MacLane's coherence theorem expressed as a word problem

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In this draft manuscript, we reduce the coherence theorem for braided monoidal categories to the resolution of a word problem, and the construction of a category of fractions. The technique explicates the combinatorial nature of that particular coherence theorem.

1. Introduction

Let $\mathcal{B}$ denote the category of braids and $\mathcal{M}$ any braided monoidal category. Let $\mathtt{Br}(\mathcal{B}, \mathcal{M})$ denote the category of *strong* braided monoidal functors from $\mathcal{B}$ to $\mathcal{M}$ and monoidal natural transformations between them. All definitions are recalled in section 5. The coherence theorem for braided monoidal categories is usually stated as follows:

Theorem 1. The categories $\mathtt{Br}(\mathcal{B}, \mathcal{M})$ and $\mathcal{M}$ are equivalent.

We prove the coherence theorem in four independent steps.

To that purpose, we introduce the category $\mathcal{A}$

— whose objects are binary trees of $\otimes$, with leaves 0 and 1,
— whose morphisms are sequences of rewriting steps $\alpha(a,b,c)$, $\lambda(a)$, $\rho(a)$, $\gamma(a,b)$, $\alpha^{-1}(a,b,c)$, $\lambda^{-1}(a)$, $\rho^{-1}(a)$, $\gamma^{-1}(a,b)$, quotiented by the laws of braided monoidal categories.

1.1. *First step*

Consider a braided monoidal category $\mathcal{M}$, and the category $\mathtt{SBr}(\mathcal{A}, \mathcal{M})$ of *strict* braided monoidal functors and monoidal natural transformations from $\mathcal{A}$ to $\mathcal{M}$. We prove that $\mathcal{M}$ and $\mathtt{SBr}(\mathcal{A}, \mathcal{M})$ are isomorphic. More precisely, we prove that the functor

$$[-] : \mathtt{SBr}(\mathcal{A}, \mathcal{M}) \longrightarrow \mathcal{M}$$

which associates to a strict natural transformation

$$\theta : (F, F_2, F_0) \stackrel{\cdot}{\longrightarrow} (G, G_2, G_0)$$

the morphism

$$\theta_1 : F(1) \longrightarrow G(1)$$

in $\mathcal{M}$, is reversible.

To every object X in the category $\mathcal{M}$, we associate a strict braided monoidal functor $[\![X]\!]$ defined as follows:

$$[\![X]\!](0) = e \qquad [\![X]\!](1) = X \qquad [\![X]\!](a \otimes b) = [\![X]\!](a) \otimes [\![X]\!](b)$$

and

$$[\![X]\!](\alpha) = \alpha \qquad [\![X]\!](\lambda) = \lambda \qquad [\![X]\!](\rho) = \rho \qquad [\![X]\!](\gamma) = \gamma$$

This defines a functor because the image of a commutative diagrams in $\mathcal{A}$ is always commutative in $\mathcal{M}$. The functor is strict braided monoidal by construction.

To every morphism $f : X \longrightarrow Y$ in the category $\mathcal{M}$, we associate a family of morphisms $[\![f]\!]$ indexed by objects of $\mathcal{A}$, as follows:

$$[\![f]\!]_0 = \mathrm{id}_e \qquad [\![f]\!]_1 = f \qquad [\![f]\!]_{a\otimes b} = [\![f]\!]_a \otimes [\![f]\!]_b$$

The family defines a natural transformation $[\![f]\!] : [\![X]\!] \longrightarrow [\![Y]\!]$ because the maps α, λ, ρ, and γ are supposed natural in $\mathcal{M}$. The natural transformation $[\![f]\!]$ is monoidal by construction.

The map $[\![-]\!]$ is functorial from $\mathcal{M}$ to $\mathsf{SBr}(\mathcal{A}, \mathcal{M})$ and it is easy to see that it defines the inverse of the evaluation functor $[-] : \mathsf{SBr}(\mathcal{A}, \mathcal{M}) \longrightarrow \mathcal{M}$. We conclude.

