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DIFFERENTIATION OF SOME FUNCTIONALS OF RISK PROCESSES.  
APPLICATION TO RUIN THEORY AND TO DETERMINATION OF OPTIMAL  
RESERVE ALLOCATION FOR MULTIDIMENSIONAL RISK PROCESSES.

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**DIFFERENTIATION OF SOME FUNCTIONALS OF RISK PROCESSES. APPLICATIONS TO RUIN THEORY AND TO DETERMINATION OF OPTIMAL RESERVE ALLOCATION FOR MULTIDIMENSIONAL RISK PROCESSES.**

STÉPHANE LOISEL,\* *Université Lyon 1*

**Abstract**

For general risk processes, the expected time-integrated negative part of the process on a fixed time interval is introduced and studied. Differentiation theorems are stated and proved. They make it possible to derive the expected value of this risk measure, and to link it with the average total time below zero studied by Dos Reis [1], and the probability of ruin. Differentiation of other functionals of unidimensional and multidimensional risk processes with respect to the initial reserve level are carried out. Applications to ruin theory, and to the determination of the optimal allocation of the global initial reserve which minimizes one of these risk measures, illustrate the variety of application fields and the benefits deriving from an efficient and effective use of such tools.

*Keywords:* Ruin theory; Sample path properties; Optimal reserve allocation; Multidimensional risk process; Risk measures

AMS 2000 Subject Classification: Primary 60G17

Secondary 60G55; 91B30; 91B32; 62P05

**Introduction**

For unidimensional risk processes  $R_t = u + X_t$  (representing the surplus of an insurance company at time  $t$ , with initial reserve  $u$  and with  $X_t = ct - S_t$ , where  $c > 0$

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is the premium by unit time, and  $S_t$  is in the most classical case a compound Poisson process (here we do not limit to the Poisson case)), many risk measures have been considered (see for example Gerber [2], Dufresne and Gerber [3] and Picard [5]): the time to ruin  $T_u = \inf\{t > 0, u + X_t < 0\}$ , the severity of ruin  $u + X_{T_u}$ , the couple  $(T_u, u + X_{T_u})$ , the time in the red (below 0) from the first ruin to the first time of recovery  $T'_u - T_u$  where  $T'_u = \inf\{t > T_u, u + X_t = 0\}$ , the maximal ruin severity  $(\inf_{t>0} u + X_t)$ , the aggregate severity of ruin until recovery  $J(u) = \int_{T_u}^{T'_u} |u + X_t| dt, \dots$  Dos Reis [1] studied the total time in the red  $\tau(u) = \int_0^{+\infty} 1_{\{u+X_t < 0\}} dt$  using Gerber's work [2].

All these random variables are drawn from the infinite time ruin theory, or involve the behavior of the risk process between ruin times and recovery times. It seems interesting to consider risk measures based on some fixed time interval  $[0, T]$  ( $T$  may be infinite). One of the simplest penalty functions may be the expected value of the time-aggregated negative part of the risk process:

$$E(I_T) = E \left( \int_0^T 1_{\{R_t < 0\}} |R_t| dt \right).$$

Note that the probability  $P(I_T = 0)$  is the probability of non ruin within finite time  $T$ .  $I_T$  may be seen as the penalty the company will have to pay due to its insolvency until the time horizon  $T$ . From an economical point of view, it seems more consistent to consider

$$I_{g,h}(u) = \left( \int_0^T (1_{\{u+X_t \geq 0\}} g(|u + X_t|) - 1_{\{u+X_t < 0\}} h(|u + X_t|)) dt \right)$$

with  $0 \leq g \leq h$ , where  $g$  corresponds to a reward function for positive reserves, and  $h$  is a penalty function in case of insolvency. As for utility functions,  $g$  and  $h$  should be both increasing and convex in the classical case.  $g \leq h$  because usually the cost of ruin is higher than the reward of the opposite wealth level.

These risk measures may be differentiated with respect to the initial reserve  $u$ , which makes it possible to compute them quite easily as integrals of other functions of  $u$  such as the probability of ruin or the total time in the red. Moreover, they have the advantage that the integral over  $t$  and the mathematical expectation may be permuted thanks to Fubini's theorem.

Statement and proofs of differentiation theorems can be found in sections 1 and 2.

Section 3 presents examples of applications to unidimensional risk measures, in particular a closed-form formula is derived for  $E(I_\infty(u))$  in the Poisson-exponential case.

One can also use these concepts to construct risk measures for multidimensional risk processes, modelling different lines of business of an insurance company (car insurance, health insurance, ...). In this framework, determining the needed global initial reserve for the global expected penalty to be small enough requires to find the optimal allocation of this reserve. Differentiation of unidimensional risk measures are useful to find this optimal reserve allocation. All this is illustrated in section 4.

