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MINIMAL SUBMANIFOLDS WITH A PARALLEL OR A HARMONIC $p$-FORM

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Abstract

The purpose of this paper is to study the relations between the existence of minimal immersions of a Riemannian manifold $M$ into another and some structural or topological properties of $M$. The properties on $M$ which we consider are the existence of a parallel or a harmonic $p$-form.
1 Introduction

The purpose of this paper is to obtain some non existence results about minimal submanifolds. Let \((M^m, g)\) be an \(m\)-dimensional Riemannian manifold isometrically immersed by \(\phi\) in an \(n\)-dimensional Riemannian manifold \((N^n, h)\) \((n > m)\). The Gauss equation allows us to obtain rigidity results in terms of geometry of \((M^m, g)\) and \((N^n, h)\).

For example, as a first consequence of the Gauss equation, we get the following well known inequality in each point \(x\) of \((M^m, g)\)

\[
|H(x)|^2 \geq \left( \frac{1}{m} \right) \left( \frac{\text{Scal}(x)}{m} - (m-1)K^1(\phi(x)) \right)
\]

where \(|H(x)|^2\) and \(\text{Scal}(x)\) are respectively the square of the mean curvature of \(\phi\) and the scalar curvature of \((M^m, g)\) at \(x\) and \(K^1(\phi(x))\) is the largest sectional curvature of \((N^n, h)\) at \(\phi(x)\). In particular, if \(K^1\) has an upper bound and if \(\text{Scal} > m(m-1)K^1_{\text{max}}\) (where \(K^1_{\text{max}} = \max_N(\mathcal{K}^1)\)) for at least a point of \((M^m, g)\), there is no minimal immersion of \((M^m, g)\) into \((N^n, h)\).

Many other results were obtained, by assuming that \((M^m, g)\) is endowed with some particular structures or topological properties (see for instance \([1]\), \([12]\) and \([3]\)). First recall the results of Sampson (\([12]\)) and Dajczer and Rodriguez (\([3]\)). They proved that there is no minimal immersion of an \(m\)-dimensional Kaehlerian manifold \((m \geq 4)\) into a Riemannian manifold of negative constant sectional curvature. Later, El Soufi (\([5]\)) obtained a generalization of this result by assuming a pinching of the sectional curvature of \((N^n, h)\) and Hernandez (\([9]\)) obtained the same conclusion under the negativity of the complex sectional curvature of \((N^n, h)\). More recently, Petit and El Soufi (\([6]\)) extend this result in the case where \((M^m, g)\) is not necessarily Kaehlerian but has a parallel 2-form and where the isotropic curvature of \((N^n, h)\) is negative (recall that the isotropic curvature of a Riemannian manifold is the restriction of the complex sectional curvature to isotropic tangent planes (\([11]\))).

The section 1 of the present paper deals with some preliminaries. In the section 2, we consider the general case where \((M^m, g)\) has a parallel \(p\)-form and we prove (theorem 3.1) that if \((N^n, h)\) satisfies a curvature pinching condition, then there is no minimal immersion from \((M^m, g)\) into \((N^n, h)\). This is the generalization of the result of El Soufi stated in [5] for the case where \((M^m, g)\) is Kaehlerian. Note that this theorem as well as the other results recalled above are of interest only if the sectional curvature of \((N^n, h)\) is negative. However, in the theorem 3.2, we obtain the same conclusion with a new pinching condition for the case where \((N^n, h)\) is not necessarily of negative sectional curvature but has a negative smallest sectional curvature. In the theorem 3.3, we study the particular case where \((N^n, h)\) is the complex hyperbolic space \(\mathbb{CH}^n(c)\) with constant holomorphic curvature equal to \(c\) and we prove that there is no totally real minimal immersion of a Riemannian manifold \((M^m, g)\) with a parallel \(p\)-form into \(\mathbb{CH}^n(c)\).
The compact manifolds with a parallel $p$-form are a particular case of manifolds with a harmonic $p$-form (or a nonzero $p$-th Betti number $b_p(M)$). In the section 3, we prove (theorem 4.1 and theorem 4.2) that for any compact manifold $(M^m, g)$ with $b_p(M) \neq 0$ and isometrically immersed in a Riemannian manifold $(N^n, h)$, there exists at least a point $x$ of $M$ so that

