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ABOUT BREZIS-MERLE PROBLEM WITH LIPSCHITZ CONDITION.

SAMY SKANDER BAHOURA

ABSTRACT. We give blow-up analysis for a Brezis and Merle’s problem with Dirichlet condition. As an application we have a proof of a compactness result under Lipschitz condition on the prescribed scalar curvature and a weaker assumption on the regularity of the domain (smooth domain or $C^{2,\alpha}$ domain, $1 \geq \alpha > 0$).

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1. INTRODUCTION AND MAIN RESULTS

We set $\Delta = - (\partial_{11} + \partial_{22})$ on open set $\Omega$ of $\mathbb{R}^2$ with a smooth (or $C^{2,\alpha}$, $\alpha > 0$) boundary.

We consider the following equation:

\[(P) \begin{cases} \Delta u = V e^u & \text{in } \Omega \subset \mathbb{R}^2, \\ u = 0 & \text{in } \partial \Omega. \end{cases} \]

Here, we assume that:

\[0 \leq V \leq b < +\infty, \ e^u \in L^1(\Omega) \text{ and } u \in W^{1,1}_0(\Omega).\]

We can see in [7] a nice formulation of this problem $(P)$ in the sense of the distributions. This Problem arises from geometrical and physical problems see for example [1, 2, 18, 19]. The above equation was studied by many authors, with or without the boundary condition, also for Riemannian surfaces, see [1-19], where one can find some existence and compactness results. In [6] we have the following important Theorem,

**Theorem A** (Brezis-Merle [6]). For $(u_i)_i$ and $(V_i)_i$ two sequences of functions relative to $(P)$ with,

\[0 < a \leq V_i \leq b < +\infty\]

then, for all compact subset $K$ of $\Omega$ it holds,

\[\sup_K u_i \leq c,\]

with $c$ depending on $a, b, K$ and $\Omega$.

One can find in [6] an interior estimate if we assume $a = 0$, but we need an assumption on the integral of $e^{u_i}$, namely, we have:
Theorem B (Brezis-Merle [6]). For \((u_i)_i\) and \((V_i)_i\) two sequences of functions relative to the problem (P) with,

\[ 0 \leq V_i \leq b < +\infty \text{ and } \int_{\Omega} e^{u_i} dy \leq C, \]

then, for all compact subset \(K\) of \(\Omega\) it holds:

\[ \sup_K u_i \leq c, \]

with \(c\) depending on \(b, C, K\) and \(\Omega\).

We look to the uniform boundedness on all \(\bar{\Omega}\) of sequences of solutions of the Problem (P). Remark that, when \(a = 0\) the boundedness of \(\int_{\Omega} e^{u_i}\) is a necessary condition in the problem (P) as showed in [6] by the following counterexample.

Theorem C (Brezis-Merle [6]). There are two sequences \((u_i)_i\) and \((V_i)_i\) of the problem (P) with,

\[ 0 \leq V_i \leq b < +\infty \text{ and } \int_{\Omega} e^{u_i} dy \leq C, \]

such that,

\[ \sup_{\Omega} u_i \rightarrow +\infty. \]

To obtain the two first previous results (Theorems A and B) Brezis and Merle used an inequality (Theorem 1 of [6]) obtained by an approximation argument with the Fatou’s lemma and they applied the maximum principle in \(W^{1,1}_0(\Omega)\) which arises from Kato’s inequality. Also this weak form of the maximum principle is used to prove the local uniform boundedness result by comparing a certain function and the Newtonian potential. We refer to [5] for a topic about the weak form of the maximum principle.

Remarks:

1) Theorem 1 of [6], can be obtained by the usual maximum principle and Agmon regularity theorem which require \(C^2\) regularity on the domain.

2) The duality Theorem which we use require \(C^2\) regularity on the domain, see Gilbarg-Trudinger books.

Note that for the problem (P), by using the Pohozaev identity, we can prove that \(\int_{\Omega} e^{u_i}\) is uniformly bounded when \(0 < a \leq V_i \leq b < +\infty\) and \(\|\nabla V_i\|_{L^\infty} \leq A\) and \(\Omega\) starshaped, when \(a = 0\) and \(\nabla \log V_i\) is uniformly bounded, we can bound uniformly \(\int_{\Omega} V_i e^{u_i}\). In [16] Ma-Wei have proved that those results stay true for all open sets not necessarily starshaped.