### 1. Differentiation theorems

**Theorem 1.** *Let  $(X_t)_{t \in [0, T]}$  be a stochastic process with almost surely time-integrable sample paths. For  $u \in \mathbb{R}$ , denote by  $\tau(u)$  the random variable corresponding to the time spent under zero by the process  $u + X_t$  between the fixed times 0 and  $T$ :*

$$\tau(u) = \int_0^T 1_{\{u+X_t < 0\}} dt,$$

Let  $\tau_0(u)$  correspond to the time spent in zero by the process  $u + X_t$ :

$$\tau_0(u) = \int_0^T 1_{\{u+X_t = 0\}} dt.$$

Let  $I_T(u)$  represent the time-integrated negative part of the process  $u + X_t$  between 0 and  $T$ :

$$I_T(u) = \left( \int_0^T 1_{\{u+X_t < 0\}} |u + X_t| dt \right)$$

and  $f(u) = E(I_T(u))$ .

For  $u \in \mathbb{R}$ , if  $E\tau_0(u) = 0$ , then  $f$  is differentiable at  $u$ , and  $f'(u) = -E\tau(u)$ .

$I_T(u)$  is illustrated by figure 1.

*Proof.* Fix  $u \in \mathbb{R}$ . For  $\varepsilon \geq 0$ , set

$$\tau_\varepsilon(u) = \int_0^T 1_{\{|u+X_t| < \varepsilon\}} dt.$$

$\tau_\varepsilon(u)$  represents the time spent by the process  $u + X_t$  in the interval  $] - \varepsilon, \varepsilon[$  between dates 0 and T.

For each sample path (considered as a function of time  $t$ ),

$$t \rightarrow 1_{\{|u+X_t|<\varepsilon\}}$$

pointwise converges, decreasingly to

$$t \rightarrow 1_{\{u+X_t=0\}}.$$

Besides, each of the integrals of the indicator functions is bounded by T. From the monotone convergence theorem,  $\tau_\varepsilon$  is decreasing with respect to  $\varepsilon$  and surely converges to  $\tau_0$ .

From the monotone convergence theorem (for mathematical expectation this time),  $E\tau_\varepsilon \downarrow E\tau_0$  as  $\varepsilon \downarrow 0$ , because for all  $\varepsilon \geq 0$ ,  $E\tau_\varepsilon \leq T$ .

**Lemma 1.1.** For  $\varepsilon \in \mathbb{R}$ ,

$$|I_T(u + \varepsilon) - I_T(u) + \varepsilon\tau(u)| \leq |\varepsilon|\tau_\varepsilon(u)$$

*Proof of the lemma.* For  $\varepsilon > 0$ ,  $\{u + \varepsilon + X_t < 0\} \subset \{u + X_t < 0\}$ , whence  $I_T(u + \varepsilon) - I_T(u) =$

$$\begin{aligned} & \int_0^T (|u + \varepsilon + X_t| - |u + X_t|) 1_{\{u+X_t<0\}} dt - \int_0^T |u + \varepsilon + X_t| 1_{\{-\varepsilon < u+X_t < 0\}} dt \\ I_T(u + \varepsilon) - I_T(u) &= -\varepsilon \int_0^T 1_{\{u+X_t<0\}} dt - \int_0^T |u + \varepsilon + X_t| 1_{\{-\varepsilon < u+X_t < 0\}} dt \quad (1) \end{aligned}$$

On the right side of (1), the left term corresponds to  $-\varepsilon\tau(u)$ . The absolute value under the integral of the second term is less than  $\varepsilon$  on the support of the indicator function.

Hence

$$|I_T(u + \varepsilon) - I_T(u) + \varepsilon\tau(u)| < \int_0^T \varepsilon 1_{\{-\varepsilon < u+ct+S_t < 0\}} dt,$$

which proves the lemma for  $\varepsilon > 0$ . A symmetrical procedure solves the case  $\varepsilon \leq 0$ , which ends the proof of the lemma.

From lemma 1.1,

$$|EI_T(u + \varepsilon) - EI_T(u) + \varepsilon E\tau(u)| \leq |\varepsilon|E\tau_\varepsilon(u)$$

and

$$EI_T(u + \varepsilon) = EI_T(u) - \varepsilon E\tau(u) + \varepsilon v(u, \varepsilon)$$

where

$$|v(u, \varepsilon)| \leq E\tau_\varepsilon(u) \rightarrow E\tau_0(u) = 0$$

as  $\varepsilon \rightarrow 0$ , which proves that  $f$  is differentiable with respect to  $u$  and that for  $u \in \mathbb{R}$ ,  $f'(u) = -E\tau(u)$ .

**Corollary 1.** *Using notations of theorem 1, let  $X_t = ct - S_t$ , where  $S_t$  is a jump process such that, almost surely,  $S_t$  has a finite number of nonnegative jumps in every finite interval, and that  $X_t$  has a positive drift ( $X_t \rightarrow +\infty$  a.s.). Then  $f$  defined by  $f(u) = E(I_T(u))$  for  $u \in \mathbb{R}$  is differentiable on  $\mathbb{R}$ , and for  $u \in \mathbb{R}$ ,  $f'(u) = -E\tau(u)$ .*

*Proof.* Only

$$E\tau_0(u) = \int_0^T 1_{\{u+ct-S_t=0\}} dt = 0$$

has to be shown.  $R_t = u + ct - S_t$  is a process whose sample paths are almost surely increasing between two consecutive jump instants. The number of jumps is almost surely finite on the time interval  $[0, T]$ . Between two times when the process is in 0, there must be at least one jump instant.

