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MINIMAL SUBMANIFOLDS WITH A PARALLEL OR A HARMONIC $p$-FORM

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Abstract

The purpose of this paper is to study the relations between the existence of minimal immersions of a Riemannian manifold $M$ into another and some structural or topological properties of $M$. The properties on $M$ which we consider are the existence of a parallel or a harmonic $p$-form.
1 Introduction

The purpose of this paper is to obtain some non existence results about minimal submanifolds. Let $(M^m, g)$ be an $m$-dimensional Riemannian manifold isometrically immersed by $\phi$ in an $n$-dimensional Riemannian manifold $(N^n, h)$ ($n > m$). The Gauss equation allows us to obtain rigidity results in terms of geometry of $(M^m, g)$ and $(N^n, h)$. For example, as a first consequence of the Gauss equation, we get the following well known inequality in each point $x$ of $(M^m, g)$

$$|H(x)|^2 \geq (1/m)((\text{Scal}(x)/m) - (m - 1)\overline{K}^1(\phi(x)))$$  \hspace{1cm} (1)

where $|H(x)|^2$ and $\text{Scal}(x)$ are respectively the square of the mean curvature of $\phi$ and the scalar curvature of $(M^m, g)$ at $x$ and $\overline{K}^1(\phi(x))$ is the largest sectional curvature of $(N^n, h)$ at $\phi(x)$. In particular, if $\overline{K}^1$ has an upper bound and if $\text{Scal} > m(m - 1)\overline{K}^1_{\text{max}}$ (where $\overline{K}^1_{\text{max}} = \max_N(\overline{K}^1)$) for at least a point of $(M^m, g)$, there is no minimal immersion of $(M^m, g)$ into $(N^n, h)$.

Many other results were obtained, by assuming that $(M^m, g)$ is endowed with some particular structures or topological properties (see for instance [1], [12] and [3]). First recall the results of Sampson ([12]) and Dajczer and Rodriguez ([3]). They proved that there is no minimal immersion of an $m$-dimensional Kaehlerian manifold $(m \geq 4)$ into a Riemannian manifold of negative constant sectional curvature. Later, El Soufi ([5]) obtained a generalization of this result by assuming a pinching of the sectional curvature of $(N^n, h)$ and Hernandez ([9]) obtained the same conclusion under the negativity of the complex sectional curvature of $(N^n, h)$. More recently, Petit and El Soufi ([6]) extend this result in the case where $(M^m, g)$ is not necessarily Kaehlerian but has a parallel 2-form and where the isotropic curvature of $(N^n, h)$ is negative (recall that the isotropic curvature of a Riemannian manifold is the restriction of the complex sectional curvature to isotropic tangent planes ([11])).

The section 1 of the present paper deals with some preliminaries. In the section 2, we consider the general case where $(M^m, g)$ has a parallel $p$-form and we prove (theorem 3.1) that if $(N^n, h)$ satisfies a curvature pinching condition, then there is no minimal immersion from $(M^m, g)$ into $(N^n, h)$. This is the generalization of the result of El Soufi stated in [5] for the case where $(M^m, g)$ is Kaehlerian. Note that this theorem as well as the other results recalled above are of interest only if the sectional curvature of $(N^n, h)$ is negative. However, in the theorem 3.2, we obtain the same conclusion with a new pinching condition for the case where $(N^n, h)$ is not necessarily of negative sectional curvature but has a negative smallest sectional curvature. In the theorem 3.3, we study the particular case where $(N^n, h)$ is the complex hyperbolic space $\mathbb{C}H^n(c)$ with constant holomorphic curvature equal to $c$ and we prove that there is no totally real minimal immersion of a Riemannian manifold $(M^m, g)$ with a parallel $p$-form into $\mathbb{C}H^n(c)$. 

3
The compact manifolds with a parallel $p$-form are a particular case of manifolds with a harmonic $p$-form (or a nonzero $p$-th Betti number $b_p(M)$). In the section 3, we prove (theorem 4.1 and theorem 4.2) that for any compact manifold $(M^m, g)$ with $b_p(M) \neq 0$ and isometrically immersed in a Riemannian manifold $(N^n, h)$, there exists at least a point $x$ of $M$ so that

$$m \left( \frac{p-1}{p} \right) |B(x)||H(x)| \geq k(x) - \left( \frac{p-1}{p} \right) ((m-1)\bar{K}^1 + \bar{p}^1)(\phi(x))$$

and

$$m \left( \frac{p-1}{\sqrt{p}} + \frac{m-p-1}{\sqrt{m-p}} \right) |B(x)||H(x)| \geq Scal(x) - (m-2)((m-1)\bar{K}^1 + \bar{p}^1)(\phi(x))$$

where $|B(x)|$, $k(x)$ and $\bar{p}^1(\phi(x))$ denote respectively the norm of the second fundamental form of $\phi$, the smallest eigenvalue of the Ricci curvature of $(M^m, g)$ at $x$ and the largest eigenvalue of the curvature operator of $(N^n, h)$ at $\phi(x)$. El Soufi proved the first inequality for $p = 2$ in [5] and the second for $p = 2$ but only for $m = 4$. The first is optimal for the usual standard minimal embeddings of the Clifford torus and of the complex projective space in the sphere. These inequalities will be a consequence of a new lower bound of the curvature term in the Weitzenböck formula for $p$-forms (see the relation (5) and the propositions 4.1 and 4.2).

As a consequence of the previous inequalities, we deduce (corollary 4.2) that if $(M^m, g)$ is minimally immersed in $(N^n, h)$, if $\bar{p}^1$ is bounded above and if

$$\min_M (Scal) > (m-2) \left( (m-1) \max_N (\bar{K}^1) + \max_N (\bar{p}^1) \right)$$

then $(M^m, g)$ is a sphere of homology. This result can be viewed as a generalization of a theorem of Leung ([10]) which has shown that if a compact Riemannian manifold $(M^m, g)$ is minimally immersed in a unit sphere and if the scalar curvature satisfies $Scal > m(m-2)$ then it is homeomorphic to an $m$-dimensional sphere.

2 Preliminaries and notations

Let $(M^m, g)$ be an $m$-dimensional Riemannian manifold and let $\phi$ be an isometric immersion of $(M^m, g)$ into an $n$-dimensional Riemannian manifold $(N^n, h)$ ($n > m$). The inner product and the norm induced by $g$ and $h$ on the tensors will be denoted respectively by $\langle \, , \, \rangle$ and $\| \|^2$. Moreover, we denote respectively by $R$, $\rho$, $\text{Ric}$ and $Scal$ the curvature tensor, the curvature operator, the Ricci tensor and the scalar curvature of $(M^m, g)$ and by $\overline{R}$, $\overline{K}$, $\overline{\rho}$ and $\overline{Scal}$ the curvature tensor, the sectional curvature, the curvature operator and
the scalar curvature of \((N^n, h)\). We recall that for all vector field \(X, Y, Z, W \in \Gamma(TN)\), \(\overline{p}\) is defined by

\[
\langle \overline{p}(X \wedge Y), Z \wedge W \rangle = R(X, Y, Z, W)
\]

