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To cite this version:
Alexis Bès, Patrick Cégielski. Non-Maximal Decidable Structures. Rapport interne LACL 2007-06. 2007. <hal-00155282>

HAL Id: hal-00155282
https://hal.archives-ouvertes.fr/hal-00155282
Submitted on 17 Jun 2007

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Non-Maximal Decidable Structures

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June 17, 2007

Abstract
Given any infinite structure $M$ with a decidable first-order theory, we give a sufficient condition in terms of the Gaifman graph of $M$, which ensures that $M$ can be expanded with some non-definable predicate in such a way that the first-order theory of the expansion is still decidable.

LACL Technical Report 2007-06

1 Introduction

Elgot and Rabin ask in [2] whether there exist maximal decidable structures, i.e. structures $M$ with a decidable elementary theory and such that the elementary theory of any expansion of $M$ by a non-definable predicate is undecidable.

Soprunov proved in [8] (using a forcing argument) that every structure in which a regular ordering is interpretable is not maximal. A partial order $(B, <)$ is said to be regular if for every $a \in B$ there exist distinct elements $b_1, b_2 \in B$ such that $b_1 < a$, $b_2 < a$, and no element $c \in B$ satisfies both $c < b_1$ and $c < b_2$. As a corollary he also proved that there is no maximal decidable structure if we replace “elementary theory” by “weak monadic second-order theory”.

In [1] we considered a weakening of the Elgot-Rabin question, namely the question of whether all structures $M$ whose first-order theory is decidable can be expanded by some constant in such a way that the resulting structure still has a decidable theory. We answered this question negatively by proving that there exists a structure $M$ whose monadic second-order theory is decidable and such that any expansion of $M$ by a constant has an undecidable elementary theory.

In this paper we address the initial Elgot-Rabin question, and provide a criterion for non-maximality. More precisely, given any structure $M$ with a decidable first-order theory, we give in Section 3 a sufficient condition in terms of the Gaifman graph of $M$, which ensures that $M$ can be expanded with some non-definable predicate in such a way that the first-order theory of the expansion...
is still decidable. The condition is the following: for every natural number \( r \) and every finite set \( X \) of elements of the base set \(|\mathcal{M}|\) of \( \mathcal{M} \) there exists an element \( x \in |\mathcal{M}| \) such that the Gaifman distance between \( x \) and every element of \( X \) is greater than \( r \). This condition holds e.g. for the structure \((\mathbb{N}, S)\), where \( S \) denotes the graph of the successor function, and more generally for any labelled infinite graph with finite degree and whose elementary theory is decidable, i.e. any structure \( \mathcal{M} = (V, E, P_1, \ldots, P_n) \) where \( V \) is infinite, \( E \) is a binary relation of finite degree, the \( P_i \)'s are unary relations, and the elementary theory of \( \mathcal{M} \) is decidable. Unlike Soprunov’s condition, our condition expresses some limitation on the expressive power of the structure \( \mathcal{M} \).

In Section 2 we recall some important definitions and results. Section 3 deals with the main theorem. We conclude the paper with related questions.

2 Preliminaries

In the sequel we consider first-order logic with equality. We deal only with relational structures. Given a language \( \mathcal{L} \) and a \( \mathcal{L} \)-structure \( \mathcal{M} \), we denote by \(|\mathcal{M}|\) the base set of \( \mathcal{M} \). For every symbol \( R \in \mathcal{L} \) we denote by \( R^\mathcal{M} \) the interpretation of \( R \) in \( \mathcal{M} \). As usual we shall sometimes confuse symbols and their interpretation. We denote by \( FO(\mathcal{M}) \) the first-order (complete) theory of \( \mathcal{M} \), i.e. the set of first-order \( \mathcal{L} \)-sentences true in \( \mathcal{M} \). By “definable in \( \mathcal{M} \)” we mean “first-order definable in \( \mathcal{M} \) without parameters”.

We denote by \( qr(\phi) \) the quantifier rank of the formula \( \phi \), defined inductively by \( qr(\phi) = 0 \) if \( \phi \) is atomic, \( qr(\neg F) = qr(F) \), \( qr(F \land G) = \max(qr(F), qr(G)) \) for \( \alpha \in \{\land, \lor, \to\} \), and \( qr(\forall x F) = qr(\exists x F) = qr(F) + 1 \). We define \( FO_n(\mathcal{M}) \) as the set of \( \mathcal{L} \)-sentences \( F \) such that \( qr(F) \leq n \) and \( \mathcal{M} \models F \).

