Relative definability and models of Unary PCF

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Abstract. We show that the poset of degrees of relative definability in the Scott model of Unary PCF is non trivial, and that, nevertheless, the hierarchy of order extensional models of the language is reduced to a bottom element (the fully abstract model) and a top one (the Scott model itself).

1 Introduction

Finitary versions of PCF and related languages have been studied in the last decade, in order to settle the well-known "full abstraction problem" for the full language. In 1993, A. Jung and A. Stoughton [4] proposed the following crash test for a solution to that problem to be a "good" one: it should provide an algorithm for deciding observational equivalences for the finitary fragment of the language (*i.e.* the language whose unique ground type is `bool`, FPCF).

Shortly later, R. Loader proved that the observational equivalence of FPCF is undecidable [9]. An immediate corollary of this breakthrough result is that the problem of FPCF-definability in, say, the Scott model of FPCF, is undecidable. Hence, relative FPCF-definability in that model is also undecidable.

The poset of degrees of relative definability is somehow related to the existence of hierarchies of (order extensional) models. The idea is the following: given a big model of FPCF, w.r.t. a natural notion of embedding, defined via a logical relation (see Sect. 2.4) an undefinable element x of the model, and a logical relation R_x which proves that x is undefinable, one can:

- remove all elements of the model which are not invariant w.r.t. R_x,
- perform an extensional collapse on the remaining elements,

hence obtaining a new, smaller, model.

Let us consider, as an example, the following hierarchy of monotone functions $g_n : bool^n \to bool$ (generalizing a well-known example of first-order, stable and non sequential function, due to G. Berry)

$$g_n(x_1, ..., x_n) = \mathtt{tt} \Leftrightarrow \exists \boldsymbol{y} \in G_n\ \boldsymbol{x} \geq \boldsymbol{y}$$

where G_n is the set of circular permutations of $(\bot, \underbrace{\mathtt{tt}, ..., \mathtt{tt}}_{n-2}, \mathtt{ff})$

For all $n \geq 3$, g_n is FPCF-undefinable; moreover g_{n+1} is FPCF-definable relatively to g_n, and the converse does not hold.

Now, for all n, we define a logical relation R_n such that g_n (and hence all the g_{n-i}) is not invariant w.r.t R_n, and g_{n+1} (and hence all the g_{n+i}) is invariant w.r.t. R_n. This relation R_n is an instance of "sequentiality relation" (see definition 2), and in Sieber's terminology $R_n = S^{n+1}_{\{1,..,n\}\{1,..,n+1\}}$, *i.e.*, R_n contains, at ground type, the set of $(n+1)$-tuples which are either constant or does contain an occurrence of $\perp$ in one of the first n components.

Performing the two operations described above w.r.t. R_n, yields a model which does not contain g_n and contains g_{n+1}.

Hence, concerning FPCF, we have that:

- The observational equivalence is undecidable.
- The definability problem in the Scott model is undecidable.
- The relative definability problem (in the Scott model) is undecidable, and the poset of degrees of relative definability is infinite.
- There exist infinite hierarchies of standard, order extensional models.

Several authors have investigated restrictions on the syntax of FPCF which make observational equivalence decidable: V. Padovani has shown that this can be achieved by eliminating all non ground constants (the `if-then-else` in FPCF) [12]. Schmidt-Schauss [15] and independently, R. Loader [8], have proved that observational equivalence is decidable also for the "unary" version of PCF (a single ground type o with two constants $\perp$ and $\top$ and a "sequential convergence test" $\wedge : o \to o \to o$).

In a recent paper [5], J. Laird shows that Berry's model of bidomains is universal for UPCF (using the listing algorithm devised by Schmidt-Shauss).

In this paper, we address the following questions:

- Is the poset of degree of relative definability in the Scott model of UPCF trivial? (*i.e.* does it contain just the degree of definable functions, and the one of functions equivalent to the "parallel convergence test" $\vee$?)
- Is the hierarchy of (standard and order extensional) model of UPCF trivial? (*i.e.* does it contain just the Scott and the fully abstract model?)

Our first remark was that a positive answer to the first question implies a positive answer to the second one. Surprisingly enough, it turns out that the poset of degree is non trivial, and the one of models is trivial[1].

The point is that, when applying the "collapsing" technique described above to the Scott model of UPCF in order to eliminate, say, the degree of $\vee$, by picking up an appropriate logical relation (typically, Sieber's $S^3_{\{1,2\},\{1,2,3\}}$) then all the other degrees collapse too, either in the first phase (elimination of non-invariant elements) or in the second one (extensional collapse of the invariant elements).

[1] The fact that the poset of models is trivial has an alternative proof, simpler then ours, due to J. Laird [6], but less general. In fact we are able to apply our result also in order to reason about the hierarchy of models of FPCF (Sect. 4.4).

2 Preliminaries

We introduce the notion of relative definability, the language UPCF and its Scott model, and logical relations, that we use both to compare degrees of definability and models.

2.1 Degrees of Definability

Given an applied calculus L, a model $\mathcal{M}$ of L, and two elements $f \in \mathcal{M}^\tau$ and $g \in \mathcal{M}^\sigma$, we say that *f is smaller than g in the L-definability preorder of $\mathcal{M}$*, $f \preceq_L^{\mathcal{M}} g$, if there exists an L-term $M : \sigma \to \tau$ such that $[\![M]\!]^{\mathcal{M}} g = f$.