Moreover, for all vector field \(X, Y \in \Gamma(TM)\), the tensor \(\overline{R}_\phi\) will be given by

\[
\overline{R}_\phi(X, Y) = \sum_{i \leq m} \overline{R}(d\phi(X), d\phi(e_i), d\phi(Y), d\phi(e_i))
\]

where \((e_i)_{1 \leq i \leq m}\) is an orthonormal frame on \(M\). On the other hand, for all \(x \in N\), we denote respectively \(K^1(x)\) and \(K^0(x)\) the largest sectional curvature and the smallest sectional curvature at \(x\) and \(\overline{p}^1(x)\) and \(\overline{p}^0(x)\) the largest eigenvalue and the smallest eigenvalue of the curvature operator. Then, it is easy to see that

\[
\overline{p}^0(x) \leq K^0(x) \leq K^1(x) \leq \overline{p}^1(x) \quad (2)
\]

Now, let \(B\) be the second fundamental form of the immersion \(\phi\) and let \(H\) be the mean curvature vector defined by:

\[
H = \frac{1}{m} \text{trace} \, B
\]

The Gauss equation tells us that for any vector field \(X, Y, Z, W \in \Gamma(TM)\), we have

\[
R(X, Y, Z, W) = \overline{R}(X, Y, Z, W) + \langle B(X, Z), B(Y, W) \rangle - \langle B(X, W), B(Y, Z) \rangle \quad (3)
\]

For the sake of completeness, we need now to recall briefly some definitions and properties about \(p\)-forms. Let \((e_i)_{1 \leq i \leq m}\) be a local orthonormal frame. Throughout this paper, for all \(q\)-tensor \(T\), we will write \(T_{i_1 \ldots i_q}\) instead of \(T(e_{i_1}, \ldots, e_{i_q})\) and then the inner product of two \(p\)-forms \(\omega\) and \(\theta\) of \((M^m, g)\) will be

\[
\langle \omega, \theta \rangle = \frac{1}{p!} \sum_{1 \leq i_1 \ldots i_p \leq m} \omega_{i_1 \ldots i_p} \theta_{i_1 \ldots i_p}
\]

The inner product (or contraction) \(i(X)\omega\) of a \(p\)-form \(\omega\) with a vector field \(X\) on \(M\) is a \(p-1\)-form, defined by

\[
(i(X))\omega(X_1, \ldots, X_{p-1}) = \omega(X, X_1, \ldots, X_{p-1}), \quad \forall X_1, \ldots, X_{p-1} \in \Gamma(TM)
\]

More generally, if \(X_1, \ldots, X_q \in \Gamma(TM)\), then the inner product of the \(p\)-form \(\omega\) with the \(q\)-tensor \(X_1 \wedge \ldots \wedge X_q\) is the \(p-q\)-form defined by

\[
(i(X_1 \wedge \ldots \wedge X_q))\omega(Y_1, \ldots, Y_{p-q}) = \omega(X_1, \ldots, X_q, Y_1, \ldots, Y_{p-q}), \quad \forall Y_1, \ldots, Y_{p-q} \in \Gamma(TM)
\]

Recall some elementary facts about inner and exterior products. Let \(\omega\) and \(\theta\) be respectively a \(p\)-form and a \(q\)-form and let \(X\) be a vector field on \(M\), then
\[ i(X)(\omega \wedge \theta) = i(X)\omega \wedge \theta + (-1)^p \omega \wedge i(X)\theta \]

and if \( X^* \) is the dual 1-form of the vector field \( X \) with respect to \( g \), then \( i(X) \) is in fact the adjoint of left exterior multiplication by \( X^* \), that is

\[ \langle i(X)(\omega), \theta \rangle = \langle \omega, X^* \wedge \theta \rangle \]

If \( M \) is orientable, we also need the following relation between the inner product and the Hodge operator \( \star \) on \((M^m, g)\) (see for instance \([4]\))

\[ i(X)(\star \omega) = (-1)^p \star (X^* \wedge \omega) \]

On the other hand, if \( \alpha \) is a 1-form which is real valued and \( \beta \) is a 1-form which is valued in a vector bundle, we define the 2-tensor \( \alpha \vee \beta \) by

\[ (\alpha \vee \beta)(X, Y) = \alpha(X)\beta(Y) + \alpha(Y)\beta(X) \quad (4) \]

We denote now by \( d, d^*, \nabla \) and \( \nabla^* \) respectively the exterior differential and the codifferential acting on \( p \)-forms, the covariant derivative of \((M^m, g)\) extended to \( p \)-forms and its adjoint with respect to \( g \). The Hodge-de Rham Laplacian \( \Delta \) acting on \( p \)-forms is given by

\[ \Delta \omega = dd^* \omega + d^* d\omega \]

To compare this Laplacian to the “rough” Laplacian \( \nabla^* \nabla \), one has the Weitzenböck formula, reading as

\[ \Delta \omega = \nabla^* \nabla \omega + R_p(\omega), \quad \forall \omega \in \Lambda^p(M) \]

Here \( R_p \in End(\Lambda^p(M)) \) is a bundle endomorphism, given by

\[ R_p(\omega)(X_1, ..., X_p) = \sum_{ij} (-1)^i [R(e_j, X_i)\omega](e_j, X_1, ..., \widehat{X_i}, ..., X_p), \quad \forall X_1, ..., X_p \in \Gamma(TM) \]

where \((e_i)_{1 \leq i \leq m}\) is a local orthonormal frame and

\[ R(X, Y)\omega = \nabla_{[X,Y]}\omega - [\nabla_X, \nabla_Y]\omega \quad \forall X, Y \in \Gamma(TM) \]

An easy consequence of the Weitzenböck formula is the following

\[ \frac{1}{2} \Delta |\omega|^2 = \langle \Delta \omega, \omega \rangle - |\nabla \omega|^2 - \langle R_p(\omega), \omega \rangle \quad (5) \]
In the sequel, we need to explicit the expression of $\mathcal{R}_p$. A straightforward calculation gives us

$$
\langle \mathcal{R}_p(\omega), \omega \rangle = \sum_{i,j} \text{Ric}_{ij} \langle i(e_i)\omega, i(e_j)\omega \rangle - \frac{1}{2} \sum_{i,j,k,l} R_{ijkl} \langle i(e_j \wedge e_i)\omega, i(e_l \wedge e_k)\omega \rangle
$$

(6)

and the last term is zero when $p = 1$.

3 Geometry of submanifolds having a parallel $p$-form

The first result of this section is the following theorem

**Theorem 3.1** Let $(M^m, g)$ be an $m$-dimensional Riemannian manifold admitting a non-trivial parallel $p$-form ($1 \leq p \leq m$) and let $(N^n, h)$ be an $n$-dimensional Riemannian manifold ($n > m$). If for any $x \in (N^n, h)$, we have

$$(m - 1)\overline{K}^1(x) < (p - 1)\overline{p}^0(x)$$

(7)

then, there is no minimal immersion from $(M^m, g)$ into $(N^n, h)$.

**Remark 3.1:**

1. For $p = 2$ and for even dimensional manifold $(M^m, g)$, the pinching condition (7) can be reformulate as the negativity of the isotropic curvature ([6]). If $(M^m, g)$ is Kaehlerian, this condition (7) is nothing but that obtained by El Soufi in [5] (theorem 2.2).