We say that the elementary diagram of a structure \( \mathcal{M} \) is computable if there exists an injective map \( f : |\mathcal{M}| \to \mathbb{N} \) such that the range of \( f \), as well as the relations \( \{(f(a_1), \ldots, f(a_n)) \mid a_1, \ldots, a_n \in |\mathcal{M}| \text{ and } \mathcal{M} \models R(a_1, \ldots, a_n)\} \) for every relation \( R \in \mathcal{L} \), are recursive (see e.g. [7]).

Let us recall useful definitions and results related to the Gaifman graph of a structure [3] (see also [5]). Let \( \mathcal{L} \) be a relational language, and \( \mathcal{M} \) be a \( \mathcal{L} \)-structure. The Gaifman graph of \( \mathcal{M} \), which we denote by \( G(\mathcal{M}) \), is the undirected graph whose set of vertices is \(|\mathcal{M}|\), and such that for all \( x, y \in |\mathcal{M}| \), there is an edge between \( x \) and \( y \) if and only if \( x = y \) or if there exist some \( n \)-ary relational symbol \( R \in \mathcal{L} \) and some \( n \)-tuple \( t \) of elements of \(|\mathcal{M}|\) which contains both \( x \) and \( y \) and satisfies \( t \in R^\mathcal{M} \).

The distance \( d(x, y) \) between two elements \( x, y \in |\mathcal{M}| \) is defined as the usual distance in the sense of the graph \( G(\mathcal{M}) \). We denote by \( B_r(x) \) the \( r \)-sphere with center \( x \), i.e. the set of elements \( y \) of \(|\mathcal{M}|\) such that \( d(x, y) \leq r \). It should be noted that for every fixed \( r \) the binary relation “\( y \in B_r(x) \)” is definable in \( \mathcal{M} \). For every \( X \subseteq |\mathcal{M}| \) we define \( B_r(X) \) as \( B_r(X) = \bigcup_{x \in X} B_r(x) \).

A \( r \)-local formula \( \varphi(x_1, \ldots, x_n) \) is a formula whose quantifiers are all relativized to \( B_r(\{x_1, \ldots, x_n\}) \). We shall use the notation \( \varphi^{(r)} \) to indicate that \( \varphi \) is \( r \)-local.
Let us state Gaifman’s theorem about local formulas.

**Theorem 1** ([3]) Let $\vec{x} = (x_1, \ldots, x_n)$ and $\varphi(\vec{x})$ be a $\mathcal{L}$-formula. From $\varphi$ one can compute effectively a formula which is equivalent to $\varphi$ and is a boolean combination of formulas of the form:

- $\psi^{(r)}(\vec{x})$
- $\exists x_1 \ldots \exists x_s \left( \bigwedge_{1 \leq i \leq s} a^{(r)}(x_i) \wedge \bigwedge_{1 \leq i < j \leq s} d(x_i, x_j) > 2^r \right)$

where $s \leq qr(\varphi) + n$ and $r \leq 7^k$.

Moreover if $\varphi$ is a sentence then only sentences of the second kind occur in the resulting formula.

### 3 A sufficient condition for non-maximality

The aim of this section is to prove the following theorem.

**Theorem 2** Let $\mathcal{L}$ be a finite relational language, and $\mathcal{M}$ be an infinite countable $\mathcal{L}$-structure which satisfies the following conditions:

1. $\text{FO}(\mathcal{M})$ is decidable
2. every element of $|\mathcal{M}|$ is definable in $\mathcal{M}$
3. for every finite set $X \subseteq |\mathcal{M}|$ and every $r \in \mathbb{N}$, there exists $a \in |\mathcal{M}|$ such that $d(a, X) > r$.

Then there exists a unary predicate symbol $R \not\in \mathcal{L}$ and a $(\mathcal{L} \cup \{R\})$-expansion $\mathcal{M}'$ of $\mathcal{M}$ such that:

- $\text{FO}(\mathcal{M}')$ is decidable
- the set $R^{\mathcal{M}'}$ is not definable in $\mathcal{M}$.
- the elementary diagram of $\mathcal{M}'$ is computable.

Let us consider a few examples.

- The structure $\mathcal{M} = (\mathbb{N}; S)$, where $S$ denotes the graph of the function $x \mapsto x + 1$, satisfies all conditions of Theorem 2. Indeed Langford [4] proved that $\text{FO}(\mathcal{M})$ is decidable. Moreover condition 2 is easy to prove, and condition 3 is a straightforward consequence of the fact that $d(x, y) = |x - y|$ for all natural numbers $x, y$.