A *degree of L-definability in $\mathcal{M}$* is an equivalence class of the equivalence relation associated with the preorder above, and degrees are partially ordered by $\preceq_L^{\mathcal{M}}$. The poset of degrees always has a smallest element, namely the degree of definable elements.

Degrees of PCF-definability in the Scott-continuous model, often called *degrees of parallelism* have been studied for instance in [1], while degrees of PCF-definability in the model of strongly stable function (which could be called *degrees of intensionality*) have been investigated in [2, 10, 11].

Of course, if $\mathcal{M}$ has the definability property w.r.t. L (i.e. if any element of $\mathcal{M}$ is the denotation of some L-term), then the poset of degrees of L-definability in $\mathcal{M}$ is reduced to a singleton.

When the language and the model we refer to are clear from the context, we will omit "L" and "$\mathcal{M}$", in the definition and notations above, and we will speak of "degrees of definability", or even, when the model is order extensional, of "degrees of parallelism". Moreover, we use the same symbols for the constants of L and their denotations in $\mathcal{M}$.

In the rest of this section, we focus on Unary PCF. Nevertheless, all the definitions and results apply to FPCF too, changing appropriately the ground type and its standard interpretation.

2.2 Unary PCF

Unary PCF is an example of applied λ-calculus: its ground constants are $\bot, \top : o$, and the only first order constant is $\wedge : o^2 \to o$,

Let $\mathcal{M}$ be a standard model of UPCF, that is:

- $[\![o]\!] = \{\bot, \top\}$, $[\![\top]\!] = \top$ and $[\![\bot]\!] = \bot$,
- $[\![\sigma \to \tau]\!]$ is a subset of $[\![\tau]\!]^{[\![\sigma]\!]}$.

We can define an extensional order $\leq$ on $\mathcal{M}$:

- At type o, $x \leq_o y$ iff $x = \bot$ or $x = y$,
- At type $\sigma \to \tau$, $f \leq_{\sigma\to\tau} g$ iff $\forall x \in [\![\sigma]\!], fx \leq_\tau gx$.

Definition 1. *$\mathcal{M}$ is a standard order-extensional model if:*

- *$\mathcal{M}$ is a standard model,*
- *All functions are monotonic for the order $\leq$.*

The Scott model $\mathcal{E}$ of UPCF is the standard, order extensional model where $[\![\sigma \to \tau]\!] = \{f : [\![\sigma]\!] \to [\![\tau]\!] \mid f$ is $\leq$-monotonic $\}$, each $[\![\sigma]\!]$ being partially ordered by $\leq$.

2.3 Sequentiality Relations

This definition and the proposition are taken from [14].

Definition 2. *If $A \subseteq B \subseteq \{1, ..., n\}$, $S^n_{A,B}$ is the set of tuples $(x_1, ..., x_n)$ such that either $\exists i \in A, x_i = \bot$ or $\forall i, j \in B, x_i = x_j$.*

A Sieber relation is a logical relation which is an intersection of a number of $S^n_{A,B}$ at base type.

Proposition 3. *The Sieber relations are exactly the relations which contain the constants of UPCF.*

The fundamental lemma of logical relations ensures that the denotation of any UPCF term, in any model, is invariant w.r.t. all Sieber relations.

2.4 Hierarchies of Models

Definition 4. *Given two standard models $\mathcal{M}$ and $\mathcal{N}$, the relation $R_{\mathcal{M},\mathcal{N}}$ is the only logical relation which is the identity at the base type.*

Definition 5. *A logical relation R is functional if*

$$\forall \sigma, \forall f \in [\![\sigma]\!]^{\mathcal{M}}, \forall g, g' \in [\![\sigma]\!]^{\mathcal{N}}, fRg \wedge fRg' \Longrightarrow g = g'$$

If R is functional and $x \in [\![\sigma]\!]^{\mathcal{M}}$, we write $R(x)$ for the only $y \in [\![\sigma]\!]^{\mathcal{N}}$ such that xRy.

A logical relation R is onto if

$$\forall \sigma, \forall g \in [\![\sigma]\!]^{\mathcal{N}}, \exists f \in [\![\sigma]\!]^{\mathcal{M}}, fRg$$

Definition 6. *A model $\mathcal{M}$ is smaller than a model $\mathcal{N}$ if $R_{\mathcal{N},\mathcal{M}}$ is functional and onto. $\mathcal{M}$ and $\mathcal{N}$ are isomorphic if $\mathcal{M}$ is smaller than $\mathcal{N}$ and $\mathcal{N}$ is smaller than $\mathcal{M}$.*

Lemma 7. *Assume R^σ is functional and onto. Let $x, x' \in [\![\sigma]\!]^{\mathcal{E}}$, $y \in [\![\sigma]\!]^{\mathcal{M}}$. If $y = R(x)$ and $y = R(x')$ then $y = R(x \wedge x')$. If $y \leq R(x)$ and $y \leq R(x')$ then $y \leq R(x \wedge x')$.*

Proof. If $\sigma = o$, R is the identity. Assume σ is a functional type, and let a be an argument of x and x'. $R(xa) = yR(a) = R(x'a)$. By hypothesis, $R(xa \wedge x'a) = yR(a)$ so $R((x \wedge x')a) = yR(a)$. Finally, we get $R(x \wedge x') = y$. The proof is the same for $\leq$. □

Lemma 8. *If R is functional and onto, there exists a total monotonic map R^{-1} such that $\forall x \in [\![\alpha]\!]^{\mathcal{M}}, x = R(R^{-1}(x))$.*