$$\frac{m}{\sqrt{p}} \left(\frac{p-1}{p}\right) |B(x)||H(x)| \geq k(x) - \left(\frac{p-1}{p}\right) \left((m-1)K^1 + \bar{p}^1(\phi(x))\right)$$

and

$$m \left(\frac{p-1}{\sqrt{p}} + \frac{m-p-1}{\sqrt{m-p}}\right) |B(x)||H(x)| \geq Scal(x) - (m-2)((m-1)K^1 + \bar{p}^1(\phi(x))$$

where $|B(x)|$, $k(x)$ and $\bar{p}^1(\phi(x))$ denote respectively the norm of the second fundamental form of $\phi$, the smallest eigenvalue of the Ricci curvature of $(M^m, g)$ at $x$ and the largest eigenvalue of the curvature operator of $(N^n, h)$ at $\phi(x)$. El Soufi proved the first inequality for $p = 2$ in [5] and the second for $p = 2$ but only for $m = 4$. The first is optimal for the usual standard minimal embeddings of the Clifford torus and of the complex projective space in the sphere. These inequalities will be a consequence of a new lower bound of the curvature term in the Weitzenböck formula for $p$-forms (see the relation (5) and the propositions 4.1 and 4.2).

As a consequence of the previous inequalities, we deduce (corollary 4.2) that if $(M^m, g)$ is minimally immersed in $(N^n, h)$, if $\bar{p}^1$ is bounded above and if

$$\min_M(Scal) > (m-2) \left((m-1) \max_N(K^1) + \max_N(\bar{p}^1)\right)$$

then $(M^m, g)$ is a sphere of homology. This result can be viewed as a generalization of a theorem of Leung ([10]) which has shown that if a compact Riemannian manifold $(M^m, g)$ is minimally immersed in a unit sphere and if the scalar curvature satisfies $Scal > m(m-2)$ then it is homeomorphic to an $m$-dimensional sphere.

## 2 Preliminaries and notations

Let $(M^m, g)$ be an $m$-dimensional Riemannian manifold and let $\phi$ be an isometric immersion of $(M^m, g)$ into an $n$-dimensional Riemannian manifold $(N^n, h)$ ($n > m$). The inner product and the norm induced by $g$ and $h$ on the tensors will be denoted respectively by $\langle \, , \rangle$ and $| |^2$. Moreover, we denote respectively by $R$, $\rho$, $Ric$ and $Scal$ the curvature tensor, the curvature operator, the Ricci tensor and the scalar curvature of $(M^m, g)$ and by $\mathcal{R}$, $\mathcal{K}$, $\mathcal{p}$ and $\mathcal{Scal}$ the curvature tensor, the sectional curvature, the curvature operator and
the scalar curvature of \((N^n, h)\). We recall that for all vector field \(X, Y, Z, W \in \Gamma(TN)\), \(\bar{p}\) is defined by

\[
\langle \bar{p}(X \wedge Y), Z \wedge W \rangle = \bar{R}(X, Y, Z, W)
\]

Moreover, for all vector field \(X, Y \in \Gamma(TM)\), the tensor \(\bar{R}_\phi\) will be given by

\[
\bar{R}_\phi(X, Y) = \sum_{i \leq m} \bar{R}(d\phi(X), d\phi(e_i), d\phi(Y), d\phi(e_i))
\]

where \((e_i)_{1 \leq i \leq m}\) is an orthonormal frame on \(M\). On the other hand, for all \(x \in N\), we denote respectively , \(K^1(x)\) and \(K^0(x)\) the largest sectional curvature and the smallest sectional curvature at \(x\) and \(\rho^1(x)\) and \(\rho^0(x)\) the largest eigenvalue and the smallest eigenvalue of the curvature operator. Then, it is easy to see that

\[
\rho^0(x) \leq K^0(x) \leq K^1(x) \leq \rho^1(x)
\]

Now, let \(B\) be the second fundamental form of the immersion \(\phi\) and let \(H\) be the mean curvature vector defined by:

\[
H = \frac{1}{m} \text{trace } B.
\]

