Universal L^s-rate-optimality of L^r-optimal quantizers by dilatation and contraction

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Abstract

Let $r, s > 0$. For a given probability measure P on $\mathbb{R}^d$, let $(\alpha_n)_{n\geq 1}$ be a sequence of (asymptotically) $L^r(P)$- optimal quantizers. For all $\mu \in \mathbb{R}^d$ and for every $\theta > 0$, one defines the sequence $(\alpha_n^{\theta,\mu})_{n\geq 1}$ by : $\forall n \geq 1,\ \alpha_n^{\theta,\mu} = \mu + \theta(\alpha_n - \mu) = \{\mu + \theta(a - \mu),\ a \in \alpha_n\}$. In this paper, we are interested in the asymptotics of the L^s-quantization error induced by the sequence $(\alpha_n^{\theta,\mu})_{n\geq 1}$. We show that for a wide family of distributions, the sequence $(\alpha_n^{\theta,\mu})_{n\geq 1}$ is L^s-rate-optimal. For the Gaussian and the exponential distributions, one shows how to choose the parameter θ such that $(\alpha_n^{\theta,\mu})_{n\geq 1}$ satisfies the empirical measure theorem and probably be asymptotically L^s-optimal.

1 Introduction

Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space and let $X : (\Omega, \mathcal{A}, \mathbb{P}) \longrightarrow \mathbb{R}^d$ be a random variable with distribution $\mathbb{P}_X = P$. Let $\alpha \subset \mathbb{R}^d$ be a subset (a codebook) of size n. A Borel partition $C_a(\alpha)_{a\in\alpha}$ of $\mathbb{R}^d$ satisfying

$$C_a(\alpha) \subset \{x \in \mathbb{R}^d : |x - a| = \min_{b\in\alpha_n} |x - b|\},$$

where $|\cdot|$ denotes a norm on $\mathbb{R}^d$ is called a Voronoi partition of $\mathbb{R}^d$ (with respect to α and $|\cdot|$). The random variable $\widehat{X}^\alpha$ taking values in the codebook α defined by

$$\widehat{X}^\alpha = \sum_{a\in\alpha} a \mathbf{1}_{\{X\in C_a(\alpha)\}}.$$

is called a Voronoi quantization of X. In other words, it is the nearest neighbour projection of X onto the codebook (also called grid) α.

The n-$L^r(P)$-optimal quantization problem for P (or X) consists in the study of the best approximation of X by a Borel function taking at most n values. For $X \in L^r(\mathbb{P})$ this leads to the following optimization problem:

$$e_{n,r}(X) = \inf \{\|X - \widehat{X}^\alpha\|_r, \alpha \subset \mathbb{R}^d,\ \mathrm{card}(\alpha) \leq n\}$$

with

$$\|X - \widehat{X}^\alpha\|_r^r = \mathbb{E}\big(d(X, \alpha)\big)^r = \int_{\mathbb{R}^d} d(x, \alpha)^r dP(x).$$

Then we can write

$$e_{n,r}(X) = e_{n,r}(P) = \inf_{\substack{\alpha\subset\mathbb{R}^d\\ \text{card}(\alpha)\le n}} \left(\int_{\mathbb{R}^d} d(x,\alpha)^r dP(x)\right)^{1/r}. \tag{1.1}$$

We remind in what follows some definitions and results that will be used throughout the paper.

- For all $n \geq 1$, the infimum in (1.1) is reached at one (at least) grid $\alpha^\star$; $\alpha^\star$ is then called a L^r-optimal n-quantizer. In addition, if card(supp$(P)) \geq n$ then card$(\alpha^\star) = n$ (see [3] or [6]). Moreover the quantization error, $e_{n,r}(X)$, decreases to zero as n goes to infinity and the so-called Zador's Theorem mentionned below gives its convergence rate provided a moment assumption on X.

- Let $X \sim P$ and let $P = P_a + P_s$ be the Lebesgue decomposition of P with respect to the Lebesgue measure λ_d, where P_a denotes the absolutely continuous part and P_s the singular part of P.

  Zador Theorem (see [3]) : Suppose $\mathbb{E}|X|^{r+\eta} < +\infty$ for some $\eta > 0$. Then

  $$\lim_{n\to+\infty} n^{r/d}(e_{n,r}(P))^r = Q_r(P).$$

  with

  $$Q_r(P) = J_{r,d}\left(\int_{\mathbb{R}^d} f^{\frac{d}{d+r}} d\lambda_d\right)^{\frac{d+r}{d}} = J_{r,d}\,\|f\|_{\frac{d}{d+r}} \in [0,+\infty),$$
  $$J_{r,d} = \inf_{n\geq 1} e^r_{n,r}(U([0,1]^d)) \in (0,+\infty),$$

  where $U([0,1]^d)$ denotes the uniform distribution on the set $[0,1]^d$ and $f = \frac{dP_a}{d\lambda_d}$. Note that the moment assumption : $\mathbb{E}|X|^{r+\eta} < +\infty$ ensure that $\|f\|_{\frac{d}{d+r}}$ is finite. Furthermore, $Q_r(P) > 0$ if and only if P_a does not vanish.

- A sequence of n-quantizers $(\alpha_n)_{n\geq 1}$ is

  - $L^r(P)$-**rate-optimal** (or **rate-optimal** for X, $X \sim P$) if

    $$\limsup_{n\to+\infty} n^{1/d}\int_{\mathbb{R}^d} d(x,\alpha_n)^r dP(x) < +\infty,$$

  - **asymptotically** $L^r(P)$-**optimal** if

    $$\lim_{n\to+\infty} n^{r/d}\int_{\mathbb{R}^d} d(x,\alpha_n)^r dP(x) = Q_r(P)$$

  - $L^r(P)$-**optimal** if for all $n \geq 1$,

    $$e^r_{n,r}(P) = \int_{\mathbb{R}^d} d(x,\alpha_n)^r dP(x).$$

- **Empirical measure theorem** (see [3]) : Let $X \sim P$. Suppose P is absolutely continuous with respect to λ_d and $\mathbb{E}|X|^{r+\eta} < +\infty$ for some $\eta > 0$. Let $(\alpha_n)_{n\geq 1}$ be an asymptotically $L^r(P)$-optimal sequence of quantizers. Then

$$\frac{1}{n}\sum_{a\in\alpha_n}\delta_a \xrightarrow{w} P_r \tag{1.2}$$

where $\xrightarrow{w}$ denotes the weak convergence and for every Borel set A of $\mathbb{R}^d$, P_r is defined by

$$P_r(A) = \frac{1}{C_{f,r}}\int_A f(x)^{\frac{d}{d+r}}d\lambda_d(x), \text{ with } \quad C_{f,r} = \int_{\mathbb{R}^d} f(x)^{\frac{d}{d+r}}d\lambda_d(x). \tag{1.3}$$

- In [4] is established the following proposition.

  Proposition : Let $X \sim P$, with $P_a \neq 0$, such that $\mathbb{E}|X|^{r+\eta} < \infty$, for some $\eta > 0$. Let (α_n) be an $L^r(P)$-optimal sequence of quantizers, $b \in \big(0, 1/2\big)$ and let $\psi_b : \mathbb{R}^d \longrightarrow \mathbb{R}_+ \cup \{+\infty\}$ be the maximal function defined by

$$\psi_b(x) = \sup_{n\geq 1}\frac{\lambda_d(B(x, bd(x,\alpha_n)))}{P(B(x, bd(x,\alpha_n)))}. \tag{1.4}$$

  Then for every $x \in \mathbb{R}^d$,

$$\forall n \geq 1, \quad n^{1/d}d(x,\alpha_n) \leq C(b)\psi_b(x)^{1/(d+r)} \tag{1.5}$$

  where $C(b)$ denotes a real constant not depending on n.

- The next proposition is established in [2]. It is used to compute the L^r-optimal quantizers for the exponential distribution.

  Proposition Let $r > 0$ and let X be an exponentially distributed random variable with scale parameter $\lambda > 0$. Then for every $n \geq 1$, the L^r-optimal quantizer $\alpha_n = (\alpha_{n1}, \cdots, \alpha_{nn})$ is unique and given by

$$\alpha_{nk} = \frac{a_n}{2} + \sum_{i=n+1-k}^{n-1} a_i, \quad 1 \leq k \leq n, \tag{1.6}$$

  where $(a_k)_{k\geq 1}$ is a $\mathbb{R}_+$-valued sequence defined by the following implicit recursive equation:

$$a_0 := +\infty, \quad \phi(-a_{k+1}) := \phi(a_k), \; k \geq 0$$

  with $\phi(x) := \int_0^{x/2}|u|^{r-1}\text{sign}(u)e^{-u}du$ (convention : $0^0 = 1$).
  Furthermore, the sequence $(a_k)_{k\geq 1}$ decreases to zero and for every $k \geq 1$,

$$a_k = \frac{r+1}{k}\big(1 + \frac{c_r}{k} + \mathrm{O}(\frac{1}{k^2})\big)$$

  for some real constant c_r.

NOTATIONS

- Let α_n be a set of n points of $\mathbb{R}^d$. For every $\mu \in \mathbb{R}^d$ and every $\theta > 0$ we denote $\alpha_n^{\theta,\mu} = \mu + \theta(\alpha_n - \mu) = \{\mu + \theta(a - \mu), \; a \in \alpha_n\}$.

- Let $f : \mathbb{R}^d \longrightarrow \mathbb{R}^d$ be a Borel function and let $\mu \in \mathbb{R}^d, \theta > 0$. One notes by $f_{\theta,\mu}$ (or f_θ if $\mu = 0$) the function defined by $f_{\theta,\mu}(x) = f(\mu + \theta(x-\mu)),\ x \in \mathbb{R}^d$.

- If $X \sim P$, $P_{\theta,\mu}$ will stand for the probability measure of the random variable $\frac{X-\mu}{\theta} + \mu,\ \theta > 0, \mu \in \mathbb{R}^d$. In other words, it is the distribution image of P by $x \longmapsto \frac{x-\mu}{\theta} + \mu$. Note that if $P = f \cdot \lambda_d$ then $P_{\theta,\mu} = f_{\theta,\mu} \cdot \lambda_d$.

- If A is a matrix A' stands for its transpose.

- Set $x = (x_1, \cdots, x_d);\ y = (y_1, \cdots, y_d) \in \mathbb{R}^d$; we denote $[x, y] = [x_1, y_1] \times \cdots \times [x_d, y_d]$.

Definition 1.1. *A sequence of quantizers $(\beta_n)_{n\geq1}$ is called a **dilatation** of the sequence $(\alpha_n)_{n\geq1}$ with **scaling number** θ and **translating number** μ if, for every $n \geq 1,\ \beta_n = \alpha_n^{\theta,\mu}$, with $\theta > 1$. If $\theta < 1$, one defines likewise the **contraction** of the sequence $(\alpha_n)_{n\geq1}$ with **scaling number** θ and **translating number** μ.*

2 Lower estimate

Let $r, s > 0$. Consider an asymptotically $L^r(P)$-optimal sequence of quantizers $(\alpha_n)_{n\geq1}$. For every $\mu \in \mathbb{R}^d$ and every $\theta > 0$, we construct the sequence $(\alpha_n^{\theta,\mu})_{n\geq1}$ and try to lower bound asymptotically the L^s-quantization error induced by this sequence. This estimation provides a necessary condition of rate-optimality for the sequence $(\alpha_n^{\theta,\mu})_{n\geq1}$. Obviously, in the particular case where $\theta = 1$ and $\mu = 0$ we get the same result as in [4], a paper we will essentially draw on.