This implies that the number of visits of 0 is almost surely finite (it is less than  $N_T + 1$  where  $N_T$  is the number of jumps between 0 and T).

So  $E\tau_0 = 0$  and the result comes from theorem 1.

**Proposition 1.** *More generally, all processes for which the distribution of  $R_t$  is diffuse for all  $t \in \mathbb{R}^+ - N$  satisfy the condition  $E\tau_0 = 0$ , if  $N$  is a null subset of  $\mathbb{R}^+$  for the Lebesgue measure.*

*Theorem 1 is also verified for this wide class of processes.*

*Proof.* For  $T \in \bar{\mathbb{R}}$ , from Fubini's theorem,

$$E\tau_0(T) \leq E \left( \int_0^{+\infty} 1_{\{R_t=0\}} dt \right) = \int_0^{+\infty} P(R_t = 0) dt$$

which provides the expected result.

**Theorem 2.** *Let  $g \in \mathcal{C}^1(\mathbb{R}^+, \mathbb{R}^+)$  be a convex function, such that  $g(0) = 0$ . Let  $X_t$  be a stochastic process such that, for  $u \in \mathbb{R}$ ,  $t \rightarrow g(u + X_t)1_{\{u+X_t < 0\}}$  is almost surely*

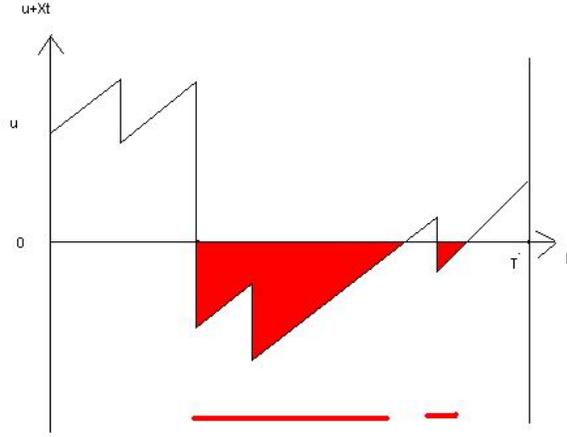


FIGURE 1: The area in red represents  $I_T(u) = \int_0^T 1_{\{u+X_t < 0\}} |u + X_t| dt$

integrable with respect to  $t$ . Let  $I_g$  be the function from  $\mathbb{R}^+$  into the space of nonnegative random variables, and defined by

$$I_g(u) = \left( \int_0^T 1_{\{u+X_t < 0\}} g(-(u + X_t)) dt \right)$$

for  $u \in \mathbb{R}$  and let  $f(\cdot) = EI_g(\cdot)$ .

For  $u \in \mathbb{R}$ , if  $f(u) < +\infty$ ,  $EI_{g'}(u) < +\infty$  and  $E\tau_0(u) = 0$ , then  $f$  is differentiable at point  $u$ , and

$$f'(u) = -E \left( \int_0^T 1_{\{u+X_t < 0\}} g'(|u + X_t|) dt \right)$$

*Proof.* Fix  $u \in \mathbb{R}$ . For  $\varepsilon > 0$ ,  $\{u + \varepsilon + X_t < 0\} \subset \{u + X_t < 0\}$ , whence

$$\begin{aligned} \frac{I_g(u + \varepsilon) - I_g(u)}{\varepsilon} &= \int_0^T \frac{g(|u + \varepsilon + X_t|) - g(|u + X_t|)}{\varepsilon} 1_{\{u+X_t < 0\}} dt \\ &\quad - \int_0^T \frac{g(|u + \varepsilon + X_t|)}{\varepsilon} 1_{\{-\varepsilon < u+X_t < 0\}} dt \end{aligned}$$

For  $t \in [0, T]$ ,

$$\frac{g(-(\varepsilon + X_t)) - g(-X_t)}{-\varepsilon} 1_{\{u+X_t < 0\}} \uparrow g'(-X_t) 1_{\{u+X_t < 0\}}$$

almost surely as  $\varepsilon \downarrow 0$ , from the increase of the rates of increase of convex functions.

From the monotone convergence theorem, for  $t \in [0, T]$ ,

$$E \left( \frac{g(-(u + \varepsilon + X_t)) - g(-(u + X_t))}{\varepsilon} 1_{\{u + X_t < 0\}} \right) \rightarrow -g'(-(u + X_t)) 1_{\{u + X_t < 0\}}$$

