pmod{p^{n+1}}.
\]

But recall that:

\[
\Gamma_{\delta}(F)(\kappa^s - 1) = \sum_{\alpha_i \in \mathbb{Z}_p^*} \beta_i \omega^\delta(\alpha_i) < \alpha_i >^s.
\]
Assertion 1) follows easily. Now, let’s suppose that \( F(T) \equiv 0 \pmod{\omega_n(T)} \) for some \( n \geq 1 \). Then:

\[
\forall a \in \{0, \ldots, p^n - 1\}, \sum_{\alpha_i \equiv a \pmod{p^n}} \beta_i = 0.
\]

This implies that:

\[
\forall a \in \{0, \ldots, p^{n-1} - 1\}, \sum_{\alpha_i \in \mathbb{Z}_p, \log_p(\alpha_i)/\log_p(\kappa) \equiv a \pmod{p^{n-1}}} \omega^\delta(\alpha_i) \beta_i = 0.
\]

But recall that:

\[
\Gamma_\delta(F(T)) = \sum_{\alpha_i \in \mathbb{Z}_p^*} \beta_i \omega^\delta(\alpha_i)(1 + T)^{\frac{\log_p(\alpha_i)}{\log_p(\kappa)}}.
\]

Thus \( \Gamma_\delta(F(T)) \equiv 0 \pmod{\omega_{n-1}(T)} \).

**Proposition 2.5** Let \( F(T) \in \Lambda \). There exists an unique power series \( \Gamma_\delta(F(T)) \in \Lambda \) such that:

\[
\forall s \in \mathbb{Z}_p, \forall n \geq 0, \Gamma_\delta(F(\kappa^s - 1)) \equiv D^{k_n(s, \delta)}(F)(0) \pmod{p^{n+1}}.
\]

**Proof** Let \( (F_N(T))_{N \geq 0} \) be a sequence of elements in \( A \) such that:

\[
\forall N \geq 0, F(T) \equiv F_N(T) \pmod{\omega_N(T)}.
\]

Fix \( N \geq 1 \). Then:

\[
\forall m \geq N, F_m(T) \equiv F_N(T) \pmod{\omega_N(T)}.
\]

Therefore, by Lemma 2.4, we have:

\[
\forall m \geq N, \Gamma_\delta(F_m(T)) \equiv \Gamma_\delta(F_N(T)) \pmod{\omega_{N-1}(T)}.
\]

This implies that the sequence \( (\Gamma_\delta(F_N(T)))_{N \geq 1} \) converges in \( \Lambda \) to some power series \( G(T) \in \Lambda \). Observe that, since \( \Lambda \) is compact, we have:

\[
\forall N \geq 1, G(T) \equiv \Gamma_\delta(F_N(T)) \pmod{\omega_{N-1}(T)}.
\]

In particular:

\[
\forall N \geq 1, G(\kappa^s - 1) \equiv \Gamma_\delta(F_N)(\kappa^s - 1) \pmod{p^N}.
\]
Thus, applying Lemma 2.4, we get:

$$\forall N \geq 1, G(\kappa^s - 1) \equiv D^{k_{N-1}(s,\delta)}(F_N)(0) \pmod{p^N}.$$  

But:

$$\forall N \geq 1, D^{k_{N-1}(s,\delta)}(F(T)) \equiv D^{k_{N-1}(s,\delta)}(F_N(T)) \pmod{(p^N, \omega_N(T))}.$$  

Therefore:

$$\forall N \geq 1, G(\kappa^s - 1) \equiv D^{k_{N-1}(s,\delta)}(F)(0) \pmod{p^N}.$$  

Now, set $\Gamma_\delta(F(T)) = G(T)$. The Proposition follows easily. ♦

3 Some properties of the $p$-adic Leopoldt transform

We need the following fundamental result:

**Proposition 3.1** Let $\delta \in \mathbb{Z}/(p-1)\mathbb{Z}$ and let $F(T) \in \Lambda$. Let $m, n \in \mathbb{N}\setminus\{0\}$. then:

$$\Gamma_\delta(F(T)) \equiv 0 \pmod{(\pi^n, \omega_{m-1}(T))} \Leftrightarrow \gamma_\delta U(F(T)) \equiv 0 \pmod{(\pi^n, \omega_m(T))}.$$  

**Proof** A similar result has been obtained by S. Rosenberg ([3], Lemma 8). We begin by proving that $\Gamma_\delta$ is a continuous $O_K$-linear map. By Lemma 2.1 this comes from the following fact:

Let $F(T) \in \Lambda$. Let $n \geq 1$ and assume that $F(T) \equiv 0 \pmod{\omega_n(T)}$, then $\Gamma_\delta(F(T)) \equiv 0 \pmod{\omega_{n-1}(T)}$.