1.2. *Second step*

We introduce the notion of *contractible* category. A category Ξ is contractible when:

— there exists at most one morphism between two objects of Ξ,
— every morphism of Ξ is reversible.

In other words, a category Ξ is contractible when it is a preorder category and a groupoid.

Consider a contractible subcategory Ξ of a category $\mathcal{C}$, full on objects. Write $a \simeq_{\Xi}^{obj} b$ for two objects a and b of $\mathcal{C}$, when there exists a morphism $a \longrightarrow b$ in Ξ. Write $f \simeq_{\Xi}^{map} g$ for two morphisms $f : a \longrightarrow b$ and $g : c \longrightarrow d$ when there exists two morphisms $a \longrightarrow c$ and $b \longrightarrow d$ in Ξ making the following diagram commute:

$$\begin{tikzcd} a \arrow[r, "f"] \arrow[d] & b \arrow[d] \\ c \arrow[r, "g"] & d \end{tikzcd}$$

The relations $\simeq_{\Xi}^{obj}$ and $\simeq_{\Xi}^{map}$ are equivalence relations on objects and morphisms of $\mathcal{C}$, respectively. We call *orbit* of an object a or morphism f its equivalence class wrt. $\simeq_{\Xi}^{obj}$ or $\simeq_{\Xi}^{map}$. The quotient category $\mathcal{C}/\Xi$ is defined as follows:

— its objects are the orbits of objects,
— its morphisms $M \longrightarrow N$ are the orbits of maps,
— its identities and composition are induced from $\mathcal{C}$.

The two categories $\mathcal{C}$ and $\mathcal{C}/\Xi$ are equivalent. Indeed, there exists a “projection” functor

$$F : \mathcal{C} \longrightarrow \mathcal{C}/\Xi$$

which maps every morphism $f : M \longrightarrow N$ to its orbit, and an “embedding” functor

$$G : \mathcal{C}/\Xi \longrightarrow \mathcal{C}$$

depending on the choice, for every orbit wrt. $\simeq_{\Xi}^{obj}$, of an object in that class. Clearly, $FG = \mathrm{id}_{\mathcal{C}/\Xi}$, and there exists a natural transformation $GF \stackrel{\cdot}{\longrightarrow} \mathrm{id}_{\mathcal{C}}$.

1.3. *Third step*

Consider a braided monoidal category $\mathcal{M}$, and the hom-category $\mathtt{Br}(\mathcal{A},\mathcal{M})$ of strong braided monoidal functors and monoidal natural transformations from $\mathcal{A}$ to $\mathcal{M}$. We prove that the categories $\mathtt{Br}(\mathcal{A},\mathcal{M})$ and $\mathtt{SBr}(\mathcal{A},\mathcal{M})$ are equivalent.

Define Ξ as the subcategory of $\mathtt{Br}(\mathcal{A},\mathcal{M})$ containing all reversible maps $\theta : (F, F_2, F_0) \stackrel{\cdot}{\longrightarrow} (G, G_2, G_0)$ such that $\theta_1 = \mathrm{id}_{F(1)}$. Every identity of $\mathtt{Br}(\mathcal{A},\mathcal{M})$ is element of Ξ. The main observation is that the category Ξ is contractible. Indeed, once the equality $\theta_1 = \mathrm{id}_{F(1)}$ is fixed, the component morphisms θ_0 and $\theta_{a\otimes b}$ of the natural transformation θ are uniquely determined by the commutative diagrams

$$\begin{tikzcd} F(a)\otimes F(b) \arrow[r, "F_2(a,b)"] \arrow[d, "\theta_a\otimes\theta_b"'] & F(a\otimes b) \arrow[d, "\theta_{a\otimes b}"] \\ G(a)\otimes G(b) \arrow[r, "G_2(a,b)"] & G(a\otimes b) \end{tikzcd} \qquad \begin{tikzcd} e \arrow[r, "F_0"] \arrow[d, equal] & F(0) \arrow[d, "\theta_0"] \\ e \arrow[r, "G_0"] & G(0) \end{tikzcd}$$

Moreover, every identity of $\mathtt{Br}(\mathcal{A},\mathcal{M})$ is element of Ξ. We may therefore consider the category $\mathtt{Br}(\mathcal{A},\mathcal{M})/\Xi$, which we know is equivalent to $\mathtt{Br}(\mathcal{A},\mathcal{M})$.