2. From the relation (2), we see that this theorem is of interest only if the sectional curvature of $(N^n, h)$ is negative. For the hyperbolic space $\mathbb{H}^n$, the condition (7) is always satisfied for $p < m$ and then there is no minimal immersion of a manifold having a parallel $p$-form ($1 \leq p \leq m - 1$) into $\mathbb{H}^n$. However, the embeddings of $\mathbb{H}^m$ in $\mathbb{H}^n$ ($m < n$) are totally geodesic, and taking the volume form of $\mathbb{H}^m$, we see that (7) is not satisfied for $p = m$.

In the following theorem, we obtain the same conclusion as in the theorem 3.1 with a new pinching condition where $(N^n, h)$ is not necessarily of negative sectional curvature (in fact, there is no condition on $\overline{K}^1$).

**Theorem 3.2** Let $(M^m, g)$ be an $m$-dimensional Riemannian manifold admitting a non-trivial parallel $p$-form ($1 \leq p \leq m$) and let $(N^n, h)$ be an $n$-dimensional Riemannian manifold ($n > m$). If for any $x \in (N^n, h)$, we have
\[ \overline{\text{Scal}}(x) < (n - m)(n + m - 1)\overline{K}^0(x) + (p(p - 1) + (m - p)(m - p - 1))\overline{\rho}^0(x) \quad (8) \]

then, there is no minimal immersion from \((M^m, g)\) into \((N^n, h)\).

**Remark 3.2:** We will see in the proof that this theorem is of interest only if the smallest sectional curvature is negative that is \(\overline{K}^0(x) < 0\) for all \(x \in N\). For instance, let us consider the space \(N^n = \mathbb{R}^r \times S^s\) where \(n = r + s\). Then \(\overline{\text{Scal}} = -r(r - 1) + s(s - 1), \overline{\rho}^0 = \overline{K}^0 = 1\) and \(\overline{\rho}^0 = \overline{K}^0 = -1\). Now, let \((M^m, g)\) be a Riemannian manifold of even dimension \(m\) and let \(p = m/2\). Then we have \(\overline{\text{Scal}} - (n - m)(n + m - 1)(\overline{K}^0 - (p(p - 1) + (m - p)(m - p - 1))\overline{\rho}^0 = 2(s^2 + rs - s - m^2/4)\) and it is easy to see that if \(r\) and \(m\) are great enough (for instance for a fixed \(s\) put \(m = r\) great enough) then the condition \((8)\) is satisfied and the conclusion of the theorem 3.2 holds for this example.

**Proof of Theorem 3.1:** Let \(\phi\) be a minimal immersion of \((M^m, g)\) into \((N^n, h)\) and assume that \(M\) has a nontrivial parallel \(p\)-form \(\omega\). Then \(\langle R^g(\omega), \omega \rangle = 0\) and from (6) we deduce

\[ 0 = \sum_{i,j} Ric_{ij} \langle i(e_i), \omega, i(e_j) \rangle - \frac{1}{2} \sum_{i,j,k,l} R_{ijkl} \langle i(e_i) \wedge i(e_j), \omega, i(e_k) \wedge \omega \rangle \quad (9) \]

and the last term is zero for \(p = 1\). Now from the Gauss formula (3), we obtain

\[ \sum_{i,j} Ric_{ij} \langle i(e_i), \omega, i(e_j) \rangle = \sum_{i,j} (\overline{R}_{ij}) \langle i(e_i), \omega, i(e_j) \rangle + \sum_{i,j} \langle B_{ij}, H \rangle \langle i(e_i), \omega, i(e_j) \rangle - \sum_{i,j,k} \langle B_{ik}, B_{jk} \rangle \langle i(e_i), \omega, i(e_j) \rangle \quad (10) \]

and

\[ \sum_{i,j,k,l} R_{ijkl} \langle i(e_i) \wedge i(e_j), \omega, i(e_k) \wedge \omega \rangle = \sum_{i,j,k,l} \overline{R}_{ijkl} \langle i(e_i) \wedge i(e_j), \omega, i(e_k) \wedge \omega \rangle 
+ \sum_{i,j,k,l} \langle B_{ik}, B_{jl} \rangle \langle i(e_i) \wedge i(e_j), \omega, i(e_k) \wedge \omega \rangle - \sum_{i,j,k,l} \langle B_{il}, B_{jk} \rangle \langle i(e_i) \wedge i(e_j), \omega, i(e_k) \wedge \omega \rangle 
= \sum_{i,j,k,l} \overline{R}_{ijkl} \langle i(e_i) \wedge i(e_j), \omega, i(e_k) \wedge \omega \rangle + 2 \sum_{i,j,k,l} \langle B_{ik}, B_{jl} \rangle \langle i(e_i) \wedge i(e_j), \omega, i(e_k) \wedge \omega \rangle \quad (11) \]

by reporting (10) and (11) in (9), we get
\[ 0 = \sum_{i,j} (R_{\alpha})_{ij} \langle i(e_i)\omega, i(e_j)\omega \rangle + m \sum_{i,j} \langle B_{ij}, H \rangle \langle i(e_i)\omega, i(e_j)\omega \rangle \\
- \sum_{i,j,k} \langle B_{ik}, B_{jk} \rangle \langle i(e_i)\omega, i(e_j)\omega \rangle - \frac{1}{2} \sum_{i,j,k,l} R_{ijkl} \langle i(e_j \wedge e_i)\omega, i(e_l \wedge e_k)\omega \rangle \\
- \sum_{i,j,k,l} \langle B_{ik}, B_{jl} \rangle \langle i(e_i \wedge e_j)\omega, i(e_l \wedge e_k)\omega \rangle \quad (12) \]

and the two last terms are zero for \( p = 1 \). Now, put \( B^+(\omega) = \sum_{i \leq m} i(e_i^*)\omega \wedge B(e_i, \cdot) \). The computation of the square of the norm of \( B^+(\omega) \) gives

\[
|B^+(\omega)|^2 = \frac{1}{p!} \sum_{1 \leq i_1 \ldots i_p \leq m} \langle (i(e_i^*)\omega \wedge B(e_i, \cdot))_{i_1 \ldots i_p}, (i(e_i^*)\omega \wedge B(e_j, \cdot))_{i_1 \ldots i_p} \rangle \\
= \frac{1}{p!} \sum_{1 \leq i_1 \ldots i_p \leq m} (-1)^{s+t} \langle B_{ii}, B_{jj} \rangle \omega_{i_1 \ldots i_p \wedge \cdot \wedge \cdot} \omega_{j_1 \ldots j_p \wedge \cdot \wedge \cdot}
\]