From Fubini's theorem,

$$E \left( \int_0^T \frac{g(u + \varepsilon + X_t) - g(u + X_t)}{\varepsilon} 1_{\{u + X_t < 0\}} dt \right) \rightarrow -EI_{g'}(u)$$

as  $\varepsilon \downarrow 0$ , where

$$I_{g'}(u) = \int_0^T g'(-(u + X_t)) 1_{\{u + X_t < 0\}} dt$$

Hence

$$|f(u + \varepsilon) - f(u) + \varepsilon EI_{g'}(u) + \varepsilon w(u, \varepsilon)| \leq E \left( \int_0^T g(-(u + \varepsilon + X_t)) 1_{\{-\varepsilon < u + X_t < 0\}} dt \right)$$

with  $w(u, \varepsilon) \rightarrow 0$  as  $\varepsilon \downarrow 0$ , and

$$|f(u + \varepsilon) - f(u) + \varepsilon EI_{g'}(u) + \varepsilon w(u, \varepsilon)| \leq \varepsilon E \tau_\varepsilon(u) E \left( \sup_{t \in [0, \varepsilon]} g'(t) \right)$$

$$EI_g(u + \varepsilon) = EI_g(u) - \varepsilon E \tau(u) + \varepsilon (v(u, \varepsilon) - w(u, \varepsilon))$$

where

$$|v(u, \varepsilon)| \leq KE \tau_\varepsilon(u) \rightarrow KE \tau_0(u) = 0$$

as  $\varepsilon \downarrow 0$ , which proves that  $f$  is right-differentiable at point  $u$  and that

$$f'_r(u) = E \left( \int_0^T g'(-(u + X_t)) 1_{\{u + X_t < 0\}} dt \right).$$

With similar reasoning,  $f$  is left-differentiable and  $f'_l = f'_r$ , which ends the proof.

## 2. Differentiation of the average time in the red and other generalizations

Recall that the time in the red is the time spent by the wealth process below 0, between time 0 and some fixed time horizon  $T$ :

$$\tau(u) = \int_0^T 1_{\{u + X_t < 0\}} dt.$$

$T$  is first supposed to be finite.

**Theorem 3.** *Let  $X_t = ct - S_t$ , where  $S_t$  is a jump process satisfying hypothesis (H1): almost surely,  $S_t$  has a finite number of nonnegative jumps in every finite interval, and for each  $t$ , the distribution of  $S_t$  is absolutely continuous.*

*For example,  $S_t$  might be a compound Poisson process with a continuous jump size distribution. Consider  $T < +\infty$  and define  $h$  by  $h(u) = E(\tau(u))$  for  $u \in \mathbb{R}$ .  $h$  is differentiable on  $\mathbb{R}_*^+$ , and for  $u > 0$ ,*

$$h'(u) = -\frac{1}{c}EN^0(u),$$

where  $N^0(u) = \text{Card}(\{t \in [0, T], u + ct - S_t = 0\})$ .

*Proof.* Almost surely in  $\omega$ , the number of jumps  $N(T)$ , and so  $N^0(u)$ , is finite. Consider a sample path  $(X_t(\omega))_{0 \leq t \leq T}$ . Let  $R_t = u + X_t$  and denote by  $T_i$  the  $i^{\text{th}}$  jump instant. Define

$$\varepsilon_0(\omega) = \inf_{n \leq N(T), R_{T_n} > 0} R_{T_n}$$

If  $N^0(u) = 0$ , then define

$$\varepsilon^+ = \inf(\{u + X_t, 0 \leq t \leq T\} \cap \mathbb{R}^+)$$

and

$$\varepsilon^- = -\sup(\{u + X_t, 0 \leq t \leq T\} \cap \mathbb{R}^-).$$

$\varepsilon^-$  and  $\varepsilon^+$  are almost surely positive. If  $|\varepsilon| < \inf(\varepsilon^+, \varepsilon^-)$ , then  $\tau(u - \varepsilon) - \tau(u) = 0$ , and the following reasoning remains valid.

Otherwise, for  $1 \leq i \leq N^0(u)$ , denote by  $t_i$  the instant of the  $i^{\text{th}}$  visit of  $R_t$  in 0, and by  $t'_i$  the instant of the first jump of  $R_t$  after  $t_i$ . The sample paths of the process  $R_t$  are almost surely right-continuous, and the probability that  $R_T = 0$  is zero. So one may consider

$$\varepsilon_1(\omega) = \min\left(\min_{1 \leq i \leq N^0(u)} c(t'_i - t_i), c(T - t_{N^0(u)})\right).$$

Then, for  $0 < \varepsilon < \min(\varepsilon_0(\omega), \varepsilon_1(\omega))$ ,

$$\{0 < u + ct - S_t < \varepsilon\} = \bigcup_{i=1}^{N^0(u)} \{]t_i, t_i + \varepsilon/c[\}$$

and so

$$\tau(u - \varepsilon) - \tau(u) = \int_0^T (1_{\{u - \varepsilon + ct - S_t < 0\}} - 1_{\{u - \varepsilon + ct - S_t < 0\}}) dt$$

$$= \int_0^T 1_{\{0 \leq u+ct-S_t < \varepsilon\}} dt = \sum_{k=1}^{N^0(u)} \frac{\varepsilon}{c}$$

Hence

$$\frac{\tau(u - \varepsilon) - \tau(u)}{\varepsilon} \rightarrow \frac{1}{c} N^0(u)$$

almost surely as  $\varepsilon \rightarrow 0$ . Moreover, between two consecutive jumps of  $R_t$ , the difference between the two integrals is less than  $\frac{\varepsilon}{c}$  in absolute value, whence

$$\int_{T_i}^{T_{i+1}} 1_{\{0 \leq u+ct-S_t < \varepsilon\}} dt \leq \frac{\varepsilon}{c}.$$

