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Circular law for non-central random matrices

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Abstract

Let \((X_{jk})_{j,k \geq 1}\) be an infinite array of i.i.d. complex random variables, with mean 0 and variance 1. Let \(\lambda_1, \ldots, \lambda_n, n\) be the eigenvalues of \((\frac{1}{\sqrt{n}} X_{jk})_{1 \leq j, k \leq n}\). The strong circular law theorem states that with probability one, the empirical spectral distribution \(\frac{1}{n} (\delta_{\lambda_1,1} + \cdots + \delta_{\lambda_n,n})\) converges weakly as \(n \to \infty\) to the uniform law over the unit disc \(\{z \in \mathbb{C}; |z| \leq 1\}\). In this short note, we provide an elementary argument that allows to add a deterministic matrix \(M\) to \((X_{jk})_{1 \leq j, k \leq n}\) provided that \(\text{Tr}(MM^*) = O(n^2)\) and \(\text{rank}(M) = O(n^\alpha)\) with \(\alpha < 1\). Conveniently, the argument is similar to the one used for the non–central version of Wigner’s and Marchenko-Pastur theorems.

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1 Introduction

For any square \(n \times n\) matrix \(A\) with complex entries, let the complex eigenvalues \(\lambda_1(A), \ldots, \lambda_n(A)\) of \(A\) be labeled so that \(|\lambda_1(A)| \geq \cdots \geq |\lambda_n(A)|\). The empirical spectral distribution of \(A\) is the discrete probability measure \(\mu_A := \frac{1}{n} \sum_{k=1}^n \delta_{\lambda_k(A)}\). We denote by \(s_1(A) \geq \cdots \geq s_n(A)\) the singular values of \(A\), i.e. the eigenvalues of the positive semi-definite Hermitian matrix \(\sqrt{AA^*}\) where \(A^*\) is the conjugate–transpose of \(A\). The operator norm is \(s_1(A) = \max_{\|x\|_2=1} \|Ax\|_2\) and the square Hilbert-Schmidt norm is \(\|A\|^2 := s_1(A)^2 + \cdots + s_n(A)^2 = \text{Tr}(AA^*) = \sum_{j,k=1}^n |A_{j,k}|^2\). Weyl’s inequality \(|\lambda_1(A)|^2 + \cdots + |\lambda_n(A)|^2 \leq s_1(A)^2 + \cdots + s_n(A)^2\) ensures that the second moment of \(\mu_A\) is always bounded above by \(\frac{1}{n}\|A\|^2\). The following result was recently obtained by Tao and Vu [19, Corollary 1.15].

**Theorem 1.1** (Circular law for central random matrices). Let \((X_{jk})_{j,k \geq 1}\) be i.i.d. complex random variables. Let \((M_{jk})_{j,k \geq 1}\) be deterministic complex numbers. For every integer \(n \geq 1\), set \(X_n = (X_{jk})_{1 \leq j, k \leq n}\) and \(M_n = (M_{jk})_{1 \leq j, k \leq n}\). If

- \(\mathbb{E}[|X_{1,1}|^2] = 1\) and \(\mathbb{E}[X_{1,1}] = 0\)
- \(\|M_n\|^2 = O(n^2)\) and \(\text{rank}(M_n) = O(n^\alpha)\) for some \(\alpha < 1\)
then with probability one, \( \mu_1(\frac{1}{\sqrt{n}}(X_n + M_n)) \) tends weakly as \( n \to \infty \) to the uniform distribution on the unit disc \( \{ z \in \mathbb{C}; |z| \leq 1 \} \) (known as the circular law).

The aim of this note is to provide an alternative and elementary argument which reduces theorem 1.1 to the central case where \( M_n \equiv 0 \) for every \( n \). Conveniently, the approach is close in spirit to the one used by Bai \[3\] for the derivation of Wigner’s and Marchenko-Pastur theorems for non-central random matrices.