Theorem 2.1. *Let $r, s \in (0, +\infty)$, and let X be a random variable taking values in $\mathbb{R}^d$ with distribution P such that $P_a = f.\lambda_d \not\equiv 0$. Suppose that $\mathbb{E}|X|^{r+\eta} < \infty$ for some $\eta > 0$. Let $(\alpha_n)_{n\geq1}$ be an asymptotically $L^r(P)$-optimal sequence of quantizers. Then, for every $\theta > 0$ and every $\mu \in \mathbb{R}^d$,*

$$\liminf_{n\to+\infty} n^{s/d} \, \|X - \widehat{X}^{\alpha_n^{\theta,\mu}}\|_s^s \geq Q_{r,s}^{Inf}(P, \theta), \tag{2.1}$$

with

$$Q_{r,s}^{Inf}(P, \theta) = \theta^{s+d} J_{s,d} \left(\int_{\mathbb{R}^d} f^{\frac{d}{d+r}} d\lambda_d \right)^{s/d} \int_{\{f>0\}} f_{\theta,\mu} f^{-\frac{s}{d+r}} d\lambda_d.$$

Proof. Let $m \geq 1$ and

$$f_m^{\theta,\mu} = \sum_{k,l=0}^{m2^m-1} \frac{l}{2^m} \mathbf{1}_{E_k^m \cap G_l^m};$$

with

$$E_k^m = \left\{ \frac{k}{2^m} \leq f < \frac{k+1}{2^m} \right\} \cap B(0, m) \text{ and } G_l^m = \left\{ \frac{l}{2^m} \leq f_{\theta,\mu} < \frac{l+1}{2^m} \right\} \cap B(0, m).$$

The sequence $(f_m^{\theta,\mu})_{m\geq1}$ is non-decreasing and

$$\lim_{m\to+\infty} f_m^{\theta,\mu} = f_{\theta,\mu} \qquad \lambda_d \text{ p.p.}$$

Let

$$I_m = \{(k,l) \in \{0,\cdots,m2^m-1\}^2 : \lambda_d(E_k^m) > 0; \lambda_d(G_l^m) > 0\}.$$

For every $(k,l) \in I_m$ there exists compact sets K_k^m and L_l^m such that :

$$K_k^m \subset E_k^m \;,\; L_l^m \subset G_l^m \;,\; \lambda_d(E_k^m \backslash K_k^m) \leq \frac{1}{m^4 2^{2m+1}} \text{ and } \lambda_d(G_l^m \backslash L_l^m) \leq \frac{1}{m^4 2^{2m+1}}.$$

Then

$$\begin{aligned} (E_k^m \cap G_l^m)\backslash(K_k^m \cap L_l^m) &= E_k^m \cap G_l^m \cap ((K_k^m)^c \cup (L_l^m)^c) \\ &\subset (E_k^m \backslash K_k^m) \cup (G_l^m \backslash L_l^m). \end{aligned}$$

Consequently,

$$\begin{aligned} \lambda_d(E_k^m \cap G_l^m \backslash K_k^m \cap L_l^m) &\leq \lambda_d(E_k^m \backslash K_k^m) + \lambda_d(G_l^m \backslash L_l^m) \\ &\leq \frac{1}{m^4 2^{2m+1}} + \frac{1}{m^4 2^{2m+1}} \\ &= \frac{1}{m^4 2^{2m}}. \end{aligned}$$

For every $m \geq 1$ and every $(k,l) \in I_m$, set

$$A_{k,l}^m := K_k^m \cap L_l^m,$$

$$\tilde{f}_m^{\theta,\mu} := \sum_{k,l=0}^{m2^m-1} \frac{l}{2^m} \mathbf{1}_{A_{k,l}^m},$$

and

$$\tilde{f}_m := \sum_{k,l=0}^{m2^m-1} \frac{k}{2^m} \mathbf{1}_{A_{k,l}^m}.$$

We get

$$\{f_m^{\theta,\mu} \neq \tilde{f}_m^{\theta,\mu}\} \subset \bigcup_{k,l \in \{0,\cdots,m2^m-1\}} \left((E_k^m \cap G_l^m) \backslash A_{k,l}^m\right).$$

Therefore, for every $m \geq 1$,

$$\lambda_d(\{f_m^{\theta,\mu} \neq \tilde{f}_m^{\theta,\mu}\}) \leq \sum_{k,l=0}^{m2^m-1} \frac{1}{m^4 2^{2m}} = \frac{1}{m^2}$$

and finally

$$\sum_{m \geq 1} \mathbf{1}_{\{f_m^{\theta,\mu} \neq \tilde{f}_m^{\theta,\mu}\}} < \infty \qquad \lambda_d \text{ p.p.}$$

As a consequently $\lambda_d(dx)$-p.p, $f_m^{\theta,\mu}(x) = \tilde{f}_m^{\theta,\mu}(x)$ for large enough m. Then $\tilde{f}_m^{\theta,\mu} \overset{\lambda_d \ p.p.}{\longrightarrow} f_{\theta,\mu}$ when $m \to +\infty$. Since in addition $A_{k,l}^m \subset E_k^m \cap G_l^m$ we obtain

$$\tilde{f}_m^{\theta,\mu} \leq f_m^{\theta,\mu} \leq f_{\theta,\mu}. \tag{2.2}$$

Moreover, for every $n \geq 1$,

$$\begin{aligned} n^{s/d} \, \|X - \widehat{X}^{\alpha_n^{\theta,\mu}}\|_s^s &= n^{s/d} \int_{\mathbb{R}^d} d(z, \mu + \theta(\alpha_n + \mu))^s f(z) \lambda_d(dz) \\ &\geq n^{s/d} \int_{\mathbb{R}^d} \min_{a \in \alpha_n} |z - (\mu + \theta(a - \mu))|^s f(z) \lambda_d(dz) \\ &\geq \theta^s n^{s/d} \int_{\mathbb{R}^d} \min_{a \in \alpha_n} |(z - \mu)/\theta + \mu - a|^s f(z) \lambda_d(dz). \end{aligned}$$

Making the change of variable $x := (z - \mu)/\theta + \mu$ yields:

$$\begin{aligned} n^{s/d} \, \|X - \widehat{X}^{\alpha_n^{\theta,\mu}}\|_s^s &\geq \theta^{s+d} n^{s/d} \int_{\mathbb{R}^d} d(x, \alpha_n)^s f_{\theta,\mu}(x) \lambda_d(dx) \\ &\geq \theta^{s+d} n^{s/d} \int_{\mathbb{R}^d} d(x, \alpha_n)^s \tilde{f}_m^{\theta,\mu} \lambda_d(dx) \qquad \text{(by (2.2))} \\ &= \theta^{s+d} n^{s/d} \sum_{k,l=0}^{m2^m-1} \frac{l}{2^m} \int_{A_{k,l}^m} d(x, \alpha_n)^s \lambda_d(dx). \end{aligned} \tag{2.3}$$

Let $m \geq 1$ and $(k,l) \in I_m$. Define the closed sets $\tilde{A}_{k,l}^m$ by $\tilde{A}_{k,l}^m = \emptyset$ if $\lambda_d(\tilde{A}_{k,l}^m) = 0$ and otherwise by

$$\tilde{A}_{k,l}^m = \{x \in \mathbb{R}^d : d(x, A_{k,l}^m) \leq \varepsilon_m\}$$

where $\varepsilon_m \in (0,1]$ is chosen so that

$$\int_{\tilde{A}_{k,l}^m} f^{\frac{d}{d+r}} d\lambda_d \leq \big(1 + 1/m\big) \int_{A_{k,l}^m} f^{\frac{d}{d+r}} d\lambda_d.$$

Since $\tilde{A}_{k,l}^m$ is compact ($\tilde{A}_{k,l}^m \subset B(0, m+1)$) $\forall (k,l)$, there exists (ref. [1], *Lemma 4.3*) a finite " firewall" set $\beta_{k,l}^m$ such that

$$\forall n \geq 1, \quad \forall x \in \tilde{A}_{k,l}^m, \quad d(x, \alpha_n \cup \beta_{k,l}^m) = d(x, (\alpha_n \cup \beta_{k,l}^m) \cap \tilde{A}_{k,l}^m).$$

The last inequality is in particular satisfied for all $x \in A_{k,l}^m$ since $A_{k,l}^m \subset \tilde{A}_{k,l}^m$.
Now set $\beta^m = \bigcup_{k,l} \beta_{k,l}^m$ and $n_{k,l}^m = \text{card}((\alpha_n \cup \beta^m) \cap \tilde{A}_{k,l}^m)$. The empirical measure theorem (see (1.2)) yields

$$\limsup_n \frac{\text{card}(\alpha_n \cap \tilde{A}_{k,l}^m)}{n} = \frac{\int_{\alpha_n \cap \tilde{A}_{k,l}^m} f^{\frac{d}{d+r}} d\lambda_d}{\int f^{\frac{d}{d+r}} d\lambda_d} \leq \frac{\int_{\tilde{A}_{k,l}^m} f^{\frac{d}{d+r}} d\lambda_d}{\int f^{\frac{d}{d+r}} d\lambda_d}.$$

Moreover

$$\frac{n_{k,l}^m}{n} \sim \frac{\text{card}(\alpha_n \cap \tilde{A}_{k,l}^m)}{n} \qquad \text{when } \; n \to +\infty$$

then

$$\liminf_{n \to +\infty} \frac{n}{n_{k,l}^m} \geq \frac{\int f^{\frac{d}{d+r}} d\lambda_d}{\int_{\tilde{A}_{k,l}^m} f^{\frac{d}{d+r}} d\lambda_d} \geq \frac{m}{m+1} \frac{\int f^{\frac{d}{d+r}} d\lambda_d}{\int_{A_{k,l}^m} f^{\frac{d}{d+r}} d\lambda_d}. \tag{2.4}$$

In the other hand,

$$\begin{aligned}\int_{A^m_{k,l}} d(x,\alpha_n)^s\lambda_d(dx) &\geq \int_{A^m_{k,l}} d(x,(\alpha_n\cup\beta^m_{k,l})\cap\tilde{A}^m_{k,l})^s\lambda_d(dx)\\ &= \lambda_d(A^m_{k,l})\int d(x,(\alpha_n\cup\beta^m_{k,l})\cap\tilde{A}^m_{k,l})^s\mathbf{1}_{A^m_{k,l}}(x)\frac{\lambda_d(dx)}{\lambda_d(A^m_{k,l})}\\ &\geq \lambda_d(A^m_{k,l})e^s_{n^m_{k,l},s}(U(A^m_{k,l})),\end{aligned}$$

where $U(A)=\mathbf{1}_A/\lambda_d(A)$ denotes the uniform distribution in the Borel set A when $\lambda_d(A)\neq 0$. Then we can write for every $(k,l)\in I_m$,

$$\liminf_{n\to+\infty} n^{s/d}\int_{A^m_{k,l}} d(x,\alpha_n)^s\lambda_d(dx)\geq \lambda_d(A^m_{k,l})\liminf_n\left(\frac{n}{n^m_{k,l}}\right)^{s/d}\liminf_n n^{s/d}e^s_{n,s}(U(A^m_{k,l})),$$

since

$$\liminf_n n^{s/d}e^s_{n,s}(U(A^m_{k,l}))\geq J_{s,d}\cdot\lambda_d(A^m_{k,l})^{s/d}.$$

Owing to Equation (2.4), one has

$$\liminf_{n\to+\infty} n^{s/d}\int_{A^m_{k,l}} d(x,\alpha_n)^s\lambda_d(dx)\geq\lambda_d(A^m_{k,l})\left(\frac{m}{m+1}\frac{\int f^{\frac{d}{d+r}}d\lambda_d}{\int_{A^m_{k,l}} f^{\frac{d}{d+r}}d\lambda_d}\right)^{s/d} J_{s,d}\cdot\lambda_d(A^m_{k,l})^{s/d}.$$

However, on the sets $A^m_{k,l}$, the statement $\frac{1}{f}\geq\left(\frac{k+1}{2^m}\right)^{-1}$ holds since $f<\frac{k+1}{2^m}$ on E^m_k. Hence

$$\liminf_{n\to+\infty} n^{s/d}\int_{A^m_{k,l}} d(x,\alpha_n)^s\lambda_d(dx)\geq J_{s,d}\left(\frac{m+1}{m}\int f^{\frac{d}{d+r}}\lambda_d(dx)\right)^{s/d}\left(\frac{k+1}{2^m}\right)^{-\frac{d}{d+r}\cdot\frac{s}{d}}\lambda_d(A^m_{k,l}).$$

It follows from Equation (2.3) and the super-additivity of the liminf that for every $m\geq 1$,

$$\begin{aligned}\liminf_n n^{s/d}\,\|X-\widehat{X}^{\alpha^{\theta,\mu}_n}\|^s_s &\geq \theta^{s+d}J_{s,d}\left(\frac{m+1}{m}\int f^{\frac{d}{d+r}}\lambda_d(dx)\right)^{s/d}\sum_{k,l=0}^{m2^m-1}\frac{l}{2^m}\left(\frac{k+1}{2^m}\right)^{-\frac{s}{d+r}}\lambda_d(A^m_{k,l})\\ &\geq \theta^{s+d}J_{s,d}\left(\frac{m+1}{m}\int f^{\frac{d}{d+r}}\lambda_d(dx)\right)^{s/d}\int_{\{f>0\}}\tilde{f}^{\theta,\mu}_m(\tilde{f}_m+2^{-m})^{-\frac{s}{d+r}}d\lambda_d.\end{aligned}$$

Finally, applying Fatou's Lemma yields

$$\liminf_{n\to+\infty} n^{s/d}\,\|X-\widehat{X}^{\alpha^{\theta,\mu}_n}\|^s_s\geq\theta^{s+d}J_{s,d}\left(\int_{\mathbb{R}^d} f^{\frac{d}{d+r}}d\lambda_d\right)^{s/d}\int_{\{f>0\}} f_{\theta,\mu}f^{-\frac{s}{d+r}}d\lambda_d.$$

□

3 Upper estimate

Let $r,s>0$. Let $(\alpha_n)_{n\geq1}$ be an (asymptotically) $L^r(P)$ - optimal sequence of quantizers. In this section we will provide some sufficient conditions of $L^s(P)$-rate-optimality for the sequence $(\alpha^{\theta,\mu}_n)_{n\geq1}$.