We prove that the categories $\mathtt{Br}(\mathcal{A},\mathcal{M})/\Xi$ and $\mathtt{SBr}(\mathcal{A},\mathcal{M})$ are isomorphic. We need to prove that for every strong monoidal functor (G, G_2, G_0) from $\mathcal{B}$ to $\mathcal{M}$, there exists a strict monoidal functor F, and a map $\theta : F \stackrel{\cdot}{\longrightarrow} (G, G_2, G_0)$ in Σ. Consider a family of isomorphisms $\theta_a : F_a \longrightarrow G_a$

$$\theta_0 = G_0 \qquad \theta_1 = \mathrm{id}_e \qquad \theta_{a\otimes b} = G_2(a,b)\circ(\theta_a\otimes\theta_b)$$

indexed by objects of $\mathcal{A}$. Then, associate to every morphism $f : a \longrightarrow b$ in $\mathcal{A}$, the morphism

$$F(f) = \theta_b^{-1}\circ G(f)\circ\theta_a$$

in $\mathcal{M}$. A close look at the diagram shows that this defines a strict braided monoidal functor $F : \mathcal{A} \longrightarrow \mathcal{M}$ and a monoidal natural transformation $\theta : F \stackrel{\cdot}{\longrightarrow} G$ in Σ. We conclude.

1.4. *Fourth step*

After steps 1. 2. and 3. the proof of coherence reduces to the comparison of two "free categories":

— the "formal" braided monoidal category $\mathcal{A}$ of $\otimes$-trees and rewriting paths, quotiented by the commutativities of braided monoidal categories,
— the "geometric" braided monoidal category $\mathcal{B}$ of braids.

This is the most interesting and difficult part of the proof, in fact the first and only time combinatorics plays a role. We prove

1. that the subcategory of $\mathcal{A}$ consisting only of α, ρ and λ maps, and their inverses, is contractible,
2. that the quotient category is isomorphic to the braid category $\mathcal{B}$, and equivalent to $\mathcal{A}$ through strong braided monoidal functors.

The first and second assertions are established in sections 3 and 4 respectively.

1.5. *Conclusion*

The categories $\mathtt{Br}(\mathcal{A}, \mathcal{M})$ and $\mathcal{M}$ are equivalent by steps 1. 2. and 3. and the categories $\mathtt{Br}(\mathcal{A}, \mathcal{M})$ and $\mathtt{Br}(\mathcal{B}, \mathcal{M})$ are equivalent by step 4. This proves theorem 1.

2. A word problem

Let Σ be the category

— whose objects are $\otimes$-trees with leaves 0 and 1,
— whose morphisms are sequences of $\alpha(a, b, c)$, $\lambda(a)$ and $\rho(a)$, quotiented by the monoidal law (1) the commutativity laws (5) (6) and the additional commutativity triangles (2).

$$\begin{tikzcd} a \otimes b \arrow[r, "f\otimes b"] \arrow[d, "a\otimes g"'] & a' \otimes b \arrow[d, "a'\otimes g"] \\ a \otimes b' \arrow[r, "f\otimes b'"] & a' \otimes b' \end{tikzcd} \qquad \text{for} \qquad \begin{array}{l} a \xrightarrow{f} a' \\ b \xrightarrow{g} b' \end{array} \tag{1}$$

$$\begin{tikzcd} e \otimes (a \otimes b) \arrow[r, "\alpha"] \arrow[d, "\lambda"'] & (e \otimes a) \otimes b \arrow[d, "\lambda\otimes b"] \\ a \otimes b \arrow[r, equal] & a \otimes b \end{tikzcd} \qquad \begin{tikzcd} a \otimes (b \otimes e) \arrow[r, "\alpha"] \arrow[d, "a\otimes\rho"'] & (a \otimes b) \otimes e \arrow[d, "\rho"] \\ a \otimes b \arrow[r, equal] & a \otimes b \end{tikzcd} \tag{2}$$