This note was motivated by the study of random Markov matrices, including the Dirichlet Markov Ensemble \[8, 7\], for which a circular law theorem is conjectured. The initial version of this note was written before the apparition of \[19\], and provided for the first time a non-central version of the circular law theorem. The initial version was based on potential theoretic tools. For convenience, the present version makes use instead of the replacement principle borrowed from \[19\].

Theorem 1.1 belongs to a sequence of works by many authors, including Mehta \[13\], Girko \[10\], Silverstein \[12\], Bai \[2\], Edelman \[9\], Sniady \[17\], Bai and Silverstein \[4\], Pan and Zhou \[14\], Götze and Tikhomirov \[11\], and Tao and Vu \[18\].

Remark 1.2 (Constant case). Consider the case where the entries of \( M_n \) are all equal to 1 in theorem 1.1. We have then \( \text{rank}(M_n) = 1 \) and \( s_1(M_n) = n \). Suppose additionally that \( X_{1,1} \) has finite fourth moment. Then, by Bai and Yin theorem \[5\], with probability one, \( \lim_{n \to \infty} s_1\left(\frac{1}{\sqrt{n}}(X_n + M_n)\right) = 2 \), and thus \( \frac{1}{\sqrt{n}}(X_n + M_n) \) is a random bounded perturbation of the rank one symmetric matrix \( \frac{1}{\sqrt{n}}M_n \) which has spectrum \( \lambda_1(M_n) = \cdots = \lambda_2(M_n) = 0 \) and \( \lambda_1(M_n) = \sqrt{n} \).

From this observation, Silverstein \[10\] has shown, via perturbation techniques such as Bauer-Fike and Gerschgorin theorems, that with probability one,

\[
\left| \lambda_2\left(\frac{1}{\sqrt{n}}(X_n + M_n)\right) \right| \leq 2 + o(1) \quad \text{and} \quad \left| \lambda_1\left(\frac{1}{\sqrt{n}}(X_n + M_n)\right) - \sqrt{n} \right| \leq 2 + o(1).
\]

See also the work of Andrew \[1\]. Also, with probability one, as \( n \to \infty \), the spectral radius \( |\lambda_1\left(\frac{1}{\sqrt{n}}(X_n + M_n)\right)| \) blows up while \( \mu_1(\frac{1}{\sqrt{n}}(X_n + M_n)) \) remains weakly localized.

2 Reduction to the central case

In order to show that theorem 1.1 reduces to the central case where \( M_n \equiv 0 \) for every \( n \geq 1 \), it suffices to check the assumptions of the replacement principle of theorem 3.1 with \( A_n := \frac{1}{\sqrt{n}}X_n \) and \( B_n := \frac{1}{\sqrt{n}}(X_n + M_n) \). By the strong law of large numbers and the assumption on \( \|M_n\| \), with probability one,

\[
\frac{1}{n}\|A_n\|^2 + \frac{1}{n}\|B_n\|^2 = O(1).
\]

Next, by theorem 3.1 and the first Borel-Cantelli lemma, for all \( z \in \mathbb{C} \), with probability one, the random matrices \( A_n - zI_n \) and \( B_n - zI_n \) are invertible for large enough \( n \). Let us define, for large enough \( n \), the quantity

\[
\Delta_{n,z} := \frac{1}{n}\log|\det(A_n - zI_n)| - \frac{1}{n}\log|\det(B_n - zI_n)|.
\]
If we set \( \mu_{n,z} := \mu \sqrt{(A_n - zI_n)(A_n - zI_n)} \) and \( \nu_{n,z} := \mu \sqrt{(B_n - zI_n)(B_n - zI_n)} \), then

\[
\Delta_{n,z} = \int_0^\infty \log(t) d(\mu_{n,z} - \nu_{n,z})(t).
\]