Proof. Lemma 7 allows us to define $R^{-1}(y)$ as the smallest x such that $R(x) = y$. R^{-1} is monotonic: if $x \leq y$, $x \leq R(R^{-1}(x))$ and $x \leq R(R^{-1}(y))$. With lemma 7, we get $x \leq R(R^{-1}(x) \wedge R^{-1}(y))$, and the monotonicity of R gives $R(R^{-1}(x)) \geq R(R^{-1}(x) \wedge R^{-1}(y))$. This yields $x = R(R^{-1}(x) \wedge R^{-1}(y))$, so $R^{-1}(x) \leq R^{-1}(x) \wedge R^{-1}(y)$, and finally $R^{-1}(x) \leq R^{-1}(y)$. □

Proposition 9. *If $\mathcal{M}$ and $\mathcal{N}$ are standard order-extensional models of UPCF and $\mathcal{N}$ is fully abstract, $\mathcal{N}$ is smaller than $\mathcal{M}$*

Proof. We write $R = R_{\mathcal{M},\mathcal{N}}$. At ground type, R is a bijection. Assume that R is functional and onto at types σ and τ.

Assume $fR^{\sigma\rightarrow\tau}g$ and $fR^{\sigma\rightarrow\tau}g'$. Let $x \in [\![\sigma]\!]^{\mathcal{N}}$. Since R^{σ} is onto, there exists $y \in [\![\sigma]\!]^{\mathcal{M}}$ such that $R(y) = x$. We get $gx = g(R(y)) = R(fy) = g'(R(y)) = g'x$, which entails $g = g'$: $R^{\sigma\rightarrow\tau}$ is functional.

Let $g \in [\![\sigma \rightarrow \tau]\!]^{\mathcal{N}}$. Since N is fully abstract, there exists a closed term G such that $[\![G]\!]^{\mathcal{N}} = g$. By the fundamental lemma of logical relations

$$[\![G]\!]^{\mathcal{M}} R^{\sigma\rightarrow\tau} [\![G]\!]^{\mathcal{N}}$$

This yields $[\![G]\!]^{\mathcal{M}} R^{\sigma\rightarrow\tau} g$: $R^{\sigma\rightarrow\tau}$ is onto. □

3 Some Degrees in the Big Model of UPCF

We know that the poset of degrees of UPCF-definability in the Scott model has a smallest element $\bot_{deg}$ (the degree of definable elements). A biggest degree $\top_{deg}$ (the degree of the "parallel convergence test" $\underline{\lambda} xy.\ x \vee y$), also exists:

Lemma 10. *All elements of the Scott model of UPCF are definable relatively to $\underline{\lambda} xy.\ x \vee y$.*

Proof. We make an induction on the types.

Let $f : \sigma_1 \rightarrow ... \rightarrow \sigma_n \rightarrow o$. Let $(x_{i1}, ..., x_{in})_{i=1...m}$ be the trace of f (the smallest tuples such that f yields $\top$).

If $x : \sigma_i$, the function $\leq_x$ mapping y to $\top$ if $x \leq y$ and to $\bot$ otherwise is definable: if $(a_{i1}, ..., a_{il})_{i=1...k}$ is the trace of x, and $a_{ij} = [\![A_{ij}]\!]$ (by hypothesis), then $\lambda y.(yA_{11}...A_{1l}) \wedge ... \wedge (yA_{k1}...A_{kl})$ defines $\leq_x$.

One easily checks that f is defined by

$$\lambda y_1...y_n.((\leq_{x_{11}} y_1) \wedge ... \wedge (\leq_{x_{1n}} y_n)) \vee ... \vee ((\leq_{x_{m1}} y_1) \wedge ... \wedge (\leq_{x_{mn}} y_n))$$

□

In the Scott model of Unary PCF, one can easily show that any first order function belongs either to the smallest degree (*i.e.* it is definable), or to the biggest one (*i.e.* it allows to ($UPCF$-)define any other element of the model).

Lemma 11. *If* $\psi : o^n \to o$ *is undefinable, then there exists* $u_1, \ldots, u_n$ *where* $u_i \in \{x, y, \top, \bot\}$, *such that* $[\![\lambda fxy.(fu_1 \ldots u_n)]\!]\,\psi = \vee$.

Proof. By hypothesis, ψ is not a constant function. Suppose there is a unique minimal sequence $\boldsymbol{c} \in \{\top, \bot\}^n$ such that $\psi(\boldsymbol{c}) = \top$. Take $u_i = \top$ if $c_i = \bot$, x_i otherwise. Then $[\![\lambda x_1 \ldots x_n.u_1 \wedge \ldots \wedge u_n]\!] = \psi$, a contradiction.

Let $\boldsymbol{c}^1, \boldsymbol{c}^2 \in \{\top, \bot\}^n$ be minimal distinct sequences such that $\psi(\boldsymbol{c}^1) = \psi(\boldsymbol{c}^2) = \top$. Take $u_i = x$ if $(c_i^1, c_i^2) = (\top, \bot)$, y if $(c_i^1, c_i^2) = (\bot, \top)$, d if $c_i^1 = c_i^2 = [\![d]\!]$. Check that $[\![\lambda fxy.(fu_1 \ldots u_n)]\!]\,\psi = \vee$. □

In other words, all first order types possess the 2-DEG property. We show that 2-DEG is not preserved at higher types, by constructing two intermediate degrees.