Indeed, let $(F_N(T))_{N \geq 0}$ be a sequence of elements in $A$ such that:

$$\forall N \geq 0, F(T) \equiv F_N(T) \pmod{\omega_N(T)}.$$  

By the proof of Proposition 2.3:

$$\forall N \geq 1, \Gamma_\delta(F(T)) \equiv \Gamma_\delta(F_N(T)) \pmod{\omega_{N-1}(T)}.$$  

Now, by Lemma 2.4:

$$\Gamma_\delta(F_n(T)) \equiv 0 \pmod{\omega_{n-1}(T)}.$$
The assertion follows.

Now, since $\Gamma_\delta, \gamma_\delta, U$ are continuous $O_K$-linear maps, it suffices to prove the Proposition in the case where $F(T) \in A$. Write $F(T) = \sum_{i=1}^r \beta_i (1 + T)^{\alpha_i}$, $\beta_1, \ldots, \beta_r \in O_K$, $\alpha_1, \ldots, \alpha_r \in \mathbb{Z}_p$. Let $I \subset \{\alpha_1, \ldots, \alpha_r\}$ be a set of representatives of the classes of $\alpha_1, \ldots, \alpha_r$ for the relation $\sim$.

For $x \in I$, $x \not\equiv 0 \pmod{p}$, set:

$$\beta_x = \sum_{\alpha_i \sim x} \beta_i \frac{\alpha_i}{x}$$

We get:

$$(p - 1) \gamma_\delta U(F(T)) = \sum_{\eta \in \mu \setminus 1, x \in I, x \in \mathbb{Z}_p^*} \eta^{-\delta} \beta_x (1 + T)^{\eta x}.$$ 

Now observe that:

$$\Gamma_\delta(F(T)) = \Gamma_\delta \gamma_\delta U(F(T)) = \sum_{x \in I, x \in \mathbb{Z}_p^*} \beta_x \omega^\delta(x) (1 + T)^{\log_p(x)/\log_p(\kappa)}.$$ 

Therefore $\Gamma_\delta(F(T)) \equiv 0 \pmod{\pi^n, \omega_m-1(T)}$ if and only if:

$$\forall a \in \{0, \ldots, p^{m-1} - 1\}, \quad \sum_{x \in I, x \in \mathbb{Z}_p^*, \log_p(x)/\log_p(\kappa) \equiv a \pmod{p^{m-1}}} \beta_x \omega^\delta(x) \equiv 0 \pmod{\pi^n}.$$ 

Now, observe that for $a \in \{0, \ldots, p^m - 1\}$, there exists at most one $\eta \in \mu_{p-1}$ such that $[\eta x]_m = a$, and if such a $\eta$ exists it is equal to $\omega(a) \omega^{-1}(x)$. Therefore $\Gamma_\delta(F(T)) \equiv 0 \pmod{\pi^n, \omega_m-1(T)}$ if and only if:

$$\forall a \in \{0, \ldots, p^m - 1\}, \quad \sum_{x \in I, x \in \mathbb{Z}_p^*, \exists \eta \in \mu_{p-1}, [\eta x]_m = a} \beta_x n_x^{-\delta} \equiv 0 \pmod{\pi^n}.$$ 

This latter property is equivalent to $\gamma_\delta U(F(T)) \equiv 0 \pmod{\pi^n, \omega_m(T)}$. $\Diamond$

Now, we can list the basic properties of $\Gamma_\delta$:

**Proposition 3.2** Let $\delta \in \mathbb{Z} / (p - 1)\mathbb{Z}$.

1) $\Gamma_\delta : \Lambda \to \Lambda$ is a surjective and continuous $O_K$-linear map.

2) $\forall F(T) \in A$, $\Gamma_\delta(F(T)) = \Gamma_\delta \gamma_\delta U(F(T))$.

3) $\forall a \in \mathbb{Z}_p^*$, $\Gamma_\delta(F((1 + T)^a - 1)) = \omega^\delta(a) (1 + T)^{\log_p(a)/\log_p(\kappa)} \Gamma_\delta(F(T))$.