By the strong law of large numbers and the assumption on \( \|M_n\| \), for all \( z \in \mathbb{C} \), with probability one, there exists \( a > 0 \) such that

\[
\max(s_1(A_n - zI_n), s_1(B_n - zI_n)) \leq n^a
\]

for large enough \( n \). On the other hand, by theorem (3.4) and the first Borel-Cantelli lemma, for all \( z \in \mathbb{C} \), with probability one, there exists \( b > 0 \) such that

\[
\min(s_n(A_n - zI_n), s_n(B_n - zI_n)) \geq n^{-b}
\]

for large enough \( n \). Therefore, with \( \alpha_n := n^{-b} \) and \( \beta_n := n^a \), and large enough \( n \),

\[
\Delta_{n,z} = \int_{\alpha_n}^{\beta_n} \log(t) d(\mu_{n,z} - \nu_{n,z})(t).
\]

Let \( F_{n,z} \) and \( G_{n,z} \) be the cumulative distribution functions of the real probability measures \( \mu_{n,z} \) and \( \nu_{n,z} \). By lemma 3.3 and the assumption on rank\( (M_n) \), for almost all \( z \in \mathbb{C} \), with probability one, there exists \( \varepsilon > 0 \) such that

\[
\|F_{n,z} - G_{n,z}\|_\infty = O(n^{-\varepsilon}).
\]

Therefore, by lemma 3.2 we obtain, for almost all \( z \in \mathbb{C} \), with probability one,

\[
|\Delta_{n,z}| \leq (\log(\beta_n) - \log(\alpha_n))\|F_{n,z} - G_{n,z}\|_\infty = o(1).
\]

3 Tools

This section gathers some tools used in our proof of theorem 1.1. By Green’s theorem, for any complex polynomial \( P \) and smooth compactly supported \( f : \mathbb{C} \to \mathbb{R} \),

\[
\int_\mathbb{C} f \, d\mu = \frac{1}{2\pi} \int_\mathbb{C} \Delta f \log |P| \, dx dy
\]

where \( \mu := \delta_{\lambda_1} + \cdots + \delta_{\lambda_n} \) is the counting measure of the roots \( \lambda_1, \ldots, \lambda_n \) of \( P \) in \( \mathbb{C} \). Used for characteristic polynomials of random matrices, this identity provides, via dominated convergence arguments, the following theorem, see [19, Theorem 2.1].

**Theorem 3.1 (Replacement principle).** Let \( (A_n)_{n \geq 1} \) and \( (B_n)_{n \geq 1} \) be two sequences of complex random matrices where \( A_n, B_n \) are \( n \times n \), without any assumptions. If

- with probability one \( \frac{1}{n} \|A_n\|^2 + \frac{1}{n} \|B_n\|^2 = O(1) \)
- for almost all \( z \in \mathbb{C} \), with probability one, the random matrices \( A_n - zI_n \) and \( B_n - zI_n \) are invertible for large enough \( n \)
for almost all \( z \in \mathbb{C} \), with probability one,
\[
\lim_{n \to \infty} \left( \frac{1}{n} \log |\det(A_n - zI_n)| - \frac{1}{n} \log |\det(B_n - zI_n)| \right) = 0
\]
then with probability one, \( \mu_{A_n} - \mu_{B_n} \) tends weakly to zero as \( n \to \infty \).

The following lemma is a special case of the integration by parts formula for the Lebesgue-Stieltjes integral (with atoms). We give a short proof for convenience.

**Lemma 3.2** (Integration by parts). If \( a_1, \ldots, a_n, b_1, \ldots, b_n \in [\alpha, \beta] \subset \mathbb{R} \), and \( F_{\mu} \) and \( F_{\nu} \) are the cumulative distribution functions of \( \mu = \frac{1}{n}(\delta_{a_1} + \cdots + \delta_{a_n}) \) and \( \nu = \frac{1}{n}(\delta_{b_1} + \cdots + \delta_{b_n}) \) respectively, then for any smooth \( f : [\alpha, \beta] \to \mathbb{R} \),
\[
\int_{\alpha}^{\beta} f(x) \, d\mu(x) - \int_{\alpha}^{\beta} f(x) \, d\nu(x) = \int_{\alpha}^{\beta} f'(x)(F_{\mu}(x) - F_{\nu}(x)) \, dx.
\]
In particular, when \( f \) is non decreasing,
\[
\left| \int_{\alpha}^{\beta} f(x) \, d\mu(x) - \int_{\alpha}^{\beta} f(x) \, d\nu(x) \right| \leq (f(\beta) - f(\alpha))\|F_{\mu} - F_{\nu}\|_{\infty}.
\]

