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ON THE BASE LOCUS OF THE LINEAR SYSTEM OF GENERALIZED THETA FUNCTIONS

CHRISTIAN PAULY

Abstract. Let \( \mathcal{M}_r \) denote the moduli space of semi-stable rank-\( r \) vector bundles with trivial determinant over a smooth projective curve \( C \) of genus \( g \). In this paper we study the base locus \( B_r \subset \mathcal{M}_r \) of the linear system of the determinant line bundle \( \mathcal{L} \) over \( \mathcal{M}_r \), i.e., the set of semi-stable rank-\( r \) vector bundles without theta divisor. We construct base points in \( B_{g+2} \) over any curve \( C \), and base points in \( B_4 \) over any hyperelliptic curve.

1. Introduction

Let \( C \) be a complex smooth projective curve of genus \( g \) and let \( \mathcal{M}_r \) denote the coarse moduli space parametrizing semi-stable rank-\( r \) vector bundles with trivial determinant over the curve \( C \). Let \( \mathcal{L} \) be the determinant line bundle over the moduli space \( \mathcal{M}_r \) and let \( \Theta \subset \text{Pic}^{g-1}(C) \) be the Riemann theta divisor in the degree \( g-1 \) component of the Picard variety of \( C \). By [BNR] there is a canonical isomorphism \( |\mathcal{L}|^* \sim |r\Theta| \), under which the natural rational map \( \varphi_\mathcal{L} : \mathcal{M}_r \rightarrow |r\Theta| \) is identified with the so-called theta map \( \theta : \mathcal{M}_r \rightarrow |r\Theta|, \ E \mapsto \theta(E) \subset \text{Pic}^{g-1}(C) \).

The underlying set of \( \theta(E) \) consists of line bundles \( L \in \text{Pic}^{g-1}(C) \) with \( h^0(C, E \otimes L) > 0 \). For a general semi-stable vector bundle \( E \), \( \theta(E) \) is a divisor. If \( \theta(E) = \text{Pic}^{g-1}(C) \), we say that \( E \) has no theta divisor. We note that the indeterminacy locus of the theta map \( \theta \), i.e., the set of bundles \( E \) without theta divisor, coincides with the base locus \( B_r \subset \mathcal{M}_r \) of the linear system \( |\mathcal{L}| \).

Over the past years many authors [A], [B2], [H], [Hi], [P], [R], [S] have studied the base locus \( B_r \) of \( |\mathcal{L}| \) and their analogues for the powers \( |\mathcal{L}^k| \). For a recent survey of this subject we refer to [B1].

It is natural to introduce for a curve \( C \) the integer \( r(C) \) defined as the minimal rank for which there exists a semi-stable rank-\( r(C) \) vector bundle with trivial determinant over \( C \) without theta divisor (see also [B1] section 6). It is known [A] that \( r(C) \geq 3 \) for any curve \( C \) and that \( r(C) \geq 4 \) for a generic curve \( C \). Our main result shows the existence of vector bundles of low ranks without theta divisor.

Theorem 1.1. We assume that \( g \geq 2 \). Then we have the following bounds.

1. \( r(C) \leq g + 2 \).
2. \( r(C) \leq 4 \), if \( C \) is hyperelliptic.

The first part of the theorem improves the upper bound \( r(C) \leq \frac{(g+1)(g+2)}{2} \) given in [A]. The statements of the theorem are equivalent to the existence of a semi-stable rank-\((g+2)\) (resp.
rank-4) vector bundle without theta divisor — see section 2.1 (resp. 2.2). The construction of these vector bundles uses ingredients which are already implicit in \([1]\).

Theorem 1.1 seems to hint towards a dependence of the integer \(r(C)\) on the curve \(C\).

Notations: If \(E\) is a vector bundle over \(C\), we will write \(H^i(E)\) for \(H^i(C, E)\) and \(h^i(E)\) for \(\dim H^i(C, E)\). We denote the slope of \(E\) by \(\mu(E) := \frac{\deg E}{\rk E}\), the canonical bundle over \(C\) by \(K\) and the degree \(d\) component of the Picard variety of \(C\) by \(\text{Pic}^d(C)\).

## 2. Proof of Theorem 1.1

### 2.1. Semi-stable rank-(\(g + 2\)) vector bundles without theta divisor

We consider a line bundle \(L \in \text{Pic}^{2g+1}(C)\). Then \(L\) is globally generated, \(h^0(L) = g + 2\) and the evaluation bundle \(E_L\), which is defined by the exact sequence
\[
0 \rightarrow E_L^* \rightarrow H^0(L) \otimes \mathcal{O}_C \xrightarrow{ev} L \rightarrow 0,
\]
is stable (see e.g. \([3]\)), with \(\deg E_L = 2g + 1\), \(\rk E_L = g + 1\) and \(\mu(E_L) = 2 - \frac{1}{g+1}\).