4) Let $\kappa'$ be another topological generator of $1 + p\mathbb{Z}_p$ and let $\Gamma_{\delta}'$ be the $p$-adic Leopoldt transform associated to $\kappa'$ and $\delta$. Then:

$$\forall F(T) \in A, \quad \Gamma_{\delta}'(F(T)) = \Gamma_\delta(F)((1 + T)^{\log_p(\kappa)/\log_p(\kappa')} - 1).$$
5) Let $F(T) \in \Lambda$. Then $\mu(\Gamma_\delta(F(T))) = \mu(\gamma_{-\delta}U(F(T)))$ and:

$$\forall N \geq 1, \lambda(\Gamma_\delta(F(T))) \geq p^{N-1} \iff \lambda(\gamma_{-\delta}U(F(T))) \geq p^N.$$ 

**Proof** The assertions 1),2),3),4) come from the fact that $\Gamma_\delta, \gamma_{-\delta}, U$ are continuous and that these assertions are true for pseudo-polynomials. The assertion 5) is a direct application of Proposition 3.1. ♦

Let’s recall the following remarkable result due to W. Sinnott:

**Proposition 3.3** Let $r_1(T), \ldots, r_s(T) \in \mathbb{F}_q(T) \cap \mathbb{F}_q[[T]]$. Let $c_1, \ldots, c_s \in \mathbb{Z}_p \setminus \{0\}$ and suppose that:

$$\sum_{i=1}^{s} r_i((1 + T)^{c_i} - 1) = 0.$$ 

Then:

$$\forall a \in \mathbb{Z}_p, \sum_{c_i = a \pmod{Q^*}} r_i((1 + T)^{c_i} - 1) \in \mathbb{F}_q.$$ 

**Proof** See [8], Proposition 1. ♦

Let’s give a first application of this latter result:

**Proposition 3.4** Let $\delta \in \mathbb{Z}/(p - 1)\mathbb{Z}$ and let $F(T) \in K(T) \cap \Lambda$.

1) If $\delta$ is odd or if $\delta = 0$, then:

$$\mu(\Gamma_\delta(F(T))) = \mu(U(F(T)) + (-1)^\delta U(F((1 + T)^{-1} - 1))).$$

2) If $\delta$ is even and $\delta \neq 0$, then:

$$\mu(\Gamma_\delta(F(T))) = \mu(U(F(T)) + U(F((1 + T)^{-1} - 1)) - 2U(F(0))).$$

**Proof** The case $\delta = 0$ has already been obtained by Sinnott ([7], Theorem 1). We prove 1), the proof of 2) is quite similar. Now, observe that 1) is a consequence of Proposition 3.2 and the following fact:

Let $F(T) \in K(T) \cap \Lambda$, then $\mu(\gamma_{-\delta}(F(T))) = \mu(F(T) + (-1)^\delta F((1 + T)^{-1} - 1))$.

Let’s prove this fact. Let $r(T) \in \Lambda$, observe that:

$$\gamma_{-\delta}(r(T)) = (-1)^\delta \gamma_{-\delta}(r((1 + T)^{-1} - 1)).$$

We can assume that $F(T) + (-1)^\delta F((1 + T)^{-1} - 1) \neq 0$. Write:

$$F(T) + (-1)^\delta F((1 + T)^{-1} - 1) = \pi^m G(T),$$

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where \( m \in \mathbb{N} \), and \( G(T) \in \Lambda \setminus \pi \Lambda \). Note that \( G(T) \in K(T) \). We must prove that \( \gamma - \delta(G(T)) \not\equiv 0 \pmod{\pi} \). Suppose that it is not the case, i.e. \( \gamma - \delta(G(T)) \equiv 0 \pmod{\pi} \). Then:

\[
G(0) \equiv 0 \pmod{\pi}.
\]

Furthermore, by Proposition 3.3, there exists \( c \in O_K \) such that:

\[
G(T) + (-1)^iG((1 + T)^{-1} - 1) \equiv c \pmod{\pi}.
\]

But, we must have \( c \equiv 0 \pmod{\pi} \). Observe that:

\[
G(T) = (-1)^iG((1 + T)^{-1} - 1).
\]

Therefore we get \( G(T) \equiv 0 \pmod{\pi} \) which is a contradiction. ♦

**Lemma 3.5** Let \( F(T) \in \mathbb{F}_q(T) \cap \mathbb{F}_q[[T]] \). Then \( F(T) \) is a pseudo-polynomial if and only if there exists some integer \( n \geq 0 \) such that \((1+T)^{n}F(T) \in \mathbb{F}_q[T] \).

**Proof** Assume that \( F(T) \) is a pseudo-polynomial. We can suppose that \( F(T) \neq 0 \). Write:

\[
F(T) = \sum_{i=1}^{r} c_i(1 + T)^{a_i},
\]

where \( c_1, \ldots, c_r \in \mathbb{F}_q^* \), \( a_1, \ldots, a_r \in \mathbb{Z}_p \) and \( a_i \neq a_j \) for \( i \neq j \). Since \( F(T) \in \mathbb{F}_q(T) \) there exist \( m, n \in \mathbb{N} \setminus \{0\} \), \( m > \text{Max}\{v_p(a_i - a_j), i \neq j\} \), such that:

\[
(T^{q^n} - T)^{q^m}F(T) \in \mathbb{F}_q[T].
\]

Thus:

\[
\sum_{i=1}^{r} c_i(1 + T)^{a_i+q^{n+m}} - \sum_{i=1}^{r} c_i(1 + T)^{a_i+q^m} \in \mathbb{F}_q[T].
\]

Observe that:
- \( \forall i, j \in \{1, \ldots, r\} \), \( a_i + q^{n+m} \neq a_j + q^m \);
- \( a_i + q^m = a_j + q^m \Leftrightarrow i = j \).