In [9] Chen-Li have proved that if \(a = 0\) and \(\nabla \log V_i\) is uniformly bounded, then we have the compactness result directly. Ma-Wei in [16], extend this result in the case where \(a > 0\).

If we assume \(V\) more regular, we can have another type of estimates called \(\sup + \inf\) type inequalities. It was proved by Shafrir see [17] that, if \((u_i)_i, (V_i)_i\) are two sequences of functions solutions of the previous
equation without assumption on the boundary and, $0 < a \leq V_i \leq b < +\infty$, then we have the following interior estimate:

$$C \left( \frac{a}{b} \right) \sup_K u_i + \inf_{\Omega} u_i \leq c = c(a, b, K, \Omega).$$

One can see in [10] an explicit value of $C \left( \frac{a}{b} \right) = \sqrt{\frac{a}{b}}$. In his proof Shafrir has used a blow-up function, the Stokes formula and an isoperimetric inequality see [2]. For Chen-Lin, they have used the blow-up analysis combined with some geometric type inequality for the integral curvature.

Now, if we suppose $(V_i)_i$ uniformly Lipschitzian with $A$ its Lipschitz constant then $C(a/b) = 1$ and $c = c(a, b, A, K, \Omega)$ see Brezis-Li-Shafrir [4]. This result was extended for Hölderian sequences $(V_i)_i$ by Chen-Lin, see [10]. Also, one can see in [14] an extension of the Brezis-Li-Shafrir result to compact Riemannian surfaces without boundary. One can see in [15] explicit form, $(8\pi m, m \in \mathbb{N}^*)$ exactly, for the numbers in front of the Dirac masses when the solutions blow-up. Here, the notion of isolated blow-up point is used.

In [8] we have some a priori estimates on the 2 and 3-spheres $S_2, S_3$.

Here we give the behavior of the blow-up points on the boundary and a proof of Brezis-Merle Problem with Lipschitz condition.

The Brezis-Merle Problem (see [6]) is:

**Problem.** Suppose that $V_i \to V$ in $C^0(\overline{\Omega})$ with $0 \leq V_i$. Also, we consider a sequence of solutions $(u_i)$ of $(P)$ relative to $(V_i)$ such that,

$$\int_{\Omega} e^{u_i} \, dx \leq C,$$

is it possible to have:

$$\|u_i\|_{L^\infty} \leq C?$$

Here, we give a characterization of the behavior of the blow-up points on the boundary and also, in particular we extend Chen-Li theorems, indeed, the result of Chen-Li holds for analytic domains and our result holds for smooth of $C^{2,\alpha}$ domains. For the behavior of the blow-up points on the boundary, the following condition is enough,

$$0 \leq V_i \leq b.$$

The condition $V_i \to V$ in $C^0(\overline{\Omega})$ is not necessary, but for the proof of the compactness for the Brezis-Merle problem we assume that:

$$\|\nabla V_i\|_{L^\infty} \leq A.$$

Our main results are:

**Theorem 1.1.** Assume that $\max_{\Omega} u_i \to +\infty$, where $(u_i)$ are solutions of the problem $(P)$ with:
0 ≤ V_i ≤ b and \( \int_{\Omega} e^{u_i} dx \leq C, \ \forall \ i, \)

then, after passing to a subsequence, there is a function \( u \), there is a number \( N \in \mathbb{N} \) and there are \( N \) points \( x_1, \ldots, x_N \in \partial \Omega \), such that,

\[
\partial_{\nu} u_i \to \partial_{\nu} u + \sum_{j=1}^{N} \alpha_j \delta_{x_j}, \ \alpha_j \geq 4\pi, \ \text{in the sense of measures on } \partial \Omega.
\]

\[
u_j \to u \ \text{in } C^1_{\text{loc}}(\overline{\Omega} - \{x_1, \ldots, x_N\}).
\]

**Theorem 1.2.** Assume that \( (u_i) \) are solutions of \( (P) \) relative to \( (V_i) \) with the following conditions:

\[
0 \leq V_i \leq b, \ |\nabla V_i|_{L^\infty} \leq A \quad \text{and} \quad \int_{\Omega} e^{u_i} \leq C,
\]

we have,

\[
|u_i|_{L^\infty} \leq c(b, A, C, \Omega).
\]

In the previous theorem we have a proof of the global a priori estimate which concern the problem \( (P) \).