where the indices with \( \hat{\cdot} \) are omitted. Then

\[
|B^+(\omega)|^2 \\
= \frac{1}{p!} \sum_{1 \leq i_1 \ldots i_p \leq m} \langle B_{ii}, B_{jj} \rangle \omega_{i_1 \ldots i_p \wedge \cdot \wedge \cdot} \omega_{j_1 \ldots j_p \wedge \cdot \wedge \cdot} \\
+ \frac{1}{p!} \sum_{1 \leq i_1 \ldots i_p \leq m} (-1)^{s+t} \langle B_{ii}, B_{jj} \rangle \omega_{i_1 \ldots i_p \wedge \cdot \wedge \cdot} \omega_{j_1 \ldots j_p \wedge \cdot \wedge \cdot} \\
= \frac{1}{(p-1)!} \sum_{1 \leq i_1 \ldots i_{p-1} \leq m} \langle B_{ik}, B_{jk} \rangle \omega_{i_1 \ldots i_{p-1} \wedge \cdot \wedge \cdot} \omega_{j_1 \ldots j_{p-1} \wedge \cdot \wedge \cdot} \\
- \frac{1}{p!} \sum_{1 \leq i_1 \ldots i_p \leq m} \langle B_{ii}, B_{jj} \rangle \omega_{i_1 \ldots i_p \wedge \cdot \wedge \cdot} \omega_{j_1 \ldots j_p \wedge \cdot \wedge \cdot} \\
- \frac{1}{p!} \sum_{1 \leq i_1 \ldots i_p \leq m} \langle B_{ii}, B_{jj} \rangle \omega_{i_1 \ldots i_p \wedge \cdot \wedge \cdot} \omega_{j_1 \ldots j_p \wedge \cdot \wedge \cdot}
\]
\[ \sum_{i,j,k} \langle B_{ik}, B_{jk} \rangle \langle i(e_i) \omega, i(e_j) \omega \rangle - \frac{1}{(p-2)!} \sum_{1 \leq i_1 \cdots i_{p-2} \leq m} \langle B_{ik}, B_{jl} \rangle \omega_{i_1 \cdots i_{p-2}} \omega_{jki_1 \cdots i_{p-2}} \]

\[ \sum_{i,j,k} \langle B_{ik}, B_{jk} \rangle \langle i(e_i) \omega, i(e_j) \omega \rangle + \frac{1}{(p-2)!} \sum_{1 \leq i_1 \cdots i_{p-2} \leq m} \langle B_{ik}, B_{jl} \rangle \omega_{i_1 \cdots i_{p-2}} \omega_{jki_1 \cdots i_{p-2}} \]

\[ \sum_{i,j,k} \langle B_{ik}, B_{jk} \rangle \langle i(e_i) \omega, i(e_j) \omega \rangle + \sum_{i,j,k,l} \langle B_{ik}, B_{jl} \rangle \langle i(e_j \wedge e_i) \omega, i(e_l \wedge e_k) \omega \rangle \]

Note that if \( M \) is Kaehlerian and \( \omega \) is the Kaehler form then \( |B^+(\omega)|^2 = |B^+|^2 \), where \( B^+ \) is the holomorphic part of \( B \) (i.e. \( B^+(X,Y) = \frac{1}{2} (B(X,Y) + B(JX,JY)) \) where \( \omega(X,Y) = \langle JX, Y \rangle \)).

Now, combining the above relation with (12), we obtain

\[ |B^+(\omega)|^2 = m \sum_{i,j} \langle B_{ij}, H \rangle \langle i(e_i) \omega, i(e_j) \omega \rangle + \sum_{i,j} (\bar{R}_\phi)_{ij} \langle i(e_i) \omega, i(e_j) \omega \rangle - \frac{1}{2} \sum_{i,j,k,l} \bar{R}_{ijkl} \langle i(e_j \wedge e_i) \omega, i(e_l \wedge e_k) \omega \rangle \] (13)

where the last term is zero if \( p = 1 \). Putting \( X^{i_1 \cdots i_p} = \sum_{i \leq m} \omega_{i_1 \cdots i_{p-2}} e_i e_i \) and \( \theta^{i_1 \cdots i_{p-2}} = \frac{1}{2} \sum_{1 \leq i,j \leq m} \omega_{i_1 \cdots i_{p-2}} e_i e_j^* \), we have

\[ \sum_{i,j} (\bar{R}_\phi)_{ij} \langle i(e_i) \omega, i(e_j) \omega \rangle = \frac{1}{(p-1)!} \sum_{i_1,\ldots,i_{p-1}} \bar{R}_\phi(X^{i_1 \cdots i_p}, X^{i_1 \cdots i_p}) \]

\[ \leq \frac{(m-1)}{(p-1)!} K^1(\phi(x)) \sum_{i_1,\ldots,i_{p-1}} |X^{i_1 \cdots i_p}|^2 = p(m-1)K^1(\phi(x))|\omega|^2 \] (14)

and

\[ \frac{1}{2} \sum_{i,j,k,l} \bar{R}_{ijkl} \langle i(e_j \wedge e_i) \omega, i(e_l \wedge e_k) \omega \rangle = \frac{2}{(p-2)!} \sum_{i_1,\ldots,i_{p-2}} \bar{p}(\theta^{i_1 \cdots i_{p-2}}, \theta^{i_1 \cdots i_{p-2}}) \]

\[ \geq \frac{2}{(p-2)!} \bar{p}(\phi(x)) \sum_{i_1,\ldots,i_{p-2}} |\theta^{i_1 \cdots i_{p-2}}|^2 \]

\[ = \frac{1}{(p-2)!} \bar{p}(\phi(x)) \sum_{i_1,\ldots,i_p} \omega^{i_1 \cdots i_p} = p(p-1) \bar{p}(\phi(x))|\omega|^2 \] (15)
consequently, for all $p \in \{1, ..., m\}$, it follows from (13) and the fact that $H = 0$

$$| \mathcal{B}^+(\omega)|^2(x) \leq p(m - 1) \mathcal{K}_1^p(\phi(x))|\omega|^2 - p(p - 1) \mathcal{P}_1^p(\phi(x))|\omega|^2$$

From this we conclude that if $(p - 1) \mathcal{P}_1^p(x) > (m - 1) \mathcal{K}_1^p(x)$ holds for any $x \in N$, then there is no minimal immersion from $(M^m, g)$ into $(N^n, h)$. \qed

**Remark 3.3:** In the theorem 3.1, the nonpositivity of the curvature operator of $(N^n, h)$ is not required. In ([2]) Corlette proved (theorem 3.1) a similar result for harmonic maps $\phi$ from $(M^m, g)$ to $(N^n, h)$ (in fact he assumes that $\phi$ is a twisted harmonic map which is not necessary) by assuming only the nonpositivity of the curvature operator of $(N^n, h)$ without pinching condition. Indeed, under this hypothesis he proved that if $(M^m, g)$ is compact and has a parallel $p$-form then $\nabla^*(d\phi \wedge \omega) = 0$ (here $\nabla$ is the pullback of the Levi-Civita connection of $TN$). Note that Corlette doesn’t obtain a result of non existence. Moreover if $\phi$ is a minimal isometric immersion, then $\phi$ is a harmonic map and the theorem of Corlette can be proved without the compacity of $M$ and a short computation shows that $\nabla^*(d\phi \wedge \omega) = -\mathcal{B}^+(\omega)$.