Let $\phi \in \mathcal{E}^{(o^3 \to o) \to (o^3 \to o)}$ be the function defined by

$$\phi(f) = \begin{cases} \underline{\lambda} xyz.\ x \vee y \vee z & \text{if } f = \underline{\lambda} xyz.\ x \vee y \\ f & \text{otherwise} \end{cases}$$

Proposition 12. $\bot_{deg} \prec \phi \prec \top_{deg}$.

Proof. First of all, ϕ is monotone since there is no element of $\mathcal{E}^{(o^3 \to o)}$ strictly in between $\underline{\lambda} xyz.\ x \vee y$ and $\underline{\lambda} xyz.\ x \vee y \vee z$.

Concerning $\bot_{deg} \prec \phi$, we are going to show that ϕ is non-invariant w.r.t. a particular $UPCF$-relation: $S^3_{\{1,2\}\{1,2,3\}}$. Let $f = \underline{\lambda} xyz.\ x \wedge y$, $g = \underline{\lambda} xyz.\ x \vee y$, and let us use S as a shorthand for $S^3_{\{1,2\}\{1,2,3\}}$.

First of all, it is easy to see that $(f, g, g) \in S$, since whenever (x_1, x_2, x_3), (y_1, y_2, y_3), $(z_1, z_2, z_3) \in S$ are such that $f\ x_1\ y_1\ z_1 = \top$ and $g\ x_2\ y_2\ z_2 = \top$, then either $x_1 = x_2 = \top$ or $y_1 = y_2 = \top$. In the former case we conclude that $x_3 = \top$, in the latter that $y_3 = \top$; in both cases, $g\ x_3\ y_3\ z_3 = \top$, and hence $(f, g, g) \in S$.

Next we prove that the ϕ-image of (f, g, g) is not in S, and hence that ϕ is not definable.

Let $h = \underline{\lambda} xyz.\ x \vee y \vee z$. The ϕ-image of (f, g, g) is (f, h, h); the following diagram shows that $(f, h, h) \notin S$:

$$\begin{array}{l} f\ h\ h \\ \top\ \bot\ \bot \in S \\ \top\ \bot\ \bot \in S \\ \underline{\bot\ \top\ \bot \in S} \\ \top\ \top\ \bot \notin S \end{array}$$

In order to show that $\phi \prec \top_{deg}$, we show that $\top_{deg} \not\preceq \phi$. We prove by induction on (n, m) that there is no normal η-long term M with n free occurrences of f, m occurrences of $\wedge$, such that $[\![\lambda fxy.M]\!]\,\phi = \vee$:

- $[\![\lambda fxy.x]\!]\,\phi$, $[\![\lambda fxy.y]\!]\,\phi \neq \vee$,
- if $M = M_1 \wedge M_2$, then:
  - either $[\![\lambda fxy.M_1]\!]\,\phi\ \bot\bot = \top$ and $[\![\lambda fxy.M_1]\!]\,\phi = \vee$,

- or $[\![\lambda fxy.M_2]\!]\,\phi\ \bot\bot = \top$ and $[\![\lambda fxy.M_2]\!]\,\phi = \vee$.

– if $M = (f(\lambda z_1 z_2 z_3.w))\, v_1\, v_2\, v_3$ where a $[\![\lambda fxy.v_i]\!]\,\phi$ is undefinable, then the latter equals the only undefinable of type $o \to (o \to o)$, that is, $\vee$.

– if $M = (f(\lambda z_1 z_2 z_3.w))\, N_1\, N_2\, N_3$ where no N_i contains an occurrence of f and $[\![(\lambda fxyz_1z_2z_3.w)]\!]\,\phi$ is undefinable, then by lemma 11 there exists $(t_1, t_2, u_1, u_2, u_3) \in \{x, y, \top, \bot\}^5$ such that:

$$[\![(\lambda fxy.w[t_1/x, t_2/y, u_1/z_1, u_2/z_2, u_3/z_3])]\!]\,\phi = \vee$$

Since all substituted terms are of ground type, the term in the left-hand side is in normal, η-long form, and contains $n-1$ occurrences of f.

– if $M = (f\, P\, N_1\, N_2\, N_3)$ where P, N_1, N_2, N_3 contain no occurrences of f, then for all $(t,u) \in \{\top, \bot\}^2$ and for $P' = P[t/x, u/y]$, $N_i' = N[t/x, u/y]$, we have:

$$\begin{aligned}[\![\lambda fxy.M]\!]\,\phi\,[\![t]\!]\,[\![u]\!] &= \phi\,[\![P']\!]\,[\![N_1']\!]\,[\![N_2']\!]\,[\![N_3']\!] \\ &= [\![P']\!]\,[\![N_1']\!]\,[\![N_2']\!]\,[\![N_3']\!] \\ &= [\![(P'\, N_1'\, N_2'\, N_3')]\!] \\ &= [\![\lambda xy.(P\, N_1\, N_2\, N_3)]\!]\,[\![t]\!]\,[\![u]\!]\end{aligned}$$

In other words $[\![\lambda fxy.M]\!]\,\phi = [\![\lambda fxy.(P\, N_1\, N_2\, N_3)]\!]\,\phi = \vee$ where the normal form of $(P\, N_1\, N_2\, N_3)$ contains no occurrence of f.

