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Eigenvalues of the transversal Dirac Operator on Kähler Foliations

Georges Habib
Institut Élie Cartan, Université Henri Poincaré, Nancy I, B.P. 239
54506 Vandœuvre-Lès-Nancy Cedex, France
habib@iecn.u-nancy.fr

Abstract
In this paper, we prove Kirchberg-type inequalities for any Kähler spin foliation. Their limiting-cases are then characterized as being transversal minimal Einstein foliations. The key point is to introduce the transversal Kählerian twistor operators.

1 Introduction

On a compact Riemannian spin manifold \((M^n, g_M)\), Th. Friedrich [Fri80] showed that any eigenvalue \(\lambda\) of the Dirac operator satisfies
\[
\lambda^2 \geq \frac{n}{4(n-1)} S_0, \quad (1.1)
\]
where \(S_0\) denotes the infimum of the scalar curvature of \(M\). The limiting case in (1.1) is characterized by the existence of a Killing spinor. As a consequence \(M\) is Einstein. K.D. Kirchberg [Kir86] established that, on such manifolds any eigenvalue \(\lambda\) satisfies the inequalities
\[
\lambda^2 \geq \left\{ \begin{array}{ll}
\frac{m+1}{4m} S_0 & \text{if } m \text{ is odd,} \\
\frac{m}{4(m-1)} S_0 & \text{if } m \text{ is even.}
\end{array} \right.
\]
On a compact Riemannian spin foliation \((M, g_M, F)\) of codimension \(q\) with a bundle-like metric \(g_M\) such that the mean curvature \(\kappa\) is a basic coclosed
1-form, S.D. Jung [Jun01] showed that any eigenvalue $\lambda$ of the transversal Dirac operator satisfies

$$\lambda^2 \geq \frac{q}{4(q - 1)} K_0^\nabla,$$

(1.2)

where $K_0^\nabla = \inf_{M} (\sigma^\nabla + |\kappa|^2)$, here $\sigma^\nabla$ denotes the transversal scalar curvature with the transversal Levi-Civita connection $\nabla$. The limiting case in (1.2) is characterized by the fact that $\mathcal{F}$ is minimal ($\kappa = 0$) and transversally Einstein (see Theorem 3.1). The main result of this paper is the following:

**Theorem 1.1** Let $(M, g_M, \mathcal{F})$ be a compact Riemannian manifold with a Kähler spin foliation $\mathcal{F}$ of codimension $q = 2m$ and a bundle-like metric $g_M$. Assume that $\kappa$ is a basic coclosed 1-form, then any eigenvalue $\lambda$ of the transversal Dirac operator satisfies:

$$\lambda^2 \geq \frac{m + 1}{4m} K_0^\nabla \quad \text{if } m \text{ is odd},$$

(1.3)

and

$$\lambda^2 \geq \frac{m}{4(m - 1)} K_0^\nabla \quad \text{if } m \text{ is even}.$$  

(1.4)

The limiting case in (1.3) is characterised by the fact that the foliation is minimal and by existence of a transversal Kählerian Killing spinor (see Theorem 4.3). We refer to Theorem 4.4 for the equality case in (1.4).

We point out that Inequality (1.3) was proved by S. D. Jung [JK03] with the additional assumption that $\kappa$ is transversally holomorphic. The author would like to thank Oussama Hijazi for his support.

## 2 Foliated manifolds

In this section, we summarize some standard facts about foliations. For more details, we refer to [Tom88], [Jun01].

Let $(M, g_M)$ be a $(p + q)$-dimensional Riemannian manifold and a foliation $\mathcal{F}$ of codimension $q$ and let $\nabla^M$ be the Levi-civita connection associated with $g_M$. We consider the exact sequence

$$0 \longrightarrow L \overset{\iota}{\longrightarrow} TM \overset{\pi}{\longrightarrow} Q \longrightarrow 0,$$

where $L$ is the tangent bundle of $TM$ and $Q = TM/L \cong L^\perp$ the normal bundle. We assume $g_M$ to be a bundle-like metric on $Q$, that means the induced metric $g_Q$ verifies the holonomy invariance condition,

$$L_X g_Q = 0, \quad \forall X \in \Gamma(L),$$

2
where $\mathcal{L}_X$ is the Lie derivative with respect to $X$. Let $\nabla$ be the connection on $Q$ defined by:

$$\nabla_X s = \begin{cases} 
\pi [X, Y_s], & \forall X \in \Gamma(L), \\
\pi (\nabla^M_X Y_s), & \forall X \in \Gamma(L^\perp), 
\end{cases}$$

where $s \in \Gamma(Q)$ and $Y_s$ is the unique vector of $\Gamma(L^\perp)$ such that $\pi(Y_s) = s$. The connection $\nabla$ is metric and torsion-free. The curvature of $\nabla$ acts on $\Gamma(Q)$ by:

$$R^\nabla (X, Y) s = \nabla_X \nabla_Y s - \nabla_Y \nabla_X s - \nabla_{[X,Y]} s, \quad \forall X, Y \in \chi(M).$$

The transversal Ricci curvature is defined by:

$$\rho^\nabla : \Gamma(Q) \rightarrow \Gamma(Q) \quad X \longmapsto \rho^\nabla (X) = \sum_{j=1}^q R^\nabla (X, e_j) e_j.$$

Also, we define the transversal scalar curvature:

$$\sigma^\nabla = \sum_{i=1}^q g_Q (\rho^\nabla (e_i), e_i) = \sum_{i,j=1}^q R^\nabla (e_i, e_j, e_j, e_i),$$

where $\{e_i\}_{i=1,\ldots,q}$ is a local orthonormal frame of $Q$ and $R^\nabla (X, Y, Z, W) = g_Q (R^\nabla (X, Y) Z, W)$, for all $X, Y, Z, W \in \Gamma(Q)$. The foliation $\mathcal{F}$ is said to be transversally Einstein if and only if

$$\rho^\nabla = \frac{1}{q} \sigma^\nabla \text{Id},$$

with constant transversal scalar curvature. The mean curvature of $Q$ is given by:

$$\kappa (X) = g_Q (\tau, X), \quad \forall X \in \Gamma(Q),$$

where $\tau = \sum_{l=1}^p II (e_l, e_l)$, with $\{e_l\}_{l=1,\ldots,p}$ is a local orthonormal frame of $\Gamma(L)$ and $II$ is the second fundamental form of $\mathcal{F}$ defined by:

$$II : \Gamma(L) \times \Gamma(L) \rightarrow \Gamma(Q) \quad (X, Y) \longmapsto II (X, Y) = \pi (\nabla^M_X Y).$$

We define basic $r$-forms by:

$$\Omega^r_B (\mathcal{F}) = \{ \Phi \in \Lambda^r T^* M | X_\perp \Phi = 0 \quad \text{and} \quad X_\perp d\Phi = 0, \quad \forall X \in \Gamma(L) \},$$

3
where \( d \) is the exterior derivative and \( X_\bot \) is the interior product. Any \( \Phi \in \Omega^r_B(\mathcal{F}) \) can be locally written as
\[
\sum_{1 \leq j_1 < \cdots < j_r \leq q} \beta_{j_1, \ldots, j_r} dy_{j_1} \wedge \cdots \wedge dy_{j_r},
\]
where \( \frac{\partial}{\partial x_l} \beta_{j_1, \ldots, j_r} = 0, \ \forall l = 1, \ldots, p. \) With the local expression of basic \( r \)-forms, one can verify that \( \kappa \) is closed if \( \mathcal{F} \) is isoparametric (\( \kappa \in \Omega^1_B(\mathcal{F}) \)). For all \( r \geq 0 \),
\[
d(\Omega^r_B(\mathcal{F})) \subset \Omega^{r+1}_B(\mathcal{F}).
\]
We denote by \( d_B = d|_{\Omega_B(\mathcal{F})} \) where \( \Omega_B(\mathcal{F}) \) is the tensor algebra of \( \Omega^r_B(\mathcal{F}) \).
We have the following formulas:
\[
d_B = \sum_{i=1}^{q} e_i^* \wedge \nabla e_i \quad \text{and} \quad \delta_B = - \sum_{i=1}^{q} e_i \nabla e_i + \kappa_\bot,
\]
where \( \delta_B \) is the adjoint operator of \( d_B \) with respect to the induced scalar product and \( \{e_i\}_{i=1, \ldots, q} \) is a local orthonormal frame of \( Q \).