Definition 3.1. *Let $\theta > 0,\ \mu \in \mathbb{R}^d$ and let P be a probability distribution such that $P = f \cdot \lambda_d$. The couple (θ, μ) is said P-**admissible** if*

$$\{f > 0\} \subset \mu(1-\theta) + \theta\{f > 0\} \qquad \lambda_d\text{-}p.p. \tag{3.1}$$

Theorem 3.1. *Let $r, s \in (0, +\infty), s < r$ and let X be a random variable taking values in $\mathbb{R}^d$ with distribution P such that $P = f \cdot \lambda_d$. Suppose that (θ, μ) is P-admissible, for $\theta > 0; \mu \in \mathbb{R}$, and $\mathbb{E}|X|^{r+\eta} < \infty,$ for some $\eta > 0$. Let $(\alpha_n)_{n\geq 1}$ be an asymptotically L^r-optimal sequence. If*

$$\int_{\{f>0\}} f_{\theta,\mu}^{\frac{r}{r-s}} f^{-\frac{s}{r-s}} d\lambda_d < +\infty \tag{3.2}$$

then, $(\alpha_n^{\theta,\mu})_{n\geq 1}$ is $L^s(P)$-rate-optimal and

$$\limsup_{n\to+\infty} n^{s/d} \, \|X - \widehat{X}^{\alpha_n^{\theta,\mu}}\|_s^s \leq \theta^{s+d} \, (Q_r(P))^{s/r} \left(\int_{\{f>0\}} f_{\theta,\mu}^{\frac{r}{r-s}} f^{-\frac{s}{r-s}} d\lambda_d \right)^{1-\frac{s}{r}}. \tag{3.3}$$

Remark 3.1. *Note that if $\theta = 1$ and $\mu = 0$ then*

$$\int_{\{f>0\}} f_{\theta,\mu}^{\frac{r}{r-s}} f^{-\frac{s}{r-s}} d\lambda_d = \int_{\{f>0\}} f^{\frac{r}{r-s}} f^{-\frac{s}{r-s}} d\lambda_d = \int_{\{f>0\}} f d\lambda_d = 1.$$

In this case the theorem is trivial since $\|X - \widehat{X}^{\alpha_n}\|_s \leq \|X - \widehat{X}^{\alpha_n}\|_r$.

Proof. Let P^θ denotes the distribution of the random variable θX. P^θ is absolutely continuous with respect to λ_d, with $p.d.f\ g_\theta(x) = \theta^{-d} f(\frac{x}{\theta})$.

For every $n \geq 1$,

$$\begin{aligned} n^{s/d} \, \|X - \widehat{X}^{\alpha_n^{\theta,\mu}}\|_s^s &= n^{s/d} \int_{\mathbb{R}^d} d(x, \alpha_n^{\theta,\mu})^s dP(x) \\ &= n^{s/d} \int_{\{f>0\}} \min_{a\in\alpha_n} |x - \mu(1-\theta) - \theta a|^s f(x) d\lambda_d(x). \end{aligned}$$

Making the change of variable $z := x - \mu(1-\theta)$ yields

$$\begin{aligned} n^{s/d} \, \|X - \widehat{X}^{\alpha_n^{\theta,\mu}}\|_s^s &= n^{s/d} \int_{\{f>0\}-\mu(1-\theta)} d(z, \theta\alpha_n)^s f(z + \mu(1-\theta)) d\lambda_d(z) \\ &\leq n^{s/d} \int_{\theta\{f>0\}} d(z, \theta\alpha_n)^s f(z + \mu(1-\theta)) g_\theta^{-1}(z) dP^\theta(z) \\ &\leq n^{s/d} \left(\int_{\mathbb{R}^d} d(z, \theta\alpha_n)^r dP^\theta(z) \right)^{s/r} \left(\int_{\theta\{f>0\}} \left(f(z + \mu(1-\theta)) g_\theta^{-1}(z) \right)^{\frac{r}{r-s}} dP^\theta(z) \right)^{\frac{r-s}{r}} \\ &\leq \left(n^{r/d} \|\theta X - \widehat{\theta X}^{\theta\alpha_n}\|_r^r \right)^{s/r} \left(\int_{\theta\{f>0\}} f(z + \mu(1-\theta))^{\frac{r}{r-s}} g_\theta^{-\frac{s}{r-s}}(z) d\lambda_d(z) \right)^{\frac{r-s}{r}} \end{aligned} \tag{3.4}$$

where we used the P-admissibility of (θ, μ) in the first inequality. The second inequality derives from *Hölder* inequality applied with $p = r/s > 1$ and $q = 1 - s/r$.

Moreover

$$\|\theta X - \widehat{\theta X}^{\theta\alpha_n}\|_r^r = \mathbb{E}\big(\min_{a\in\alpha_n} |\theta X - \theta a|^r\big) = \theta^r \|X - \widehat{X}^{\alpha_n}\|_r^r. \tag{3.5}$$

Then

$$n^{s/d}\,\|X - \widehat{X}^{\alpha_n^{\theta,\mu}}\|_s^s \le \theta^s \left(n^{r/d}\|X - \widehat{X}^{\alpha_n}\|_r^r\right)^{s/r} \left(\int_{\theta\{f>0\}} f(z+\mu(1-\theta))^{\frac{r}{r-s}} g_\theta^{-\frac{s}{r-s}}(z)d\lambda_d(z)\right)^{\frac{r-s}{r}}.$$

Owing to the asymptotically $L^r(P)$-optimality of (α_n) and making again the change of variable $x := z/\theta$ yields

$$\begin{aligned}
\limsup_{n\to+\infty} n^{s/d}\,\|X - \widehat{X}^{\alpha_n^{\theta,\mu}}\|_s^s &\le \theta^s\,(Q_r(P))^{s/r}\left(\theta^{\frac{ds}{r-s}}\int_{\theta\{f>0\}} f(z+\mu(1-\theta)))^{\frac{r}{r-s}} f(z/\theta)^{-\frac{s}{r-s}}d\lambda_d(z)\right)^{\frac{r-s}{r}}\\
&= \theta^s\,(Q_r(P))^{s/r}\left(\theta^{\frac{rd}{r-s}}\int_{\{f>0\}} f_{\theta,\mu}(x)^{\frac{r}{r-s}} f(x)^{-\frac{s}{r-s}}d\lambda_d(x)\right)^{\frac{r-s}{r}}\\
&= \theta^{s+d}\,(Q_r(P))^{s/r}\left(\int_{\{f>0\}} f_{\theta,\mu}(x)^{\frac{r}{r-s}} f(x)^{-\frac{s}{r-s}}d\lambda_d(x)\right)^{\frac{r-s}{r}}.
\end{aligned}$$

□

When $s > r$, the next theorem provides a less accurate asymptotic upper bound than the previous one since, beyond the restriction on the distribution of X, we need now the sequence (α_n) to be (exactly) $L^r(P)$-optimal.

Theorem 3.2. *Let $r, s \in (0, +\infty),\ s > r,\ \theta > 0$ and let X be a random variable taking values in $\mathbb{R}^d$ with distribution P such that $P = f\cdot\lambda_d$. Suppose that $\mathbb{E}|X|^{r+\eta} < \infty$ for some $\eta > 0$ and $P_{\theta,\mu} \ll P$ $\big($i.e $P_{\theta,\mu}$ is absolutely continuous with respect to P $\big)$ for some $\mu \in \mathbb{R}^d$. Let $(\alpha_n)_{n\ge1}$ be an $L^r(P)$-optimal sequence of quantizers and suppose that the maximal function (see (1.4)) satisfies*

$$\psi_b^{s/(d+r)} \in\ L^1(P_{\theta,\mu}) \ \textit{ for some }\ b\in(0,1/2). \tag{3.6}$$

Then,

$$\limsup_n n^{s/d}\,\|X - \widehat{X}^{\alpha_n^{\theta,\mu}}\|_s^s \le \theta^{s+d}C(b)\int f_{\theta,\mu}f^{-\frac{s}{d+r}}d\lambda_d < +\infty \tag{3.7}$$

where $C(b)$ is a positive real constant not depending on θ and n.

Notice that this theorem does not require that (θ, μ) is P-admissible.

Proof. One deduces from differentiation of measures that

$$f^{-\frac{s}{d+r}} \ \le\ \psi_b^{\frac{s}{d+r}} \qquad P_{\theta,\mu}\text{-a.s.}$$

Then, under Assumption (3.6),

$$\int f^{-\frac{s}{d+r}}dP_{\theta,\mu} = \int f_{\theta,\mu}f^{-\frac{s}{d+r}}d\lambda_d < +\infty.$$

For all $n \geq 1$,

$$\begin{aligned} n^{s/d}\, \|X - \widehat{X}^{\alpha_n^{\theta,\mu}}\|_s^s &= n^{s/d} \int_{\mathbb{R}^d} d(z, \alpha_n^{\theta,\mu})^s f(z) d\lambda_d(z) \\ &= n^{s/d}\theta^s \int_{\mathbb{R}^d} \min_{a\in\alpha_n} |(z-\mu)/\theta + \mu - a|^s f(z) d\lambda_d(z) \end{aligned}$$

We make the change of variable $x := (z-\mu)/\theta + \mu$. Then

$$\begin{aligned} n^{s/d}\, \|X - \widehat{X}^{\alpha_n^{\theta,\mu}}\|_s^s &= n^{s/d}\theta^{s+d} \int_{\mathbb{R}^d} d(x, \alpha_n)^s f(\mu + \theta(x-\mu)) d\lambda_d(x) \\ &= n^{s/d}\theta^s \int_{\mathbb{R}^d} d(x, \alpha_n)^s dP_{\theta,\mu}(x). \end{aligned}$$

Besides, the following inequalities are established in [4] :

$$\begin{aligned} \limsup_n n^{s/d} d(\cdot, \alpha_n)^s &\leq C(b) f^{-\frac{s}{d+r}} \\ \text{and} \qquad n^{s/d} d(\cdot, \alpha_n)^s &\leq C(b)\psi_b^{\frac{s}{d+r}} \qquad P\text{-a.s (hence } P_{\theta,\mu}\text{-a.s., since } P_{\theta,\mu} \ll P). \end{aligned}$$

Under Assumption (3.6) we can apply the Lebesgues dominated convergence theorem to the above inequalities, which yields

$$\begin{aligned} \limsup_n n^{s/d} \int d(x, \alpha_n)^s dP_{\theta,\mu}(x) &\leq \int \limsup_n n^{s/d} d(x, \alpha_n)^s dP_{\theta,\mu}(x) \\ &\leq C(b) \int f^{-\frac{s}{d+r}} dP_{\theta,\mu}(x). \\ &= \theta^d C(b) \int f_{\theta,\mu}(x) f^{-\frac{s}{d+r}}(x) d\lambda_d(x). \end{aligned}$$

□

For a given distribution, Assumption (3.6) is not easy to verify. But when $s \neq r + d$, the lemma and corollaries below provide a sufficient condition so that Assumption (3.6) is satisfied. The next subject extends the results obtained in ([4]). For further details we then refer to ([4]).