- The same holds for any structure of the form $\mathcal{M} = (\mathbb{N}; S, P_1, \ldots, P_n)$ where the $P_i$’s denote unary predicates and $\text{FO}(\mathcal{M})$ is decidable. Note that expanding a structure by unary predicates does not change its Gaifman graph.
• More generally Theorem 2 applies to any infinite labelled graph with finite degree, more precisely to any structure of the form $\mathcal{M} = (V; E, P_1, \ldots, P_n)$ where $V$ is infinite, $E$ is a binary relation with finite degree, the $P_i$’s denote unary predicates, $FO(\mathcal{M})$ is decidable, and every element of $V$ is definable in $\mathcal{M}$. In this case the Gaifman graph of $\mathcal{M}$ has finite degree, which implies condition 3. Note that Theorem 2 also applies to some structures for which the degree of the Gaifman graph is infinite – see the last example.

• The structure $\mathcal{M} = (\mathbb{N}; <)$ does not satisfy condition 3 of Theorem 2 since $d(x, y) \leq 1$ for all $x, y \in \mathbb{N}$. Observe that $FO(\mathcal{M})$ is decidable [4], and moreover $\mathcal{M}$ is not maximal: consider e.g. the structure $\mathcal{M}' = (\mathbb{N}; <, +)$ where $+$ denotes the graph of addition; $FO(\mathcal{M}')$ is decidable [6], and $+$ is not definable in $\mathcal{M}$ since in $\mathcal{M}$ one can only define finite or co-finite subsets of $\mathbb{N}$.

One can prove actually that for every infinite structure $\mathcal{M}$ in which some linear ordering of elements of $|\mathcal{M}|$, condition 3 does not hold. However the next example shows that Theorem 2 can be applied to some structures in which an infinite linear ordering is interpretable.

• Consider the disjoint union of $\omega$ copies of $(\mathbb{N}; <)$ equipped with a successor relation between copies, i.e. the structure $\mathcal{M} = (\mathbb{N} \times \mathbb{N}; <, Suc)$ where

- $(x, y) < (x', y')$ if and only if $(x = x'$ and $y < y')$;
- $Suc((x, y), (x', y'))$ if and only if $x' = x + 1$

then $\mathcal{M}$ satisfies the conditions of Theorem 2: the first condition comes from the fact that $FO(\mathcal{M})$ reduces to $FO(\mathbb{N}; <)$ and the two other conditions are easy to check.

Let us explain informally the structure of the proof of Theorem 2. Given $\mathcal{M}$ which satisfies the conditions of Theorem 2, we define $R^M$ by marking gradually elements of $|\mathcal{M}|$, some in $R^M$ and some in its complement. More precisely we define by induction on $n$ the sequence $(X_n)_{n \in \mathbb{N}}$ with $X_n = (R_n, S_n, T_n, F_n)$ where $R_n$ corresponds to a set of elements of $R^M$ (we will say “marked positively”), $S_n$ corresponds to a set of elements marked in the complement of $R^M$ (we will say “marked negatively”), $T_n$ roughly corresponds to a set of spheres whose elements are marked in the complement of $R^M$, and $F_n$ denotes the set of formulas of quantifier rank $\leq n$ which will be true in $\mathcal{M'}$. At each step $n$, the partial marking $X_n$ ensures that any subsequent marking will lead to a set $R^M$ not definable by any formula of quantifier rank $n$. Moreover $X_n$ also fixes $FO_n(\mathcal{M}')$. Finally $R^M$ will be defined as the union of the sets $R_n$. In the construction we impose some sparsity condition on $R^M$; this condition ensures that there are few elements of $R^M$ in each $r$–sphere, which allows to express with $\mathcal{L}$-sentences whether a $r$–sphere of $\mathcal{M}$ can be marked conveniently, and then use the condition that $FO(\mathcal{M})$ is decidable in order to extend the marking in an effective way.
Proof of Theorem 2.

Assume that $\mathcal{M}$ is a $L$–structure which satisfies the conditions of the theorem. Let $R \not\in L$ be a unary predicate symbol. For every $X \subseteq |\mathcal{M}|$ we shall denote by $\mathcal{M}(X)$ the $(L \cup \{R\})$–expansion of $\mathcal{M}$ defined by interpreting $R$ by $X$.