3 The transversal Dirac operator on Kähler Foliations

In this section, we start by recalling some facts on Riemannian foliations which could be found in [GK91a], [GK91b], [AG97], [Jun01]. For completeness, we also sketch a straightforward proof of Inequality ((1.2)) established in [Jun01] and end by recalling well-known facts (see [Kir86], [Kir96], [Hij94a], [Hij94b], [JK03]) on Kähler spin foliations.

On a foliated Riemannian manifold \((M, g_M, \mathcal{F})\), a transversal spin structure is a pair \((\text{Spin} Q, \eta)\) where \( \text{Spin} Q \) is a \( \text{Spin}_q \)-principal fibre bundle over \( M \) and \( \eta \) a 2-fold cover such that the following diagram commutes:

\[
\begin{array}{ccc}
\text{Spin} Q \times \text{Spin} q & \overset{\eta \otimes \text{Ad}}{\longrightarrow} & \text{Spin} Q \\
\downarrow \quad \eta \downarrow & & \downarrow \eta \\
\text{SO} Q \times \text{SO} q & \longrightarrow & \text{SO} Q
\end{array}
\]

The maps \( \text{Spin} Q \times \text{Spin} q \longrightarrow \text{Spin} Q \) and \( \text{SO} Q \times \text{SO} q \longrightarrow \text{SO} Q \), are respectively the actions of \( \text{Spin}_q \) and \( \text{SO}_q \) on the principal fibre bundles \( \text{Spin} Q \) and
SOQ. In this case, \( \mathcal{F} \) is called a transversal spin foliation. We define the foliated spinor bundle by:

\[
S(\mathcal{F}) := \text{Spin}_{\rho} \Sigma_q,
\]

where \( \rho : \text{Spin}_{\rho} \rightarrow \text{Aut}(\Sigma_q) \) is the complex spin representation and \( \Sigma_q \) is a \( \mathbb{C} \) vector space of dimension \( N \) with \( N = 2[\frac{q}{2}] \). Recall that the Clifford multiplication \( M \) on \( S(\mathcal{F}) \) is given by:

\[
M : \Gamma(Q) \times \Gamma(S(\mathcal{F})) \rightarrow \Gamma(S(\mathcal{F})),
\]

\[
(X, \Psi) \mapsto X \cdot \Psi.
\]

There is a natural Hermitian product on \( S(\mathcal{F}) \) such that, for all \( X, Y \in \Gamma(Q) \), the following relations are true:

\[
\langle X \cdot \Psi, \Phi \rangle = -\langle \Psi, X \cdot \Phi \rangle,
\]

\[
X(\langle \Psi, \Phi \rangle) = \langle \nabla_X \Psi, \Phi \rangle + \langle \Psi, \nabla_X \Phi \rangle,
\]

\[
\nabla_Y (X \cdot \Psi) = (\nabla_Y X) \cdot \Psi + X \cdot (\nabla_Y \Psi),
\]

where \( \nabla \) is the Levi-Civita connection on \( S(\mathcal{F}) \) and \( \Psi, \Phi \in \Gamma(S(\mathcal{F})) \).

The transversal Dirac operator \( \text{GK91a, GK91b} \) is locally given by:

\[
D_{\text{tr}} \Psi = \sum_{i=1}^{q} e_i \cdot \nabla_{e_i} \Psi - \frac{1}{2} \kappa \cdot \Psi,
\]

for all \( \Psi \in \Gamma(S(\mathcal{F})) \). We can easily prove using Green’s theorem \( \text{YT90} \) that this operator is formally self adjoint. Furthermore, in \( \text{GK91b} \) it is proved that if \( \mathcal{F} \) is isoparametric and \( \delta_B \kappa = 0 \), then we have the Schrödinger-Lichnerowicz formula:

\[
D_{\text{tr}}^2 \Psi = \nabla^{\star}_{\text{tr}} \nabla_{\text{tr}} \Psi + \frac{1}{4} K_{\sigma}^\nabla \Psi,
\]

where \( K_{\sigma}^\nabla = \sigma^\nabla + |\kappa|^2 \) and

\[
\nabla^{\star}_{\text{tr}} \nabla_{\text{tr}} \Psi = -\sum_{i=1}^{q} \nabla_{e_i,e_i}^2 \Psi + \nabla_{\sigma} \Psi,
\]

with \( \nabla^2_{X,Y} = \nabla_X \nabla_Y - \nabla_{\nabla_X Y} \), for all \( X, Y \in \Gamma(TM) \). Denote by \( \mathcal{P} \) the transversal twistor operator defined by