**Proof of Theorem 3.2:** From the relation (13) and the inequality (15) of the previous proof, we deduce that if $\phi$ is a minimal immersion, we have

$$| \mathcal{B}^+(\omega)|^2 \leq \sum_{i,j} (\mathcal{R}_\phi)_{ij} \langle i(e_i)\omega, i(e_j)\omega \rangle - p(p - 1) \mathcal{P}_1^p(\phi(x))|\omega|^2$$

Since $M$ is locally oriented, we can define locally the Hodge operator $\ast$ and the $m - p$-form $\ast \omega$. But $| \mathcal{B}^+(\ast \omega)|^2$ is independent on the choice of the orientability and is consequently globally defined. Then we have

$$| \mathcal{B}^+(\omega)|^2 + | \mathcal{B}^+(\ast \omega)|^2 \leq \sum_{i,j} (\mathcal{R}_\phi)_{ij} \left( \langle i(e_i)\omega, i(e_j)\omega \rangle + \langle i(e_i) \ast \omega, i(e_j) \ast \omega \rangle \right)$$

$$- (p(p - 1) + (m - p)(m - p - 1)) \mathcal{P}_1^p(\phi(x))|\omega|^2 \quad (16)$$

Now using the properties about inner and exterior products recalled in the first section, we have

$$\sum_{i,j} \left( (\mathcal{R}_\phi)_{ij} \langle i(e_i)\omega, i(e_j)\omega \rangle + (\mathcal{R}_\phi)_{ij} \langle i(e_i) \ast \omega, i(e_j) \ast \omega \rangle \right) =$$

$$\sum_{i,j} \left( (\mathcal{R}_\phi)_{ij} \langle i(e_i)\omega, i(e_j)\omega \rangle + (\mathcal{R}_\phi)_{ij} \langle e_i^* \wedge \omega, e_j^* \wedge \omega \rangle \right) =$$

$$\sum_{i,j} \left( (\mathcal{R}_\phi)_{ij} \langle i(e_i)\omega, i(e_j)\omega \rangle + (\mathcal{R}_\phi)_{ij} \langle i(e_j)(e_i^* \wedge \omega), \omega \rangle \right) =$$
\[\sum_{i,j} (R_{\phi})_{ij} (i(e_i) \omega, i(e_j) \omega) + \sum_i (R_{\phi})_{ii} |\omega|^2 - \sum_{i,j} (R_{\phi})_{ij} (e_i^* \wedge i(e_j) \omega, \omega) = \sum_i (R_{\phi})_{ii} |\omega|^2\]  

(17)

A straightforward computation gives

\[\sum_i (R_{\phi})_{ii} \leq \overline{\text{Scal}}(\phi(x)) - (n - m) (n + m - 1) K^0(\phi(x))\]

and from (16) and (17) we deduce that for all \(x \in M\)

\[0 \leq \overline{\text{Scal}}(\phi(x)) - \left((n - m) (n + m - 1) K^0(x) + (p(p - 1) + (m - p)(m - p - 1)) \overline{\rho}^0(x)\right) (\phi(x))\]

Consequently if

\[\overline{\text{Scal}}(x) < (n - m) (n + m - 1) K^0(x) + (p(p - 1) + (m - p)(m - p - 1)) \overline{\rho}^0(x)\]

for all \(x \in N\), there is no minimal immersion from \((M^m, g)\) into \((N^n, h)\). Using the inequalities (2), we can easily see that the above condition implies

\[\left(\frac{m(m - 1)}{p(p - 1) + (m - p)(m - p - 1)}\right) K^0(x) < \overline{\rho}^0(x) \leq K^0(x)\]

for all \(x \in N\). And the theorem 3.2 is of interest only for \(K^0 < 0\).

To finish this section, we study the particular case where the ambient space is the complex hyperbolic space \(\mathbb{CH}^n(c)\) with constant holomorphic curvature equal to \(c (c < 0)\). Let us recall that in this case the curvature tensor of \(\mathbb{CH}^n(c)\) has the expression

\[
\mathcal{R}(X, Y, Z, W) = \frac{c}{4} (\langle X \wedge Y, Z \wedge W \rangle + \langle X \wedge Y, JZ \wedge JW \rangle + 2 \langle X, JY \rangle \langle Z, JW \rangle) \quad (18)
\]

for all \(X, Y, Z, W \in \Gamma(T\mathbb{CH}^n(c))\). Here \(J\) denotes the complex structure of \(\mathbb{CH}^n(c)\). For any isometric immersion \(\phi\) of a Riemannian manifold \((M^m, g)\) into \(\mathbb{CH}^n(c)\), we define the \((1,1)\)-tensor \(J_\phi\) on \(M\) by \(J_\phi X = \sum_{i \leq m} (Jd\phi(X), d\phi(e_i)) e_i, \forall X \in \Gamma(TM)\) and for all orthonormal frame \((e_i)_{1 \leq i \leq m}\). Recall that the immersion \(\phi\) is said to be totally real if \(J_\phi \equiv 0\).

For this kind of immersions we have the

**Theorem 3.3** Let \((M^m, g)\) be an \(m\)-dimensional Riemannian manifold admitting a non-trivial parallel \(p\)-form \((p \geq 1)\). Then there is no minimal totally real immersion of \((M^m, g)\) into \(\mathbb{CH}^n(c)\).
Proof: Let $\phi$ be a minimal immersion of $(M^m, g)$ into $\mathbb{H}^n$ and assume that $M$ has a non trivial parallel $p$-form $\omega$. Then from (13) we have

$$|B^+(\omega)|^2 = \sum_{i,j} (R_{\phi})_{ij} \langle i(e_i)\omega, i(e_j)\omega \rangle - \frac{1}{2} \sum_{i,j,k,l} R_{ijkl} \langle i(e_j \wedge e_i)\omega, i(e_l \wedge e_k)\omega \rangle$$

Now the conclusion follows from a straightforward computation. Using (18) the above equality becomes

$$|B^+(\omega)|^2 = \frac{c}{4} (p(m-p)|\omega|^2 + 3 \sum_{i,j \leq m} \langle J_{\phi}e_i, J_{\phi}e_j \rangle \langle i(e_i)\omega, i(e_j)\omega \rangle - \sum_{i,j \leq m} \langle i(e_j \wedge e_i)\omega, i(e_j \wedge J_{\phi}e_i)\omega \rangle - \sum_{i,j \leq m} \langle i(e_i \wedge J_{\phi}e_i)\omega, i(e_j \wedge J_{\phi}e_j)\omega \rangle$$

where $(e_i)_{1 \leq i \leq m}$ is a local orthonormal frame.

4 Geometry of submanifolds with $b_p(M) \neq 0$

Let $(M^m, g)$ and $(N^n, h)$ be two Riemannian manifolds and assume that $(M^m, g)$ is compact. We use the same notations as in the previous sections. Moreover in this section, $k(x)$ denotes the smallest eigenvalue at $x$ of the Ricci curvature of $(M^m, g)$ and we put $k_0 = \min_M k(x)$. On the other hand, if $\overline{K}^1$ is bounded above, we will set $\overline{K}^1_{\text{max}} = \max_N (\overline{K}^1)$ and $\overline{p}^1_{\text{max}} = \max_N (\overline{p}^1)$. The first result is the following theorem.

**Theorem 4.1** Let $(M^m, g)$ be a compact Riemannian manifold of dimension $m \geq 2$ so that $b_p(M) \neq 0$ for some $p \geq 1$. Then for any isometric immersion $\phi$ from $(M^m, g)$ into an $n$-dimensional Riemannian manifold $(N^n, h)$, there exists at least a point $x \in M$ so that

$$\frac{m}{\sqrt{p}} \left( \frac{p-1}{p} \right) |B(x)||H(x)| \geq k(x) - \left( \frac{p-1}{p} \right) ((m-1)\overline{K}^1 + \overline{p}^1)(\phi(x))$$

The following corollary is an immediate consequence of this theorem.