□

By using ϕ, we are now able to define a new degree, represented by a function Φ which moreover is S-invariant: let $\Phi \in \mathcal{E}^{((o^3 \to o) \to (o^3 \to o)) \to (o^2 \to o)}$ be the function defined by

$$\Phi(\psi) = \begin{cases} \underline{\lambda} xy.\top & \text{if } \psi > \phi \\ \underline{\lambda} xy.x \vee y & \text{if } \psi = \phi \\ \underline{\lambda} xy.\bot & \text{if } \psi \not\geq \phi \end{cases}$$

Proposition 13. *The degrees of ϕ and Φ are incomparable.*

Proof. First of all, it is easy to see that Φ is actually an element of $\mathcal{E}$, i.e. a monotone function.

Concerning $\Phi \not\preceq \phi$, it is enough to remark that, if $\Phi \preceq \phi$, then $\underline{\lambda} xy.\ x \vee y \preceq \phi$.

In order to show that $\phi \not\preceq \Phi$, we prove that Φ is invariant w.r.t. $S^3_{\{1,2\}\{1,2,3\}} = S$ (and we conclude using the fundamental lemma of logical relations, since ϕ is not S-invariant).

This amounts to showing that whenever

$$\begin{array}{l} \Phi\ \ \Phi\ \ \Phi \\ \psi_1\ \psi_2\ \psi_3 \\ x_1\ x_2\ x_3 \in S \\ \underline{y_1\ y_2\ y_3 \in S} \\ \top\ \ \top\ \ \bot \end{array}$$

one has $(\psi_1, \psi_2, \psi_3) \notin S$, *i.e.* that $\psi_1, \psi_2 \geq \phi$ and $\psi_3 \not> \phi$ entail $(\psi_1, \psi_2, \psi_3) \notin S$.

Now, we decompose $\psi_3 \not> \phi$ in two (not mutually exclusive) cases, and prove $(\psi_1, \psi_2, \psi_3) \notin S$ for both of them:

– case 1: $\psi_3(\underline{\lambda} xyz.\ x \vee y) \leq \underline{\lambda} xyz.\ x \vee y \vee z$.
let $f = \underline{\lambda} xyz.\ x \wedge y$, $g = \underline{\lambda} xyz.\ x \vee y$, as before.

$$\begin{array}{llll}
\psi_1 & \psi_2 & \psi_3 & \\
f & g & g & \in S \\
\top & \bot & \bot & \in S \\
\top & \bot & \bot & \in S \\
\bot & \top & \bot & \in S \\
\hline
\top & \top & \bot & \notin S
\end{array}$$

– case 2: there exist $f_0 \in \mathcal{E}^{o \to o \to o \to o}, x_0, y_0, z_0 \in \mathcal{E}^o$ such that $\psi_3\ f_0\ x_0\ y_0\ z_0 = \bot$ and $f_0\ x_0\ y_0\ z_0 = \top$ (remark that if both (case 1) and (case 2) do not hold, then $\psi_3 > \phi$). Let $f' \in \mathcal{E}^{o \to o \to o \to o}$ be the function defined by

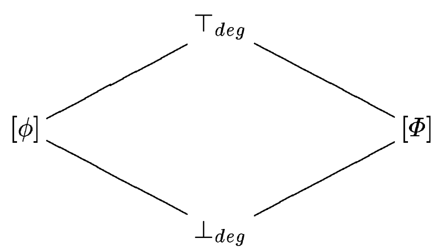
$$f'\ x\ y\ z = \begin{cases} \top \text{ if } x \geq x_0,\ y \geq y_0 \text{ and } z \geq z_0 \\ \bot \text{ otherwise} \end{cases}$$

Remark that f' is definable and $f' \leq f_0$. Showing that $(f', f', f_0) \in S$ is trivial. We can now conclude:

$$\begin{array}{llll}
\psi_1 & \psi_2 & \psi_3 & \\
f' & f' & f_0 & \in S \\
x_0 & x_0 & x_0 & \in S \\
y_0 & y_0 & y_0 & \in S \\
z_0 & z_0 & z_0 & \in S \\
\hline
\top & \top & \bot & \notin S
\end{array}$$

□

We can summarize the results of this section by the following diagram, showing a fragment of the poset of degrees:



4 Standard Order-extensional Models

In this section, we state a theorem about a fragment of the hierarchy of standard order-extensional models: all the models strictly greater than the bidomains model contain a weak version of the parallel or. We apply this result to UPCF, and give some clues about FPCF.

4.1 Bidomains

Definition 14. *We write $x \uparrow y$ if x and y are bounded. A dI-domain $(D, \sqsubseteq, \bot)$ is a Scott domain such that:*

- *Each compact element has finitely many lower bounds,*
- $\forall x, y, z \in D, y \uparrow z \Longrightarrow x \sqcap (y \sqcup z) = (x \sqcap y) \sqcup (x \sqcap z)$.