Thus, by Lemma 2.2, we get:

\[
\forall i \in \{1, \ldots, r\}, a_i + q^m \in \mathbb{N}.
\]

Therefore \((1+T)^{q^m}F(T) \in \mathbb{F}_q[T]\). The Lemma follows. ♦

Let’s give a second application of Proposition 3.3:
Proposition 3.6  Let \( \delta \in \mathbb{Z}/(p - 1)\mathbb{Z} \) and let \( F(T) \in \mathbb{F}_q(T) \cap \mathbb{F}_q[[T]] \). Suppose that there exist an integer \( r \in \{0, \cdots, (p - 3)/2\} \), \( c_1, \cdots, c_r \in \mathbb{Z}_p \setminus \{0\} \), \( G_1(T), \cdots, G_r(T) \in \mathbb{F}_q(T) \cap \mathbb{F}_q[[T]] \) and a pseudo-polynomial \( R(T) \in \mathbb{F}_q[[T]] \) such that:

\[
\gamma_\delta(F(T)) = R(T) + \sum_{i=1}^r G_i((1 + T)^{c_i} - 1). 
\]

Then, there exists an integer \( n \geq 0 \) such that:

\[
(1 + T)^n(F(T) + (-1)^\delta F((1 + T)^{-1} - 1)) \in \mathbb{F}_q[T].
\]

Proof Note that if \( \eta, \eta' \in \mu_{p-1} : \eta \equiv \eta' \pmod{\mathbb{Q}^*} \Leftrightarrow \eta = \eta' \) or \( \eta = \eta'^{-1} \). Since \( r < (p - 1)/2 \), by Proposition 3.3, there exists \( \eta \in \mu_{p-1} \) such that:

\[
\eta^\delta F((1 + T)^n - 1) + \eta^{-\delta} F((1 + T)^{-n} - 1)
\]

is a pseudo-polynomial. Therefore:

\[
F(T) + (-1)^\delta F((1 + T)^{-1} - 1)
\]

is a pseudo-polynomial.

It remains to apply Lemma 3.3. 

Let \( F(T) \in \Lambda \). We say that \( F(T) \) is a pseudo-rational function if \( F(T) \) is the quotient of two pseudo-polynomials. For example, \( \forall a \in \mathbb{Z}_p, \forall b \in \mathbb{Z}_p^* \), \( \frac{(1 + T)^{a} - 1}{(1 + T)^{b} - 1} \) is a pseudo-rational function. We finish this section by giving a generalization of [8], Theorem 1:

Theorem 3.7  Let \( \delta \in \mathbb{Z}/(p - 1)\mathbb{Z} \) and let \( F(T) \in \mathbb{F}_q(T) \cap \mathbb{F}_q[[T]] \). Then \( \Gamma_\delta(F(T)) \) is a pseudo-rational function if and only if there exists some integer \( n \geq 0 \) such that:

\[
(1 + T)^n(U(F(T)) + (-1)^\delta U(F((1 + T)^{-1} - 1))) \in \mathbb{F}_q[T].
\]

Proof Assume that \( \Gamma_\delta(F(T)) \) is a pseudo-rational function. Then, by 3) of Proposition 3.2 and Proposition 3.1, there exist \( c_1, \cdots, c_r \in \mathbb{F}_q^* \), \( a_1, \cdots, a_r \in \mathbb{Z}_p \), \( a_i \neq a_j \) for \( i \neq j \), such that:

\[
\Gamma_\delta(U(F((1 + T)^{a_i} - 1)))
\]

is a pseudo-polynomial.
This implies, again by Proposition 3.1, that:
\[
\gamma - \delta U \left( \sum_{i=1}^{r} c_i F((1 + T)^{\kappa_i} - 1) \right) \text{ is a pseudo-polynomial.}
\]

Set:
\[
G(T) = U(F(T)) + (-1)^\delta U(F((1 + T)^{-1} - 1)) \in \mathbb{F}_q(T) \cap \mathbb{F}_q[[T]].
\]