So for  $\varepsilon > 0$  small enough, with notations  $T_{N(T)+1} = T$  and  $T_0 = 0$ ,

$$\left( \frac{\tau(u - \varepsilon) - \tau(u)}{\varepsilon} \right) = \left( \sum_{i=0}^{N(T)} \frac{1}{\varepsilon} \int_{T_i}^{T_{i+1}} 1_{\{0 \leq u+ct-S_t < \varepsilon\}} dt \right)$$

$$\left( \frac{\tau(u - \varepsilon) - \tau(u)}{\varepsilon} \right) \leq \left( \sum_{i=0}^{N(T)} \frac{1}{\varepsilon} \frac{\varepsilon}{c} \right) \leq \frac{1}{c} (EN(T) + 1)$$

Hence, from the dominated convergence theorem,

$$E \left( \frac{\tau(u - \varepsilon) - \tau(u)}{\varepsilon} \right) \rightarrow \frac{1}{c} EN^0(u)$$

as  $\varepsilon \rightarrow 0$ . This proves that  $h$  is left-differentiable on  $\mathbb{R}_*^+$ , and that for  $u > 0$ ,

$$h'_l(u) = -\frac{1}{c} EN^0(u).$$

With similar reasoning,  $h$  is right-differentiable on  $\mathbb{R}_*^+$ , and  $h'_l = h'_r$ . Hence  $h$  is differentiable on  $\mathbb{R}_*^+$ , and for  $u > 0$ ,  $h'(u) = -\frac{1}{c} EN^0(u)$ .

**Remark 1.** This provides the second-order differentiate of  $EI_T(\cdot)$ , which appears to be positive.  $EI_T(\cdot)$  is thus well strictly convex, which will be very important for the minimization in section 4.

**Remark 2.** This second-order differentiate corresponds in the general case to the expectation of the local time  $L_T(0)$  in 0 of the process  $u + X_t$  up to time T:

$$L_T(0) = \lim_{\varepsilon \downarrow 0} \left( \frac{1}{2\varepsilon} \int_0^T P(|u + X_t| < \varepsilon) dt \right)$$

**Theorem 4.** *Let  $g, h$  be two convex functions in  $\mathcal{C}^1(\mathbb{R}^+, \mathbb{R}^+)$ , such that for  $x \geq 0$ ,  $g(x) \geq g(0)$  and  $h(x) \geq h(0)$ . Let  $X_t$  be a stochastic process such that  $t \rightarrow g(u + X_t)$  and  $t \rightarrow h(u + X_t)$  are almost surely integrable on  $[0, T]$ . Let  $I_g^+$  be the function from  $\mathbb{R}$  into the space of nonnegative random variables, and defined by*

$$I_g^+(u) = \int_0^T \mathbf{1}_{\{u+X_t \geq 0\}} g(u + X_t) dt$$

for  $u \geq 0$  and let  $f(\cdot) = EI_g^+(\cdot) - EI_h(\cdot)$ .

If, for  $u \in \mathbb{R}$ ,

$$EI_g^+(u), \quad EI_h(u), \quad EI_{g'}^+(u), \quad EI_{h'}(u) < +\infty,$$

and if  $E\tau_0(u) = 0$ , then  $f$  is differentiable on  $\mathbb{R}_*^+$ , and for  $u > 0$ ,

$$f'(u) = EI_{g'}^+(u) - EI_{h'}(u) - (g(0) + h(0))EL_T(0)$$

**Corollary 2.** *With the hypotheses of theorem 4, if besides  $X_t = ct - S_t$ , where  $S_t$  satisfies hypothesis (H1) of theorem 3, then the differentiate may be rewritten for  $u > 0$  as:*

$$f'(u) = EI_{g'}^+(u) - EI_{h'}(u) + \frac{(g(0) + h(0))EN^0(u)}{c}$$

where  $N^0(u) = \text{Card}(\{t \in [0, T], \quad u + ct - S_t = 0\})$ .

*Proof of corollary 2.* Immediate from theorem 4, after replacing the last term in (2) following the proof of theorem 3.

*Proof of theorem 4.* Decompose

$$I_g^+(u) - I_h(u) = -\tilde{I}_{(g-g(0))}(-u) - I_{(h-h(0))}(u) - h(0)\tau(u) + g(0)(T - \tau(u)),$$

where  $\tilde{I}_g$  is obtained from  $I_g$  by changing  $X_t$  into  $-X_t$ . From linearity of expectation and of differentiation, applying theorem 2 to  $g - g(0)$  with  $-X_t$  and to  $h - h(0)$  with  $X_t$ , and using theorem 3 end the proof of theorem 4.

**Theorem 5.** *If besides the process  $X_t$  converges almost surely to  $+\infty$  as  $t \rightarrow +\infty$ , and if for  $u \geq 0$ ,  $EI_\infty < +\infty$  and  $E\tau(u, \infty) < +\infty$ , then theorem 1 remains valid with  $T = +\infty$ .*

*Proof.* Same kind of reasoning as previously.