A stable function between two dI-domains X and Y is a Scott-continuous function f such that

$$\forall x, y \in X, x \uparrow y \Longrightarrow f(x \sqcap y) = f(x) \sqcap f(y)$$

Definition 15. $(D, \leq, \sqsubseteq, \bot)$ *is a bidomain if:*

- $(D, \leq, \bot)$ *is a Scott domain,*
- $(D, \sqsubseteq, \bot)$ *is a dI-domain,*
- *the identity between $(D, \leq, \bot)$ and $(D, \sqsubseteq, \bot)$ is continuous,*
- *if x and y are bounded in $(D, \sqsubseteq, \bot)$ then $x \wedge y = x \sqcap y$.*

Proposition 16. *The category of bidomains is cartesian closed.*

Proof. We define $D \Rightarrow D'$:

- $f \in D \Rightarrow D'$ if f is stable for $\sqsubseteq$ and continuous for $\leq$ and $\sqsubseteq$
- $f \leq g$ iff $\forall x \in D, fx \leq gx$
- $f \sqsubseteq g$ iff $\forall x, y \in D, x \sqsubseteq y \Longrightarrow fx = fy \sqcap gx$

□

Definition 17. *Let $t \in [\![\alpha_1 \to \alpha_2 \to \ldots \to o]\!]^{\mathcal{M}}$. $Tr(t)$ is the set of $((x_1, ..., x_n), y) \in ([\![\alpha_1]\!]^{\mathcal{M}} \times \ldots \times [\![\alpha_n]\!]^{\mathcal{M}}) \times [\![o]\!]^{\mathcal{M}}$ such that:*

- $y \neq \bot$
- $tx_1 ... x_n = y$,
- *For any $x'_i \in [\![\alpha_i]\!]^{\mathcal{M}}$, if $\forall i, x'_i \sqsubseteq x_i$ and $tx'_1 ... x'_n = y$, then $\forall i, x'_i = x_i$.*

Note that $Tr(t) \subseteq Tr(u)$ entails $t \sqsubseteq u$.

4.2 Bidomains and $\vee^-$

$\vee^-$ is a parallel function smaller than the usual parallel or:

Definition 18. *Let $\top \neq \bot$ be an element of the base domain. $\vee^-_\top$ is the function defined by:*

$$\vee^-_\top xy = \begin{cases} \top \text{ if } x = \top \text{ or } y = \top \\ \bot \text{ otherwise} \end{cases}$$

Note that if a model of FPCF contains $\vee^-_{\mathtt{t}}$ or $\vee^-_{\mathtt{f}}$, it contains them both. Thus we speak of $\vee^-$ meaning "any $\vee^-_{\mathtt{x}}$". For UPCF, $\vee^- = \underline{\lambda} xy.x \vee y$.

First, we state a useful lemma:

Lemma 19. *If domains of $\mathcal{M}$ and $\mathcal{B}$ are isomorphic for types smaller or equal to $\alpha \to \beta$, and $f : [\![\alpha]\!]^{\mathcal{M}} \to [\![\beta]\!]^{\mathcal{M}}$, is $\leq$-monotone and $\sqsubseteq$-stable, then $f \in [\![\alpha \to \beta]\!]^{\mathcal{M}}$.*

Proof. We write $R = R_{\mathcal{M},\mathcal{B}}$. If $a, b \in [\![\alpha]\!]^{\mathcal{M}}$, we define $a \sqsubseteq b \iff R(a) \sqsubseteq R(b)$

Let us now assume that f is $\leq$-continuous and $\sqsubseteq$-stable. Since the domains of $\mathcal{B}$ are isomorphic to the domains of $\mathcal{M}$, we can define a function $\hat{f}$ between domains of $\mathcal{B}$. $\hat{f}$ is also $\leq$-continuous and $\sqsubseteq$-stable: $\hat{f}$ is an element of $[\![\alpha \to \beta]\!]^{\mathcal{B}}$.

As $R^{\alpha\to\beta}$ is a bijection, we can use R^{-1}. By definition of $R^{\alpha\to\beta}$,

$$\forall x \in [\![\alpha]\!], (R^{-1}(\hat{f})x, \hat{f}R(x)) \in R^{\beta}$$

By definition of $\hat{f}$,

$$\forall x \in [\![\alpha]\!], (fx, \hat{f}R(x)) \in R^{\beta}$$

Since R^{β} is a bijection, $\forall x \in [\![\alpha]\!], fx = R^{-1}(f)x$. We get $f = R^{-1}(\hat{f})$, which means that $f \in [\![\alpha \to \beta]\!]^{\mathcal{M}}$. □

We can now proceed with the theorem:

Theorem 20. *If a standard order-extensional model is strictly greater than the bidomains model $\mathcal{B}$, then it contains $\vee^{-}$.*

Proof. Let $\mathcal{M}$ be a standard order-extensional model such that $\mathcal{B}$ is strictly smaller than $\mathcal{M}$. Let ω be a smallest type (for the type depth) such that $R_{\mathcal{M},\mathcal{B}}$ does not define an isomorphism between $[\![\omega]\!]^{\mathcal{M}}$ and $[\![\omega]\!]^{\mathcal{B}}$.

Since $\mathcal{B}$ is smaller than $\mathcal{M}$ and $[\![\omega]\!]^{\mathcal{M}} \neq [\![\omega]\!]^{\mathcal{B}}$, there is a $\phi \in [\![\omega]\!]^{\mathcal{M}}$ such that $\nexists \psi \in [\![\omega]\!]^{\mathcal{B}}, \phi R_{\mathcal{M},\mathcal{B}} \psi$. As the argument and result domains are isomorphic, we can write $\phi \notin [\![\sigma \to \tau]\!]^{\mathcal{B}}$. Of course, $\omega \neq o$. Let us write $\omega = \sigma \to \tau$. Since ϕ is continuous for $\leq$, we have that ϕ is not stable for $\sqsubseteq$. Either ϕ is not monotone for $\sqsubseteq$, or there exist f and g bounded in $([\![\sigma]\!], \sqsubseteq)$ such that $\phi(f \wedge g) \neq \phi f \wedge \phi g$. We prove that $\vee^{-}$ is definable in each case.