Let $P = f \cdot \lambda_d$ be an absolutely continous distribution. Let $r, s \in (0, +\infty)$ and (θ, μ) be a P-admissible couple of parameters. We will need the following hypotheses:

(H1) for all $M > 0$,

$$\sup_{z\in B(0,M)} \frac{f(\mu + \theta(z-\mu))}{f(z)} \mathbf{1}_{\{f(z)>0\}} < +\infty. \tag{3.8}$$

(H2) There exists $b \in (0, 1/2),\ M \in (0, +\infty)$ such that

$$\int_{B(0,M)^c} \left(\sup_{t\leq 2b|x|} \frac{\lambda_d(B(x,t))}{P(B(x,t))} \right)^{s/(d+r)} dP_{\theta,\mu} < +\infty. \tag{3.9}$$

(H3) $\lambda_d(\cdot \cap \text{supp}(P)) \ll P$ and $\text{supp}(P)$ is a finite union of closed convex sets.

Lemma 3.1. *Let $P = f \cdot \lambda_d$ and $r > 0$ such that $\int |x|^r P(dx) < +\infty$. Assume $(\alpha_n)_{n\geq 1}$ is a sequence of quantizers such that $\int d(x, \alpha_n)^r dP \to 0$. Let (θ, μ) be a P-admissible couple of parameters for which (**H1**) holds.*

(a) *If $p \in (0, 1)$ then for every $b > 0$, $\psi_b^p \in L^1_{loc}(P_{\theta,\mu})$.*

(b) *If $p \in (1, +\infty]$ and if furthermore **(H3)** holds then for every $b > 0$,*

$$f^{-p} \in L^1_{loc}(P) \Longrightarrow \psi_b^p \in L^1_{loc}(P_{\theta,\mu}).$$

Proof. It follows from the P-admissibility of (θ, μ) that

$$P_{\theta,\mu}(dz) = \theta^d f(\mu + \theta(z - \mu))\lambda_d(dz) = g_\theta(z)P(dz),$$

where $g_\theta(z) = \theta^d \, \frac{f(\mu+\theta(z-\mu))}{f(z)} \mathbf{1}_{\{f(z)>0\}}$. Then g_θ is locally bounded by (**H1**).

(a) If $p \in (0, 1)$, it follows from Lemma 1 in [4] that $\psi_b^p \in L^1_{\text{loc}}(P)$. Hence $\psi_b^p \in L^1_{\text{loc}}(P_{\theta,\mu})$ since g_θ is locally bounded.

(b) If $p \in (1, +\infty)$ it follows from Lemma 2 in [4] that if $f^{-p} \in L^1_{\text{loc}}(P)$ then $\psi_b^p \in L^1_{\text{loc}}(P_{\theta,\mu})$ since g_θ is locally bounded.

□

Corollary 3.1. *(Distributions with unbounded supports) Let $r > 0$, $s \in (0, +\infty)$, $s \neq r + d$ and let X be a random variable with probability measure $P = f \cdot \lambda_d$ such that $E|X|^{r+\eta} < +\infty$ for some $\eta > 0$. Let (θ, μ) be P-admissible and suppose that **(H1)**, **(H2)** hold.*

(a) *If $s \in (0, r + d)$ then Assumption (3.6) of Theorem 3.2 holds true.*

(b) *If $s \in (r + d, +\infty)$, and if furthermore, **(H3)** holds and $f^{-\frac{s}{r+d}} \in L^1_{loc}(P)$ then Assumption (3.6) of Theorem 3.2 holds true.*

Proof. Let $x_0 \in \text{supp}(P)$. We know from [1] that $d(x_0, \alpha_n) \to 0$. Then following the lines of the proof of Corollary 2 in [4] one has for $|x| > N = |x_0| + \sup_{n\geq 1} d(x_0, \alpha_n)$, $d(x, \alpha_n) \leq 2|x|$ for every $n \geq 1$. Thus for every $b > 0$, $x \in B(0, N)^c$,

$$\psi_b(x) \leq \sup_{t\leq 2b|x|} \frac{\lambda_d(B(x, t))}{P(B(x, t))}.$$

Now, coming back to the core of our proof, it follows from **(H2)** that (for b coming from **(H2)**),

$$\int_{B(0,M\vee N)^c} \psi_b^{s/(d+r)} dP_{\theta,\mu} < +\infty.$$

Since

$$\int \psi_b^{s/(d+r)} dP_{\theta,\mu} = \int_{B(0,M\vee N)} \psi_b^{s/(d+r)} dP_{\theta,\mu} + \int_{B(0,M\vee N)^c} \psi_b^{s/(d+r)} dP_{\theta,\mu},$$

it remains to show that the first term in the right hand side of this last equality is finite.

(a) If $s \in (0, r + d)$ it follows from Lemma 3.1, (a) that the first term in the right hand side of the above equality is finite. As a consequence, $\psi^{\frac{s}{r+d}} \in L^1(P_{\theta,\mu})$.

(b) If $s > r + d$, the first term in the right hand side of the above equality still finite owing to Lemma 3.1, (b). Consequently, Assumption (3.6) of Theorem 3.2 holds true provided **(H3)** holds and $f^{-\frac{s}{r+d}} \in L^1_{loc}(P)$.

□

We next give two useful criterions ensuring that Hypothesis **(H2)** holds. The first one is useful for distributions with radial tails and the second one for distributions which does not satisfy this last assumption.

Criterion 3.1. *Let X be a random variable with probability measure such that $P = f \cdot \lambda_d$ and $E|X|^{r+\eta} < +\infty$ for some $\eta > 0$.*
(a) *Let $r, s > 0$ and $f = h(|\cdot|)$ on $B_{|\cdot|}(0, N)^c$ with $h : (R, +\infty) \to \mathbb{R}_+,\ R \in \mathbb{R}_+$, a decreasing function and $|\cdot|$ any norm on $\mathbb{R}^d$. Suppose that (θ, μ) is a couple of P-admissible parameters such that*

$$\int f(cx)^{-\frac{s}{d+r}} dP_{\theta,\mu}(x) < +\infty \tag{3.10}$$

*for some $c > 1$. Then **(H2)** holds.*
(b) *Let $r, s > 0$. If $supp(P) \subset [R_0, +\infty)$ for some $R_0 \in \mathbb{R}$ and $f_{|(R_0', +\infty)}$ decreasing for $R_0' \geq R_0$. Assume furthermore that (θ, μ) is a couple of P-admissible parameters such that (3.10) is satisfied for some $c > 1$. Then Hypothesis **(H2)** holds.*

Note that (b) follows from (a), for $d = 1$, and that (a) is simply deduced from the proof of Corollary 3 in [4] since it has been showed that for $b \in (0, 1/2),\ M := N/(1-2b)$ one has for every $x \in B(0, M)^c$,

$$\sup_{t \leq 2b|x|} \frac{\lambda_d(B(x,t))}{P(B(x,t))} \leq \frac{1}{f(x(1+2b))}.$$

Criterion 3.2. *Let $r, s > 0,\ P = f \cdot \lambda_d$ and $\int |x|^{r+\eta} P(dx) < +\infty$ for some $\eta > 0$. Let (θ, μ) be a P-admissible couple such that*

$$\sup_{z \neq 0} \frac{f(\mu + \theta(z - \mu))}{f(z)} \mathbf{1}_{\{f(z)>0\}} < +\infty. \tag{3.11}$$

Assume furthermore that

$$\inf_{x \in supp(P), \rho > 0} \frac{\lambda_d(supp(P) \cap B(x, \rho))}{\lambda_d(B(x, \rho))} > 0$$

and that f satisfies the local growth control assumption : there exists real numbers $\varepsilon \geq 0,\ \eta \in (0, 1/2),\ M, C > 0$ such that

$$\forall x, y \in supp(P),\ |x| \geq M,\ |y - x| \leq 2\eta|x| \implies f(y) \geq C f(x)^{1+\varepsilon}.$$

If

$$\int f(x)^{-\frac{s(1+\varepsilon)}{d+r}} dP(x) < +\infty, \tag{3.12}$$

*then **(H2)** holds. If in particular f satisfies the local growth control assumption for $\varepsilon = 0$ or for every $\varepsilon \in (0, \underline{\varepsilon}]$, with $\underline{\varepsilon} > 0$, and if*

$$\int f(x)^{-\frac{s}{d+r}} dP(x) = \int_{\{f>0\}} f(x)^{1-\frac{s}{d+r}} d\lambda_d(x) < +\infty$$

*then Hypothesis **(H2)** holds.*

Notice that Hypothesis (3.11) can be relaxed if we suppose that $f(x)^{-\frac{s(1+\varepsilon)}{d+r}} \in L^1(P_{\theta,\mu})$ instead of (3.12). The criterion follows from Corollary 4 in [4].

4 Toward a necessary and sufficient condition for $L^s(P)$-rate optimality when $s > r$

Before dealing with examples, let us make some comments about inequalities (2.1) and (3.7). Note first that the moment assumption $\mathbb{E}|X|^{r+\eta} < +\infty$ for some $\eta > 0$, ensure that $\int_{\mathbb{R}^d} f^{\frac{d}{d+r}} d\lambda_d < +\infty$ (cf [3]). Consequently, if $\int f_{\theta,\mu} f^{-\frac{s}{d+r}} d\lambda_d = +\infty$ one derives from inequality (2.1) that

$$\lim_n \|X - \widehat{X}^{\alpha_n^{\theta,\mu}}\|_s^s = +\infty.$$

Then the sequence $(\alpha_n^{\alpha,\mu})_{n\geq 1}$ is not L^s-rate-optimal.

On the other hand if $\int f_{\theta,\mu} f^{-\frac{s}{d+r}} d\lambda_d < +\infty$ one derives from Inequality (3.7) that $(\alpha_n^{\theta,\mu})_{n\geq 1}$ is L^s-rate-optimal. For $s > r$, this leads to a necessary and sufficient condition so that the sequence $(\alpha_n^{\theta,\mu})_{n\geq 1}$ (in particular the sequence $(\alpha_n)_{n\geq 1}$ by taking $\theta = 1$ and $\mu = 0$) is L^s-rate-optimal.

Remark 4.1. *Let $\mu \in \mathbb{R}^d$, $\theta, r > 0,\ s > r$ and let P be a probability distribution such that $P = f \cdot \lambda_d$. Assume (θ, μ) is P-admissible. Let $(\alpha_n)_{n\geq 1}$ be an $L^r(P)$-optimal sequence of n-quantizers and suppose that Assumption (3.6) of Theorem 3.2 holds true. Then*

$$(\alpha_n^{\theta,\mu})_{n\geq 1} \text{ is } L^s\text{-rate-optimal} \iff \int f_{\theta,\mu} f^{-\frac{s}{d+r}} d\lambda_d < +\infty. \tag{4.1}$$

Remark 4.2. *If $s < r$, the inequality (3.3) provides a sufficient condition so that the sequence $(\alpha_n^{\theta,\mu})_{n\geq 1}$ is L^s-rate-optimal, which is : $\int f_{\theta,\mu}^{\frac{r}{r-s}} f^{-\frac{s}{r-s}} d\lambda_d < +\infty$ (always satisfied by $(\alpha_n)_{n\geq 1}$ itself).*

Now, for $s \neq r$, is it possible to find a $\theta = \theta^\star$ for which the sequence $(\alpha_n^{\theta,\mu})_{n\geq 1}$ is asymptotically $L^s(P)$-optimal? (when $s < r$ this is the only question of interest since we know that $(\alpha_n)_{n\geq 1}$ is $L^s(P)$-rate-optimal for every $s < r$).

For a fixed r, b and μ, we can write from inequalities (3.3) and (3.7) :

$$\limsup_n n^{s/d} \|X - \widehat{X}^{\alpha_n^{\theta,\mu}}\|_s^s \leq Q_{r,s}^{\text{Sup}}(P, \theta) \tag{4.2}$$

with

$$Q_{r,s}^{\text{Sup}}(P,\theta) = \begin{cases} \theta^{s+d} \left(Q_r(P)\right)^{s/r} \left(\int_{f>0} f_{\theta,\mu}^{\frac{r}{r-s}} f^{-\frac{s}{r-s}} d\lambda_d\right)^{1-\frac{s}{r}} & \text{if } s < r \\ \theta^{s+d} C(b) \int f_{\theta,\mu} f^{-\frac{s}{d+r}} d\lambda_d & \text{if } s > r. \end{cases}$$

One knows that for a given $s > 0$, we have for all $n \geq 1$,

$$e_{n,s}^s(X) \leq \|X - \widehat{X}^{\alpha_n^{\theta,\mu}}\|_s^s.$$

Then for every $\theta > 0$,

$$Q_s(P) \leq Q_{r,s}^{\text{Sup}}(P, \theta).$$

Consequently for a fixed $s > 0$, in order to have the best estimation of Zador's constant in L^s, we must minimize over θ, the quantity $Q_{r,s}^{\text{Sup}}(P,\theta)$. In that way, we may hope to reach the sharp rate of convergence in Zador Theorem and so construct an asymptotically L^s-optimal sequence.