Now, by Proposition 3.3, there exist \(d_1, \ldots, d_\ell, b_1, \ldots, b_\ell \in \mathbb{Z}_p^\star\), \(b_i \neq b_j\) for \(i \neq j\), \(\eta_1, \ldots, \eta_\ell \in \mu_{p-1}\), with \(\forall i, j \in \{1, \ldots, \ell\}, \eta_i \kappa_i \equiv \eta_j \kappa_j \pmod{\mathbb{Q}^\star}\), and \(\eta_i \kappa_i \neq \eta_j \kappa_j\) for \(i \neq j\), such that:
\[
\sum_{i=1}^{\ell} d_i G((1 + T)^{\eta_i \kappa_i} - 1) \text{ is a pseudo-polynomial.}
\]

For \(i = 1, \ldots, \ell\), write:
\[
\eta_i \kappa_i = \eta_1 \kappa_1 x_i,
\]
where \(x_i \in \mathbb{Q}^\star \cap \mathbb{Z}_p^\star\), and \(x_i \neq x_j\) for \(i \neq j\). Since \(G(T) = (-1)^\delta G((1 + T)^{-1} - 1)\), we can assume that \(x_1, \ldots, x_\ell\) are positives. Now, we get:
\[
\sum_{i=1}^{\ell} d_i G((1 + T)^{x_i} - 1) \text{ is a pseudo-polynomial.}
\]

Therefore, there exist \(N_1, \ldots, N_\ell \in \mathbb{N} \setminus \{0\}\), \(N_i \neq N_j\) for \(i \neq j\), such that:
\[
\sum_{i=1}^{\ell} d_i G((1 + T)^{N_i} - 1) \text{ is a pseudo-polynomial.}
\]

Now, by Lemma 3.5, there exists some integer \(N \geq 0\) such that:
\[
(1 + T)^N \left( \sum_{i=1}^{\ell} d_i G((1 + T)^{N_i} - 1) \right) \in \mathbb{F}_q[T].
\]

But, since \(G(T) \in \mathbb{F}_q(T) \cap \mathbb{F}_q[[T]]\), \(d_1, \ldots, d_\ell \in \mathbb{F}_q^\star\), \(N_1, \ldots, N_\ell \in \mathbb{N} \setminus \{0\}\) and \(N_i \neq N_j\) for \(i \neq j\), this implies that there exist some integer \(n \geq 0\) such that \((1 + T)^n G(T) \in \mathbb{F}_q[T]. \diamondsuit \)
4 Application to Kubota-Leopoldt $p$-adic L-functions

Let $\theta$ be a Dirichlet character of the first kind, $\theta \neq 1$ and $\theta$ even. We denote by $f(T, \theta)$ the Iwasawa power series attached to the $p$-adic L-function $L_p(s, \theta)$ (see [9], Theorem 7.10). Write:

$$\theta = \chi \omega^{\delta+1},$$

where $\chi$ is of conductor $d$, $d \geq 1$ and $d \not\equiv 0 \pmod{p}$, and $\delta \in \mathbb{Z}/(p-1)\mathbb{Z}$. Set $\kappa = 1 + pd$ and $K = \mathbb{Q}_p(\chi)$. We set:

$$F_\chi(T) = \sum_{a=1}^{d} \chi(a)(1 + T)^a \frac{\gamma_\alpha}{1 - (1 + T)^d}.$$

Let’s give the basic properties of $F_\chi(T)$:

**Lemma 4.1**

1) If $d \geq 2$, $F_\chi(T) \in \Lambda$.
2) If $d = 1$, $\forall \alpha \in \mathbb{Z}/(p-1)\mathbb{Z}$, $\alpha \neq 1$, $\gamma_\alpha(F_\chi(T)) \in \Lambda$.
3) $U(F_\chi(T)) = F_\chi(T) - \chi(p)F_\chi((1 + T)^p - 1)$.
4) If $d \geq 2$, $F_\chi((1 + T)^{-1} - 1) = \varepsilon F_\chi(T)$, where $\varepsilon = 1$ if $\chi$ is odd and $\varepsilon = -1$ if $\chi$ is even.
5) If $d = 1$, $F_\chi((1 + T)^{-1} - 1) = -1 - F_\chi(T)$.

**Proof** 1), 4) and 5) are obvious.
2) For $d = 1$, we have:

$$F_\chi(T) = -1 + \sum_{a=0}^{p-1} (1 + T)^a.$$ 

Set:

$$G(T) = (1 - (1 + T)^p)\gamma_\alpha(F_\chi(T)).$$

Note that:

$$\forall \eta \in \mu_{p-1}, \frac{1 - (1 + T)^p}{1 - (1 + T)^{np}} \equiv \eta^{-1} \pmod{\omega_1(T)}.$$
Therefore:

\[(p - 1)G(T) \equiv \sum_{\eta \in \mu_{p-1}} \eta^{a-1} \sum_{a=0}^{p-1} (1 + T)^{\eta a} \pmod{\omega_1(T)}.
\]

Thus:

\[(p - 1)G(T) \equiv \sum_{\eta \in \mu_{p-1}} \eta^{a-1} \sum_{b=0}^{p-1} (1 + T)^b \pmod{\omega_1(T)}.
\]

Since \(\alpha \neq 1\), we get:

\[G(T) \equiv 0 \pmod{\omega_1(T)}.
\]

Therefore \(\gamma_\alpha(F_\chi(T)) \in \Lambda\).