The proof of Chen-Li and Ma-Wei [9,16], use the moving-plane method for the case \( \nabla \log V_i \) uniformly bounded near the boundary (and \( C^{2,\alpha} \) domain, \( 1 \geq \alpha > 0 \)) and for analytic domain for the case \( \nabla V_i \) uniformly bounded.

To prove Theorem 1.2, we argue by contradiction and use Theorem 1.1.

2. **Proof of the theorems**

**Proof of theorem 1.1:**

We have:

\[
u_j \in W^{1,1}_0(\Omega).
\]

Since \( e^{u_i} \in L^1(\Omega) \) by the corollary 1 of Brezis-Merle’s paper (see [6]) we have \( e^{u_i} \in L^k(\Omega) \) for all \( k > 2 \) and the elliptic estimates of Agmon and the Sobolev embedding (see [1]) imply that:

\[
u_j \in W^{2,k}(\Omega) \cap C^{1,\epsilon}(\overline{\Omega}).
\]

We denote by \( \partial_{\nu} u_i \) the inner normal derivative. By the maximum principle we have, \( \partial_{\nu} u_i \geq 0 \).

By the Stokes formula we have,

\[
\int_{\partial \Omega} \partial_{\nu} u_i d\sigma \leq C,
\]
We use the weak convergence in the space of Radon measures to have the existence of a nonnegative Radon measure \( \mu \) such that,

\[
\int_{\partial \Omega} \partial_v u_i \varphi d\sigma \rightarrow \mu(\varphi), \quad \forall \ \varphi \in C^0(\partial \Omega).
\]

We take an \( x_0 \in \partial \Omega \) such that, \( \mu(x_0) < 4\pi \). For \( \epsilon > 0 \) small enough set \( I_\epsilon = B(x_0, \epsilon) \cap \partial \Omega \). We choose a function \( \eta_\epsilon \) such that,

\[
\begin{aligned}
\eta_\epsilon &\equiv 1, \text{ on } I_\epsilon, \ 0 < \epsilon < \delta/2, \\
\eta_\epsilon &\equiv 0, \text{ outside } I_{2\epsilon}, \\
0 \leq \eta_\epsilon &\leq 1, \\
||\nabla \eta_\epsilon||_{L^\infty(I_{2\epsilon})} &\leq \frac{C_0(\Omega, x_0)}{\epsilon}.
\end{aligned}
\]

We take a \( \tilde{\eta}_\epsilon \) such that,

\[
\begin{aligned}
\Delta \tilde{\eta}_\epsilon &= 0 \quad \text{in } \Omega \subset \mathbb{R}^2, \\
\tilde{\eta}_\epsilon &= \eta_\epsilon \quad \text{in } \partial \Omega.
\end{aligned}
\]

Remark: We use the following steps in the construction of \( \eta_\epsilon \):

We take a cutoff function \( \eta_0 \) in \( B(0, 2) \) or \( B(x_0, 2) \):

1- We set \( \eta_0(x) = \eta_0(|x - x_0|/\epsilon) \) in the case of the unit disk it is sufficient.

2- Or, in the general case: we use a chart \( (f, \tilde{\Omega} = f(B_r(0))) \), for \( r > 0 \) small enough and \( f(0) = x_0 \) and we take \( \mu_\epsilon(x) = \eta_0(f(|x|/\epsilon)) \) to have connected sets \( I_\epsilon \) and we take \( \eta_\epsilon(y) = \mu_\epsilon(f^{-1}(y)) \). Because \( f, f^{-1} \) are Lipschitz, \( |f(x) - x_0| \leq k_2|x| \leq 1 \) for \( |x| \leq 1/k_2 \) and \( |f(x) - x_0| \geq k_1|x| \geq 2 \) for \( |x| \geq 2/k_1 > 1/k_2 \), the support of \( \eta \) is in \( I_{(2/k_1)\epsilon} \).