Theorem 2. The category Σ verifies the three following properties:

— the category has pushouts,
— every morphism is epi,
— from every object a, there exists a morphism $a \longrightarrow b$ to a normal object b.

Proof. The proof appears in (Melliès, 2002). It is motivated by works of rewriters like Lévy (Lévy, 1978; Huet and Lévy, 1979) and algebraists like Dehornoy (Dehornoy, 1998). □

Definition 1. An object a in a category $\mathcal{C}$ is called normal when id_a is the only morphism outgoing from a.

3. Calculus of fractions

3.1. *Definition*

For every category $\mathcal{C}$ and class Σ of morphisms in $\mathcal{C}$, there exists a universal solution to the problem of "reversing" the morphisms in Σ. More explicitly, there exists a category $\mathcal{C}[\Sigma^{-1}]$ and a functor

$$P_\Sigma : \mathcal{C} \longrightarrow \mathcal{C}[\Sigma^{-1}]$$

such that:

— the functor P_Σ maps every morphism of Σ to reversible morphism,
— if a functor $F : \mathcal{C} \longrightarrow \mathcal{M}$ makes every morphism of Σ reversible, then F factors as $F = G \circ P_{\mathcal{C}}\Sigma$ for a unique functor $G : \mathcal{C}[\Sigma^{-1}] \longrightarrow \mathcal{M}$.

The class Σ is a ***calculus of left fraction*** in the category $\mathcal{C}$, see (Gabriel and Zisman, 1967), when

— Σ contains the identities of $\mathcal{C}$,
— Σ is closed under composition,
— each diagram $Y \xleftarrow{s} X \xrightarrow{f} Z$ where $s \in \Sigma$ may be completed into a commutative diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "s"'] & Y \arrow[d, dotted, "t"] \\ Z \arrow[r, dotted, "g"] & X' \end{tikzcd} \qquad \text{where } t \in \Sigma$$

— each commutative diagram

$$X' \xrightarrow{s} X \overset{f}{\underset{g}{\rightrightarrows}} Y \qquad \text{where } s \in \Sigma$$

may be completed into a commutative diagram

$$X \overset{f}{\underset{g}{\rightrightarrows}} Y \xrightarrow{t} Y' \qquad \text{where } t \in \Sigma$$

The property of left fraction calculus enables an elegant definition of the category $\mathcal{C}[\Sigma^{-1}]$ and functor $P_\Sigma : \mathcal{C} \longrightarrow \mathcal{C}[\Sigma^{-1}]$. The category $\mathcal{C}[\Sigma^{-1}]$ is defined as follows.

Its objects are the objects of $\mathcal{C}$, and its morphisms $X \longrightarrow Y$ are the equivalence classes of pairs (f, s) of morphisms of $\mathcal{C}$

$$X \xrightarrow{f} \cdot \xleftarrow{s} Y \qquad \text{where } s \in \Sigma$$

under the equivalence relation which identifies two pairs $X \xrightarrow{f} Z \xleftarrow{s} Y$ and $X \xrightarrow{g} Z' \xleftarrow{t} Y$ when there exists a pair $X \xrightarrow{h} Z'' \xleftarrow{u} Y$ and two morphisms $Z \xrightarrow{f'} Z'' \xleftarrow{g'} Z'$ forming a commutative diagram

$$\begin{tikzcd} & Z \arrow[d, "f'"] & \\ X \arrow[ur, "f"] \arrow[r, "h"] \arrow[dr, "g"'] & Z'' & Y \arrow[ul, "s"'] \arrow[l, "u"'] \arrow[dl, "t"] \\ & Z' \arrow[u, "g'"'] & \end{tikzcd} \qquad \text{where } u \in \Sigma$$