Throughout the proof we shall use the following interesting consequences of conditions 1 and 2:

- the elementary diagram of $\mathcal{M}$ is computable. Indeed since $L$ is finite we can enumerate all formulas $\varphi(x)$ with one free variable. Let us denote by $(\varphi_i(x))_{i \geq 0}$ such an enumeration. Then the application $f : |\mathcal{M}| \rightarrow \mathbb{N}$ which maps every element $e$ of $|\mathcal{M}|$ to the least integer $i$ such that $\varphi_i$ defines $e$ is injective; moreover the range of $f$, and the relations $\{(f(a_1), \ldots, f(a_n)) : \mathcal{M} \models Q(a_1, \ldots, a_n)\}$ for every symbol $Q$ of $L$, are recursive.

- if $\psi(x)$ is a formula with one free variable and $\mathcal{M} \models \exists x \psi(x)$ then one can find in an effective way the first integer $i$ who belongs to the range of $f$ and such that $\mathcal{M} \models \exists x (\varphi_i(x) \land \psi(x))$. That is, one can find effectively some element $x \in |\mathcal{M}|$ for which $\psi(x)$ holds in $\mathcal{M}$.

- every finite or co-finite subset $A \subseteq |\mathcal{M}|$ is definable in $\mathcal{M}$. This will allow to use shortcuts such as “$x \in A$” when we write formulas in the language $L$.

We now define by induction on $n$ the sequence $(X_n)_{n \in \mathbb{N}}$ such that for every $n$, $X_n = (R_n, S_n, T_n, F_n)$ where

1. $R_n, S_n, T_n$ are finite subsets of $|\mathcal{M}|$;
2. $R_n \cap S_n = \emptyset$;
3. $F_n$ is a set of $(L \cup \{R\})$–sentences with quantifier rank $\leq n$;
4. $d(R_n, R_{n+1} \setminus R_n) \geq 7^{n+1}$;
5. $d(x, y) \geq 7^{n+1}$ for every pair of distinct elements of $R_{n+1} \setminus R_n$;
6. for every $R' \subseteq |\mathcal{M}|$ such that $R_n \subseteq R'$ and

   $$R' \cap ((S_n \cup \bigcup_{i \leq n} B_{7^n}(T_i)) \setminus R_n) = \emptyset,$$

   $R'$ is not definable by any $L$–formula of quantifier rank $\leq n$;
7. For every $R' \subseteq |\mathcal{M}|$ such that $R_n \subseteq R'$,

   $$R' \cap ((S_n \cup \bigcup_{i \leq n} B_{7^n}(T_i)) \setminus R_n) = \emptyset,$$
\[ d(R', R' \setminus R_n) \geq 7^{n+1}, \]
and \( d(x, y) \geq 7^{n+1} \) whenever \( x, y \) are distinct elements of \( R' \setminus R_n \), we have
\[ FO_n(M(R')) = F_n. \]

**Induction hypothesis:** assume that \((X_i)_{i<n}\) is defined and satisfies the required conditions.

Let us define \( X_n \). The definition consists in two main steps: during the first step we extend the marking in order to ensure that \( R^M' \) will not be definable by any formula with quantifier rank \( n \); this is the easiest step, and it uses condition (3) of the theorem. During the second step, we extend again the marking in order to fix \( FO_n(M') \).

We set \( r = 7^n \).

**First step:** during this step we mark a finite number of elements in order to ensure that \( R^M' \) will not be definable by any \( L^- \)formula with quantifier rank \( n \).

Since we deal with a finite relational language, there exist up to equivalence finitely many formulas with quantifier rank \( n \). From \( L \) one can compute an integer \( k_n \) and a finite set of \( L^- \)formulas \( \{ \alpha_{n,i}(x) : 1 \leq i \leq k_n \} \) such that every \( L^- \)formula with quantifier rank \( n \) is equivalent to a disjunction of some of the \( \alpha_{n,i} \)'s, and moreover such that the formulas \( \alpha_{n,i} \) are incompatible. For \( i = 1, \ldots, k_n \), let us denote by \( E_{n,i} \) the subset of \( |M| \) defined by \( \alpha_{n,i}(x) \). By construction the sequence \((E_{n,1}, \ldots, E_{n,k_n})\) is a partition of \( |M| \), and every subset of \( |M| \) definable by a formula of quantifier rank \( n \) is a finite union of some of the subsets \( E_{n,i} \).