\[
\begin{aligned}
\eta_\epsilon &\equiv 1, \text{ on } f(I_{(1/k_1)\epsilon}), \ 0 < \epsilon < \delta/2, \\
\eta_\epsilon &\equiv 0, \text{ outside } f(I_{(2/k_1)\epsilon}), \\
0 \leq \eta_\epsilon &\leq 1, \\
||\nabla \eta_\epsilon||_{L^\infty(I_{(2/k_1)\epsilon})} &\leq \frac{C_0(\Omega, x_0)}{\epsilon}.
\end{aligned}
\]

3- Also, we can take: \( \mu_\epsilon(x) = \eta_0(|x|/\epsilon) \) and \( \eta_\epsilon(y) = \mu_\epsilon(f^{-1}(y)) \), we extend it by 0 outside \( f(B_1(0)) \). We have \( f(B_1(0)) = D_1(x_0), f(B_1(0)) = D_\epsilon(x_0) \) and \( f(B_1^+) = D_\epsilon^+(x_0) \) with \( f \) and \( f^{-1} \) smooth diffeomorphism.

\[
\begin{aligned}
\eta_\epsilon &\equiv 1, \text{ on the connected set } J_\epsilon = f(I_\epsilon), \ 0 < \epsilon < \delta/2, \\
\eta_\epsilon &\equiv 0, \text{ outside } J'_\epsilon = f(I_{2\epsilon}), \\
0 \leq \eta_\epsilon &\leq 1, \\
||\nabla \eta_\epsilon||_{L^\infty(J'_\epsilon)} &\leq \frac{C_0(\Omega, x_0)}{\epsilon}.
\end{aligned}
\]

And, \( H_1(J'_\epsilon) \leq C_1 H_1(I_{2\epsilon}) = C_1 4\epsilon \), because \( f \) is Lipschitz. Here \( H_1 \) is the Hausdorff measure.
We solve the Dirichlet Problem:

\[
\begin{aligned}
\Delta \tilde{\eta}_e &= \Delta \eta_e \quad \text{in } \Omega \subset \mathbb{R}^2, \\
\tilde{\eta}_e &= 0 \quad \text{in } \partial \Omega.
\end{aligned}
\]

and finally we set \( \tilde{\eta}_e = -\bar{\eta}_e + \eta_e \). Also, by the maximum principle and the elliptic estimates we have:

\[
||\nabla \tilde{\eta}_e||_{L^\infty} \leq C \left(||\eta_e||_{L^\infty} + ||\nabla \eta_e||_{L^\infty} + ||\Delta \eta_e||_{L^\infty}\right) \leq \frac{C_1}{\epsilon^2},
\]

with \( C_1 \) depends on \( \Omega \).

We use the following estimate, see \([3, 7, 19]\),

\[
||\nabla u_i||_{L^q} \leq C_q, \quad \forall \ i \quad \text{and} \quad 1 < q < 2.
\]

We deduce from the last estimate that, \( (u_i) \) converge weakly in \( W_0^{1,q}(\Omega) \), almost everywhere to a function \( u \geq 0 \) and \( \int_{\Omega} e^{u} < +\infty \) (by Fatou’s lemma). Also, \( V_i \) weakly converge to a nonnegative function \( V \) in \( L^\infty \).

The function \( u \) is in \( W_0^{1,q}(\Omega) \) solution of:

\[
\begin{aligned}
\Delta u &= V e^u \quad \text{in } \Omega \subset \mathbb{R}^2, \\
u &= 0 \quad \text{in } \partial \Omega.
\end{aligned}
\]

According to the corollary 1 of Brezis-Merle result, see \([6]\), we have \( e^{ku} \in L^1(\Omega) \), \( k > 1 \). By the elliptic estimates, we have \( u \in C^1(\bar{\Omega}) \).

For two vectors \( v, w \) of \( \mathbb{R}^2 \) we denote by \( v \cdot w \) the inner product of \( v \) and \( w \).