The Gauss equation tells us that for any vector field \(X, Y, Z, W \in \Gamma(TM)\), we have

\[
R(X, Y, Z, W) = \bar{R}(X, Y, Z, W) + \langle B(X, Z), B(Y, W) \rangle - \langle B(X, W), B(Y, Z) \rangle
\]

For the sake of completeness, we need now to recall briefly some definitions and properties about \(p\)-forms. Let \((e_i)_{1 \leq i \leq m}\) be a local orthonormal frame. Throughout this paper, for all \(q\)-tensor \(T\), we will write \(T_{i_1...i_q}\) instead of \(T(e_{i_1}, ..., e_{i_q})\) and then the inner product of two \(p\)-forms \(\omega\) and \(\theta\) of \((M^m, g)\) will be

\[
\langle \omega, \theta \rangle = \frac{1}{p!} \sum_{1 \leq i_1, ..., i_p \leq m} \omega_{i_1...i_p} \theta_{i_1...i_p}
\]

The inner product (or contraction) \(i(X)\omega\) of a \(p\)-form \(\omega\) with a vector field \(X\) on \(M\) is a \(p - 1\)-form, defined by

\[
(i(X))\omega(X_1, ..., X_{p-1}) = \omega(X, X_1, ..., X_{p-1}), \ \forall X_1, ..., X_{p-1} \in \Gamma(TM)
\]

More generally, if \(X_1,...X_q \in \Gamma(TM)\), then the inner product of the \(p\)-form \(\omega\) with the \(q\)-tensor \(X_1 \wedge ... \wedge X_q\) is the \(p - q\)-form defined by

\[
(i(X_1 \wedge ... \wedge X_q)\omega)(Y_1, ..., Y_{p-q}) = \omega(X_q, ..., X_1, Y_1, ..., Y_{p-q}), \ \forall Y_1, ..., Y_{p-q} \in \Gamma(TM)
\]

Recall some elementary facts about inner and exterior products. Let \(\omega\) and \(\theta\) be respectively a \(p\)-form and a \(q\)-form and let \(X\) be a vector field on \(M\), then

\[\]
\[ i(X)(\omega \wedge \theta) = i(X)\omega \wedge \theta + (-1)^p \omega \wedge i(X)\theta \]

and if \( X^* \) is the dual 1-form of the vector field \( X \) with respect to \( g \), then \( i(X) \) is in fact the adjoint of left exterior multiplication by \( X^* \), that is

\[ \langle i(X)(\omega), \theta \rangle = \langle \omega, X^* \wedge \theta \rangle \]

If \( M \) is orientable, we also need the following relation between the inner product and the Hodge operator \( \ast \) on \( (M^m, g) \) (see for instance [4])

\[ i(X)(\ast \omega) = (-1)^p \ast (X^* \wedge \omega) \]

On the other hand, if \( \alpha \) is a 1-form which is real valued and \( \beta \) is a 1-form which is valued in a vector bundle, we define the 2-tensor \( \alpha \vee \beta \) by

\[ (\alpha \vee \beta)(X, Y) = \alpha(X)\beta(Y) + \alpha(Y)\beta(X) \]  

We denote now by \( d, d^*, \nabla \) and \( \nabla^* \) respectively the exterior differential and the codifferential acting on \( p \)-forms, the covariant derivative of \( (M^m, g) \) extended to \( p \)-forms and its adjoint with respect to \( g \). The Hodge-de Rham Laplacian \( \Delta \) acting on \( p \)-forms is given by

\[ \Delta \omega = dd^* \omega + d^* d\omega \]

To compare this Laplacian to the “rough” Laplacian \( \nabla^* \nabla \), one has the Weitzenböck formula, reading as

\[ \Delta \omega = \nabla^* \nabla \omega + R_p(\omega), \quad \forall \omega \in \Lambda^p(M) \]