We shall mark elements in order that for some \( i \), the subset \( E_{n,i} \) contains at least an element marked positively and another element marked negatively. This will ensure that condition 6 is satisfied. More precisely, for \( i = 1, \ldots, k_n \), we mark positively (respectively negatively) at most one new element of \( E_{n,i} \).

We define the sets \( R'_{n,i} \) (resp. \( S'_{n,i} \)) such that \( R'_{n,i} \) contains the set of new elements to mark positively (resp. negatively) in \( E_{n,i} \) (each of the sets \( R'_{n,i} \) and \( S'_{n,i} \) is either empty or reduced to a singleton). We proceed as follows:

- if there exists some element of \( E_{n,i} \) which is not marked yet, and moreover all marked elements of \( E_{n,i} \) are marked positively, then we mark negatively the first unmarked element of \( E_{n,i} \).

Formally, assume that the sets \( R'_{n,j} \) and \( S'_{n,j} \) have been defined for every \( j < i \), and let
\[ Z_{n,i} = R_{n-1} \cup \bigcup_{j<i} R'_{n,j} \cup S_{n-1} \cup \bigcup_{j<i} S'_{n,j} \cup \bigcup_{1<n} B_1(T_1) \]
If
\[ \mathcal{M} \models \exists x(\alpha_{n,i}(x) \land x \notin Z_{n,i}) \]
and moreover
\[ \mathcal{M} \models (E_{n,i} \cap Z_{n,i}) \subseteq (R_{n-1} \cup \bigcup_{j<i} R'_{n,j}) \]
(this set-theoretic property is expressible as a \( \mathcal{L} \)-sentence) then we set
\[ S'_{n,i} \]
as the singleton set consisting in the first \( x \) such that
\[ \mathcal{M} \models \exists x(\alpha_{n,i}(x) \land x \notin Z_{n,i}). \]
Otherwise we set \( S'_{n,i} = \emptyset \).

- Then, if all currently marked elements of \( E_{n,i} \) are marked negatively, and moreover there exists some unmarked element \( x \) of \( E_{n,i} \) at distance \( \geq 7^{n+1} \) from already marked elements, then we mark positively the first such element \( x \).

  Formally, let
\[ Z'_{n,i} = Z_{n,i} \cup S'_{n,i} \]
If
\[ \mathcal{M} \models (E_{n,i} \cap (R_{n-1} \cup \bigcup_{j<i} R'_{n,j})) = \emptyset \]
and moreover
\[ \mathcal{M} \models \exists x(\alpha_{n,i}(x) \land d(x, Z'_{n,i}) \geq 7^{n+1}) \]
then let \( R'_{n,i} \) be the singleton set consisting in the first such \( x \). Otherwise we set \( R'_{n,i} = \emptyset \).

Note that the previous procedure is effective (see the remarks at the beginning of the proof).

Second step: during this step we extend the marking in order to fix \( FO_n(\mathcal{M}') \).

Up to equivalence, there exist finitely many \( (\mathcal{L} \cup \{R\}) \)-formulas \( F \) such that \( qr(F) = n \). By Proposition 1 every such formula \( F \) is equivalent to a boolean combination of formulas of the form
\[ \exists x_1 \ldots \exists x_s ( \bigwedge_{1 \leq i \leq s} \alpha^{(r)}(x_i) \land \bigwedge_{1 \leq i < j \leq s} d(x_i, x_j) > 2^r). \]
Consider an enumeration \( G_{n,1}, \ldots, G_{n,m} \) of all formulas of the previous form which arise when we apply Theorem 1 to formulas \( F \) such that \( qr(F) = n \).

During this step we shall fix which formulas \( G_{n,j} \) will be true in \( \mathcal{M}' \), which will suffice (using again Theorem 1) to fix which formulas \( F \) with quantifier rank \( n \) will be true in \( \mathcal{M}' \).
The first idea is to check, for every $j$, whether there exists $R' \subseteq |M|$ which extends in a convenient way the current marking and such that $M(R') = G_{n,j}$. If the answer is positive, then we shall extend our marking just enough to ensure that every subsequent extension of the marking will satisfy $M' = G_{n,j}$. If the answer is negative, then we do not extend the marking, and then every subsequent extension of the marking will satisfy $M' = \neg G_{n,j}$.

We define by induction on $j \leq m_n$ the sets $R''_{n,j}$ and $T'_{n,j}$, such that $R''_{n,j}$ contains new elements to mark positively, and $T'_{n,j}$ contains the centers of new $r$-spheres whose elements are marked negatively.