For μ well chosen, the examples below show that, for the Gaussian and the exponential distribution, the minimum $\theta^\star$ exists and the sequence $(\alpha_n^{\theta^\star,\mu})_{n\geq 1}$ satisfies the empirical measure theorem and is suspected to be asymptotically L^s-optimal.

5 Examples

Let $(\alpha_n)_{n\geq 1}$ be an $L^r(P)$-optimal sequence of quantizers for a given probability distribution P, and consider the sequence $(\alpha_n^{\theta,\mu})_{n\geq 1}$. For a fixed μ and s, we try to solve the following minimization problem

$$\theta^\star = \arg\min_{\theta>0} \big\{Q^{\text{Sup}}_{r,s}(P,\theta),\ (\alpha_n^{\theta,\mu})_{n\geq 1}\ L^s(P)\text{-rate-optimal}\big\}. \tag{5.1}$$

In all examples, C will denote a generic real constant (not depending on θ) which may change from line to line. The choice of μ depends on the probability measure and it is not clear how to choose it. In practice, we shall set $\mu = \mathbb{E}(X)$ when X is a symmetric random variable otherwise we will usually set $\mu = 0$.

5.1 The multivariate Gaussian distribution

5.1.1 Optimal dilatation and contraction

Proposition 5.1. *Let $r, s > 0$ and let $P = \mathcal{N}(m;\Sigma),\ m \in \mathbb{R}^d, \Sigma \in \mathcal{S}^+(d,\mathbb{R})$.*

(a) *If $s \in (r, r+d) \cup (r+d, +\infty)$, the sequence $(\alpha_n^{\theta,m})_{n\geq 1}$ is $L^s(P)$-rate-optimal iff $\theta \in \big(\sqrt{s/(d+r)}, +\infty\big)$ and*

$$\theta^\star = \sqrt{(s+d)/(r+d)} \in (1, +\infty)$$

is the unique solution of (5.1) *on the set* $\big(\sqrt{s/(d+r)}, +\infty\big)$.

(b) *If $s \in (0, r)$, the sequence $(\alpha_n^{\theta,m})_{n\geq 1}$ is $L^s(P)$-rate-optimal if $\theta \in \big(\sqrt{s/r}, +\infty\big)$ and*

$$\theta^\star = \sqrt{(s+d)/(r+d)} \in (0, 1)$$

is the unique solution of (5.1) *on the set* $\big(\sqrt{s/r}, +\infty\big)$.

Proof. Since the multivariate Gaussian distribution is symmetric, one sets $\mu = m$. Keep in mind that the probability density function f of P is given for every $x \in \mathbb{R}^d$ by,

$$f(x) = \big((2\pi)^d \det \Sigma\big)^{-\frac{1}{2}} e^{-\frac{1}{2}(x-m)'\Sigma^{-1}(x-m)}.$$

Note first that Hypothesis **(H1)** is obviously satisfied from the continuity of $\frac{f(m+\theta(z-m))}{f(z)}\mathbf{1}_{\{f(z)>0\}}$ on every $\bar{B}(0,M),\ M > 0$.

(a) Let $s \in (r, d+r)$. For every $\theta > 0, \mu \in \mathbb{R}^d$, the couple (θ,μ) is P-admissible ($f > 0$) and f is radial since $f(x) = \varphi(|x-m|_\Sigma)$ with $\varphi : (0,+\infty) \longmapsto \mathbb{R}_+$ defined by

$$\varphi(\xi) = \big((2\pi)^d \det\Sigma\big)^{-1/2} \exp(-\frac{1}{2}|\xi|^2), \ \text{ with } |x|_\Sigma = |\Sigma^{-\frac{1}{2}}x|.$$

Let $\theta > \sqrt{s/(r+d)}$. Then Assumption (3.10) holds for every $c \in (1, \theta\sqrt{\frac{r+d}{s}})$. Consequently, it follows from Corollary 3.1, (a) that Assumption (3.6) of Theorem 3.2 holds.

If $s > d+r$, the required additional hypotheses **(H3)** and $f^{-\frac{s}{r+d}} \in L^1_{loc}(P)$ are clearly satisfied since $P = f\cdot\lambda_d$ (and f^{-1} is continuous ensuring that $\lambda_d(\cdot \cap \text{supp}(P)) \ll P$) and $f^{-\frac{s}{r+d}}$ is continuous on every $\bar{B}(0,M),\ M>0$. Then it follows from Corollary 3.1, (b) that Assumption (3.6) of Theorem 3.2 holds.

In the other hand

$$\begin{aligned}\int_{\mathbb{R}^d} f_{\theta,m}(x)f(x)^{-\frac{s}{d+r}}dx &= \int_{\mathbb{R}^d} f(m+\theta(x-m))f(x)^{-\frac{s}{d+r}}dx \\ &= C\int_{\mathbb{R}^d} e^{-\frac{1}{2}(\theta^2-\frac{s}{d+r})(x-m)'\Sigma^{-1}(x-m)}dx\end{aligned}$$

so that

$$\int_{\mathbb{R}^d} f_{\theta,m}(x)f(x)^{-\frac{s}{d+r}}<+\infty \quad \text{iff} \quad \theta>\sqrt{\frac{s}{d+r}}.$$

Now we are in position to solve the problem (5.1). Let $\theta\in\left(\sqrt{s/(d+r)},+\infty\right)$,

$$\begin{aligned}\theta^{s+d}\int_{\mathbb{R}^d} f_{\theta,m}(x)f(x)^{-\frac{s}{d+r}}dx &= \left((2\pi)^d det\ \Sigma\right)^{-\frac{1}{2}(1-\frac{s}{d+r})}\ \theta^{s+d}\int_{\mathbb{R}^d} e^{-\frac{1}{2}(\theta^2-\frac{s}{d+r})(x-m)'\Sigma^{-1}(x-m)}dx \\ &= \left((2\pi)^d det\ \Sigma\right)^{-\frac{s}{d+r}}\ \theta^{s+d}\left(\theta^2-\frac{s}{d+r}\right)^{-\frac{d}{2}}.\end{aligned}$$

For $\theta\in\left(\sqrt{s/(d+r)},+\infty\right)$, we want to minimize the function h defined by

$$h(\theta)=\theta^{s+d}\left(\theta^2-\frac{s}{d+r}\right)^{-\frac{d}{2}}.$$

The function h is differentiable on $\left(\sqrt{s/(d+r)},+\infty\right)$ with derivative

$$h'(\theta)=s\theta^{d+s-1}\left(\theta^2-\frac{s}{d+r}\right)^{-1-d/2}\left(\theta^2-\frac{s+d}{r+d}\right).$$

One easily checks that h reaches its unique minimum on $\left(\sqrt{s/(d+r)},+\infty\right)$ at $\theta^\star=\sqrt{(s+d)/(r+d)}$.

(b) Let $s<r$ and consider the inequality (3.3). We get

$$\int f_{\theta,m}^{\frac{r}{r-s}}(x)f^{-\frac{s}{r-s}}(x)dx=C\int_{\mathbb{R}^d} e^{-\frac{1}{2}\frac{r}{r-s}(\theta^2-\frac{s}{r})(x-m)'\Sigma^{-1}(x-m)}dx.$$

So if $\theta\in\left(\sqrt{s/r},+\infty\right)$ then $\int f_{\theta,m}^{\frac{r}{r-s}}(x)f^{-\frac{s}{r-s}}(x)dx<+\infty$. This proves the first assertion.

To prove the second assertion, let $\theta\in\left(\sqrt{s/r},+\infty\right)$. Then

$$\begin{aligned}\theta^{d+s}\left(\int f_{\theta,m}^{\frac{r}{r-s}}(x)f^{-\frac{s}{r-s}}(x)dx\right)^{1-\frac{s}{r}} &= C\ \theta^{s+d}\left(\int_{\mathbb{R}^d} e^{-\frac{1}{2}\frac{r}{r-s}(\theta^2-\frac{s}{r})(x-m)'\Sigma^{-1}(x-m)}dx\right)^{1-\frac{s}{r}} \\ &= C\ \theta^{s+d}\left(\theta^2-\frac{s}{r}\right)^{-\frac{d}{2r}(r-s)}.\end{aligned}$$

We proceede as before setting

$$h(\theta)=\theta^\alpha\left(\theta^2-\frac{s}{r}\right)^\beta,\ \text{with } \alpha=d+s \text{ and } \beta=-\frac{d}{2r}(r-s).$$

For all $\theta \in \left(\sqrt{s/r}, +\infty\right)$,

$$h'(\theta) = \theta^{\alpha-1}\left(\theta^2 - \frac{s}{r}\right)^{\beta-1}\left((\alpha+2\beta)\theta^2 - \frac{\alpha s}{r}\right).$$

The sign of h' depends on the sign of $\left((\alpha+2\beta)\theta^2 - \frac{\alpha s}{r}\right)$. Moreover $\alpha + 2\beta = \frac{s}{r}(d+r) > 0$ then h' vanishes at $\theta^\star = \sqrt{(s+d)/(r+d)}$, is negative on the set $\left(\sqrt{s/r}, \theta^\star\right)$ and positive on $\left(\theta^\star, +\infty\right)$. Therefore h reaches its minimum on $\left(\sqrt{s/r}, +\infty\right)$ at the unique point $\theta^\star$.

$\square$

Remark 5.1. *Let $X \sim \mathcal{N}(m;\Sigma)$.*
If $s < r$, then $\theta^\star < 1$. Hence, $(\alpha_n^{\theta^\star,m})_{n\geq 1}$ is a contraction of $(\alpha_n)_{n\geq1}$ with scaling number $\theta^\star$ and translating number m. In the other hand, if $s > r$, then $\theta^\star > 1$. In this case the sequence $(\alpha_n^{\theta^\star,m})_{n\geq 1}$ is a dilatation of $(\alpha_n)_{n\geq1}$ with scaling number $\theta^\star$ and translating number m. Also note that $\theta^\star$ does not depend on the covariance matrix Σ.

What we do expect from the resulting sequence $(\alpha_n^{\theta^\star,m})_{n\geq 1}$? The proposition below shows that it satisfies the empirical measure theorem (keep in mind that this theorem is satisfied by asymptotically optimal quantizers although the converse is not true in general).

Proposition 5.2. *Let $r, s > 0$ and let $P = \mathcal{N}(m;\Sigma)$. Assume $(\alpha_n)_{n\geq1}$ is asymptotically $L^r(P)$-optimal. Then the sequence $(\alpha_n^{\theta^\star,m})_{n\geq 1}$ (as defined before with $\theta^\star = \sqrt{(s+d)/(r+d)}$) satisfies the empirical measure theorem.*

In other words, for every $a, b \in \mathbb{R}^d$,

$$\frac{1}{n}\textit{card}(\{x \in \alpha_n^{\theta^\star,m} \cap [a,b]\}) \longrightarrow \frac{1}{C_{f,s}}\int_{[a,b]} f(x)^{\frac{d}{d+s}}dx.$$

Proof. For all $n \geq 1$,

$$\{x \in \alpha_n^{\theta^\star,m} \cap [a,b]\} = \{x \in \alpha_n \cap [(a-m)/\theta^\star + \mu, (b-m)/\theta^\star] + m\}.$$

Since $(\alpha_n)_{n\geq1}$ is asymptotically L^r-optimal; by applying the empirical measure theorem to the sequence $(\alpha_n)_{n\geq1}$, we obtain:

$$\frac{1}{n}\text{card}(\{x \in \alpha_n \cap [(a-m)/\theta^\star + m, (b-m)/\theta^\star + m]\}) \longrightarrow \frac{1}{C_{f,r}}\int_{[(a-m)/\theta^\star+m,(b-m)/\theta^\star+m]} f(x)^{\frac{d}{d+r}}dx.$$

It remains to verify that

$$\frac{1}{C_{f,r}}\int_{[(a-m)/\theta^\star+m,(b-m)/\theta^\star+m]} f(x)^{\frac{d}{d+r}}dx = \frac{1}{C_{f,s}}\int_{[a,b]} f(x)^{\frac{d}{d+s}}dx.$$