Let us assume that ϕ is not $\sqsubseteq$-monotone. We choose f and g such that $f \sqsubseteq g$ and $\phi f \not\sqsubseteq \phi g$. If $\tau = o$, we have $\phi f = \top$ and $\phi g = \bot$. But, since $f \sqsubseteq g$ entails $f \leq g$, ϕ is not $\leq$-monotone. Thus, τ is functional: let us write $\tau = \tau_1 \to .. \to \tau_n \to o$.

As $\phi f \not\sqsubseteq \phi g$, $Tr(\phi f) \not\subseteq Tr(\phi g)$. We choose $((y_1...y_n), \top) \in Tr(\phi f) \backslash Tr(\phi g)$ (where $\top$ can be any element in $[\![o]\!]$ except $\bot$) Since ϕ is $\leq$-monotone, $f \leq g$, and $\phi f y_1...y_n = \top$, $\phi g y_1...y_n = \top$. Therefore, there is a $((x_1...x_n), \top) \in Tr(\phi g)$ such that $\forall i, x_i \sqsubseteq y_i$. Since $((y_1, ..., y_n), \top) \notin Tr(\phi g)$, there is an i such that $x_i \sqsubset y_i$, which entails $\phi f x_1...x_n = \bot$. We have chosen f, g, x_i and y_i such that:

- $\phi f x_1...x_n = \bot$,
- $\phi g x_1...x_n = \top$,
- $\phi f y_1...y_n = \top$.

As $f \sqsubseteq g$, and by lemma 19, the function $m : o \to \sigma$ defined by $m\bot = f$ and $m\top = g$ is in $\mathcal{M}$. As $x_i \sqsubseteq y_i$, and by lemma 19, the function $\chi_i : o \to \sigma$ defined by $\chi_i\bot = x_i$ and $\chi_i\top = y_i$ is in $\mathcal{M}$. One easily checks that

$$\underline{\lambda} ab.\phi(\psi a)(\chi_1 b)...(\chi_n b) = \vee_{\top}^{-}$$

Let us now assume that ϕ is $\sqsubseteq$-monotone, and that there exist f, g, h such that $f \sqsubseteq h$, $g \sqsubseteq h$, and $\phi(f \sqcap g) \neq (\phi f) \sqcap (\phi g)$. Since ϕ is $\sqsubseteq$-monotone, $\phi(f \sqcap g) \sqsubseteq (\phi f) \sqcap (\phi g)$, and we get $\phi(f \sqcap g) \sqsubset (\phi f) \sqcap (\phi g)$. This yields:

$$\phi(f \sqcap g) < (\phi f) \sqcap (\phi g)$$

We can choose $x_1...x_n$ (where n can be 0) and $\top \in [\![o]\!]$ ($\top \neq \bot$) such that $\phi(f \sqcap g)x_1...x_n = \bot$ and $((\phi f) \sqcap (\phi g))x_1...x_n = \top$. We can define a stable function ψ by:

- $\psi\bot\bot = f \sqcap g$,
- $\psi\top\bot = f$,
- $\psi\bot\top = g$,
- $\psi\top\top = h$.

Being $\leq$-continuous and $\sqsubseteq$-stable, ψ is in $\mathcal{M}$. One easily checks that

$$\underline{\lambda} ab.\phi(\psi ab)x_1...x_n = \vee_{\top}^{-}$$

□

4.3 Unary PCF

We apply the theorem to the case of UPCF.

Proposition 21. *Any standard order-extensional model of UPCF is smaller than the Scott model $\mathcal{E}$.*

Proof. Let $\mathcal{M}$ be a standard order-extensional model. We write R for $R_{\mathcal{E},\mathcal{M}}$.

At type o, R is a bijection. Assume that R^σ and R^τ are functional and onto, and let us prove that $R^{\sigma\to\tau}$ is also functional and onto. Seeing R as a partial function, we write $R(x)$ for the only y such that xRy. R^{-1} is the function defined in lemma 8 at types σ and τ.

Let $f \in [\![\sigma \to \tau]\!]^{\mathcal{E}}$, $g, g' \in [\![\sigma \to \tau]\!]^{\mathcal{M}}$ such that $fR^{\sigma\to\tau}g$ and $fR^{\sigma\to\tau}g'$. Let $x \in [\![\sigma]\!]^{\mathcal{M}}$. By definition of $R^{\sigma\to\tau}$,

$$gx = g(R(R^{-1}(x))) = R(f(R^{-1}(x)) = g'(R(R^{-1}(x))) = g'x$$

Thus, $R^{\sigma\to\tau}$ is functional.

Let $g \in [\![\sigma \to \tau]\!]^{\mathcal{M}}$. If x has an image by R, we define $\hat{f}(x) = R^{-1}(g(R(x)))$. Since the domains of $\mathcal{E}$ are lattices[2], we can define

$$f(x) = \bigvee_{R^{-1}(y) \leq x} \hat{f}(R^{-1}(y))$$

The monotonicity of R and R^{-1} yields the monotonicity of f. If $x \in [\![\sigma]\!]^{\mathcal{E}}$, $R(fx) = g(R(x))$. As $R^{\sigma} = \{(x, R(x))\}$, $f R^{\sigma \to \tau} g$: $R^{\sigma \to \tau}$ is onto. □

Theorem 22. *The model $\mathcal{B}$ of bidomains of UPCF is fully abstract.*

Proof. See[5] □

The following is an easy consequence of lemma 10:

Lemma 23. *If $\vee^{-}$ is in $\mathcal{M}$, then $\mathcal{M} \approx \mathcal{E}$.*

Corollary 24. *There are only two standard order-extensional models of UPCF: $\mathcal{B}$ and $\mathcal{E}$.*

Proof. Combine these results with theorem 20. □

4.4 Finitary PCF

We show that there are infinitely many degrees above $[\vee^{-}]$ and infinitely many models smaller than $\mathcal{E}$ that contain $\vee^{-}$.