We proceed as follows. Fix $j$, and assume that the sets $R''_{n,i}$ and $T'_{n,i}$ have been defined for every $i < j$. We have

$$G_{n,j} : \exists x_1 \ldots \exists x_s \left( \bigwedge_{1 \leq i \leq s} \alpha^{(r)}_{n,j}(x_i) \land \bigwedge_{1 \leq i < j \leq s} d(x_i, x_j) > 2r \right)$$

for some $r$-local formula $\alpha^{(r)}_{n,j}$ (formally $s$ depend on $n$ and $j$, but we omit the subscripts for the sake of readability).

Let $R^+_{n,j} = R_{n-1} \cup \bigcup_{i < k_n} R'_{n,i} \cup \bigcup_{i < j} R''_{n,i}$, and let $R^-_{n,j}$ be the set of elements currently marked negatively, that is

$$R^-_{n,j} = (S_{n-1} \cup \bigcup_{i < k_n} S'_{n,i} \cup \bigcup_{i < n} B(7)(T_i) \cup \bigcup_{i < j} B(7)(T'_{n,i})) \setminus R^+_{n,j}.$$

Let $P_{n,j} = R^+_{n,j} \cup R^-_{n,j}$.

We want to check whether there exists $R' \subseteq |M|$ such that

1. $M(R') = G_{n,j}$;
2. $R^+_{n,j} \subseteq R'$ and $R^-_{n,j} \cap R' = 0$ (i.e. $R'$ extends the current marking);
3. $d(R^+_{n,j}, R' \setminus R^+_{n,j}) \geq 7^{n+1}$;
4. $d(x, y) \geq 7^{n+1}$ for every pair of distinct elements of $R' \setminus R^+_{n,j}$.

Let us denote by $(\ast)$ the conjunction of these four conditions. Let us prove that one can express $(\ast)$ with a $\mathcal{L}$-sentence.

Assume first that there exists $R'$ which satisfies $(\ast)$. Let $x_1, \ldots, x_s \in |M|$ be such that

$$M(R') = (\bigwedge_{1 \leq i \leq s} \alpha^{(r)}_{n,j}(x_i) \land \bigwedge_{1 \leq i < j \leq s} d(x_i, x_j) > 2r)$$

Conditions 3 and 4 of $(\ast)$ imply that each sphere $B_r(x_i)$ contains at most one element of $R' \setminus R^+_{n,j}$, and moreover that if such an element exists, it is the unique
element of $R'$ in $B_r(x_i)$. Thus we can assume without loss of generality that there exist $t \leq s$ and $y_1, \ldots, y_t \in |M|$ such that

$$B_r(x_i) \cap (R' \setminus R^+_{n,j}) = \{y_i\}$$

for every $i \leq t$, and

$$B_r(x_i) \cap (R' \setminus R^+_{n,j}) = \emptyset$$

for every $i > t$. Condition (3) yields $d(R^+_{n,j}, y_i) \geq 7^{n+1}$ for every $i$, and condition (4) yields $d(y_i, y_j) \geq 7^{n+1}$ for all distinct integers $i, j$.

Let us consider first the $r$-spheres $B_r(x_i)$ for $i \leq t$. By definition of $x_i$ we have $M(R') \models \alpha^{(r)}_{n,j}(x_i)$. Now $y_i$ is the unique element of $R' \cap B_r(x_i)$ thus we have $M \models \alpha'_{n,j}(x_i, y_i)$ where $\alpha'_{n,j}(x_i, y_i)$ is obtained from $\alpha^{(r)}_{n,j}(x_i)$ by replacing every atomic formula of the form $R(z)$ by $(z = y_i)$.

Now consider the $r$-spheres $B_r(x_i)$ for $i > t$. By definition we have $M(R') \models \alpha^{(r)}_{n,j}(x_i)$, and $B_r(x_i)$ contains no element of $R' \setminus R^+_{n,j}$. Thus we have $M \models \gamma_{n,j}(x_i)$ where $\gamma_{n,j}(x_i)$ is obtained from $\alpha^{(r)}_{n,j}(x_i)$ by replacing every atomic formula of the form $R(z)$ by $(z \in B_r(x_i) \cap R^+_{n,j})$.