3.2. *Application*

By theorem 2, the class Σ is a calculus of fraction in the category Σ. But we can prove a bit more. A consequence of the pushout property, and definition of a normal object, is that any two morphisms $a \longrightarrow e$ and $b \longrightarrow e$ to a normal object e are equal in Σ. So, consider two morphisms in $\Sigma[\Sigma^{-1}]$, represented as pairs in Σ

$$a \xrightarrow{f} c \xleftarrow{s} b \qquad a \xrightarrow{g} d \xleftarrow{t} b$$

Any two morphisms $b \longrightarrow e$ and $c \longrightarrow e$ to a normal object e in Σ, make the following diagram commute in Σ:

$$\begin{tikzcd}
 & b \arrow[d, "f'"] & \\
a \arrow[ur, "f"] \arrow[r, "h"] \arrow[dr, "g"'] & e & b \arrow[ul, "s"'] \arrow[l, "u"'] \arrow[dl, "t"] \\
 & d \arrow[u, "g'"'] &
\end{tikzcd}$$

We obtain

Theorem 3. The category $\Sigma[\Sigma^{-1}]$ is contractible.

4. From word problem to coherence theorem

In section 3 we have proved that the category $\Sigma[\Sigma^{-1}]$ consisting of $\alpha(a,b,c)$, $\lambda(a)$ $\rho(a)$ steps and their inverse, is contractible.

— In step 1. we prove that $\Sigma[\Sigma^{-1}]$ is a subcategory of $\mathcal{A}$, full on objects.
— In step 2. we prove that the category $\mathcal{A}/\Sigma[\Sigma^{-1}]$ is braided monoidal, and embeds in $\mathcal{A}$ through strong braided monoidal functors,
— in step 3. we prove that the category $\mathcal{A}/\Sigma[\Sigma^{-1}]$ is isomorphic to the usual braid category $\mathcal{B}$.

4.1. *First step*

The two triangles (2) are commutative in $\mathcal{A}$, as in every monoidal category, see exercise VII.§1.1. in (Mac Lane, 1991). This ensures the existence of a functor $\Sigma \longrightarrow \mathcal{A}$. By definition, this functor induces a functor $\Sigma[\Sigma^{-1}] \longrightarrow \mathcal{A}$. This functor is injective on objects. It is also faithful for the obvious reason that $\Sigma[\Sigma^{-1}]$ is contractible. Thus, $\Sigma[\Sigma^{-1}] \longrightarrow \mathcal{A}$ defines an embedding of the contractible category $\Sigma[\Sigma^{-1}]$ into $\mathcal{A}$. The embedding is full on objects. This enables to consider the quotient category $\mathcal{A}/\Sigma[\Sigma^{-1}]$.

4.2. *Second step*

The category $\mathcal{A}/\Sigma[\Sigma^{-1}]$ becomes strict monoidal with the following definition. Let $f : a \longrightarrow b$ and $g : c \longrightarrow d$ be two morphisms of $\mathcal{A}/\Sigma[\Sigma^{-1}]$ represented by two morphisms $f_0 : a_0 \longrightarrow b_0$ and $g_0 : c_0 \longrightarrow d_0$ in the category $\mathcal{A}$. Define the tensor product $f \otimes g : a \otimes b \longrightarrow c \otimes d$ as the projection of the tensor product $f_0 \otimes g_0 : a_0 \otimes c_0 \longrightarrow b_0 \otimes d_0$ and the unit object e as the orbit of the unit in $\mathcal{A}$. Correctness follows from monoidality of $\otimes$ in $\mathcal{A}$. Monoidality of $\otimes$ is not difficult to prove, and diagrams (5) (6) are obvious.