We can write,

\[
\Delta((u_i - u)\tilde{\eta}_e) = (V_i e^{u_i} - V e^u)\tilde{\eta}_e - 2\nabla(u_i - u) \cdot \nabla \tilde{\eta}_e. \tag{1}
\]

We use the interior estimate of Brezis-Merle, see \([6]\),

**Step 1:** Estimate of the integral of the first term of the right hand side of (1).

We use the Green formula between \( \tilde{\eta}_e \) and \( u_i \), we obtain,

\[
\int_{\Omega} V e^{u} \tilde{\eta}_e dx = \int_{\partial \Omega} \partial_{\nu} u_i \tilde{\eta}_e \leq C' ||\partial_{\nu} u||_{L^\infty} = C\epsilon \tag{2}
\]

We have,

\[
\begin{aligned}
\Delta u_i &= V_i e^{u_i} \quad \text{in } \Omega \subset \mathbb{R}^2, \\
u_i &= 0 \quad \text{in } \partial \Omega.
\end{aligned}
\]

We use the Green formula between \( u_i \) and \( \tilde{\eta}_e \) to have:
\[
\int_{\Omega} V_i e^{u_i} \hat{\eta} \, dx = \int_{\partial\Omega} \partial_n u_i \hat{\eta} \, d\sigma \rightarrow \mu(\eta_i) \leq \mu(J^\prime_s) \leq 4\pi - \epsilon_0, \quad \epsilon_0 > 0
\]  

(3)

From (2) and (3) we have for all \( \epsilon > 0 \) there is \( i_0 = i_0(\epsilon) \) such that, for \( i \geq i_0, \)

\[
\int_{\Omega} \left| (V_i e^{u_i} - V e^u) \hat{\eta} \right| \, dx \leq 4\pi - \epsilon_0 + C\epsilon
\]

(4)

**Step 2:** Estimate of integral of the second term of the right hand side of (1).

Let \( \Sigma_\epsilon = \{ x \in \Omega, d(x, \partial\Omega) = \epsilon^3 \} \) and \( \Omega_{\epsilon^3} = \{ x \in \Omega, d(x, \partial\Omega) \geq \epsilon^3 \}, \epsilon > 0. \) Then, for \( \epsilon \) small enough, \( \Sigma_\epsilon \) is a manifold.

The measure of \( \Omega - \Omega_{\epsilon^3} \) is \( k_2 \epsilon^3 \leq \text{meas}(\Omega - \Omega_{\epsilon^3}) = \mu_L(\Omega - \Omega_{\epsilon^3}) \leq k_1 \epsilon^3. \) Here \( \mu_L \) is the Lebesgue measure.

**Remark:** for the unit ball \( B(0, 1) \), our new manifold is \( B(0, 1 - \epsilon^3) \).

(Proof of this fact; let’s consider \( d(x, \partial\Omega) = d(x, z_0), z_0 \in \partial\Omega \), this imply that \( (d(x, z_0))^2 \leq (d(x, z))^2 \) for all \( z \in \partial\Omega \) which it is equivalent to \( (z - z_0) \cdot (2x - z - z_0) \leq 0 \) for all \( z \in \partial\Omega \), let’s consider a chart around \( z_0 \) and \( \gamma(t) \) a curve in \( \partial\Omega \), we have;

\[
(\gamma(t) - \gamma(t_0) \cdot (2x - \gamma(t) - \gamma(t_0)) \leq 0 \text{ and it is clear that, } \gamma'(t_0) \cdot (x - \gamma(t_0)) = 0, \text{ which imply that } x = z_0 - s\nu_0 \text{ where } \nu_0 \text{ is the outward normal of } \partial\Omega \text{ at } z_0)
\]

With this fact, we can say that \( S = \{ x, d(x, \partial\Omega) \leq \epsilon \} = \{ x = z_0 - s\nu_0, z_0 \in \partial\Omega, \quad -\epsilon \leq s \leq \epsilon \}. \) It is sufficient to work on \( \partial\Omega \). Let’s consider a charts \( \{ z, D = B(z, 4\epsilon_z), \gamma_z \} \) with \( z \in \partial\Omega \) such that \( \bigcup_k B(z_k, \epsilon_k) \) is cover of \( \partial\Omega \). One can extract a finite cover \( \{ B(z_k, \epsilon_k) \}, k = 1, ..., m, \) by the area formula the measure of \( S \cap B(z_k, \epsilon_k) \) is less than a \( k\epsilon \) (a \( \epsilon \)-rectangle). For the reverse inequality, it is sufficient to consider one chart around one point on the boundary.