**Remark 3.** These conditions of integrability are fulfilled if the time spent below 0 for a single ruin is integrable.

Denote by  $\psi(u)$  the probability of ruin in infinite time with initial reserve  $u$ .

**Theorem 6.** *Theorem 3 remains valid with  $T = +\infty$  if besides  $X_t$  has a positive drift and if  $\tau(u)$  is integrable for all  $u > 0$ . Besides, in the compound Poisson case, for  $u > 0$ ,*

$$h'(u) = -\frac{1}{c} \frac{\psi(0)}{1 - \psi(0)} \psi(u)$$

*Proof.* For  $T \in \bar{\mathbb{R}}$ , denote

$$\tau(u, T) = \int_0^T \mathbf{1}_{\{0 < u + X_t < \varepsilon\}} dt.$$

$(N^0(u, n))_{n \geq 0}$  is a nondecreasing sequence of random variables which surely converges to  $N^0(u, +\infty)$ , possibly infinite.

Let us show that  $EN^0(u, +\infty) < +\infty$ .

Almost surely,  $u + X_t \rightarrow +\infty$  as  $t \rightarrow +\infty$ . Hence, almost surely,  $N^0(u, +\infty) < +\infty$  and is equal to the number of ruins:

$$N^0(u, \infty) = \text{Card}(\{t > 0, \quad u + ct - S_t < 0 \text{ and } u + ct^- - S_{t^-} > 0\})$$

Indeed, after each ruin, there is a recovery because  $X_t$  converges almost surely to  $+\infty$  as  $t$  goes to  $+\infty$ , and the number of jumps which lead exactly to the value 0 is finite almost surely. Besides, in the compound Poisson case, the number of ruins has the following distribution:

$$P(N^0(u, \infty) = n) = \psi(u)\psi(0)^{n-1}(1 - \psi(0))$$

for  $n \geq 1$  and  $P(N^0(u, \infty) = 0) = 1 - \psi(u)$ . So  $N^0(u, \infty)$  follows a zero-modified geometric distribution :  $P(N^0(u, \infty) = 0) = 1 - \psi(u)$  and for  $n > 0$ ,

$$P(N^0(u, \infty) = n | N^0(u, \infty) > 0) = \psi(0)^{n-1}(1 - \psi(0))$$

Hence  $N^0(u, \infty)$  is integrable and

$$EN^0(u, \infty) = \psi(u) \frac{\psi(0)}{1 - \psi(0)}$$

For all  $\omega$  and for  $\varepsilon > 0$ , the function

$$(T, \omega) \rightarrow \frac{\tau(u + \varepsilon, T) - \tau(u, T)}{\varepsilon}(\omega)$$

is increasing with respect to  $T$ , and its limit expectation is equal to  $-\frac{1}{c}EN^0(u, T)$  as  $\varepsilon \downarrow 0$ . From the monotone convergence theorem,

$$E \lim_{\varepsilon \downarrow 0} \left( \frac{\tau(u + \varepsilon, \infty) - \tau(u, \infty)}{\varepsilon} \right) = -\frac{1}{c}EN^0(u, \infty)$$

**Remark 4.** In infinite time, the probability of ruin may be regarded as the expectation of the local time in 0 of the process (up to multiplication by a constant number).

### 3. Applications to the unidimensional case

**Theorem 7.** In the Poisson-Exponential( $1/\mu$ ) case,  $\psi(u) = (1 + \mu R)e^{-Ru}$ , with  $R = \frac{1}{\mu} \left(1 - \frac{\lambda\mu}{c}\right)$ . Hence, for  $T = +\infty$ ,

$$E\tau(u) = \frac{1 + \mu R}{cR} \mu R e^{-Ru}$$

and

$$EI_\infty(u) = \frac{1 + \mu R}{cR^2} \mu R e^{-Ru}$$

*Proof.* This comes from a mere integration of the well-known formula for  $\psi(u)$ , as the considered functions tend to 0 as  $u \rightarrow +\infty$ . Besides, as  $\mu$  is the average claim amount,  $R = \mu - \frac{\lambda}{c}$  and  $\rho = \frac{c - \lambda\mu}{c}$ .

This method provides a way to get back the average total time in the red from the integration of the probability of ruin. Dos Reis [1] derived this result for  $E\tau(u, \infty)$  by considering the number of ruins, and using the distributions of the length of the first period in the red (until recovery), and of those of the following periods in the red, which had been derived by Gerber [2].

**Remark 5.** Of course, it is possible to derive  $EI_\infty(u)$  for Gamma-distributed or phase-type-distributed claim amounts, as we know the probability of ruin in these cases. It is not developed here to keep it concise.