\[
\mathcal{P} : \Gamma(S(\mathcal{F})) \xrightarrow{\nabla_{\text{tr}}} \Gamma(Q^* \otimes S(\mathcal{F})) \xrightarrow{\pi} \Gamma(\ker \mathcal{M}),
\]
where $\pi$ is the orthogonal projection on the kernel of the Clifford multiplication $\mathcal{M}$. With respect to a local orthonormal frame $\{e_1, \cdots, e_q\}$, for all $\Psi \in \Gamma(S(\mathcal{F}))$, one has

$$
P\Psi = \sum_{i=1}^q e_i^* \otimes (\nabla_{e_i}\Psi + \frac{1}{q} e_i \cdot D_{e_i}\Psi + \frac{1}{2q} e_i \cdot \kappa \cdot \Psi).$$  \hspace{1cm} (3.2)

For any spinor field $\Psi$, one can easily show that

$$
\sum_{i=1}^q e_i \cdot P_{e_i}\Psi = 0.
$$  \hspace{1cm} (3.3)

Now we give a simple proof of the following theorem:

**Theorem 3.1** [Jun01] Let $(M, g_M, \mathcal{F})$ be a compact Riemannian manifold with a spin foliation $\mathcal{F}$ of codimension $q$ and a bundle-like metric $g_M$ with $\kappa \in \Omega^1_B(\mathcal{F})$. Assume that $\delta_B\kappa = 0$ and let $\lambda$ be an eigenvalue of the transversal Dirac operator, then

$$
\lambda^2 \geq \frac{q}{4(q-1)}K_0^\nabla.
$$  \hspace{1cm} (3.4)

**Proof.** For all $\Psi \in \Gamma(S(\mathcal{F}))$, we have using Identities (3.2), (3.3) (3.1),

$$
|P\Psi|^2 = |\nabla^{tr}\Psi|^2 - \frac{1}{q} |D_{e_i}\Psi|^2 - \frac{1}{q} \Re(D_{e_i}\Psi, \kappa \cdot \Psi) - \frac{1}{4q} |\kappa|^2 |\Psi|^2.
$$

For any spinor field $\Phi$, we have that $(\Phi, \kappa \cdot \Phi) = -(\kappa \cdot \Phi, \Phi) = -(\Phi, \kappa \cdot \Phi)$, so the scalar product $(\Phi, \kappa \cdot \Phi)$ is a pure imaginary function. Hence for any eigenspinor $\Psi$ of the transversal Dirac operator, we obtain

$$
\int_M |P\Psi|^2 + \frac{1}{4q} \int_M |\kappa|^2 |\Psi|^2 = \int_M |\nabla^{tr}\Psi|^2 - \frac{1}{q} \int_M \lambda^2 |\Psi|^2,
$$

from which we deduce (3.4) with the help of the Schrödinger-Lichnerowicz formula. Finally, we can easily prove in the limiting case that $\mathcal{F}$ is minimal i.e. $\kappa = 0$, and transversally Einstein. \hfill \Box

A foliation $\mathcal{F}$ is called Kähler if there exists a complex parallel orthogonal structure $J : \Gamma(Q) \to \Gamma(Q)$ $(\dim Q = q = 2m)$. Let $\Omega$ be the associated Kähler, i.e., for all $X, Y \in \Gamma(Q)$, $\Omega(X, Y) = g_Q(J(X), Y) = -g_Q(X, J(Y))$. The Kähler form can be locally expressed as

$$
\Omega = \frac{1}{2} \sum_{i=1}^q e_i \cdot J(e_i) = -\frac{1}{2} \sum_{i=1}^q J(e_i) \cdot e_i.
$$
and for all $X \in \Gamma(Q)$, we have $[\Omega, X] := \Omega \cdot X - X \cdot \Omega = 2J(X)$. Under the action of the Kähler form, the spinor bundle splits into an orthogonal sum

$$S(\mathcal{F}) = \bigoplus_{r=0}^{m} S_r(\mathcal{F}),$$

where $S_r(\mathcal{F})$ is an eigenbundle associated with the eigenvalue $i\mu_r = i(2r - m)$ of the Kähler form $\Omega$. Moreover, the spinor bundle of a Kähler spin foliation carries a parallel anti-linear map $j$ satisfying the relations:

$$j^2 = (-1)^{m(m+1)/2} Id,$$

$$[X, j] = 0,$$

$$(j\Psi, j\Phi) = (\Phi, \Psi),$$

and we have $j\Psi_r = (j\Psi)_{m-r}$. For all $X \in \Gamma(Q)$, we have

$$p_+(X) \cdot S_r(\mathcal{F}) \subset S_{r+1}(\mathcal{F}) \quad \text{and} \quad p_-(X) \cdot S_r(\mathcal{F}) \subset S_{r-1}(\mathcal{F}),$$

where $p_{\pm}(X) = \frac{X \pm iJ(X)}{2}$. We define the operator $\tilde{D}_{tr}$ by

$$\tilde{D}_{tr}\Psi = \sum_{i=1}^{q} J(e_i) \cdot \nabla_{e_i} \Psi - \frac{1}{2} J(\kappa) \cdot \Psi.$$

The local expression of $\tilde{D}_{tr}$ is independent of the choice of the local frame and by Green’s theorem [YT90], we prove that this operator is self-adjoint.

On a Kähler spin foliation, the operators $D_{tr}$ and $\tilde{D}_{tr}$ satisfy:

$$[\Omega, D_{tr}] = 2\tilde{D}_{tr},$$

$$[\Omega, \tilde{D}_{tr}] = -2D_{tr},$$

$$[\Omega, D_{tr}^2] = 0,$$

$$D_{tr}\tilde{D}_{tr} + \tilde{D}_{tr}D_{tr} = 0,$$

$$\tilde{D}_{tr}^2 = D_{tr}^2.$$ (3.9)

We should point out that Equations (3.7), (3.8) and (3.9) are true under the assumptions that $\mathcal{F}$ is isoparametric and $\delta_B \kappa = 0$. Now we define the two operators $D_+$ and $D_-$ by

$$D_+ = \frac{1}{2}(D_{tr} - i\tilde{D}_{tr}) \quad \text{and} \quad D_- = \frac{1}{2}(D_{tr} + i\tilde{D}_{tr}).$$ (3.10)

Furthermore, $D_{tr}$ splits into $D_+$ and $D_-$, and we have the two exact sequences:

$$\Gamma(S_m(\mathcal{F})) \xrightarrow{D_+} \ldots \Gamma(S_r(\mathcal{F})) \xrightarrow{D_+} \Gamma(S_{r-1}(\mathcal{F})) \xrightarrow{D_-} \ldots \Gamma(S_0(\mathcal{F})),$$ (3.11)

$$\Gamma(S_0(\mathcal{F})) \xrightarrow{D_+} \ldots \Gamma(S_r(\mathcal{F})) \xrightarrow{D_+} \Gamma(S_{r+1}(\mathcal{F})) \xrightarrow{D_-} \ldots \Gamma(S_m(\mathcal{F})).$$ (3.12)
4 Eigenvalues of the transversal Dirac operator

In this section, we prove Kirchberg-type inequalities by using the transversal Kählerian twistor operators on Kähler spin foliations. We refer to [Kir90], [Kir92].

Definition 4.1 On a Kähler spin foliation, we define the transversal Kählerian twistor operators by

\[ \mathcal{P}^r : \Gamma(S_r(\mathcal{F})) \xrightarrow{\nabla_{e_i}} \Gamma(Q^* \otimes S_r(\mathcal{F})) \xrightarrow{\mathcal{M}_r} \Gamma(\ker \mathcal{M}_r), \]

where \( \mathcal{M}_r \) is the transversal Clifford multiplication defined by

\[ \mathcal{M}_r : \Gamma(Q^* \otimes S_r(\mathcal{F})) \rightarrow \Gamma(S_{r-1}(\mathcal{F})) \oplus \Gamma(S_{r+1}(\mathcal{F})) \]

\[ X \otimes \Psi_r \mapsto p^- (X) \cdot \Psi_r \oplus p^+ (X) \cdot \Psi_r. \]

For all \( r \in \{0, \ldots, m\} \) and \( \Psi_r \in \Gamma(S_r(\mathcal{F})) \), we have

\[ \mathcal{P}^r \Psi_r = \sum_{i=1}^{q} e_i^* \otimes (\nabla_{e_i} \Psi_r + a_r p^- (e_i) \cdot \mathcal{D}^- \Psi_r + b_r p^+ (e_i) \cdot \mathcal{D}^+ \Psi_r), \]