A cohomology class \(e \in \text{Ext}^1(LK^{-1}, E_L) = H^1(E_L \otimes KL^{-1})\) determines a rank-(\(g + 2\)) vector bundle \(\mathcal{E}_e\) given as an extension
\[
0 \rightarrow E_L \rightarrow \mathcal{E}_e \rightarrow LK^{-1} \rightarrow 0.
\]

**Proposition 2.1.** For any non-zero class \(e\), the rank-(\(g + 2\)) vector bundle \(\mathcal{E}_e\) is semi-stable.

**Proof.** Consider a proper subbundle \(A \subset \mathcal{E}_e\). If \(A \subset E_L\), then \(\mu(A) \leq \mu(E_L) = 2 - \frac{1}{g+1}\) by stability of \(E_L\), so the subbundles of \(E_L\) cannot destabilize \(\mathcal{E}_e\). If \(A \not\subset E_L\), we introduce \(S = A \cap E_L \subset E_L\) and consider the exact sequence
\[
0 \rightarrow S \rightarrow A \rightarrow LK^{-1}(-D) \rightarrow 0,
\]
where \(D\) is an effective divisor. If \(\rk S = g + 1\) or \(S = 0\), we easily conclude that \(\mu(A) < \mu(\mathcal{E}_e) = 2\). If \(\rk S < g + 1\) and \(S \neq 0\), then stability of \(E_L\) gives the inequality \(\mu(S) < \mu(E_L) = 2 - \frac{1}{g+1}\). We introduce the integer \(\delta = 2\rk S - \deg S\). Then the previous inequality is equivalent to \(\delta \geq 1\). Now we compute
\[
\mu(A) = \frac{\deg S + \deg LK^{-1}(-D)}{\rk S + 1} \leq \frac{2\rk S - \delta + 3}{\rk S + 1} = 2 + \frac{1 - \delta}{\rk S + 1} \leq 2 = \mu(\mathcal{E}_e),
\]
which shows the semi-stability of \(\mathcal{E}_e\). \(\square\)

We tensorize the exact sequence \((1)\) with \(L\) and take the cohomology
\[
0 \rightarrow H^0(E_L^* \otimes L) \rightarrow H^0(L) \otimes H^0(L) \xrightarrow{\mu} H^0(L^2) \rightarrow 0.
\]

Note that \(h^1(E_L^* \otimes L) = h^0(E_L \otimes KL^{-1}) = 0\) by stability of \(E_L\). The second map \(\mu\) is the multiplication map and factorizes through \(\text{Sym}^2 H^0(L)\), i.e.,
\[
\Lambda^2 H^0(L) \subset H^0(E_L^* \otimes L) = \ker \mu.
\]

By Serre duality a cohomology class \(e \in \text{Ext}^1(LK^{-1}, E_L) = H^1(E_L \otimes KL^{-1}) = H^0(E_L^* \otimes L)^*\) can be viewed as a hyperplane \(H_e \subset H^0(E_L^* \otimes L)\). Then we have the following
Proposition 2.2. If $\Lambda^2 H^0(L) \subset H_e$, then the vector bundle $E_e$ satisfies

$$h^0(E_e \otimes \lambda) > 0, \quad \forall \lambda \in \text{Pic}^{g-3}(C).$$

Proof. We tensorize the exact sequence (1) with $\lambda \in \text{Pic}^{g-3}(C)$ and take the cohomology

$$0 \to H^0(E_L \otimes \lambda) \to H^0(E_e \otimes \lambda) \to H^0(LK^{-1}\lambda) \xrightarrow{\cup e} H^1(E_L \otimes \lambda) \to \cdots$$

Since $\text{deg} LK^{-1}\lambda = g$, we can write $LK^{-1}\lambda = \mathcal{O}_C(D)$ for some effective divisor $D$. It is enough to show that $h^0(E_e \otimes \lambda) > 0$ holds for $\lambda$ general. Hence we can assume that $h^0(LK^{-1}\lambda) = h^0(\mathcal{O}_C(D)) = 1$.