**Corollary 4.1** Let $(M^m, g)$ be a compact Riemannian manifold of dimension $m \geq 2$ minimally immersed in an $n$-dimensional Riemannian manifold $(N^n, h)$ ($n > m$). If $\overline{p}^1$ is bounded above and if for an integer $p$ so that $1 \leq p \leq m/2$, we have
\[ k_0 > \left( \frac{p - 1}{p} \right) \left( (m - 1)K_\text{max}^1 + \rho_\text{max}^1 \right) \]

then \( b_q(M) = 0 \) for \( q \in \{1, \ldots, p\} \).

**Remark 4.1:** The inequality (19) is an equality at each point for the standard embedding of \( \mathbb{S}^p (\sqrt{p/m}) \times \mathbb{S}^{k_1} (\sqrt{k_1/m}) \times \cdots \times \mathbb{S}^{k_q} (\sqrt{k_q/m}) \) into \( \mathbb{S}^{m+r} \) where \( p + k_1 + \cdots + k_q = m \) with \( k_i \geq p \) (1 \( \leq i \leq q \)). For the case \( p = 2 \), (19) is also an equality at each point for the standard embedding from \( \mathbb{C}P^q \) with holomorphic curvature \( 2q/(q + 1) \) into \( \mathbb{S}^{q^2 + 2q} \).

This theorem 4.1 is a generalization of a result obtained by El Soufi (see theorem 3.1 of [5]) in the particular case \( p = 2 \).

To prove the theorem 4.1 we need the following proposition which gives an estimate of the term \( \langle R_p(\omega), \omega \rangle \) for any \( p \)-form \( \omega \).

**Proposition 4.1** Let \( (M^m, g) \) be an \( m \)-dimensional compact Riemannian manifold isometrically immersed in an \( n \)-dimensional Riemannian manifold \( (N^n, h) \). Then for any \( p \in \{1, \ldots, m\} \) and for any \( p \)-form \( \omega \) of \( M \), we have for all \( x \in M \)

\[ \langle R_p(\omega), \omega \rangle \geq p \left( pk(x) - (p - 1) \left( (m - 1)K_1^1 + \rho_1^1 \right)(\phi(x)) - m \left( \frac{p - 1}{\sqrt{p}} \right) |H(x)| |B(x)| \right) |\omega|^2 \]

Before proving this proposition, we introduce the following \( p \)-tensor associated to any \( p \)-form \( \omega \) of \( M \) and the isometric immersion \( \phi \)

\[ B^-(\omega) = \frac{1}{(p - 2)!} \sum_{i,i_1,\ldots,i_{p-2} \leq m} \left( i(e_i \wedge e_{i_1} \wedge \cdots e_{i_{p-2}}) \omega \right) \land B(e_{i_1} \ldots) \otimes (e_{i_1}^* \wedge \cdots e_{i_{p-2}}^*) \]

where \( (e_i)_{1 \leq i \leq m} \) is an orthonormal frame at a point \( x \in M \) and \( \land \) denotes the symmetric product defined in the preliminaries (see (4)). It will be convenient to choose the norm \( |B^-(\omega)| \) so that

\[ |B^-(\omega)|^2 = \frac{1}{(p - 2)!} \sum_{jki_1\ldots i_{p-2} \leq m} \left| B^-(\omega)_{jki_1\ldots i_{p-2}} \right|^2 \]

Such a tensor has been introduced for the first time in [8] where the second fundamental form is replaced by the Hessian of a function. First note that if \( \omega \) is a volume form at a point \( p \) of \( M \) where we have define the Hodge operator \( \star \) so that \( \omega = \star 1 \), we deduce from (20) shown in the proof of the proposition 4.1, that
\[
\frac{1}{2n}|B^-(v_g)|^2 = \frac{n-1}{n} \sum_{ijk\leq n} \langle B_{ik}, B_{jk} \rangle \langle i(e_i) \times 1, i(e_j) \times 1 \rangle
\]

\[
- \frac{1}{n} \sum_{ijkl\leq n} \langle B_{ij}, B_{kl} \rangle \langle i(e_i \wedge e_i) \times 1, i(e_j \wedge e_j) \times 1 \rangle
\]

\[
= \frac{n-1}{n} \sum_{ijk\leq n} \langle B_{ik}, B_{jk} \rangle \langle e_i^* \times e_j^* , e_k^* \times e_j^* \rangle - \frac{1}{n} \sum_{ijkl\leq n} \langle B_{ij}, B_{kl} \rangle \langle e_i^* \wedge e_j^* , e_k^* \wedge e_j^* \rangle
\]

\[
= |B|^2 - n|H|^2 = |B - H \otimes g|^2
\]

In other words we obtain the square of the umbilicity tensor. Moreover if \(\alpha\) denotes the Kaehler form of a Kaehlerian manifold, a straightforward calculation gives

\[
|B^-(\alpha)|^2 = 4|B^-|^2
\]

where \(B^-\) is the anti-holomorphic part of \(B\) (i.e. \(B^-(X, Y) = \frac{1}{2}(B(X, Y) - B(JX, JY))\)) where \(\alpha(X, Y) = \langle JX, JY \rangle\).

On the other hand we will see in the proof of theorem 4.1 that \(B^-(\omega)\) is vanishing identically for the standard embedding of \(\mathbb{S}^p(\sqrt{p/m}) \times \mathbb{S}^q(\sqrt{k_1/m}) \times \ldots \times \mathbb{S}^q(\sqrt{k_q/m})\) into \(\mathbb{S}^{m+r}\) where \(p + k_1 + \ldots + k_q = m\) with \(k_i \geq p\) (1 \(\leq i \leq q\).

**Proof of Proposition 4.1:** For \(p = 1\) and from the relation (6), the equality of the proposition is obvious. Suppose now that \(p \geq 2\). Let \(x \in M\) and let \((e_i)_{1 \leq i \leq m}\) be a local orthonormal frame in a neighborhood of \(x\). We have

\[
\frac{(p-2)!}{2} |B^-(\omega)|^2 = \frac{1}{2} \sum_{i,j,i_1,\ldots,i_{p-2}} |B^-(\omega)_{i,j,i_1,\ldots,i_{p-2}}|^2 =
\]

\[
= \frac{1}{2} \sum_{i,j,k,l,i_1,\ldots,i_{p-2}} \langle i(e_i \wedge e_i \wedge \ldots \wedge e_{i_{p-2}}) \omega \wedge B(e_1, \ldots) \rangle_{kl} \langle i(e_j \wedge e_i \wedge \ldots \wedge e_{i_{p-2}}) \omega \wedge B(e_j, \ldots) \rangle_{kl}
\]

\[
= \sum_{i,j,k,l,i_1,\ldots,i_{p-2}} \langle i(e_i \wedge e_i \wedge \ldots \wedge e_{i_{p-2}}) \omega \rangle_k \langle i(e_j \wedge e_i \wedge \ldots \wedge e_{i_{p-2}}) \omega \rangle_k \langle B_{dl}, B_{jk} \rangle
\]

\[
+ \sum_{i,j,k,l,i_1,\ldots,i_{p-2}} \langle i(e_i \wedge e_i \wedge \ldots \wedge e_{i_{p-2}}) \omega \rangle_k \langle i(e_j \wedge e_i \wedge \ldots \wedge e_{i_{p-2}}) \omega \rangle_k \langle B_{dl}, B_{ik} \rangle
\]

\[
= \sum_{i,j,k,l,i_1,\ldots,i_{p-2}} \langle B_{dl}, B_{jk} \rangle \omega_{ik1 \ldots i_{p-2}} \omega_{jkl1 \ldots i_{p-2}} + \sum_{i,j,k,l,i_1,\ldots,i_{p-2}} \langle B_{dl}, B_{ik} \rangle \omega_{ik1 \ldots i_{p-2}} \omega_{i_1 \ldots i_{p-2}}
\]

\[
= (p-1)! \sum_{i,j,k} \langle B_{ik}, B_{jk} \rangle \langle i(e_i) \omega, i(e_j) \omega \rangle - (p-2)! \sum_{i,j,k} \langle B_{ij}, B_{kl} \rangle \langle i(e_i \wedge e_i) \omega, i(e_k \wedge e_j) \omega \rangle
\]