Let us call $\vee^{n}$ the function of type $o^{n} \to o$ defined by:

$$\vee^{n} x_1 \ldots x_n = \begin{cases} t & \text{if } n \text{ of } n-1 \text{ of the } x_i \text{ are equal to } t \\ f & \text{if } x_1 = \ldots = x_n = f \\ \bot & \text{otherwise} \end{cases}$$

We write $x \wedge y = if\ x\ then\ (if\ y\ then\ t\ else\ \bot)\ else\ (if\ y\ then\ \bot\ else\ f)$. One easily checks that $\vee^{n} = \lambda x_1 ... x_n . \vee^{n-1} (x_1 \wedge x_2) x_3 ... x_n$, which entails $\vee^{n} \preceq \vee^{n-1}$. Let us prove that $\vee^{n} \prec \vee^{n-1}$, with the n-ary relation S^n defined by

$$S^n = S^n_{\{1 \ldots n\},\{1 \ldots n\}} \cap \left(\bigcap_{A \subseteq \{1 \ldots n-1\}} S^n_{A,A} \right)$$

As one can check, the tuples $(x_1, ..., x_n)$ in S^n are exactly such that:

- there is no $i, j < n$ such that $x_i = t$ and $x_j = f$
- the tuple is not $tt...tf$ nor $ff...ft$.

First, $\vee^{n}$ is not invariant by S^{n+1} as shown by:

[2] Apart from this, the proof would be valid for FPCF

$$
\begin{array}{cccccc}
\vee^n & \vee^n & \vee^n & \ldots & \vee^n & \vee^n \\
\bot & t & t & \ldots & t & f \\
t & \bot & t & & t & f \\
t & t & \bot & & t & f \\
\vdots & & & \ddots & \vdots & \vdots \\
t & t & t & \ldots & \bot & f \\
\hline
t & t & t & \ldots & t & f
\end{array}
$$

Let us now prove that $\vee^n$ is invariant by S^n. Then we conclude with the fundamental lemma of logical relations. Let $x_{i,j}$ and y_i such that:

$$
\begin{array}{ccc}
\vee^n & \ldots & \vee^n \\
x_{11} & \ldots & x_{n1} \\
\vdots & & \vdots \\
x_{1n} & \ldots & x_{nn} \\
\hline
y_1 & \cdots & y_n
\end{array}
$$

and $(y_1, ..., y_n) \notin S^n$.

- If there exist $i, i' < n$ such that $y_i = t$ and $y_{i'} = f$ then for all j but one, $x_{ij} = t$ and $x_{i'j} = f$, which entails $(x_{1j}...x_{nj}) \notin S^n$.
- If $\forall i < n, y_i = t$ and $y_n = f$, one can check that there exists a j_0 such that $\forall i < n, x_{ij_0} = t$. Since for all j, $x_{nj} = f$, $(x_{1j_0}, ..., x_{nj_0}) \notin S^n$.
- If $\forall i < n, y_i = f$ and $y_n = t$, for all j but one that $\forall i < n, x_{ij} = f$ and $x_{nj} = t$, which entails $(x_{1j}, ..., x_{nj}) \notin S^n$.

Note that for $n \geq 3$, $\vee^- = \underline{\lambda} xy. \vee^n tt \ldots txy$, hence we have defined a sequence of undefinable elements of $\mathcal{E}$:

$$\vee^- \prec ... \prec \vee^4 \prec \vee^3 \prec \vee^2 = \vee$$

These elements being first-order functions, they cannot vanish with an extensional collapse. Thus, we have defined an infinite hierarchy of standard order extensional models of FPCF:

$$\ldots \subset \mathcal{E}^4 \subset \mathcal{E}^3 \subset \mathcal{E}$$

These models are not greater than $\mathcal{B}$, but there might be corresponding models above $\mathcal{B}$.

5 Conclusion

We have shown that the poset of degrees of parallelism in the Scott model of UPCF is non-trivial, and that the poset of extensional models of the language is reduced to the fully abstract and the Scott ones.

Some open questions arise naturally:

– Decidability of the definability (and relative definability) problem in the Scott model of UPCF.
– Existence of infinitely many degrees of relative definability in the Scott models of UPCF.

A broader framework for this work is the study of the three related issues below[3] for a given extensional, finitary applied λ-calculus L, in order to explore the boundary decidable/undecidable with respect to the set of constants of L.

(a) Decidability of the definability (and relative definability) problem in the Scott model.
(b) Existence of a (finitary and "non-syntactic") fully abstract model.
(c) Decidability of the observational equivalence.

For FPCF, we know that (c), and hence (a) and (b), are false [9].
For UPCF, (b) is true [5] and (a) open.
For the simply typed λ-calculus without constants (replacing "Scott model" with "full set-theoretic model") (c) is true [12], (a) false [7, 3], (b) open.
For finitary, parallel PCF, (a) is true [13].

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[3] The second one is stated informally; (c) is weaker than (b) and (b) weaker than (a).

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