We write,

\[
\int_{\Omega} |\nabla (u_i - u) \cdot \nabla \hat{\eta}_i| \, dx = \int_{\Omega_{\epsilon^3}} |\nabla (u_i - u) \cdot \nabla \hat{\eta}_i| \, dx + \int_{\Omega - \Omega_{\epsilon^3}} |\nabla (u_i - u) \cdot \nabla \hat{\eta}_i| \, dx.
\]

(5)

**Step 2.1:** Estimate of \( \int_{\Omega - \Omega_{\epsilon^3}} |\nabla (u_i - u) \cdot \nabla \hat{\eta}_i| \, dx \).

First, we know from the elliptic estimates that \( ||\nabla \hat{\eta}_i||_{L^q} \leq C_1/\epsilon^2, \) \( C_1 \) depends on \( \Omega \)

We know that \( |\nabla u_i| \) is bounded in \( L^q, 1 < q < 2, \) we can extract from this sequence a subsequence which converge weakly to \( h \in L^q. \) But, we know that we have locally the uniform convergence to \( |\nabla u| \) (by Brezis-Merle’s theorem), then, \( h = |\nabla u| \) a.e. Let \( q' \) be the conjugate of \( q. \)

We have, \( \forall f \in L^q(\Omega) \)

\[
\int |\nabla u_i| f \, dx \rightarrow \int |\nabla u| f \, dx
\]

If we take \( f = 1_{\Omega - \Omega_{\epsilon^3}}, \) we have:
for $\epsilon > 0$ \( i_1 = i_1(\epsilon) \in \mathbb{N}, \ i \geq i_1, \ \int_{\Omega_{-\epsilon,3}} |\nabla u_i| \leq \int_{\Omega_{\epsilon,3}} |\nabla u| + \epsilon^3. \)

Then, for $i \geq i_1(\epsilon),$ 

\[
\int_{\Omega_{-\epsilon,3}} |\nabla u_i| \leq \text{meas}(\Omega - \Omega_{\epsilon,3})||\nabla u||_{L^\infty} + \epsilon^3 = \epsilon^3(k_1||\nabla u||_{L^\infty} + 1).
\]

Thus, we obtain,

\[
\int_{\Omega_{-\epsilon,3}} |\nabla u_i| \cdot \nabla \tilde{\eta}_{\epsilon} \, dx \leq \epsilon C_1(2k_1||\nabla u||_{L^\infty} + 1) \tag{6}
\]

The constant $C_1$ does not depend on $\epsilon$ but on $\Omega.$

**Step 2.2:** Estimate of $\int_{\Omega_{\epsilon,3}} |\nabla(u_i - u) \cdot \nabla \tilde{\eta}_{\epsilon}| \, dx.$

We know that, $\Omega_{\epsilon} \subset \subset \Omega,$ and (because of Brezis-Merle’s interior estimates) $u_i \to u$ in $C^1(\Omega_{\epsilon,3}).$ We have,

\[
||\nabla (u_i - u)||_{L^\infty(\Omega_{\epsilon,3})} \leq \epsilon^3, \text{ for } i \geq i_3 = i_3(\epsilon).
\]

We write,

\[
\int_{\Omega_{\epsilon,3}} |\nabla (u_i - u) \cdot \nabla \tilde{\eta}_{\epsilon}| \, dx \leq ||\nabla (u_i - u)||_{L^\infty(\Omega_{\epsilon,3})}||\nabla \tilde{\eta}_{\epsilon}||_{L^\infty} \leq C_1 \epsilon \text{ for } i \geq i_3,
\]

For $\epsilon > 0,$ we have for $i \in \mathbb{N}, \ i \geq \max\{i_1, i_2, i_3\},$

\[
\int_{\Omega} |\nabla (u_i - u) \cdot \nabla \tilde{\eta}_{\epsilon}| \, dx \leq \epsilon C_1(2k_1||\nabla u||_{L^\infty} + 2) \tag{7}
\]

From (4) and (7), we have, for $\epsilon > 0,$ there is $i_3 = i_3(\epsilon) \in \mathbb{N}, i_3 = \max\{i_0, i_1, i_2\}$ such that,

\[
\int_{\Omega} |\Delta (u_i - u) \tilde{\eta}_{\epsilon}| \, dx \leq 4\pi - \epsilon_0 + \epsilon C_1(2k_1||\nabla u||_{L^\infty} + 2 + C) \tag{8}
\]

We choose $\epsilon > 0$ small enough to have a good estimate of (1).