The parallel with the Brownian case is also interesting. The local time of a standard Brownian motion  $W_t$  in  $x$  is defined by

$$L_t(x) = \lim_{\varepsilon \downarrow 0} \frac{1}{4\varepsilon} \int_0^t 1_{\{|W_s - x| < \varepsilon\}} ds$$

This provides a density for the occupation time  $\Gamma_t(B)$  of a Borelian set  $B$  between 0 and  $t$ :

$$\Gamma_t(B) = \int_B 2L_t(x) dx$$

Paul Lévy's Brownian local time representation Theorem with downcrossings states that

**Theorem 8.** (*Paul Lévy*)

$$2L_t(0) = \lim_{\varepsilon \downarrow 0} \varepsilon D_t(\varepsilon)$$

where  $D_t(\varepsilon)$  is the number of downcrossings of the interval  $[0, \varepsilon]$  by the process  $W_s$  between 0 and  $t$ .

This well-known theorem might be viewed as a limit case of theorem 3.

#### 4. Multidimensional risk measures and optimal allocation

For a unidimensional risk process, one classical goal is to determine the minimal initial reserve  $u_\varepsilon$  needed for the probability of ruin to be less than  $\varepsilon$ .

In a multidimensional framework, modelling the evolution of the different lines of business of an insurance company by a multirisk process  $(u_1 + X_t^1, \dots, u_n + X_t^n)$  ( $u_i + X_t^i$  corresponds to the wealth of the  $i^{\text{th}}$  line of business at time  $t$ ), one could look for the global initial reserve  $u$  which ensures that the probability of ruin  $\psi$  satisfies

$$\psi(u_1, \dots, u_n) \leq \varepsilon$$

for the optimal allocation  $(u_1, \dots, u_n)$  such that

$$\psi(u_1, \dots, u_n) = \inf_{v_1 + \dots + v_n = u} \psi(v_1, \dots, v_n)$$

with

$$\psi(u_1, \dots, u_n) = P(\exists i \in [1, n], \exists t > 0, u + X_t^i < 0).$$

Instead of the probability of crossing some barriers, it may be more interesting to minimize the sum of the expected cost of the ruin for each line of business until time  $T$ , which may be represented by the expectation of the sum of integrals over time of the negative part of the process. In both cases, finding the global reserve needed requires to determine the optimal allocation. It has just been shown in the previous sections how to compute  $E(I_T)$  for one line of business, and the linearity of the expectation makes it possible to compute the sum for  $n$  dependent lines of business just as in the independent case. The structure of dependence between lines of business has no impact on this risk measure. This may be considered as a problem of optimal allocation of resource under budget constraints as in economics, the goal being to maximize the utility function given by the opposite of the sum of the  $E(I_T^i)$ .

#### 4.1. Minimizing the penalty function

Recall that what has to be minimized is

$$A(u_1, \dots, u_n) = \sum_{i=1}^n EI_T^i$$

where

$$EI_T^i = E \left[ \int_0^T |R_t^i| 1_{\{R_t^i < 0\}} dt \right]$$

with  $R_t^i = u_i + X_t^i$  under the constraint  $u_1 + \dots + u_n = u$ . This does not depend on the dependence structure between the lines of business because of the linearity of the expectation. Denote  $v_i(u_i)$  the differentiate of  $EI_T^i$  with respect to  $u_i$ . Using the Lagrange multipliers implies that if  $(u_1, \dots, u_n)$  minimizes  $A$ , then  $v_k(u_k) = v_1(u_1)$  for all  $1 \leq k \leq n$ . Compute  $v_i(u_i)$ :

$$v_i(u_i) = \left( E \left[ \int_0^T |R_t^i| 1_{\{R_t^i < 0\}} dt \right] \right)' = -E\tau^i = - \int_0^T P[\{R_t^i < 0\}] dt$$

where  $\tau^i$  represents the time spent in the red between 0 and  $T$  for line of business  $i$ .

The differentiation theorem of the previous section justifies the previous derivation. The sum of the average times spent under 0 is a decreasing function of the  $u_i$ . So  $A$  is strictly convex. On the compact space

$$\mathcal{S} = \{(u_1, \dots, u_n) \in (\mathbb{R}^+)^n, \quad u_1 + \dots + u_n = u\},$$

A admits a unique minimum. The optimal allocation is thus the following: there is a subset  $J \subset [1, n]$  such that for  $i \notin J$ ,  $u_i = 0$ , and for  $i, j \in J$ ,  $E\tau_i = E\tau_j$ . The interpretation is quite intuitive: the safest lines of business do not require any reserve, and the other ones share the global reserve in order to get equal average times in the red for those lines of business.

Relaxing nonnegativity, on  $\{u_1 + \dots + u_n = u\}$ , if  $(u_1, \dots, u_n)$  is an extremum point for  $A$ , then for the  $n$  lines of business, the average times spent under 0 are equal to one another. If it is a minimum for the sum of the times spent below 0 for each line of business, then the average number of visits are proportional to the  $c_i$ , and in infinite time the ruin probabilities are in fixed proportions. However the existence of a minimum is not guaranteed, because  $(u_1, \dots, u_n)$  is no longer compact. It would be more tractable with the average time in the red or with minimization on the  $c_i$ , because some factors penalize very negative  $u_i$  in these problems.