Finally, we have proved that
\[ \frac{1}{2} |\mathcal{B}^- (\omega) |^2 = (p - 1) \sum_{i, j, k} \langle B_{ik}, B_{jk} \rangle \langle i(e_i) \omega, i(e_j) \omega \rangle - \sum_{i, j, k, l} \langle B_{ik}, B_{jl} \rangle \langle i(e_i \wedge e_j) \omega, i(e_k \wedge e_l) \omega \rangle \]

(20)

Now, combining this with the relations (10) and (11) and using the expression of \( \langle R_p(\omega), \omega \rangle \) (see (6)), we get

\[ \frac{1}{2} |\mathcal{B}^- (\omega) |^2 = \langle R_p(\omega), \omega \rangle - p \sum_{i, j} \text{Ric}_{ij} \langle i(e_i) \omega, i(e_j) \omega \rangle \]

\[ + (p - 1) \sum_{i, j} (\mathcal{R}_p)_{ij} \langle i(e_i) \omega, i(e_j) \omega \rangle + \frac{1}{2} \sum_{i, j, k, l} \mathcal{R}_{ijkl} \langle i(e_j \wedge e_k) \omega, i(e_l) \omega \rangle \]

\[ + m(p - 1) \sum_{i, j} \langle B_{ij}, H \rangle \langle i(e_i) \omega, i(e_j) \omega \rangle \]

(21)

From the hypotheses on the curvature of \( N \) and by techniques already used in the proof of the theorem 3.1 (see (14) and (15)) we deduce that

\[ \frac{1}{2} |\mathcal{B}^- (\omega) |^2 \leq \langle R_p(\omega), \omega \rangle - p \sum_{i, j} \text{Ric}_{ij} \langle i(e_i) \omega, i(e_j) \omega \rangle \]

\[ + p(p - 1) \left( (m - 1) \mathcal{K}^1 (x) + p^1 (x) \right) |\omega|^2 + m(p - 1) \sum_{i, j} \langle B_{ij}, H \rangle \langle i(e_i) \omega, i(e_j) \omega \rangle \]

(22)

Now, let us estimate the last term. For this, assume that at the point \( x \in M, (e_i)_{1 \leq i \leq m} \) diagonalizes the symmetric tensor \( \langle B(X, Y), H \rangle \). We have

\[ \sum_{i, j} \langle B_{ij}, H \rangle \langle i(e_i) \omega, i(e_j) \omega \rangle = \frac{1}{(p - 1)!} \sum_{i, i_1, ..., i_p-1} \langle B_{ii}, H \rangle \omega_{i_1 ... i_p-1}^2 \]

\[ = \frac{1}{p!} \sum_{i, i_1, ..., i_p-1} \left( \langle B_{ii}, H \rangle + \langle B_{i_1i_1}, H \rangle + ... + \langle B_{i_{p-1}i_{p-1}}, H \rangle \right) \omega_{i_1 ... i_p-1}^2 \]

\[ \leq \frac{1}{p!} \sum_{i, i_1, ..., i_p-1} \left( |B_{ii}| + |B_{i_1i_1}| + ... + |B_{i_{p-1}i_{p-1}}| \right) |H| \omega_{i_1 ... i_p-1}^2 \]

\[ \leq \frac{\sqrt{p}}{p!} \sum_{i, i_1, ..., i_p-1} |B||H| \omega_{i_1 ... i_p-1}^2 \]

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Finally we have proved
\[
\sum_{i,j} \langle B_{ij}, H \rangle \langle i(e_i) \omega, i(e_j) \omega \rangle \leq \sqrt{p} |B||H| |\omega|^2
\]  \hfill (23)

Since $Ric \geq kg$, we have
\[
\sum_{i,j} Ric_{ij} \langle i(e_i) \omega, i(e_j) \omega \rangle \geq pk|\omega|^2,
\]
and (23), the inequality of the proposition 4.1.

The proof of the theorem 4.1 is now an immediate consequence of the proposition 4.1

**Proof of theorem 4.1:** Since $b_p(M) \neq 0$, there exists a nontrivial $p$-form $\omega$ so that $\Delta \omega = 0$. And from the Weitzenböck formula (5), we deduce
\[
\int_M \langle \mathcal{R}_p(\omega), \omega \rangle \leq 0 \hfill (24)
\]

Now, applying the estimate of the proposition 4.1 we get the desired inequality.

We can show a similar result to the theorem 4.1, with the scalar curvature $Scal$ of $(M^m, g)$ instead of the Ricci curvature.

**Theorem 4.2** Let $(M^m, g)$ be a compact Riemannian manifold of dimension $m \geq 2$ so that $b_p(M) \neq 0$ for a $p \geq 1$. Then for any isometric immersion $\phi$ from $(M^m, g)$ into an $n$-dimensional Riemannian manifold $(N^n, h)$, there exists at least a point $x \in M$ so that
\[
m \left( \frac{p - 1}{\sqrt{p}} + \frac{m - p - 1}{\sqrt{m - p}} \right) |B(x)||H(x)| \geq Scal(x) - (m - 2)((m - 1)\bar{K}^l + \bar{p}^l)(\phi(x)) \hfill (25)
\]

We immediately deduce the following

**Corollary 4.2** Let $(M^m, g)$ be a compact Riemannian manifold of dimension $m \geq 2$ minimally immersed into an $n$-dimensional Riemannian manifold $(N^n, h)$ ($n > m$). If $\bar{p}^l$ is bounded above and if
\[
\min_M Scal > (m - 2) \left( (m - 1)\bar{K}^l_{\text{max}} + \bar{p}^l_{\text{max}} \right)
\]
then for any $p \in \{1, ..., m\}$, we have $b_p(M) = 0$.

**Remark 4.2:** If $p = m/2$, we can improve (25) to obtain
\[
m(m - 2)|H(x)|^2 \geq Scal(x) - (m - 2)((m - 1)\bar{K}^l + \bar{p}^l)(\phi(x)) \hfill (26)
\]
The theorem 4.2 and the inequality (26) was obtained by El Soufi (theorem 3.1 and theorem 3.2 of [5]) in the particular case where \( m = 4 \) and \( p = 2 \).

On the other hand, note that for \( p \neq m/2 \), the theorem 4.1 is not a consequence of the theorem 4.2.

For the same reasons as in the proof of theorem 3.2, we can choose locally an orientation on \( M \) and define locally the Hodge operator \(*\). But for all \( p\)-form \( \omega \) of \( M \), the quantity \( \langle R_{m-p}(\omega), \omega \rangle \) is globally defined.