The category $\mathcal{A}/\Sigma[\Sigma^{-1}]$ is braided monoidal. Let a and b be two objects of the quotient category $\mathcal{A}/\Sigma[\Sigma^{-1}]$ represented by two objects a_0 and b_0 in the category $\mathcal{A}$. Define the braiding $\gamma(a,b) : a \otimes b \longrightarrow b \otimes a$ in the quotient category as the orbit of the braiding

$\gamma(a_0, b_0) : a_0 \otimes b_0$. Correctness of the definition and naturality of γ follow from naturality of γ in $\mathcal{A}$. Diagrams (7) (8) (9) follow from the definition of γ in $\mathcal{A}/\Sigma[\Sigma^{-1}]$.

Moreover,

— the "projection" functor $F : \mathcal{A} \longrightarrow \mathcal{A}/\Sigma[\Sigma^{-1}]$ is strict braided monoidal. Diagrams (10) (11) are trivial, and diagram (12) follows from the definition of the braiding γ in $\mathcal{A}/\Sigma[\Sigma^{-1}]$.

— the "embedding" functor $(G, G_2, G_0) : \mathcal{B}/\Xi \longrightarrow \mathcal{A}$ is strong braided monoidal, with the morphisms $G_0 : e \longrightarrow G(e')$ and $G_2(a, b) : G(a) \otimes G(b) \longrightarrow G(a \otimes b)$ determined as morphisms of Σ. Diagrams (10) (11) hold because every maps are in $\Sigma[\Sigma^{-1}]$, and diagram (12) holds because of the definition of the braiding in $\mathcal{A}/\Sigma[\Sigma^{-1}]$.

4.3. *Third step*

We prove that $\mathcal{A}/\Sigma[\Sigma^{-1}]$ and $\mathcal{B}$ are isomorphic categories. The objects of $\mathcal{A}/\Sigma[\Sigma^{-1}]$ are the natural numbers. The morphisms of $\mathcal{A}/\Sigma[\Sigma^{-1}]$ are sequences of $\rho(m, n)$ and $\rho^{-1}(m, n)$ steps, modulo monoidality, and commutativity of the triangles induced by (8) (9):

$$\begin{tikzcd} m \otimes n \otimes p \arrow[r, "\gamma(m\otimes n, p)"] \arrow[d, "m\otimes\gamma(n,p)"'] & p \otimes m \otimes n \arrow[d, equal] \\ m \otimes p \otimes n \arrow[r, "\gamma(m,p)\otimes n"] & p \otimes m \otimes n \end{tikzcd} \qquad \begin{tikzcd} m \otimes n \otimes p \arrow[r, "\gamma(m, n\otimes p)"] \arrow[d, "\gamma(m,n)\otimes p"'] & n \otimes p \otimes m \arrow[d, equal] \\ n \otimes m \otimes p \arrow[r, "n\otimes\gamma(m,p)"] & n \otimes p \otimes m \end{tikzcd} \tag{3}$$

We construct a strict monoidal functor from $\Sigma[\Sigma^{-1}]$ to $\mathcal{B}$ by interpreting each $\gamma(m, n)$ by the braiding $\gamma_{\mathcal{B}}(m, n) : m + n \longrightarrow n + m$ in $\mathcal{B}$ consisting in permuting n braids over m braids. The functor is full. The only difficult point to prove is that it is faithfull.

This reduces to proving that the hexagonal diagram generating equality of "braids" in $\mathcal{B}$ commutes (already) in the category $\mathcal{A}/\Sigma[\Sigma^{-1}]$:

$$\begin{tikzcd}[column sep=small] & m \otimes n \otimes p \arrow[dl, "\gamma(m,n)\otimes p"'] \arrow[dr, "m\otimes\gamma(n,p)"] & \\ n \otimes m \otimes p \arrow[d, "n\otimes\gamma(m,p)"'] & & m \otimes p \otimes n \arrow[d, "\gamma(m,p)\otimes n"] \\ n \otimes p \otimes m \arrow[dr, "\gamma(n,p)\otimes m"'] & & p \otimes m \otimes n \arrow[dl, "p\otimes\gamma(m,n)"] \\ & p \otimes n \otimes m & \end{tikzcd} \tag{4}$$

The diagram is a nice "geometric" consequence of the commutative triangles 3, and monoidality of $\mathcal{A}/\Sigma[\Sigma^{-1}]$.