Here \( R_p \in End(\Lambda^p(M)) \) is a bundle endomorphism, given by

\[ R_p(\omega)(X_1, ..., X_p) = \sum_{ij} (-1)^i[R(e_j, X_i)\omega](e_j, X_1, ..., \widehat{X_i}, ..., X_p), \quad \forall X_1, ..., X_p \in \Gamma(TM) \]

where \((e_i)_{1 \leq i \leq m}\) is a local orthonormal frame and

\[ R(X, Y)\omega = \nabla_{[X,Y]}\omega - [\nabla_X, \nabla_Y]\omega \quad \forall X, Y \in \Gamma(TM) \]

An easy consequence of the Weitzenböck formula is the following

\[ \frac{1}{2} \Delta |\omega|^2 = \langle \Delta \omega, \omega \rangle - |\nabla \omega|^2 - \langle R_p(\omega), \omega \rangle \]  

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In the sequel, we need to explicit the expression of $R_p$. A straightforward calculation gives us

$$\langle R_p(\omega), \omega \rangle = \sum_{i,j} Ric_{ij} \langle i(e_i)\omega, i(e_j)\omega \rangle - \frac{1}{2} \sum_{i,j,k,l} R_{ijkl} \langle i(e_j \wedge e_i)\omega, i(e_l \wedge e_k)\omega \rangle$$

and the last term is zero when $p = 1$.

3 Geometry of submanifolds having a parallel $p$-form

The first result of this section is the following theorem

**Theorem 3.1** Let $(M^m, g)$ be an $m$-dimensional Riemannian manifold admitting a non-trivial parallel $p$-form ($1 \leq p \leq m$) and let $(N^n, h)$ be an $n$-dimensional Riemannian manifold ($n > m$). If for any $x \in (N^n, h)$, we have

$$(m-1)K^1(x) < (p-1)\rho^0(x)$$

then, there is no minimal immersion from $(M^m, g)$ into $(N^n, h)$.

**Remark 3.1:**

1. For $p = 2$ and for even dimensional manifold $(M^m, g)$, the pinching condition (7) can be reformulate as the negativity of the isotropic curvature ([6]). If $(M^m, g)$ is Kaehlerian, this condition (7) is nothing but that obtained by El Soufi in [5] (theorem 2.2).

2. From the relation (2), we see that this theorem is of interest only if the sectional curvature of $(N^n, h)$ is negative. For the hyperbolic space $\mathbb{H}^n$, the condition (7) is always satisfied for $p < m$ and then there is no minimal immersion of a manifold having a parallel $p$-form ($1 \leq p \leq m-1$) into $\mathbb{H}^n$. However, the embeddings of $\mathbb{H}^m$ in $\mathbb{H}^n$ ($m < n$) are totally geodesic, and taking the volume form of $\mathbb{H}^m$, we see that (7) is not satisfied for $p = m$.

In the following theorem, we obtain the same conclusion as in the theorem 3.1 with a new pinching condition where $(N^n, h)$ is not necessarily of negative sectional curvature (in fact, there is no condition on $\overline{K}^1$).

**Theorem 3.2** Let $(M^m, g)$ be an $m$-dimensional Riemannian manifold admitting a non-trivial parallel $p$-form ($1 \leq p \leq m$) and let $(N^n, h)$ be an $n$-dimensional Riemannian manifold ($n > m$). If for any $x \in (N^n, h)$, we have
\[ \text{Scal}(x) < (n-m)(n+m-1)\overline{K}^0(x) + (p(p-1) + (m-p)(m-p-1))\overline{\rho}^0(x) \quad (8) \]

then, there is no minimal immersion from \((M^m, g)\) into \((N^n, h)\).

**Remark 3.2:** We will see in the proof that this theorem is of interest only if the smallest sectional curvature is negative that is \(\overline{K}^0(x) < 0\) for all \(x \in N\). For instance, let us consider the space \(N^n = \mathbb{R}^r \times S^s\) where \(n = r + s\). Then \(\text{Scal} = -r(r-1) + s(s-1), \overline{K}^0 = 1\) and \(\overline{\rho}^0 = -1\). Now, let \((M^m, g)\) be a Riemannian manifold of even dimension \(m\) and let \(p = m/2\). Then we have \(\text{Scal} - (n-m)(n+m-1)\overline{K}^0 - (p(p-1) + (m-p)(m-p-1))\overline{\rho}^0 = 2(s^2 + rs - s - m^2/4)\) and it is easy to see that if \(r\) and \(m\) are great enough (for instance for a fixed \(s\) put \(m = r\) great enough) then the condition (8) is satisfied and the conclusion of the theorem 3.2 holds for this example.