Remind that

$$f(x) = \left((2\pi)^d \text{det}\,\Sigma\right)^{-\frac{1}{2}} e^{-\frac{1}{2}(x-m)'\Sigma^{-1}(x-m)}$$

and $see(1.3)\big)$

$$C_{f,r} = \int_{\mathbb{R}^d} f(x)^{\frac{d}{d+r}}dx.$$

Hence, for all $r > 0$,

$$C_{f,r} = \big((2\pi)^d \det \Sigma\big)^{\frac{r}{2(r+d)}} \left(\frac{d+r}{d}\right)^{\frac{d}{2}}.$$

By making the change of variable $x = m + \theta^\star(z-m)$, one gets :

$$\frac{1}{C_{f,r}} \int_{[(a-m)/\theta^\star+m,(b-m)/\theta^\star+m]} f(z)^{\frac{d}{d+r}} dz \quad = \quad \frac{1}{C_{f,r}} (\theta^\star)^{-d} \int_{[a,b]} f((x-m)/\theta^\star + m)^{\frac{d}{d+r}} dx.$$

It is easy to check that

$$\big(f((x-m)/\theta^\star + m\big)^{\frac{d}{d+r}} = \big(f(x)\big)^{\frac{d}{d+s}} \big((2\pi)^d \det \Sigma\big)^{-\frac{1}{2}(\frac{d}{d+r} - \frac{d}{d+s})}$$

and that

$$\frac{1}{C_{f,r}} (\theta^\star)^{-d} \big((2\pi)^d \det\Sigma\big)^{-\frac{1}{2}(\frac{d}{d+r} - \frac{d}{d+s})} = \big((2\pi)^d \det \Sigma\big)^{-\frac{s}{2(s+d)}} \left(\frac{d+s}{d}\right)^{-\frac{d}{2}}.$$

The last term is simply equal to $\frac{1}{C_{f,s}}$. We then deduce that

$$\frac{1}{C_{f,r}} \int_{[(a-m)/\theta^\star+m,(b-m)/\theta^\star+m]} f(x)^{\frac{d}{d+r}} dx = \frac{1}{C_{f,s}} \int_{[a,b]} f(x)^{\frac{d}{d+s}} dx.$$

$\square$

We have just built a sequence $(\alpha_n^{\theta^\star,m})_{n\geq 1}$ verifying the empirical measure theorem. The question we ask know is : is this sequence asymptotically L^s-optimal? The next proposition shows that the lower bound in (2.1) is in fact reached by considering the sequence $(\alpha_n^{\theta^\star,m})_{n\geq 1}$.

Proposition 5.3. *Let $s > 0$ and let $\theta = \theta^\star = \sqrt{(s+d)/(r+d)}$. Then, the constant in the asymptotic lower bound for the L^s error induced by the sequence $(\alpha_n^{\theta^\star,m})_{n\geq 1}$ (see (2.1)) satisfies :*

$$Q_{r,s}^{Inf}(P, \theta^\star) = Q_s(P). \tag{5.2}$$

Proof. Keep in mind that if $P \sim \mathcal{N}(m;\Sigma)$ then, for all $r > 0$,

$$\big(Q_r(P)\big)^{1/r} = \big(J_{r,d}\big)^{1/r} \sqrt{2\pi} \left(\frac{d+r}{d}\right)^{\frac{d+r}{2r}} \big(\det \Sigma\big)^{\frac{1}{2d}}.$$

We have in one hand

$$\begin{aligned}
\left(\int_{\mathbb{R}^d} f^{\frac{d}{d+r}}(x) d(x)\right)^{s/d} &= \left(\big((2\pi)^d \det \Sigma\big)^{-\frac{1}{2}\frac{d}{d+r}} \int_{\mathbb{R}^d} e^{-\frac{1}{2}\frac{d}{d+r}(x-m)'\Sigma^{-1}(x-m)} dx\right)^{s/d} \\
&= \left(\big((2\pi)^d \det \Sigma\big)^{\frac{1}{2}\frac{r}{d+r}} \Big(\frac{d+r}{d}\Big)^{\frac{d}{2}}\right)^{s/d}
\end{aligned}$$

and in the other hand

$$\begin{aligned}
\int_{\mathbb{R}^d} f_{\theta^\star,\mu}(x) f^{-\frac{s}{d+r}}(x) d(x) &= \big((2\pi)^d \det \Sigma\big)^{-\frac{1}{2} - \frac{s}{d+r}} \int_{\mathbb{R}^d} e^{-\frac{1}{2}\frac{d}{d+r}(x-m)'\Sigma^{-1}(x-m)} dx \\
&= \big((2\pi)^d \det \Sigma\big)^{-\frac{s}{d+r}} \Big(\frac{d+r}{d}\Big)^{\frac{d}{2}}.
\end{aligned}$$

Combining these two results yields

$$\begin{aligned}
Q^{\mathrm{Inf}}_{r,s}(P,\theta^\star) &= (\theta^\star)^{s+d} J_{s,d}\left(\int_{\mathbb{R}^d} f^{\frac{d}{d+r}}d\lambda_d\right)^{s/d}\int_{\mathbb{R}^d} f_{\theta^\star,\mu}f^{-\frac{s}{d+r}}d\lambda_d \\
&= J_{s,d}\left(\frac{s+d}{r+d}\right)^{\frac{d+s}{2}}\left((2\pi)^d \mathrm{det}\,\Sigma\right)^{\frac{s}{2d}}\left(\frac{r+d}{d}\right)^{\frac{d+s}{2}} \\
&= J_{s,d}\left(\frac{s+d}{d}\right)^{\frac{d+s}{2}}\left((2\pi)^d \mathrm{det}\,\Sigma\right)^{\frac{s}{2d}} \\
&= Q_s(P).
\end{aligned}$$

□

After some elementary calculations, it follows from the proposition above and inequalities (2.1),(4.2), the corollary below :

Corollary 5.1. *Let $X \sim \mathcal{N}(m;\Sigma)$ and $\theta^\star = \sqrt{(s+d)/(r+d)}$. Then,*

$$Q_s(P)^{1/s} \le \liminf_{n\to\infty} n^{1/d}\,\|X-\widehat{X}^{\alpha_n^{\theta^\star,m}}\|_s \le \limsup_{n\to\infty} n^{1/d}\,\|X-\widehat{X}^{\alpha_n^{\theta^\star,m}}\|_s \le Q^{Sup}_{r,s}(P,\theta^\star)^{1/s} \quad (5.3)$$

with

$$Q^{Sup}_{r,s}(P,\theta^\star)^{1/s} = \begin{cases} \left(\frac{s+d}{d}\right)^{\frac{s+d}{2s}} J_{r,d}^{\frac{1}{r}}\left((2\pi)^d \mathit{det}\,\Sigma\right)^{\frac{1}{2d}} & \textit{if } s<r \\ \left(\frac{s+d}{d}\right)^{\frac{d}{2}}\sqrt{\frac{s+d}{r+d}}\,C(b)\,\left((2\pi)^d \mathit{det}\,\Sigma\right)^{\frac{1}{2(d+r)}} & \textit{if } s>r. \end{cases}$$

Remark 5.2. (a) *If $s>r$, we cannot prove the asymptotically $L^s(P)$-optimality of $(\alpha_n^{\theta^\star,m})_{n\ge1}$ using (3.7) since the constant $C(b)$ is not explicit.*

(b) *When $s<r$, the corollary above shows that the upper bound in (3.3) does not reach the Zador constant. Then our upper estimate does not allow us to show that the sequence $(\alpha_n^{\theta^\star,m})_{n\ge1}$ is asymptotically $L^s(P)$-optimal.*

Moreover, using $H\ddot{o}lder$ inequality (with $p=r/(r-s)$ and $q=r/s$), we have for every $\theta>0$,

$$\begin{aligned}
\int_{\mathbb{R}^d} f_{\theta,\mu}(x)f^{-\frac{s}{d+r}}(x)d\lambda_d(x) &= \int_{\mathbb{R}^d} f_{\theta,\mu}(x)f^{-s/r}(x)f^{\frac{sd}{r(d+r)}}(x)d\lambda_d(x) \\
&\le \left(\int_{\mathbb{R}^d} f_{\theta,\mu}^{\frac{r}{r-s}}(x)f^{-\frac{s}{r-s}}(x)d\lambda_d(x)\right)^{\frac{r-s}{r}}\left(\int_{\mathbb{R}^d} f^{\frac{d}{d+r}}(x)d\lambda_d\right)^{\frac{s}{r}}.
\end{aligned}$$

and (for $\theta=\theta^\star$)

$$\int_{\mathbb{R}^d} f_{\theta^\star,\mu}(x)f^{-\frac{s}{d+r}}(x)d\lambda_d(x) = \left(\int_{\mathbb{R}^d} f_{\theta^\star,\mu}^{\frac{r}{r-s}}(x)f^{-\frac{s}{r-s}}(x)d\lambda_d(x)\right)^{\frac{r-s}{r}}\left(\int_{\mathbb{R}^d} f^{\frac{d}{d+r}}(x)d\lambda_d\right)^{\frac{s}{r}}. \quad (5.4)$$

Hence, according to (5.2), one gets for every $s<r$,

$$(\theta^\star)^{s+d}J_{s,d}\left(\int_{\mathbb{R}^d} f_{\theta^\star,\mu}^{\frac{r}{r-s}}(x)f^{-\frac{s}{r-s}}(x)d\lambda_d(x)\right)^{\frac{r-s}{r}}\|f\|^{s/r}_{\frac{d}{d+r}} = Q_s(P). \quad (5.5)$$

Thus, to reach the Zador constant in (3.3) we must rather have $J_{s,d}$ instead of $J_{r,d}$ (which will be coherent since for all $s<r$, $J_{s,d}^{1/s} \le J_{r,d}^{1/r}$), that is,

$$\limsup_{n\to\infty} n^{1/d}\,\|X-\widehat{X}^{\alpha_n^{\theta,\mu}}\|_s \le \theta^{s+d} J_{s,d}\left(\int_{\mathbb{R}^d} f_{\theta,\mu}^{\frac{r}{r-s}}(x) f^{-\frac{s}{r-s}}(x)\,d\lambda_d(x)\right)^{\frac{r-s}{r}} \|f\|_{\frac{d}{d+r}}^{s/r}.$$

.

5.1.2 Numerical experiments

For numerical example, supppose that $d=1$ and $r \in \{1,2,4\}$. Let $X \sim \mathcal{N}(0,1)$ and, for a fixed n, let $\alpha_{n,r}=\{x_{1,r},\cdots,x_{n,r}\}$ be the n-L^r-optimal grid for X (obtained by a Newton-Raphson zero search). For every $n \in \{20,50,\cdots,900\}$ and for $(s,r)=(1,2)$ and $(4,2)$, we make a linear regression of $\alpha_{n,r}$ onto $\alpha_{n,s}$:

$$x_{i,s} \simeq \hat{a}_{sr} x_{i,r} + \hat{b}_{sr}, \quad i=1,\cdots,n.$$

Table 1 provides the regression coefficients we obtain for different values of n. We note that when n increases, the coefficients $\hat{a}_{sr}$ tend to the value $\sqrt{(s+1)/(r+1)}=\theta^\star$ whereas the coefficients $\hat{b}_{sr}$ almost vanish. For example, for $n=900$ and for $(r,s)=(2,1)$ (resp. $(2,4)$) we get $\hat{a}_{sr}=0.8170251$ (resp. 1.2900417). The expected values are $\sqrt{2/3}=0.8164966$ (resp. $\sqrt{5/3}=1.2909944$). The absolute errors are then 5.285×10^{-4} $\left(\text{resp. } 9.527\times 10^{-4}\right)$. We remark that the error mainly comes from the tail of the distribution.

n	$\hat{a}_{12}$	$\hat{b}_{12}$	ϵ		$\hat{a}_{42}$	$\hat{b}_{42}$	ϵ
20	0.8250096	1.826E-14	0.0003025		1.2761027	- 3.650E-12	0.0008607
50	0.8211387	- 1.021E-13	0.0006870		1.2828110	3.733E-10	0.0020110
100	0.8193424	8.693E-14	0.0009909		1.2859567	4.059E-09	0.0029445
300	0.8177506	- 1.045E-11	0.0013601		1.2887640	0.0000004	0.0041021
700	0.8171428	- 7.219E-11	0.0015111		1.2898393	- 0.0000089	0.0048006
800	0.8170775	- 6.725E-11	0.0015247		1.2900041	0.0000216	0.0040577
900	0.8170251	4.564E-11	0.0015346		1.2900417	- 0.0000141	0.0048182

Table 1: Regression coefficients for the Gaussian.

The previous numerical results, in addition to Equation (5.2), strongly suggest that the sequence $(\alpha_n^{\theta^\star,m})_{n\ge1}$ is in fact asymptotically $L^s(P)$-optimal. This leads to the following conjecture.