**Proof.** One can assume by continuity that \( a_1, \ldots, a_n, b_1, \ldots, b_n \) are all different. We reorder \( a_1, \ldots, a_n, b_1, \ldots, b_n \) into \( c_1 \leq \cdots \leq c_{2n} \). For every \( 1 \leq k \leq 2n \), set \( \varepsilon_k = +1 \) if \( c_k \in \{a_1, \ldots, a_n\} \) and \( \varepsilon_k = -1 \) if \( c_k \in \{b_1, \ldots, b_n\} \). We have
\[
\int_{\alpha}^{\beta} f(x) \, d\mu(x) - \int_{\alpha}^{\beta} f(x) \, d\nu(x) = \frac{1}{n} \sum_{k=1}^{n} (f(a_i) - f(b_i)) = \frac{1}{n} \sum_{k=1}^{2n} \varepsilon_k f(c_k).
\]
By an Abel transform, we get by denoting \( S_k = \varepsilon_1 + \cdots + \varepsilon_k \),
\[
\sum_{k=1}^{2n} \varepsilon_k f(c_k) = -\sum_{k=1}^{2n-1} S_k(f(c_{k+1}) - f(c_k)) + S_{2n} f(c_{2n}).
\]
Since \( F_{\mu} - F_{\nu} \) is constant and equal to \( S_k \) on \([c_k, c_{k+1}]\),
\[
S_k(f(c_{k+1}) - f(c_k)) = \int_{c_k}^{c_{k+1}} f'(x)(F_{\mu}(x) - F_{\nu}(x)) \, dx.
\]
It remains to notice that \( S_{2n} = F_{\mu}(c_{2n}) - F_{\nu}(c_{2n}) = 0 \). \( \square \)

The following lemma is a direct consequence of interlacing inequalities for singular values obtained by Thompson [20] in 1976. It was also obtained by Bai [3] and generalized by Benaych-Georges and Rao [9]. It is worthwhile to mention that it gives neither an upper bound for \( s_1(B), \ldots, s_k(B) \) nor a lower bound for \( s_{n-k+1}(B), \ldots, s_n(B) \) where \( k := \text{rank}(A - B) \), even in the case \( k = 1 \).

**Lemma 3.3** (Rank inequality). Let \( A \) and \( B \) be two \( n \times m \) complex matrices. Let \( F_{\sqrt{AA^*}}, F_{\sqrt{BB^*}} \) be the cumulative distribution functions of \( \mu_{\sqrt{AA^*}} \) and \( \mu_{\sqrt{BB^*}} \). Then
\[
\|F_{\sqrt{AA^*}} - F_{\sqrt{BB^*}}\|_{\infty} \leq \frac{1}{n} \text{rank}(A - B).
\]
The following theorem is due to Tao and Vu [18, Theorem 2.1], and is inspired from the work of Rudelson and Vershynin [15].

**Theorem 3.4** (Polynomial bounds for smallest singular values). Let $L$ be a probability distribution on $\mathbb{C}$ with finite and non-zero variance. For every constants $A > 0$ and $C_1 > 0$, there exists constants $B > 0$ and $C_2 > 0$ such that for every $n \times n$ random matrix $X$ with i.i.d. entries of law $L$ and every $n \times n$ deterministic matrix $C$ with $s_1(C) \leq n^{C_1}$, we have

$$
P(s_n(X + C) \leq n^{-B}) \leq C_2n^{-A}.
$$

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**References**


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