If $h^0(E_L \otimes \lambda) > 0$, we are done. So we assume $h^0(E_L \otimes \lambda) = 0$, which implies $h^1(E_L \otimes \lambda) = 1$ by Riemann-Roch. Hence we obtain that $h^0(E_e \otimes \lambda) > 0$ if and only if the cup product map

$$\cup e : H^0(\mathcal{O}_X(D)) \to H^1(E_L \otimes \lambda) = H^0(E^*_L \otimes (L(-D))^*)$$

is zero. Furthermore $\cup e$ is zero if and only if $H^0(E^*_L \otimes (L(-D)) \subset H_e$. Now we will show the inclusion

$$h^0(E^*_L \otimes (L(-D))) \subset \Lambda^2 H^0(L).$$

We tensorize the exact sequence (1) with $L(-D)$ and take cohomology

$$0 \to H^0(E^*_L \otimes (L(-D))) \to H^0(L) \otimes H^0(L(-D)) \xrightarrow{\mu} H^0(L^2(-D)) \to \cdots$$

Since $h^0(E^*_L \otimes (L(-D))) = 1$, we conclude that $h^0(L(-D)) = 2$ and $H^0(E^*_L \otimes (L(-D)) = \Lambda^2 H^0(L(-D)) \subset \Lambda^2 H^0(L)$.

Finally the proposition follows: if $\Lambda^2 H^0(L) \subset H_e$, then by (1) $H^0(E^*_L \otimes (L(-D)) \subset H_e$ for general $D$, or equivalently $h^0(E_e \otimes \lambda) > 0$ for general $\lambda \in \text{Pic}^{g-3}(C)$.

We introduce the linear subspace $\Gamma \subset \text{Ext}^1(LK^{-1}, E_L)$ defined by

$$\Gamma := \ker (\text{Ext}^1(LK^{-1}, E_L) = H^0(E^*_L \otimes L)^* \to \Lambda^2 H^0(L^*)$$

which has dimension $\frac{ag - 1}{2} > 0$. Then for any non-zero cohomology class $e \in \Gamma$ and any $\gamma \in \text{Pic}^2(C)$ satisfying $\gamma^{g+2} = L^2 K^{-1} = \text{det} E_e$, the rank-$(g + 2)$ vector bundle

$$E_e \otimes \gamma^{-1}$$

has trivial determinant, is semi-stable by Proposition 2.1 and has no theta divisor by Proposition 2.2.

2.2. Hyperelliptic curves. In this subsection we assume that $C$ is hyperelliptic and we denote by $\sigma$ the hyperelliptic involution. The construction of a semi-stable rank-4 vector bundle without theta divisor has been given in (2) section 6 in the case $g = 2$, but it can be carried out for any $g \geq 2$ without major modification. For the convenience of the reader, we recall the construction and refer to (2) for the details and the proofs.

Let $w \in C$ be a Weierstrass point. Any non-trivial extension

$$0 \to \mathcal{O}_C(-w) \to G \to \mathcal{O}_C \to 0$$

is a stable, $\sigma$-invariant, rank-2 vector bundle with $\text{deg} G = -1$. By (2) Theorem 4 a cohomology class $e \in H^1(\text{Sym}^2 G)$ determines a symplectic rank-4 bundle

$$0 \to G \to E_e \to G^* \to 0.$$
Moreover it is easily seen that, for any non-zero class $e$, the vector bundle $E_e$ is semi-stable. By Lemma 16 the composite map

$$D_G : \mathbb{P}H^1(\text{Sym}^2 G) \rightarrow \mathcal{M}_4 \overset{\partial}{\rightarrow} |4\Theta|, \quad e \mapsto \partial(E_e)$$

is the projectivization of a linear map

$$\tilde{D}_G : H^1(\text{Sym}^2 G) \rightarrow H^0(\text{Pic}^{g-1}(C), 4\Theta).$$

The involution $i(L) = KL^{-1}$ on $\text{Pic}^{g-1}(C)$ induces a linear involution on $|4\Theta|$ with eigenspaces $|4\Theta|_{\pm}$. Note that $4\Theta \in |4\Theta|_{+}$. We now observe that $\partial(E) \in |4\Theta|_{+}$ for any symplectic rank-4 vector bundle $E$ — see e.g. [B2]. Moreover we have the equality $\tilde{\theta}(\sigma^*E) = i^*\theta(E)$ for any vector bundle $E$. These two observations imply that the linear map $\tilde{D}_G$ is equivariant with respect to the induced involutions $\sigma$ and $i$. Since $\text{im} \tilde{D}_G \subset H^0(\text{Pic}^{g-1}(C), 4\Theta)_{+}$, we obtain that one of the two eigenspaces $H^1(\text{Sym}^2 G)_{\pm}$ is contained in the kernel $\ker \tilde{D}_G$, hence give base points for the theta map. We now compute as in [Hi] using the Atiyah-Bott-fixed-point formula

$$h^1(\text{Sym}^2 G)_{+} = g - 1, \quad h^1(\text{Sym}^2 G)_{-} = 2g + 1.$$

One can work out that $H^1(\text{Sym}^2 G)_{+} \subset \ker \tilde{D}_G$. Hence any $E_e$ with non-zero $e \in H^1(\text{Sym}^2 G)_{+}$ is a semi-stable rank-4 vector bundle without theta divisor.

### References


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