(4.1)

where \( \mathcal{D}^\pm = D^\pm + \frac{1}{2} p_{\pm} (\kappa) \) with \( a_r = \frac{1}{2(r+1)} \) and \( b_r = \frac{1}{2(m-r+1)} \). For any spinor field \( \Psi_r \in \Gamma(S_r(\mathcal{F})) \), we can easily prove

\[ \sum_{i=1}^{q} e_i \cdot \mathcal{P}^r_{e_i} \Psi_r = 0. \]

(4.2)

Remark 4.2 For any non zero eigenvalue \( \lambda \) of \( D_{\mathcal{F}} \), there exists a spinor field \( \Psi \in \Gamma(S(\mathcal{F})) \) called of type \( (r, r+1) \), such that \( D_{\mathcal{F}} \Psi = \lambda \Psi \) and \( \Psi = \Psi_r + \Psi_{r+1} \), with \( r \in \{0, \ldots, m-1\} \). By using (3.10), (3.11) and (3.12) it follows that \( D^- \Psi_r = D^+ \Psi_{r+1} = 0 \), \( D^- \Psi_{r+1} = \lambda \Psi_r \), \( D^+ \Psi_r = \lambda \Psi_{r+1} \) and \( \| \Psi_r \|_{L^2} = \| \Psi_{r+1} \|_{L^2} \).

Proof. Let \( \varphi \) be an eigenspinor of \( D_{\mathcal{F}} \). There exists an \( r \) such that \( \varphi_r \) does not vanish. Let \( \Psi = \frac{1}{\lambda} D_- \varphi + D_+ \varphi_r \), one can easily get that \( D_{\mathcal{F}} \Psi = \lambda \Psi \).

Theorem 4.3 Let \((M, g_M, \mathcal{F})\) be a compact Riemannian manifold with a Kähler spin foliation \( \mathcal{F} \) of codimension \( q = 2m \) and a bundle-like metric \( g_M \).
with $\kappa \in \Omega^1_b(\mathcal{F})$ and $\delta B \kappa = 0$. Then any eigenvalue $\lambda$ of the transversal Dirac operator, satisfies
\[
\lambda^2 \geq \frac{m+1}{4m} K^\nabla_0.
\] (4.3)

If $\Psi$ is an eigenspinor of type $(r, r+1)$ associated with an eigenvalue $\lambda$ satisfying equality in (4.3), then $r = \frac{m-1}{2}$, the foliation $\mathcal{F}$ is minimal and for all $X \in \Gamma(Q)$, the spinor $\Psi$ satisfies
\[
\nabla_X \Psi + \frac{\lambda}{2(m+1)} (X \cdot \Psi - i \varepsilon J(X) \cdot \bar{\Psi}) = 0,
\] (4.4)

where $\varepsilon = (-1)^{\frac{m-1}{2}}$, and $\bar{\Psi} := (-1)^r (\Psi_r - \Psi_{r+1})$. As a consequence $m$ is odd and $\mathcal{F}$ is transversally Einstein with non negative constant transversal curvature $\sigma^{\nabla}$. 

Proof. For all $\Psi_r \in \Gamma(S_r(\mathcal{F}))$, using Identities (4.1) and (4.2), we have
\[
|\mathcal{P}(r) \Psi_r|^2 = \sum_{i=1}^q |\mathcal{P}_{e_i}^{(r)} \Psi_r|^2 = \sum_{i=1}^q (\mathcal{P}_{e_i}^{(r)} \Psi_r, \nabla_{e_i} \Psi_r)
\]
\[
= \sum_{i=1}^q (\nabla_{e_i} \Psi_r + a_r p_-(e_i) \cdot D_+ \Psi_r
\]
\[
+ b_r p_+(e_i) \cdot D_- \Psi_r, \nabla_{e_i} \Psi_r).
\]

Finally we obtain,
\[
|\mathcal{P}(r) \Psi_r|^2 = |\nabla^{tr} \Psi_r|^2 - a_r |D_+ \Psi_r|^2 - b_r |D_- \Psi_r|^2. \tag{4.5}
\]