3) For \(d = 1\), we have:

\[U(F_\chi(T)) = \sum_{a=1}^{q_0-1} \chi(a)(1 + T)^a \pmod{1 - (1 + T)^{q_0}}.
\]

Now, let \(d \geq 2\). Set \(q_0 = \kappa = 1 + pd\). Note that:

\[F_\chi(T) = \frac{\sum_{a=1}^{q_0} \chi(a)(1 + T)^a}{1 - (1 + T)^{q_0}}.
\]

Therefore:

\[U(F_\chi(T)) = \sum_{a=1}^{q_0} \chi(a)(1 + T)^a \pmod{1 - (1 + T)^{q_0}}.
\]

But:

\[F_\chi(T) - \chi(p)F_\chi((1+T)^p-1) = \frac{\sum_{a=1}^{q_0} \chi(a)(1 + T)^a}{1 - (1 + T)^{q_0}} - \chi(p)\frac{\sum_{a=1}^{d} \chi(a)(1 + T)^{pa}}{1 - (1 + T)^{q_0}}.
\]

The Lemma follows easily.

\[\text{Lemma 4.2} \quad \text{Assume that } d \geq 2. \text{ The denominator of } F_\chi(T) \text{ is } \phi_d(1 + T) \text{ where } \phi_d(X) \text{ is the } d \text{th cyclotomic polynomial and the same is true for } F_\chi(T).
\]

\[\text{Proof} \quad \text{Let } \zeta \in \mu_d. \text{ If } \zeta \text{ is not a primitive } d \text{th root of unity, then, by } [9], \text{ Lemma 4.7, we have:}
\]

\[\sum_{a=1}^{d} \chi(a)\zeta^a = 0.
\]
If $\zeta$ is a primitive $d$th root of unity, then by \[9\], Lemma 4.8, we have:

$$\sum_{a=1}^{d} \chi(a)\zeta^a \not\equiv 0 \pmod{\tilde{\pi}},$$

where $\tilde{\pi}$ is any prime of $K(\mu_d)$. ◇

**Lemma 4.3** The derivative of $\gamma_{-\delta}(F_{\chi}(T))$ is not a pseudo-polynomial modulo $\pi$.

**Proof** We first treat the case $d \geq 2$. By 3) and 4) of Lemma [4.1] and Proposition [5,6], $\gamma_{-\delta}(F_{\chi}(T))$ is not a pseudo-polynomial. But observe that $U = \mathcal{D}p^{-1}$. Thus $\mathcal{D}\gamma_{-\delta}(F_{\chi}(T))$ is not a pseudo-polynomial.

For the case $d = 1$. Set $F_{\chi}(T) = F_{\chi}(T) - 2F_{\chi}((1 + T)^{2} - 1) = 1 - \frac{1}{2 + T}$. Observe that:

- $F_{\chi}((1 + T)^{-1} - 1) = 1 - F_{\chi}(T)$,
- $U(F_{\chi}(T)) = F_{\chi}(T) - F_{\chi}((1 + T)^{p} - 1)$.

Therefore, as in the case $d \geq 2$, $\gamma_{-\delta}(F_{\chi}(T))$ is not a pseudo-polynomial. And one can conclude as in the case $d \geq 2$. ◇

**Lemma 4.4**

$$\Gamma_{\delta}\gamma_{-\delta}(F_{\chi}(T)) = f(\frac{1}{1 + T} - 1, \theta).$$

**Proof** We treat the case $d = 1$, the case $d \geq 2$ is quite similar. Set $T = e^{Z} - 1$. We get:

$$\gamma_{-\delta}(F_{\chi}(T)) = \sum_{n \geq 0, n \equiv 1 + \delta \pmod{p - 1}} \frac{B_n}{n!}Z^{n-1}.$$

Thus, by [8], Theorem 5.11, we get:

$$\forall k \in \mathbb{N}, k \equiv \delta \pmod{p - 1}, D^{k}\gamma_{-\delta}U(F_{\chi})(0) = L_{p}(-k, \theta).$$

But, by Proposition [2.3] we have for $s \in \mathbb{Z}_{p}$:

$$\Gamma_{\delta}\gamma_{-\delta}U(F_{\chi})(\kappa^{s} - 1) = \lim_{n}D^{kn, s, \delta}\gamma_{-\delta}U(F_{\chi})(0) = L_{p}(-s, \theta) = f(\kappa^{-s} - 1, \theta).$$

The Lemma follows. ◇

We can now state and prove our main result:
Theorem 4.5

1) \( f(T, \theta) \) is not a pseudo-rational function.
2) \( \lambda(f(T, \theta)) < (\frac{p-1}{2})^\phi(p-1) \), where \( \phi \) is Euler’s totient function.