We conclude that $\mathcal{A}/\Sigma[\Sigma^{-1}]$ and the category $\mathcal{B}$ of braids are isomorphic.

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5. Appendix: definitions of braided monoidal categories, functors and natural transformations

5.1. *Monoidal category*

A monoidal category $\mathcal{M}$ is a category with a bifunctor

$$\otimes : \mathcal{M} \times \mathcal{M} \longrightarrow \mathcal{M}$$

which is associative up to a natural isomorphism

$$\alpha : a \otimes (b \otimes c) \longrightarrow (a \otimes b) \otimes c$$

equipped with an element e, which is a unit up to natural isomorphisms

$$\lambda : e \otimes a \longrightarrow a \qquad \rho : a \otimes e \longrightarrow a$$

These maps must make Mac Lane's famous "pentagon" commute

$$\begin{tikzcd}
a \otimes (b \otimes (c \otimes d)) \arrow[rr, "\alpha"] \arrow[d, "a\otimes\alpha"'] & & (a \otimes b) \otimes (c \otimes d) \arrow[d, "\alpha"] \\
a \otimes ((b \otimes c) \otimes d) \arrow[r, "\alpha"] & (a \otimes (b \otimes c)) \otimes d \arrow[r, "\alpha\otimes d"] & ((a \otimes b) \otimes c) \otimes d
\end{tikzcd} \tag{5}$$

as well as the triangle:

$$\begin{tikzcd}
a \otimes (e \otimes b) \arrow[r, "\alpha"] \arrow[d, "a\otimes\lambda"'] & (a \otimes e) \otimes b \arrow[d, "\rho\otimes b"] \\
a \otimes b \arrow[r, equal] & a \otimes b
\end{tikzcd} \tag{6}$$

A monoidal category is *strict* when all α, λ and ρ are identities.

5.2. *Braided monoidal category*

A braiding for a monoidal category $\mathcal{M}$ consists of a family of isomorphisms

$$\gamma_{a,b} : \quad a \otimes b \longrightarrow b \otimes a$$

natural in a and b, which satisfy the commutativity

$$\begin{tikzcd} a \otimes e \arrow[r, "\gamma"] \arrow[d, "\rho"'] & e \otimes a \arrow[d, "\lambda"] \\ a \arrow[r, equal] & a \end{tikzcd} \tag{7}$$

and makes both following hexagons commute:

$$\begin{tikzcd} a \otimes (b \otimes c) \arrow[r, "\alpha"] \arrow[d, "a\otimes\gamma"] & (a \otimes b) \otimes c \arrow[r, "\gamma"] & c \otimes (a \otimes b) \arrow[d, "\alpha"] \\ a \otimes (c \otimes b) \arrow[r, "\alpha"] & (a \otimes c) \otimes b \arrow[r, "\gamma\otimes b"] & (c \otimes a) \otimes b \end{tikzcd} \tag{8}$$

$$\begin{tikzcd} (a \otimes b) \otimes c \arrow[r, "\alpha^{-1}"] \arrow[d, "\gamma\otimes c"] & a \otimes (b \otimes c) \arrow[r, "\gamma"] & (b \otimes c) \otimes a \arrow[d, "\alpha^{-1}"] \\ (b \otimes a) \otimes c \arrow[r, "\alpha^{-1}"] & b \otimes (a \otimes c) \arrow[r, "b\otimes\gamma"] & b \otimes (c \otimes a) \end{tikzcd} \tag{9}$$