Let $\lambda$ be an eigenvalue of $D_{tr}$ and let $\Psi$ an eigenspinor of type $(r, r+1)$. Applying Equality (4.5) to $\Psi_r$, one gets
\[
|\mathcal{P}(r) \Psi_r|^2 = |\nabla^{tr} \Psi_r|^2 - a_r \lambda^2 |\Psi_{r+1}|^2 - a_r \lambda \Re(\Psi_{r+1}, p_+(\kappa) \cdot \Psi_r)
\]
\[
- \frac{a_r}{4} |p_+ (\kappa) \cdot \Psi_r|^2 - \frac{b_r}{4} |p_- (\kappa) \cdot \Psi_r|^2.
\]

By the Schrödinger-Lichnerowicz formula and by the fact that $\Psi_r$ and $\Psi_{r+1}$ have the same $L^2$-norms, we get
\[
\int_M |\mathcal{P}(r) \Psi_r|^2 + \frac{a_r}{4} \int_M |p_+ (\kappa) \cdot \Psi_r|^2 + \frac{b_r}{4} \int_M |p_- (\kappa) \cdot \Psi_r|^2 =
\]
\[
\int_M ((1 - a_r) \lambda^2 - \frac{1}{4} K^\nabla_0 |\Psi_r|^2 - a_r \lambda \int_M \Re(\Psi_{r+1}, p_+(\kappa) \cdot \Psi_r). \tag{4.6}
\]
Similarly applying (4.5) to $\Psi_{r+1}$, we obtain
\[
\int_M |\mathcal{P}^{(r+1)} \Psi_{r+1}|^2 + \frac{a_r+1}{4} \int_M |p_+(\kappa) \cdot \Psi_{r+1}|^2 + \frac{b_{r+1}}{4} \int_M |p_-(\kappa) \cdot \Psi_{r+1}|^2 = \\
\int_M ((1-b_r+1)\lambda^2 - \frac{1}{4}K^\nabla |\Psi_{r+1}|^2 + b_{r+1}\lambda \int_M \Re^\nabla(\Psi_{r+1}, p_+(\kappa) \cdot \Psi_r),
\]
(4.7)

where $K^\nabla = \sigma^\nabla + |\kappa|^2$. In order to get rid the term $\lambda \int_M \Re^\nabla(\Psi_{r+1}, p_+(\kappa) \cdot \Psi_r)$, since the l.h.s. of (4.6) and (4.7) are non negative, dividing (4.6) by $a_r$ and (4.7) by $b_{r+1}$ then summing up, we find by substituting the values of $a_r$ and $b_{r+1}$,
\[
\lambda^2 \geq \frac{m+1}{4m}K^\nabla_0.
\]

Now, we discuss the limiting case of Inequality (4.3). Dividing (4.6) by $a_r$ and (4.7) by $b_{r+1}$ then summing up as before, and substituting $a_r, b_{r+1}$ and $\lambda$ by their values, we easily deduce that $\kappa = 0$, $\mathcal{P}^{(r)} \Psi_r = 0$ and $\mathcal{P}^{(r+1)} \Psi_{r+1} = 0$. Hence by (4.6), we find that $\lambda^2 = \frac{1}{4(1-a_r)}\sigma_0 = \frac{m+1}{4m}\sigma_0$ where $\sigma_0 = \inf_M \sigma^\nabla$, then $r = \frac{m-1}{2}$ and $m$ is odd. It remains to prove that $\Psi$ satisfies (4.4). For $r = \frac{m-1}{2}$, by definition of the Kählerian twistor operators, for all $j \in \{1, \cdots, q\}$, we obtain
\[
\nabla_{e_j} \Psi_r + \frac{\lambda}{m+1}p_-(e_j) \cdot \Psi_{r+1} = 0,
\]
and
\[
\nabla_{e_j} \Psi_{r+1} + \frac{\lambda}{m+1}p_+(e_j) \cdot \Psi_r = 0.
\]

Summing up the two equations, we get (4.4) for $X = e_j$. Using Ricci identity in (4.4), one easily proves that $\mathcal{F}$ is transversally Einstein. \hfill \Box

**Theorem 4.4** Under the same conditions as in Theorem 4.3 for $m$ even, any eigenvalue $\lambda$ of the transversal Dirac operator satisfies
\[
\lambda^2 \geq \frac{m}{4(m-1)}K^\nabla_0.
\]