Conjecture 1. *Let $P \sim \mathcal{N}(m;\Sigma)$ and let $(\alpha_n)_{n\ge1}$ be an $L^r(P)$-optimal sequence of quantizers. Then, for every $s>0$, the sequence $(\alpha_n^{\theta^\star,m})_{n\ge1}$ (with $\theta^\star=\sqrt{(s+d)/(r+d)}$) is asymptotically $L^s(P)$-optimal.*

5.2 Exponential distribution

5.2.1 Optimal dilatation and contraction

Proposition 5.4. *Let $r,s>0$ and X be an exponentially distributed random variable with rate parameter $\lambda>0$. Set $\mu=0$.*

(a) *If* $s \in \big(r, r+1\big) \cup \big(r+1, +\infty\big)$, *the sequence* $(\alpha_n^{\theta,0})_{n\geq 1}$ *is* L^s*-rate-optimal iff* $\theta \in \big(s/(r+1), +\infty\big)$ *and*

$$\theta^\star = (s+1)/(r+1)$$

is the unique solution of (5.1) *on the set* $\big(s/(r+1), +\infty\big)$.

(b) *If* $s \in (0, r)$, *the sequence* $(\alpha_n^{\theta,0})_{n\geq 1}$ *is* L^s*-rate-optimal for all* $\theta \in \big(s/r, +\infty\big)$ *and*

$$\theta^\star = (s+1)/(r+1)$$

is the unique solution of (5.1) *on* $\big(s/r, +\infty\big)$.

Proof. **(a)** Let $s \in (r, r+1)$. For all $\theta > 0, \mu \in \mathbb{R}^d$, the couple (θ, μ) is P-admissible and the function f is decreasing on $(0, +\infty)$. For $\theta > s/(r+1)$, Assumption (3.10) holds true for every $c \in \big(1, \theta(1+r)/s\big)$. Moreover, Hypothesis **(H1)** is clearly satisfied. Consequently, if follows from Corollary $3.1, (a)$ that Assumption (3.6) holds true.

If $s > r+1$, Assumption (3.6) still holds since the additionnal assumptions **(H3)** and $f^{-\frac{s}{r+1}} \in L^1_{loc}(P)$ required to apply the corollary $3.1, (b)$ are satisfied.

In the other hand, one has

$$\int_{\mathbb{R}} f(\theta x) f(x)^{-s/(r+1)} dx = C \int_0^{+\infty} e^{-\lambda(\theta - s/(r+1))x} dx < +\infty \Longleftrightarrow \theta > s/(r+1).$$

Now, let us solve the problem (5.1) For all $\theta > s/(r+1)$,

$$\begin{aligned} \theta^{s+1} \int_{\mathbb{R}} f(\theta x) f(x)^{-\frac{s}{r+1}} dx &= C\, \theta^{s+1} \int_0^{+\infty} e^{-\lambda(\theta - \frac{s}{r+1})x} dx \\ &= C\, \theta^{s+1} \left(\theta - \frac{s}{r+1}\right)^{-1}. \end{aligned}$$

Let

$$h(\theta) = \theta^{s+1} \left(\theta - \frac{s}{r+1}\right)^{-1}.$$

Then

$$h'(\theta) = s\theta^s \left(\theta - \frac{s}{r+1}\right)^{-2} \left(\theta - \frac{s+1}{r+1}\right).$$

Hence, h reaches its unique minimun on $\big(s/(r+1), +\infty\big)$ at $\theta^\star = (s+1)/(r+1)$.

(b) Let $s < r$. Then

$$\int_{\mathbb{R}} f^{\frac{r}{r-s}}(\theta x) f^{-\frac{s}{r-s}}(x) dx = C \int_{\mathbb{R}_+} e^{-x\frac{\lambda}{r-s}(r\theta - s)} dx.$$

Then, for all $\theta > s/r$, $\int_{\mathbb{R}} f^{\frac{r}{r-s}}(\theta x) f^{-\frac{s}{r-s}}(x) dx < +\infty$. This gives the first assertion.
For all $\theta > s/r$, then

$$\begin{aligned} \theta^{s+1} \left(\int_{\mathbb{R}} f_{\theta,\mu}^{\frac{r}{r-s}}(x) f^{-\frac{s}{r-s}}(x) dx\right)^{1-\frac{s}{r}} &= C\, \theta^{s+1} \left(\int_{\mathbb{R}_+} e^{-x\frac{\lambda}{r-s}(r\theta - s)} dx\right)^{\frac{r-s}{r}} \\ &= C\, \theta^{s+1} (r\theta - s)^{\frac{s-r}{r}}. \end{aligned}$$

We easily check that the function $h(\theta) = \theta^{s+1}(r\theta - s)^{\frac{s-r}{r}}$ reaches its minimum on $\big(s/r, +\infty\big)$ at the unique point $\theta^\star = (s+1)/(r+1)$.

$\square$

Remark 5.3. *Let $X \sim \mathcal{E}(\lambda)$. If $s < r$, then $\theta^\star = (s+1)/(r+1) < 1$. Hence, the sequence $(\alpha_n^{\theta^\star,0})_{n\geq 1}$ is a contraction of $(\alpha_n)_{n\geq 1}$ with scaling number $\theta^\star$. In the other hand, if $s > r$, then $\theta^\star > 1$ and then $(\alpha_n^{\theta^\star,0})_{n\geq 1}$ is a dilatation of $(\alpha_n)_{n\geq 1}$ with scaling number $\theta^\star$. Note that $\theta^\star$ does not depend on the parameter λ of the exponential distribution.*

One shows below that the sequence $(\alpha_n^{\theta^\star,0})_{n\geq 1}$, with $\theta^\star = (1+s)/(1+r)$, satisfies the empirical measure theorem.

Proposition 5.5. *Let $r, s > 0$ and let X be an exponentially distributed random variable with rate parameter $\lambda > 0$. Assume $(\alpha_n)_{n\geq 1}$ is an asymptotically L^r-optimal sequence of quantizers for X and let $(\alpha_n^{\theta^\star,0})_{n\geq 1}$ be defined as before, with $\theta^\star = (s+1)/(r+1)$. Then, the sequence $(\alpha_n^{\theta^\star,0})$ satisfies the empirical measure theorem.*

Proof. Since $(\alpha_n^{\theta^\star,0})_{n\geq 1} = (\theta^\star \alpha_n)_{n\geq 1}$, It amounts to show that

$$\frac{\mathrm{card}(\alpha_n \cap [a/\theta^\star, b/\theta^\star])}{n} \longrightarrow \frac{1}{C_{f,s}} \int_a^b f(x)^{\frac{1}{1+s}} dx$$

i.e that for all $a, b \in \mathbb{R}_+$,

$$\frac{1}{C_{f,r}} \frac{1}{\theta^\star} \int_a^b f(x/\theta^\star)^{\frac{1}{1+r}} dx = \frac{1}{C_{f,s}} \int_a^b f(x)^{\frac{1}{1+s}} dx.$$

Elementary computations show that $\forall\, r > 0$,

$$C_{f,r} = \lambda^{-\frac{r}{1+r}}(1+r).$$

so that

$$\begin{aligned}
\frac{1}{C_{f,r}} \frac{1}{\theta^\star} \int_a^b f(x/\theta^\star)^{\frac{1}{1+r}} dx &= \frac{1}{C_{f,r}} \frac{1+r}{1+s} \int_a^b \Big(\lambda e^{-x\lambda\frac{1+r}{1+s}}\Big)^{\frac{1}{1+r}} dx \\
&= \frac{1}{C_{f,r}} \frac{1+r}{1+s}\, \lambda^{\frac{1}{1+r} - \frac{1}{1+s}} \int_a^b \Big(\lambda e^{-\lambda x}\Big)^{\frac{1}{1+s}} dx \\
&= \frac{1}{C_{f,s}} \int_a^b f(x)^{\frac{1}{1+s}} dx.
\end{aligned}$$

$\square$

Is the sequence $(\alpha_n^{\theta^\star,0})_{n\geq 1}$ asymptotically L^s-optimal? The remark 5.2 is also valid for the exponential distribution. Our upper bounds in (3.3) and (3.7) do not allow us to show that $(\theta^\star \alpha_n)$ is asymptotically L^s-optimal because of the corollary below. But the numerical results strongly suggest that it is.

Corollary 5.2. *Let $X \sim \mathscr{E}(\lambda)$ and $\theta^\star = (s+1)/(r+1)$. Then,*

$$Q_s(P)^{1/s} \le \liminf_{n\to\infty} n^{1/d}\|X - \widehat{X}^{\alpha_n^{\theta^\star,0}}\|_s \le \limsup_{n\to\infty} n^{1/d}\,\|X - \widehat{X}^{\alpha_n^{\theta^\star,0}}\|_s \le Q_{r,s}^{Sup}(P,\theta^\star)^{1/s} \tag{5.6}$$

with

$$Q_{r,s}^{Sup}(P,\theta^\star)^{1/s} = \begin{cases} \frac{1}{2\lambda}(s+1)^{1+1/s}(r+1)^{-1/r} & \text{if } s < r \\ (s+1)^{1+1/s}\big((r+1)\lambda^{\frac{1}{1+r}}\big)^{-1}C(b)^{1/s} & \text{if } s > r. \end{cases}$$

Proof. We easily prove, like in proposition 5.2, that $Q_{r,s}^{\text{Inf}}(P,\theta^\star) = Q_s(P)$. The corollary follows then from (2.1) and (4.2) (keep in mind that for all $r > 0$, $J_{r,1} = \frac{1}{(r+1)2^r}$). □

5.2.2 Numerical experiments

For numerical examples, Table 2 gives the regression coefficients we obtain by regressing the L^2 grids onto the grids we get with the L^1 and L^4 norms, for different values of n. The notations are the same as in the previous example. We note that for large enough n, the coefficients $\hat{a}_{sr}$ tend to $(s+1)/(r+1) = \theta^\star$. For example, if $n = 900$, we get $\hat{a}_{12} = 0.6676880$; $\hat{a}_{42} = 1.6640023$ whereas the expected values are respectively $2/3 = 0.66666667$ and $5/3 = 1.6666667$. The absolute errors are in the order of 10^{-3}. Like the Gaussian case, we remark that the error of the estimation results mainly from the tail of the exponential distribution.

n	$\hat{a}_{12}$	$\hat{b}_{12}$	ϵ		$\hat{a}_{42}$	$\hat{b}_{42}$	ϵ
20	0.6765013	- 0.0104881	0.0019489		1.6396807	0.0288348	3.081E-33
50	0.6726145	- 0.0082123	0.0045310		1.6502245	0.0225246	1.149E-28
100	0.6706176	- 0.0062439	0.0070734		1.6556979	0.0172020	1.573E-27
300	0.6686428	- 0.0036234	0.0114628		1.6611520	0.0100523	1.508E-27
700	0.6677864	- 0.0022222	0.0146186		1.6635261	0.0061356	1.222E-25
800	0.6676880	- 0.0020482	0.0150735		1.6638043	0.0057199	2.020E-26
900	0.6676079	- 0.0019043	0.0154634		1.6640023	0.0053173	9.683E-25

Table 2: Regression coefficients for exponential distribution.

Conjecture 2. *Let X be an exponentially distributed random variable with rate parameter λ and let $(\alpha_n)_{n\ge1}$ be an L^r-optimal sequence of quantizers for X. Then for $s > 0$ and $\theta^\star = (s+1)/(r+1)$ the sequence $(\alpha_n^{\theta^\star,0})_{n\ge1}$ is asymptotically L^s-optimal.*

These exampless could suggest that a contraction (or a dilatation) parameter $\theta^\star$, solution of the minimisation problem (5.1), always leads to a sequence of quantizers satisfying the empirical measure theorem. The following example shows that this can fail.

5.3 Gamma distribution

5.3.1 Optimal dilatation and contraction

Proposition 5.6. *Let $r, s > 0$ and let X be a Gamma distribution with parameters a and λ : $X \sim \Gamma(a,\lambda)$, $a > 0$, $\lambda > 0$.*

(a) *if $s \in (r, r+1)$, the sequence $(\alpha_n^{\theta,0})_{n\geq 1}$ is L^s-rate-optimal iff $\theta \in \big(s/(r+1), +\infty\big)$ and for all $a > 0$,*

$$\theta^\star = (s+a)/(r+a)$$

is the unique solution of (5.1) *on the set* $\big(s/(r+1), +\infty\big)$.