Indeed, we have:

\[
\left\{ \begin{array}{ll}
\Delta (u_i - u) \tilde{\eta}_{\epsilon} = g_{i,\epsilon} & \text{in } \Omega \subset \mathbb{R}^2, \\
(u_i - u) \tilde{\eta}_{\epsilon} = 0 & \text{in } \partial \Omega.
\end{array} \right.
\]

with $||g_{i,\epsilon}||_{L^1(\Omega)} \leq 4\pi - \epsilon_0/2.$

We can use Theorem 1 of [6] to conclude that there is $q \geq \tilde{q} > 1$ such that:
\[ \int_{V_\epsilon(x_0)} e^{\tilde{q}|u_i-u|} dx \leq \int_{\Omega} e^{\tilde{q}|u_i-u|} \tilde{\eta} dx \leq C(\epsilon, \Omega). \]

where, \( V_\epsilon(x_0) \) is a neighborhood of \( x_0 \) in \( \bar{\Omega} \). Here we have used that in a neighborhood of \( x_0 \) by the elliptic estimates, \( 1 - C\epsilon \leq \tilde{\eta} \leq 1. \) (We can take, \( f(B_{c\epsilon}(0)) \) and we have \( B_{k_2\epsilon^3}(x_0) \subset f(B_{c\epsilon^3}(0)) \subset B_{k_1\epsilon^3}(x_0) \) for a chart \( (f, B_1(0)) \) around \( x_0 \)).

Thus, for each \( x_0 \in \partial\Omega - \{\bar{x}_1, \ldots, \bar{x}_m\} \) there is \( \epsilon_{x_0} > 0, q_{x_0} > 1 \) such that:

\[ \int_{B(x_0, \epsilon_{x_0})} e^{q_{x_0} u_i} dx \leq C, \forall \ i. \] (9)

By the elliptic estimates (see [13]) \( (u_i) \) is uniformly bounded in \( W^{2,q_1}(V_\epsilon(x_0)) \) and also, in \( C^{1}(V_\epsilon(x_0)) \). Finally, we have, for some \( \epsilon > 0 \) small enough,

\[ \|u_i\|_{C^{1,\theta}[B(x_0,\epsilon)]} \leq c_3 \forall \ i. \]

We have proved that, there is a finite number of points \( \bar{x}_1, \ldots, \bar{x}_m \) such that the sequence \( (u_i)_i \) is locally uniformly bounded (in \( C^{1,0}, \theta > 0 \)) in \( \Omega - \{\bar{x}_1, \ldots, \bar{x}_m\} \).

**Proof of theorem 1.2:**

We know that:

\[ u_i \in W^{2,k}(\Omega) \cap C^{1,\epsilon}(\bar{\Omega}). \]

We can do integration by parts. The first Pohozaev identity applied around each blow-up point see for example [16] gives:

\[ \int_{\partial\Omega_{x_k}} [(\partial_\nu u_i)\nabla u_i - \frac{1}{2}\|
abla u_i\|^2] dx = \int_{\Omega_{x_k}} \nabla V_i e^{u_i} - \int_{\partial\Omega_{x_k}} V_i e^{u_i} \nu, \]

Here \( \Omega_{x_k} \) is a neighborhood of \( x_k \) on which we can use the integration by part obtained by a chart around \( x_k \).

We use the boundary condition on \( \Omega \) and the boundedness of \( u_i \) and \( \partial_j u_i \) outside the \( x_k \), to have:

\[ \int_{\partial\Omega} (\partial_\nu u_i)^2 dx \leq c_0(b, A, C, \Omega). \] (11)

Thus we can use the weak convergence in \( L^2(\partial\Omega) \) to have a subsequence \( \partial_\nu u_i \), such that:

\[ \int_{\partial\Omega} \partial_\nu u_i \varphi dx \to \int_{\partial\Omega} \partial_\nu u \varphi dx, \forall \ \varphi \in L^2(\partial\Omega), \]

Thus, \( \alpha_j = 0, j = 1, \ldots, N \) and \( (u_i) \) is uniformly bounded.

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REFERENCES


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