The theorem 4.2 is a consequence of the following proposition

**Proposition 4.2** Let \((M^m, g)\) be an \( m\)-dimensional compact Riemannian manifold isometrically immersed in an \( n\)-dimensional Riemannian manifold \((N^n, h)\). Then for all \( p \in \{1, \ldots, m-1\} \) and for all \( p\)-form \( \omega \) on \((M^m, g)\), we have for all \( x \in M \)

\[
\langle R_p(\omega), \omega \rangle + \frac{p}{m-p} \langle R_{m-p}(\omega), \omega \rangle \geq p \left( \text{Scal}(x) - (m-2) \left( (m-1)K^1 + \overline{p}^1 \right) \phi(x) \right) \\
- m \left( \frac{p-1}{\sqrt{p}} + \frac{m-p-1}{\sqrt{m-p}} \right) |H(x)||B(x)||\omega|^2
\]

**Remark 4.3:** If \( p = m/2 \), we can improve this inequality (see the proof of the proposition 4.2) to obtain

\[
\langle R_p(\omega), \omega \rangle + \frac{p}{m-p} \langle R_{m-p}(\omega), \omega \rangle \geq p \left( \text{Scal}(x) - (m-2) \left( (m-1)K^1 + \overline{p}^1 \right) \phi(x) \right) - m(m-2)|H(x)||\omega|^2
\]

And if for \( p \geq 1 \), \( b_p(M) \neq 0 \), we get (26).

**Proof of the proposition 4.2:** From the inequality (22) we obtain that for all \( p \geq 1 \)

\[
\langle R_p(\omega), \omega \rangle \geq p \sum_{i,j} \text{Ric}_{ij} \langle i(e_i)\omega, i(e_j)\omega \rangle \\
- p(p-1) \left( (m-1)K^1 + \overline{p}^1 \right) \phi(x)|\omega|^2 \\
- m(p-1) \sum_{i,j} \langle B_{ij}, H \rangle \langle i(e_i)\omega, i(e_j)\omega \rangle
\]

Since \(*\omega\) is a \((m-p)\)-form we have also

\[
\langle R_{m-p}(\omega), \omega \rangle \geq (m-p) \sum_{i,j} \text{Ric}_{ij} \langle i(e_i)\omega, i(e_j)\omega \rangle \\
- (m-p)(m-p-1) \left( (m-1)K^1 + \overline{p}^1 \right) \phi(x)|\omega|^2
\]
\[ -m(m-p-1) \sum_{i,j} \langle B_{ij}, H \rangle \langle i(e_i) \star \omega, i(e_j) \star \omega \rangle \]  

(29)

Multiplying (29) by \( \frac{p}{m-p} \), and summing the obtained inequality with (28), we find

\[
\langle R_p(\omega), \omega \rangle + \frac{p}{m-p} \langle R_{m-p}(\star \omega), \star \omega \rangle \geq 
\]

\[ p \sum_{i,j} \left( Ric_{ij} \langle i(e_i) \omega, i(e_j) \omega \rangle + Ric_{ij} \langle i(e_i) \star \omega, i(e_j) \star \omega \rangle \right) 
- p(m-2) \left( (m-1)K + \bar{\rho} \right) (\phi(x))|\omega|^2 
- m(p-1) \sum_{i,j} \langle B_{ij}, H \rangle \langle i(e_i) \omega, i(e_j) \omega \rangle - mp \left( \frac{m-p-1}{m-p} \right) \sum_{i,j} \langle B_{ij}, H \rangle \langle i(e_i) \star \omega, i(e_j) \star \omega \rangle 
\]

By computations which are similar to (17) we get

\[
\sum_{i,j} \left( Ric_{ij} \langle i(e_i) \omega, i(e_j) \omega \rangle + Ric_{ij} \langle i(e_i) \star \omega, i(e_j) \star \omega \rangle \right) = Scal|\omega|^2 
\]

(30)

Thus

\[
\langle R_p(\omega), \omega \rangle + \frac{p}{m-p} \langle R_{m-p}(\star \omega), \star \omega \rangle \geq pScal|\omega|^2 
- p(m-2) \left( (m-1)K + \bar{\rho} \right) (\phi(x))|\omega|^2 
- m(p-1) \sum_{i,j} \langle B_{ij}, H \rangle \langle i(e_i) \omega, i(e_j) \omega \rangle 
- mp \left( \frac{m-p-1}{m-p} \right) \sum_{i,j} \langle B_{ij}, H \rangle \langle i(e_i) \star \omega, i(e_j) \star \omega \rangle 
\]

(31)

\( \text{(Note that if } p = m/2, \text{ then } p-1 = p \left( \frac{m-p-1}{m-p} \right), \text{ and we show with the same arguments as in the proof of (30) that) } \)

\[
m(p-1) \sum_{i,j} \langle B_{ij}, H \rangle \langle i(e_i) \omega, i(e_j) \omega \rangle + mp \left( \frac{m-p-1}{m-p} \right) \sum_{i,j} \langle B_{ij}, H \rangle \langle i(e_i) \star \omega, i(e_j) \star \omega \rangle = 
\]

\[ p(m-2)m|H|^2|\omega|^2 \]

and reporting this in (31), we obtain the inequality (27) of the remark 3.3.

From the estimate (23), we deduce
\[ m(p-1) \sum_{i,j} \langle B_{ij}, H \rangle \langle i(e_i)\omega, i(e_j)\omega \rangle \leq m(p-1)\sqrt{p}||H||B||\omega||^2 \]

and

\[ mp \left( \frac{m-p-1}{m-p} \right) \sum_{i,j} \langle B_{ij}, H \rangle \langle i(e_i) \star \omega, i(e_j) \star \omega \rangle \leq \frac{mp(m-p-1)}{\sqrt{m-p}}||H||B||\omega||^2 \]

now reporting these inequalities in (31), we find the inequality of the proposition. \( \square \)

The proof of the theorem is now immediate

**Proof of theorem 4.2:** Let \( \omega \) be a harmonic \( p \)-form. Since the Hodge operator commutes with the Laplacian, then \( \star \omega \) is a harmonic \( (m-p) \)-form, and we have

\[ \int_M \left( \langle R_p(\omega), \omega \rangle + \frac{p}{m-p} \langle R_{m-p}(\star \omega), \star \omega \rangle \right) \leq 0 \]

and the theorem follows from the proposition 4.2. \( \square \)

**Remark 4.4:** We can improve all the results of this section by considering the particular case of submanifolds of the complex projective space \( \mathbb{C}P^n(c) \) \((c > 0)\). We just need to compute in (21) the terms with the curvature tensor of the complex projective space. Then we obtain the same statements as previously by replacing \((m-1)K^2 + \overline{P}^1 \) by \( c/8((m-1) + 3 \| J_\phi \|^2 \) where for all \( x \in M, \| J_\phi \| (x) = \sup \{|J_\phi(X)|/X \in T_xM \text{ and } |X| = 1\} \). In particular, if \((M^m, g)\) is an \( m \)-dimensional compact Riemannian manifold minimally immersed in \( \mathbb{C}P^n(c) \) and if

\[ \min_M(\text{Scal}) > \frac{c(m-2)((m-1) + 3 \max_M(\| J_\phi \|^2))}{8} \]

then for any \( p \in \{1, \ldots, m\} \), we have \( b_p(M) = 0 \).

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**References**