If $\Psi$ is an eigenspinor of type $(r, r+1)$ associated with an eigenvalue satisfying equality in (4.8), then $r = \frac{m}{2}$, the foliation $\mathcal{F}$ is minimal and $\Psi$ satisfies for all $X \in \Gamma(Q)$,
\[
\nabla_X \Psi_{r+1} = -\frac{\lambda}{q}(X - iJX) \cdot \Psi_r.
\]

**Proof.** Let $\Psi$ an eigenspinor of type $(r, r+1)$ associated with any eigenvalue $\lambda$ of the transversal Dirac operator $D_{tr}$. Recalling Equalities (4.6) and (4.7), we have
\[
0 \leq \int_M ((1-a_r)\lambda^2 - \frac{1}{4}K^\nabla |\Psi_r|^2 - a_r\lambda \int_M \Re^\nabla(\Psi_{r+1}, p_+(\kappa) \cdot \Psi_r),
\]
(4.10)
and

\[ 0 \leq \int_M ((1 - b_{r+1}) \lambda^2 + \frac{1}{4} K_0^\nabla) |\Psi_{r+1}|^2 + b_{r+1} \lambda \int_M \Re(\Psi_{r+1}, p_+(\kappa) \cdot \Psi_r). \tag{4.11} \]

Hence if \( \lambda \int_M \Re(\Psi_{r+1}, p_+(\kappa) \cdot \Psi_r) \leq 0 \), then by (4.11)

\[ \lambda^2 \geq \frac{1}{4(1 - b_{r+1})} K_0^\nabla, \]

The antilinear isomorphism \( j \) sends \( S_r(\mathcal{F}) \) to \( S_{m-r}(\mathcal{F}) \). This allows the choice of \( \mu_r \) to be non negative (i.e. \( r \geq \frac{m}{2} \)) where \( \mu_r \) is the eigenvalue associated with \( \Psi_r \). Then a careful study of the graph of the function \( \frac{1}{1 - b_{r+1}} \), yields (4.8).

On the other hand if \( \lambda \int_M \Re(\Psi_{r+1}, p_+(\kappa) \cdot \Psi_r) > 0 \). Applying Equation (4.5) to the spinor \( j\Psi \), which is a spinor of type \( (m - (r + 1), m - r) \), we find the same inequalities as (4.10) and (4.11), then

\[ \lambda^2 > \frac{1}{1 - a_r} \frac{K_0^\nabla}{4}. \]

As before we can choose \( \mu_{m-(r+1)} \geq 0 \) (i.e. \( r \leq \frac{m}{2} - 1 \)). A careful study of the graph of the function \( \frac{1}{1 - a_r} \) gives Inequality (4.8).

Now we discuss the limiting case of (4.8). As we have seen, it could not be achieved if \( \lambda \int_M \Re(\Psi_{r+1}, p_+(\kappa) \cdot \Psi_r) > 0 \), so only the other case should be considered. By (4.7), one has

\[
\int_M |P^{(r+1)} \Psi_{r+1}|^2 + \frac{a_{r+1}}{4} \int_M |p_+(\kappa) \cdot \Psi_{r+1}|^2 \\
+ \frac{b_{r+1}}{4} \int_M |p_-(\kappa) \cdot \Psi_{r+1}|^2 - b_{r+1} \lambda \int_M \Re(\Psi_{r+1}, p_+(\kappa) \cdot \Psi_r) = \\
(1 - b_{r+1}) \int_M (\frac{m}{4(m-1)} K_0^\nabla - \frac{1}{4(1 - b_{r+1})} K_0^\sigma) |\Psi_{r+1}|^2.
\]

Since \( \frac{m}{m-1} = \inf_{r \geq \frac{m}{2}} \frac{1}{1 - b_{r+1}} \), and the l.h.s. of (4) is non negative, we deduce that \( \kappa = 0, P^{r+1} \Psi_{r+1} = 0 \) and \( \frac{m}{m-1} = \frac{1}{1 - b_{r+1}} \), so \( r = \frac{m}{2} \). It remains to show that Equation (4.9) holds. For this, take \( X = e_j \) where \( \{e_j\}_{j=1,\ldots,q} \) is a local orthonormal frame. For \( r = \frac{m}{2} \), and by definition of the Kählerian twistor operators, for all \( j \in \{1, \cdots, q\} \), we obtain

\[ \nabla_{e_j} \Psi_{r+1} + \frac{\lambda}{q} (e_j - iJe_j) \cdot \Psi_r = 0. \]
References