5.3. *Monoidal functor*

A monoidal functor $(F, F_2, F_0) : \mathcal{M} \longrightarrow \mathcal{M}'$ between monoidal categories $\mathcal{M}$ and $\mathcal{M}'$:
— an ordinary functor $F : \mathcal{M} \longrightarrow \mathcal{M}'$,
— for objects a, b in $\mathcal{M}$ morphisms in $\mathcal{M}'$:

$$F_2(a, b) : F(a) \otimes F(b) \longrightarrow F(a \otimes b)$$

which are natural in a and b,
— for the units e and e', a morphism in $\mathcal{M}'$:

$$F_0 : e \longrightarrow e'$$

making the diagrams commute:

$$\begin{tikzcd} F(a) \otimes (F(b) \otimes F(c)) \arrow[r, "\alpha'"] \arrow[d, "F(a)\otimes F_2(b,c)"'] & (F(a) \otimes F(b)) \otimes F(c) \arrow[d, "F_2(a,b)\otimes F(c)"] \\ F(a) \otimes F(b \otimes c) \arrow[d, "F_2(a,b\otimes c)"'] & F(a \otimes b) \otimes F(c) \arrow[d, "F_2(a\otimes b,c)"] \\ F(a \otimes (b \otimes c)) \arrow[r, "F(\alpha)"] & F((a \otimes b) \otimes c) \end{tikzcd} \tag{10}$$

$$\begin{tikzcd} F(b) \otimes e' \arrow[r, "\rho"] \arrow[d, "F(b)\otimes F_0"'] & F(b) \\ F(b) \otimes F(e) \arrow[r, "F_2(b,e)"] & F(b \otimes e) \arrow[u, "F(\rho)"'] \end{tikzcd} \qquad \begin{tikzcd} e' \otimes F(b) \arrow[r, "\lambda"] \arrow[d, "F_0\otimes F(b)"'] & F(b) \\ F(e) \otimes F(b) \arrow[r, "F_2(e,b)"] & F(e \otimes b) \arrow[u, "F(\lambda)"'] \end{tikzcd} \tag{11}$$

A monoidal functor f is said to be *strong* when F_0 and all $F_2(a,b)$ are isomorphisms, *strict* when F_0 and all $F_2(a,b)$ are identities.

5.4. *Braided monoidal functors*

If $\mathcal{M}$ and $\mathcal{M}'$ are braided monoidal categories, a braided monoidal functor is a monoidal functor $(F, F_2, F_0) : \mathcal{M} \longrightarrow \mathcal{M}'$ which commutes with the braidings γ and γ' in the following sense:

$$\begin{tikzcd} F(a) \otimes F(b) \arrow[r, "\gamma'"] \arrow[d, "F_2(a,b)"'] & F(b) \otimes F(a) \arrow[d, "F_2(b,a)"] \\ F(a \otimes b) \arrow[r, "F(\gamma)"] & F(b \otimes a) \end{tikzcd} \tag{12}$$

5.5. *Monoidal natural transformations*

A monoidal natural transformation $\theta : (F, F_2, F_0) \stackrel{\cdot}{\longrightarrow} (G, G_2, G_0) : \mathcal{M} \longrightarrow \mathcal{M}'$ between two monoidal functors is a natural transformation between the underlying ordinary functors $\theta : F \longrightarrow G$ making the diagrams

$$\begin{tikzcd} F(a) \otimes F(b) \arrow[r, "F_2(a,b)"] \arrow[d, "\theta_a \otimes \theta_b"'] & F(a \otimes b) \arrow[d, "\theta_{a\otimes b}"] \\ G(a) \otimes G(b) \arrow[r, "G_2(a,b)"] & G(a \otimes b) \end{tikzcd} \qquad \begin{tikzcd} e' \arrow[r, "F_0"] \arrow[d, equal] & Fe \arrow[d, "\theta_e"] \\ e' \arrow[r, "G_0"] & Ge \end{tikzcd} \tag{13}$$

commute in $\mathcal{M}'$.