The previous arguments show that $M \models G'_{n,j}$ where $G'_{n,j}$ is the $L$-sentence $G'_{n,j}$ defined as follows:

$$G'_{n,j} : \bigvee_{t \leq s} H_{n,j,t}$$

where

$$H_{n,j,t} : \exists x_1 \ldots \exists x_s \exists y_1 \ldots \exists y_t \left( \bigwedge_{1 \leq i < j \leq s} d(x_i, x_j) > 2r \land \bigwedge_{1 \leq i < j \leq t} d(y_i, y_j) > 7r \land \bigwedge_{1 \leq i \leq t} d(y_i, R^+_{n,j}) > 7r \land \bigwedge_{1 \leq i \leq t} \beta^{(r)}_{n,j}(x_i, y_i) \land \bigwedge_{t < i \leq s} \gamma^{(r)}_{n,j}(x_i) \right)$$

with

$$\beta^{(r)}_{n,j}(x_i, y_i) : y_i \in B_r(x_i) \land y_i \notin P_{n,j} \land B_r(x_i) \cap R^+_{n,j} = \emptyset \land \alpha^{(r)}_{n,j}(x_i, y_i).$$

Conversely, assume that $M \models G'_{n,j}$. Let $t, x_1, \ldots, x_s$, and $y_1, \ldots, y_t$ be such that $H_{n,j,t}$ holds in $M$. Then if we set $R' = R^+_{n,j} \cup \{y_1, \ldots, y_t\}$, one checks easily that $R'$ satisfies $(\ast)$.

Therefore we have shown that the question whether there exists $R'$ which satisfies $(\ast)$ is equivalent to the question whether $M \models G'_{n,j}$ for some $L$-formula which can be constructed effectively from $G_{n,j}$.

If $M \models \neg G'_{n,j}$ (which can be checked effectively since by our hypotheses $FO(M)$ is decidable), then we set

$$R''_{n,j} = T''_{n,j} = F''_{n,j} = \emptyset.$$
This completes the second step of the construction of $X_n$.

We can now define $X_n$ as follows: we set

\[
R_n = R_{n-1} \cup \bigcup_{i \leq k} R'_{n,i} \cup \bigcup_{j \leq m} R''_{n,j}
\]

\[
S_n = S_{n-1} \cup \bigcup_{i \leq k} S'_{n,i}
\]

and

\[
T_n = \bigcup_{j \leq m} T'_{n,j}.
\]

In order to define $F_n$, consider a formula $F$ with quantifier rank $n$. By Theorem 1, $F$ is equivalent to a formula $F'$ which is a boolean combination of formulas of the form $G_{n,j}$. Consider the truth value of $F'$ determined by setting “true” all formulas $G_{n,j} \in F'$ and “false” formulas $G_{n,j} \notin F'$. Then we define $F_n$ as the union of $F_{n-1}$ and of all formulas $F$ for which $F'$ is true.

We have defined $X_n$. There remains to show that $X_n$ satisfies all conditions required in the definition.

• Conditions (1) to (5) are easy consequences of the construction of $X_n$ (and the induction hypotheses).

• Let us consider condition (6). Let $R' \subseteq |M|$ be such that $R_n \subseteq R'$ and

\[
R' \cap (\bigcup_{i \leq n} B_{\tau_i}(T_i)) = \emptyset.
\]

Let us prove that $R'$ is not definable by any $L$–formula of quantifier rank $\leq n$. Since every subset of $|M|$ definable by a $L$–formula with quantifier rank $n$ is the union of some of the sets $E_{n,i}$, it suffices to prove that $R'$ and its complement intersect some $E_{n,i}$.

By construction, the set $X = R_n \cup S_n \cup \bigcup_{i \leq n} T_i$ is finite. Now by hypothesis $M$ satisfies condition 3 of Theorem 2, thus there exists $x \in |M|$ such that $d(X, x) > 7^n$. The element $x$ belongs to some set $E_{n,i}$. Let us prove that $R'$ and its complement intersect $E_{n,i}$.

Consider the step of the construction of $X_n$ during which we marked elements of $E_{n,i}$. Recall that just before this step the set of marked elements was

\[
Z_{n,i} = R_{n-1} \cup \bigcup_{j < i} R'_{n,j} \cup S_{n-1} \cup \bigcup_{j < i} S'_{n,j} \cup \bigcup_{i < n} B_{\tau_j}(T_i)
\]

Since $x \in E_{n,i}$ and $d(X, x) > 7^n$, the set $E_{n,i} \setminus Z_{n,i}$ is non-empty. Thus either $E_{n,i}$ already contained an element marked negatively (and in this case $S'_{n,i} = \emptyset$), or we marked one (from $E_{n,i} \setminus Z_{n,i}$) and put it in $S'_{n,i}$.