Proof

1) Assume the contrary, i.e. \( f(T, \theta) \) is a pseudo-rational function. Then \( f(\frac{1}{1+T} - 1, \theta) \) is also a pseudo-rational function. Thus \( \Gamma_{\delta\gamma - \delta} U(F)(T) \) is a pseudo-rational function.

We first treat the case \( d \geq 2 \). By Theorem 3.7, there exists an integer \( n \geq 0 \) such that \((1 + T)^n U(F)(T) + (-1)^\delta U(F)((1 + T)^{-1} - 1) \) \( \in \mathbb{F}_q[T] \). This is a contradiction by 3) and 4) of Lemma 4.1 and Lemma 4.2.

For the case \( d = 1 \). We work with \( \tilde{F}(T) = F(T) - 2F((1 + T)^2 - 1) = 1 - \frac{1}{2T+1} \). Then, by Proposition 3.2, \( \Gamma_{\delta\gamma - \delta} U(F)(T) \) is a pseudo-rational function. We get a contradiction as in the case \( d \geq 2 \).

2) Our proof is inspired by a method introduced by S. Rosenberg ([6]). We first treat the case \( d = 1 \). Note that we can assume that \( \lambda(f(T, \theta)) \geq 1 \).

Now, by Lemma 4.3:

\[
\mu(\gamma - \delta(F)(T))) = 0.
\]

Futhermore, we have:

\[
\gamma - \delta(F)(0) \equiv 0 \pmod{\pi}.
\]

Therefore, by 3) of Lemma 4.1 we get:

\[
\lambda(\gamma - \delta U(F)(T))) = \lambda(\gamma - \delta(F)(T))).
\]

Therefore we have to evaluate \( \lambda(\gamma - \delta(F)(T))) \). Set \( F(T) = \frac{1}{T} \). Since \( \delta \) is odd, we have:

\[
\gamma - \delta(F)(T)) = \gamma - \delta(F(T)).
\]

Observe that \( F((1 + T)^{-1} - 1) = 1 - F(T) \). Let \( S \subset \mu_{p-1} \) be a set of representatives of \( \mu_{p-1}/\{1, -1\} \). We have:

\[
(p-1)\gamma - \delta(F)(T)) = 2 \sum_{\eta \in S} \eta^{-\delta} F((1 + T)^\eta - 1) - \sum_{\eta \in S} \eta^{-\delta}.
\]

Set:

\[
G(T) = \prod_{\eta \in S}((1 + T)^\eta - 1)\gamma - \delta(F(T)).
\]
Then:
- $\mu(G(T)) = 0$,
- $\lambda(G(T)) = \frac{p-1}{2} + \lambda(\gamma_{-\delta}(F(T)))$.

For $S' \subset S$, write $t(S') = \sum_{x \in S'} x$. We can write:

$$G(T) = \sum_{S' \subset S} a_{S'} (1 + T)^{t(S')}$$

where $a_{S'} \in O_K$. Set:

$$N = \operatorname{Max}\{v_p(t(S') - t(S'')) \mid S', S'' \subset S, t(S') \neq t(S'')\}.$$

It is clear that:

$$p^N < \left(\frac{p-1}{2}\right)^{\phi(p-1)}.$$ 

But, by Lemma 2.2, we have:

$$\lambda(G(T)) < p^{N+1}.$$ 

Thus, by Proposition 3.2, we get:

$$\lambda(f(T, \theta)) = \lambda(f\left(\frac{1}{1+T} - 1, \theta\right)) < p^N < \left(\frac{p-1}{2}\right)^{\phi(p-1)}.$$ 

Now, we treat the general case, i.e. $d \geq 2$. Again we can assume that $\lambda(f(T, \theta)) \geq 1$. Thus as in the case $d = 1$, we get:

$$\lambda(\gamma_{-\delta} U(F_{\chi}(T))) = \lambda(\gamma_{-\delta}(F_{\chi}(T))).$$ 