(b) *if $s > r+1$ and if $a \in \big(0, \frac{s+r+1}{s}\big)$, the sequence $(\alpha_n^{\theta,0})_{n\geq 1}$ is L^s-rate-optimal for every $\theta \in \big(s/(r+1), +\infty\big)$ and*

$$\theta^\star = (s+a)/(r+a)$$

is the unique solution of (5.1) *on the set* $\big(s/(r+1), +\infty\big)$ *(Note that the assumptions imply $a \in (0, 2)$).*

(c) *if $s < r$, the sequence $(\alpha_n^{\theta,0})_{n\geq 1}$ is L^s-rate-optimal for every $\theta \in \big(s/r, +\infty\big)$ and for all $a > 0$,*

$$\theta^\star = (s+1)/(r+1)$$

is the unique solution of (5.1) *on the set* $\big(s/r, +\infty\big)$.

Proof. We set $\mu = 0$. Keep in mind that the density function is written

$$f(x) = \frac{\lambda^a}{\Gamma(a)} x^{a-1} e^{-\lambda x} \mathbf{1}_{\{x>0\}}, \text{ with } \Gamma(a) = \int_0^{+\infty} x^{a-1} e^{-x} dx.$$

(a) and **(b)**. Let $s \in (r, r+1)$ and set $R_0 = \max(0, (a-1)/\lambda)$. The function f is decreasing on $(R_0, +\infty)$ and for every $\theta > 0, \mu$, the couple (θ, μ) is P-admissible. For $\theta > s/(r+1)$, Assumption (3.10) holds true for every $c \in \big(1, \theta(1+r)/s\big)$. Moreover, Hypothesis **(H1)** clearly holds. Consequently, it follows from Corollary 3.1, (a) that Assumption (3.6) of Theoreme 3.2 holds true.

When $s > r+1$, the additionnal hypothesis $f^{-\frac{s}{r+1}} \in L^1_{loc}(P)$ holds for $a < \frac{r+1}{s} + 1$. Note that if $P = f \cdot \lambda_d$ then $\lambda_d(\text{supp}(P) \cap \{f = 0\}) = 0$ implies that $\lambda_d(\cdot \cap \text{supp}(P)) \ll P$. It follows that **(H3)** holds. In this case Assumption (3.6) of Theoreme 3.2 holds true.

For all $\theta > 0$,

$$\int_{\mathbb{R}} f(\theta x) f(x)^{-\frac{s}{1+r}} dx = \left(\frac{\lambda^a}{\Gamma(a)}\right)^{1-s/(r+1)} \int_0^{+\infty} x^{(a-1)(1-\frac{s}{r+1})} e^{-(\theta - \frac{s}{r+1})\lambda x} dx$$

and then

$$\int_{\mathbb{R}} f(\theta x) f(x)^{-\frac{s}{r+1}} dx < +\infty \quad \text{iff} \quad \theta > s/(r+1) \text{ and } a(r+1-s) + s > 0.$$

Let $\theta > s/(r+1)$. Then

$$\begin{aligned} \theta^{s+1} \int_{\mathbb{R}} f(\theta x) f(x)^{-\frac{s}{1+r}} dx &= \left(\frac{\lambda^a}{\Gamma(a)}\right)^{1-s/(r+1)} \theta^{s+1} \theta^{a-1} \int_0^{+\infty} x^{(a-1)(1-\frac{s}{1+r})} e^{-(\theta - \frac{s}{1+r})\lambda x} dx \\ &= C\, \theta^\gamma \left(\theta - \frac{s}{1+r}\right)^{-\beta}. \end{aligned}$$

with

$$\gamma = s + a \text{ and } \beta = (a-1)(1 - s/(r+1)) + 1.$$

We define on $\mathbb{R}_+^\star$ the function h by

$$h(\theta) = \theta^\gamma \left(\theta - \frac{s}{1+r}\right)^{-\beta}.$$

The function h is differentiable for all $\theta > s/(1+r)$ and

$$h'(\theta) = \theta^{\gamma-1}\left(\theta - \frac{s}{1+r}\right)^{-\beta-1}\left((\gamma-\beta)\theta - \frac{s\gamma}{1+r}\right).$$

Hence, the minimum of h is then unique on $\big(s/(r+1), +\infty\big)$ and is reached at $\theta^\star$.

Notice that the condition required for $f^{-\frac{s}{r+1}}$ to be in $L^1_{loc}(P)$ is $a < \frac{r+1}{s} + 1$ and for every $s > r+1$ one has $1 + \frac{r+1}{s} < \frac{s}{s-(r+1)}$. Combined to the condition $a(r+1-s) > 0$ yields the condition for a.

(c) Let $s < r$. Then

$$\int_{\mathbb{R}} f^{\frac{r}{r-s}}(\theta x) f^{-\frac{s}{r-s}}(x)dx = \frac{\lambda^a}{\Gamma(a)}\int_0^{+\infty} x^{a-1} e^{-\frac{\lambda x}{r-s}(r\theta - s)}dx.$$

Therefore $\int_{\mathbb{R}} f^{\frac{r}{r-s}}(\theta x) f^{-\frac{s}{r-s}}(x)dx < +\infty$ iff $\theta > s/r$.

Let $\theta > s/r$. Then

$$\begin{aligned}\theta^{1+s}\left(\int_{\mathbb{R}} f_{\theta,\mu}^{\frac{r}{r-s}}(x) f^{-\frac{s}{r-s}}(x)dx\right)^{1-\frac{s}{r}} &= C\,\theta^{s+a}\left(\int_0^{+\infty} x^{a-1}e^{-\frac{\lambda x}{r-s}(r\theta-s)}dx\right)^{\frac{r-s}{r}} \\ &= C\,\theta^{s+a}\big(r\theta - s\big)^{a\frac{s-r}{r}}.\end{aligned}$$

Considering the function h defined by $h(\theta) = \theta^{s+a}(r\theta - s)^{a\frac{s-r}{r}}$ we show that h reached its minimum on $\big(s/r, +\infty\big)$ at the unique point $\theta^\star = (s+a)/(r+a)$.

$\square$

Remark 5.4. *Let $X \sim \Gamma(a, \lambda)$. If $s < r$, then $\theta^\star = (s+a)/(r+a) < 1$. Then the sequence $(\alpha_n^{\theta^\star,0})_{n\geq 1}$ is a contraction of $(\alpha_n)_{n\geq 1}$ with scaling number $\theta^\star$. On the other hand, if $s > r$, then $\theta^\star > 1$ and the sequence $(\alpha_n^{\theta^\star,0})_{n\geq 1}$ is a dilatation of $(\alpha_n)_{n\geq 1}$ with scaling number $\theta^\star$. Moreover there is no constraint on the parameter a as long as $s < r+1$. In this case when we set $a = 1$ (exponential distribution with parameter λ) we retrieve the result of the exponential distribution. Note that $\theta^\star$ does not depend on the parameter λ. That is expected since $\Gamma(1,\lambda) = \mathscr{E}(\lambda)$ and, in the exponential case we know that the scaling number does not depend on λ.*

Let $\theta^\star = (s+a)/(r+a)$ and consider now the sequence $(\alpha_n^{\theta^\star,0})_{n\geq 1}$ defined as previously. Does this sequence verify the empirical measure theorem? If $a = 1$ we boil down to the exponential distribution. On the other hand, when $a \neq 1$, one shows below that there exists $a > 1,\ s > 0$ and $r > 0$ such that the sequence $(\alpha_n^{\theta^\star,0})_{n\geq 1}$ does not verify the empirical measure theorem.

Suppose that $(\alpha_n^{\theta^\star,0})_{n\geq 1}$ satisfies the empirical measure theorem. Then we must have, for all $u \in \mathbb{R}_+$,

$$\frac{1}{C_{f,r}}\frac{1}{\theta^\star}\int_0^u f(x/\theta^\star)^{\frac{1}{1+r}}dx = \frac{1}{C_{f,s}}\int_0^u f(x)^{\frac{1}{1+s}}dx. \tag{5.7}$$

with $f(x)=\frac{\lambda^a}{\Gamma(a)}x^{a-1}e^{-\lambda x}\mathbf{1}_{\{x>0\}}$ and $C_{f,r}=\int f(x)^{\frac{1}{1+r}}dx$ for all $r>0$.

Moreover, let $r>0$. Then,

$$\begin{aligned}
C_{f,r} &= \lambda^{\frac{a}{1+r}}\Gamma(a)^{-\frac{1}{1+r}}\int_0^{+\infty}x^{(a-1)/(r+1)}e^{-\frac{\lambda}{1+r}x}dx\\
&= \lambda^{\frac{a}{1+r}}\Gamma(a)^{-\frac{1}{1+r}}\int_0^{+\infty}x^{(r+a)/(r+1)-1}e^{-\frac{\lambda}{1+r}x}dx\\
&= \lambda^{\frac{a}{1+r}}\Gamma(a)^{-\frac{1}{1+r}}\Gamma\left(\frac{r+a}{r+1}\right)\lambda^{-\frac{r+a}{r+1}}\left(r+1\right)^{\frac{r+a}{r+1}}\\
&= \Gamma\left(\frac{r+a}{r+1}\right)\Gamma(a)^{-\frac{1}{1+r}}\lambda^{-\frac{r}{r+1}}\left(r+1\right)^{\frac{r+a}{r+1}}.
\end{aligned}$$

Equation (5.7) is written down for all $u\in\mathbb{R}_+$,

$$C(r)\left(\frac{r+a}{s+a}\right)^{\frac{r+a}{r+1}}\int_0^u x^{\frac{a-1}{r+1}}e^{-\frac{\lambda(r+a)}{(r+1)(s+a)}x}dx=C(s)\int_0^u x^{\frac{a-1}{s+1}}e^{-\frac{\lambda}{s+1}x}dx$$

with $C(r)=\Gamma\left(\frac{r+a}{r+1}\right)^{-1}\lambda^{\frac{r+a}{r+1}}\left(r+1\right)^{-\frac{r+a}{r+1}}$, $\qquad\forall\, r>0$.

Let $m\in\mathbb{N}$ and $\alpha>0$. We show by induction that, for $u>0$,

$$\int_0^u x^ne^{-\alpha x}dx=-\left(\frac{1}{\alpha}u^n+\frac{n}{\alpha^2}u^{n-1}+\frac{n(n-1)}{\alpha^3}u^{n-2}+\cdots+\frac{n!}{\alpha^n}u+\frac{n!}{\alpha^{n+1}}\right)e^{-\alpha u}+\frac{n!}{\alpha^{n+1}}.$$

Consider $a>1$ such that $\frac{a-1}{r+1}$ and $\frac{a-1}{s+1}$ are integers. Set $n=\frac{a-1}{r+1}$, $m=\frac{a-1}{s+1}$, $\alpha=\frac{\lambda(r+a)}{(r+1)(s+a)}$ and $\beta=\frac{\lambda}{s+1}$. Then Equation (5.7) is finally written down

$$\begin{aligned}
C(r)&\left(\frac{r+a}{s+a}\right)^{\frac{r+a}{r+1}}\left[\left(\frac{1}{\alpha}u^n+\frac{n}{\alpha^2}u^{n-1}+\frac{n(n-1)}{\alpha^3}u^{n-2}+\cdots+\frac{n!}{\alpha^n}u+\frac{n!}{\alpha^{n+1}}\right)e^{-\alpha u}-\frac{n!}{\alpha^{n+1}}\right]\\
&=C(s)\left[\left(\frac{1}{\beta}u^m+\frac{m}{\beta^2}u^{m-1}+\frac{m(m-1)}{\beta^3}u^{m-2}+\cdots+\frac{m!}{\beta^m}u+\frac{m!}{\beta^{m+1}}\right)e^{-\beta u}-\frac{m!}{\beta^{m+1}}\right].
\end{aligned}$$

Set $a=7,\ s=1,\ r=2,\ \lambda=1$ and $u=1$. Then $n=2, m=3, \alpha=3/8, \beta=1/2$ and this lead, after some calculations to :

$$\frac{185}{128}e^{-3/8}-\frac{79}{48}e^{-1/2}=-\frac{511}{512};$$

which is clearly not satisfied. We then deduce that for $(a,r,s)=(7,2,1)$, the sequence $(\alpha_n^{\theta^\star,0})_{n\geq1}$ does not satisfy the empirical measure theorem. Hence, we have constructed an $L^s(P)$-rate-optimal sequence which does not satisfy the empirical mesure theorem.

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