Therefore the complement of $R'$ intersects $E_{n,i}$.
Just after this step, then either $E_{n,i}$ already contained some element marked positively, or by definition of $x$ there existed an element $y$ of $E_{n,i}$ at distance $\geq 7^n$ from currently marked elements, and thus we could mark positively the first such element $y$. In both cases this ensures that $R'$ intersects $E_{n,i}$.

- Let us prove now that $X_n$ satisfies condition (7). Let $R' \subseteq |M|$ be such that $R_n \subseteq R'$, 
  \[ R' \cap (\{S_n \cup \bigcup_{i \leq n} B_{7^n}(T_i)\} \setminus R_n) = \emptyset, \]
  \[ d(R', R \setminus R_n) \geq 7^{n+1} \]
  and $d(x, y) \geq 7^{n+1}$ whenever $x, y$ are distinct elements of $R' \setminus R_n$. Let us prove that $FO_n(M(R')) = F_n$. The case of formulas with quantifier rank $< n$ follows from our induction hypotheses. Consider now formulas with quantifier rank $n$. Their truth values are completely determined by the truth values of formulas $G_{n,j}$. Thus it is sufficient to prove that for every $j$ we have $M(R') \models G_{n,j}$ if and only if $F_{n,j} = \{G_{n,j}\}$. Fix $j$, and consider the step of the construction of $X_n$ during which we dealt with the formula $G_{n,j}$. If $M \models G_{n,j}$ then in this case $F_{n,j} = \{G_{n,j}\}$, and the definition of $R'_{n,j}$ and $T_{n,j}$ imply that the formula $G_{n,j}$ holds for every $R'$ which extends (in a convenient way) the marking $(R_n, S_n, T_n)$, thus we have $M(R') \models G_{n,j}$. On the other hand if $M \not\models G'_{n,j}$, then the property (7) cannot be satisfied, and we have set $F_{n,j} = \emptyset$. In particular $R'$ does not satisfy ($\ast$). Now the hypotheses on $R'$ yield that $R'$ satisfies the three last conditions of ($\ast$), thus the first condition is not satisfied, that is $M(R') \not\models G_{n,j}$.

This concludes the proof that there exists a sequence $(X_n)_{n \geq 0}$ which satisfies all conditions required in the definition.

Now let $M'$ be the $(\mathcal{L} \cup \{R\})$–expansion of $M$ defined by 
\[ R^{M'} = \bigcup_{n \geq 0} R_n. \]

Let us prove that $M'$ satisfies the properties required in Theorem 2.

The definition of $R^{M'}$ implies that for every $n$, $R^{M'}$ is not definable by any $\mathcal{L}$–sentence with quantifier rank $n$, and moreover that $FO_n(M') = F_n$. Therefore $R^{M'}$ is not definable in $M$, and $FO(M')$ is decidable.

Let us prove that the elementary diagram of $M'$ is computable. Consider the function $f$ used for the elementary diagram of $M$; it is sufficient to prove that \{ $f(a) \mid M' \models R(a), a \in |M|$ \} is recursive. Since every element $e$ of $|M|$ is definable, there exists $n, i$ such that $E_{n,i} = \{e\}$. During the construction of $X_n$, and more precisely just before the marking of $E_{n,i}$, then either $e$ had already been marked, or $e$ is marked during this step. Thus eventually every element of $|M|$ is marked in $R^{M'}$ or in its complement. This implies that
both \( \{ f(a) \mid \mathcal{M}' \models R(a), a \in |\mathcal{M}| \} \) and \( \{ f(a) \mid \mathcal{M}' \not\models R(a), a \in |\mathcal{M}| \} \) are recursively enumerable, from which the result follows.

This concludes the proof of Theorem 2.

4 Conclusion

We gave a sufficient condition in terms of the Gaifman graph of the structure \( \mathcal{M} \) which ensures that \( \mathcal{M} \) is not maximal. A natural problem is to extend Theorem 2 to structures \( \mathcal{M} \) which do not satisfy condition (3). We currently investigate the case of labelled linear orderings, i.e. infinite structures \( (A; <, P_1, \ldots, P_n) \) where \( < \) is a linear ordering over \( A \) and the \( P_i \)'s denote unary predicates; the Gaifman distance is trivial for these structures. Another related general problem is to find a way to refine the notion of Gaifman distance.

Finally, it would also be interesting to study the complexity gap between the decision procedure for the theory of \( \mathcal{M} \) and the one for the structure \( \mathcal{M}' \) constructed in the proof of Theorem 2.

References