Now, by Lemma 4.2, we can write:

$$F_{\chi}(T) = \prod_{\eta \in S} \phi_d((1 + T)^\eta),$$

where $r_a \in O_K$ for $a \in \{0, \cdots, \phi(d) - 1\}$. Let again $S \subset \mu_{p-1}$ be a set of representatives of $\mu_{p-1}/\{1, -1\}$. By Lemma 4.1, we have:

$$(p-1)\gamma_{-\delta}(F_{\chi}(T)) = 2 \sum_{\eta \in S} \eta^{-\delta} F_{\chi}((1 + T)^\eta - 1).$$ 

Set:

$$G(T) = \left(\prod_{\eta \in S} \phi_d((1 + T)^\eta))\gamma_{-\delta}(F_{\chi}(T)).$$
We have:

\[ G(T) = \sum_{a=0}^{\phi(d)-1} \sum_{\eta \in S} \sum_{S' \subseteq S \setminus \{\eta\}} \sum_{d'=\eta \in S'} b_{S',d}(1 + T)^{a \eta + \sum_{\eta' \in S'} d'_\eta}, \]

where \( b_{S',d} \in \mathcal{O}_K \). Note that again \( \mu(G(T)) = 0 \) and that \( \lambda(G(T)) = 1 \).

Now, for \( a, b \in \{0, \cdots, \phi(d) - 1\}, \eta_1, \eta_2 \in S, S_1 \in S \setminus \{\eta_1\}, S_2 \in S \setminus \{\eta_2\} \), set:

\[ V = a\eta_1 + \sum_{\eta \in S_1} d_\eta \eta - b\eta_2 - \sum_{\eta \in S_2} d'_\eta \eta, \]

where \( \forall \eta \in S_1, d_\eta \in \{0, \cdots, \phi(d)\} \), and \( \forall \eta \in S_2, d'_\eta \in \{0, \cdots, \phi(d)\} \).

If \( \eta_1 = \eta_2 \) then we can write:

\[ V = (a - b)\eta_1 + \sum_{\eta \in S'} u_\eta \eta, \]

where \( |u_\eta| \in \{0, \cdots, \phi(d)\} \) and \( |S'| \leq \frac{p-3}{2} \).

If \( \eta_1 \neq \eta_2 \), we can write:

\[ V = a'_\eta_1 + b'_\eta_2 + \sum_{\eta \in S'} u_\eta \eta, \]

where \( |a'|, |b'|, |u_\eta| \in \{0, \cdots, \phi(d)\} \), and \( |S'| \leq \frac{p-5}{2} \). Therefore, if \( V \neq 0 \), we get:

\[ p^v p(V) < (\frac{p-1}{2})^{\phi(d-1)}. \]

Now, we can conclude as in the case \( d = 1 \). ♦

Let \( E \) be a number field and let \( E_\infty/E \) be the cyclotomic \( \mathbb{Z}_p \)-extension of \( E \). For \( n \geq 0 \), let \( A_n \) be the \( p \)th Sylow subgroup of the ideal class group of the \( n \)th layer in \( E_\infty/E \). Then, by [3], Theorem 13.13, there exist \( \mu_p(E) \in \mathbb{N}, \lambda_p(E) \in \mathbb{N} \) and \( \nu_p(E) \in \mathbb{Z} \), such that for all sufficiently large \( n \):

\[ |A_n| = p^\mu_p(E)p^n + \lambda_p(E)n + \nu_p(E). \]

Recall that it is conjectured that \( \mu_p(E) = 0 \) and if \( E \) is an abelian number field it has been proved by B. Ferrero and L. Washington ([3]).
Corollary 4.6 Let $F$ be an abelian number field of conductor $N$. Write $N = p^m d$, where $m \in \mathbb{N}$ and $d \geq 1, d \not\equiv 0 \pmod{p}$. Then:

$$\lambda_p(F) < 2 \left( \frac{p-1}{2} \phi(d) \right)^{\phi(p-1)+1}.$$ 

Proof Set, for all $n \geq 0$, $q_n = p^{n+1}d$. Then $F \subset \mathbb{Q}(\mu_{q_m})$. It is not difficult to see that (see the arguments in the proof of Theorem 7.15 in [9]):

$$\lambda_p(F) \leq \lambda_p(\mathbb{Q}(\mu_{q_m})).$$

But, note that:

$$\lambda_p(\mathbb{Q}(\mu_{q_m})) = \lambda_p(\mathbb{Q}(\mu_{q_0})).$$

Now, by [9] Proposition 13.32 and Theorem 7.13:

$$\lambda_p(\mathbb{Q}(\mu_{q_0})) \leq 2 \sum_{\theta \text{ even, } \theta \neq 1, f_0 | q_0} \lambda(f(T, \theta)).$$

It remains to apply Theorem 4.5. ♦

Note that the bound of this latter Corollary is certainly far from the truth even in the case $p = 3$ (see [4]).

References