#### 4.2. Example

In the Poisson-Exponential( $\frac{1}{\mu}$ ) case, recall that

$$EI_u = \frac{1 + \mu R}{cR^2} \mu R e^{-Ru}.$$

Consider a two-line-of-business model, with the following parameters:

$\mu_2 = 1$ ,  $c_1 = c_2 = 1$ ,  $R_1 = 0.2$ ,  $R_2 = 0.4$  and  $u := 10$ .

A mere modification of the parameter  $\mu_1$  makes the optimal allocation strongly vary. When  $\mu_1 = 1$  (Figure 2), the optimal allocation is about  $(u_1 = 6.409748867, u_2 = 3.590251133)$ . When  $\mu_1 = 10$  (Figure 3), the optimal allocation is  $(u_1 = 0, u_2 = 10)$ . In the second case, line of business 2 is much more risky than line of business 1, which justifies the transfer of the whole global initial reserve to line of business 2. Here, the Lundberg exponents are the same in both cases, and heavier claims (with a smaller frequency) are more risky, and the line of business requires a higher initial reserve. For more properties or examples about optimal reserve allocation, the interested reader may consult [4].

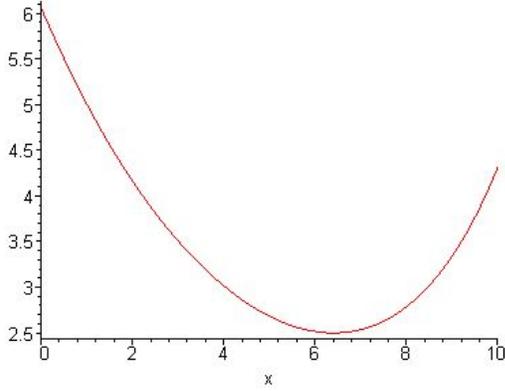


FIGURE 2:  $\mu_1 = 1$ : The two lines of business receive part of the global reserve

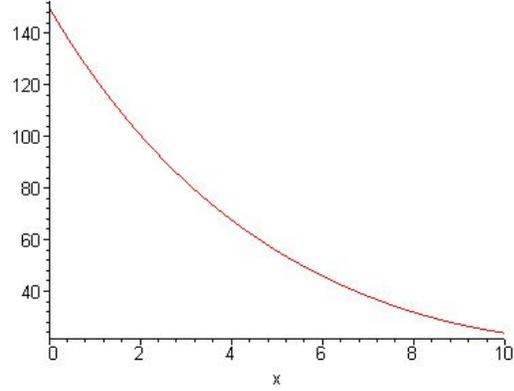


FIGURE 3:  $\mu_1 = 10$ : Line of business 2 receives the whole global initial reserve

### 4.3. Further applications

$A$  is a multidimensional risk measure which does not depend on the structure of dependence between lines of business. It is just an example of what can be considered. Another possibility would be to minimize the sum

$$B = \sum_{i=1}^n E\tau'_i(u)$$

where

$$E\tau'_i(u) = E \left( \int_0^T 1_{\{R_t^i < 0\}} 1_{\{\sum_{j=1}^n R_t^j > 0\}} dt \right).$$

$B$  takes dependence into account, and the following proposition prescribes to do the same kind of reasoning:

**Proposition 2.** *Let  $X_t = ct - S_t$ , where  $S_t$  satisfies hypothesis (H1) of theorem 3. Define  $B$  by  $B(u_1, \dots, u_n) = \sum_{j=1}^n E(\tau'_j(u))$  for  $u \in \mathbb{R}^n$ .  $B$  is differentiable on  $(\mathbb{R}_+^*)^n$ , and for  $u_1, \dots, u_n > 0$ ,*

$$\frac{\partial B}{\partial u_i} = -\frac{1}{c_i} EN_i^0(u),$$

where  $N_i^0(u) = \text{Card} \left( \{t \in [0, T], (R_t^i = 0) \cap (\sum_{j=1}^n R_t^j > 0)\} \right)$ .

It is also possible to differentiate with respect to  $c$  instead of  $u$ .

**Theorem 9.** *With notations of theorem 1, consider the case  $X_t = ct - S_t$ , where  $S_t$  satisfies hypothesis (H1) of theorem 3, and define  $\tilde{f}(c) = E(I(c))$ .*

If for all  $c$ ,  $E\tau_0(c) = 0$ , then  $\tilde{f}$  is differentiable on  $\mathbb{R}$  and for  $c \in \mathbb{R}$ ,

$$\tilde{f}'(u) = - \int_0^T tP(R_t < 0)dt.$$

It is interesting to look for the optimal allocation of the global premium  $c = c_1 + \dots + c_n$  because if  $c_i$  is small enough to make the safety loading negative, the process  $R_t^i$  tends to  $-\infty$ . Quite often, optimizing with the  $c_i$  will be easier than with the  $u_i$  for this reason. These examples illustrate how these differentiation results may be used.

The differentiation developed here is quite general and may be useful to solve many problems involving multirisk models. For a discussion about multidimensional risk measures, optimal allocation procedures, and impact of dependence between lines of business, the interested reader may consult Loisel [4].

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