**Proof of theorem 3.1:** Let \(\phi\) be a minimal immersion of \((M^m, g)\) into \((N^n, h)\) and assume that \(M\) has a nontrivial parallel \(p\)-form \(\omega\). Then \(\langle \mathcal{R}_\omega(\omega), \omega \rangle = 0\) and from (6) we deduce

\[ 0 = \sum_{i,j} Ric_{ij} \langle i(e_i)\omega, i(e_j)\omega \rangle - \frac{1}{2} \sum_{i,j,k,l} R_{ijkl} \langle i(e_j \wedge e_i)\omega, i(e_l \wedge e_k)\omega \rangle \quad (9) \]

and the last term is zero for \(p = 1\). Now from the Gauss formula (3), we obtain

\[ \sum_{i,j} Ric_{ij} \langle i(e_i)\omega, i(e_j)\omega \rangle = \sum_{i,j} (\overline{R_{\phi}})_{ij} \langle i(e_i)\omega, i(e_j)\omega \rangle + m \sum_{i,j} \langle B_{ij}, H \rangle \langle i(e_i)\omega, i(e_j)\omega \rangle - \sum_{i,j,k} \langle B_{ik}, B_{jk} \rangle \langle i(e_i)\omega, i(e_j)\omega \rangle \quad (10) \]

and

\[ \sum_{i,j,k,l} R_{ijkl} \langle i(e_j \wedge e_i)\omega, i(e_l \wedge e_k)\omega \rangle = \sum_{i,j,k,l} (\overline{R_{ijkl}}) \langle i(e_j \wedge e_i)\omega, i(e_l \wedge e_k)\omega \rangle + \sum_{i,j,k,l} \langle B_{ik}, B_{jl} \rangle \langle i(e_j \wedge e_i)\omega, i(e_l \wedge e_k)\omega \rangle - \sum_{i,j,k,l} \langle B_{il}, B_{jk} \rangle \langle i(e_j \wedge e_i)\omega, i(e_l \wedge e_k)\omega \rangle + 2 \sum_{i,j,k,l} \langle B_{ik}, B_{jl} \rangle \langle i(e_j \wedge e_i)\omega, i(e_l \wedge e_k)\omega \rangle \quad (11) \]

by reporting (10) and (11) in (9), we get
\[
0 = \sum_{i,j} (R_{ij})_{ij} \langle i(e_i) \omega, i(e_j) \omega \rangle + m \sum_{i,j} \langle B_{ij}, H \rangle \langle i(e_i) \omega, i(e_j) \omega \rangle \\
- \sum_{i,j,k} \langle B_{ik}, B_{jk} \rangle \langle i(e_i) \omega, i(e_j) \omega \rangle - \frac{1}{2} \sum_{i,j,k,l} R_{ijkl} \langle i(e_j \land e_i) \omega, i(e_l \land e_k) \omega \rangle \\
- \sum_{i,j,k,l} \langle B_{ik}, B_{jl} \rangle \langle i(e_j \land e_i) \omega, i(e_l \land e_k) \omega \rangle
\] (12)

and the two last terms are zero for \( p = 1 \). Now, put \( B^+ (\omega) = \sum_{i \leq m} i(e_i^*) \omega \land B(e_i, \omega) \). The computation of the square of the norm of \( B^+ (\omega) \) gives

\[
|B^+ (\omega)|^2 = \frac{1}{p!} \sum_{1 \leq i_1, \ldots, i_p \leq m} \langle (i(e_i^*) \omega \land B(e_i, \omega))_{i_1 \ldots i_p}, (i(e_i^*) \omega \land B(e_j, \omega))_{i_1 \ldots i_p} \rangle \\
= \frac{1}{p!} \sum_{1 \leq i_1, \ldots, i_p \leq m} (-1)^{s+t} \langle B_{i s}, B_{j t} \rangle \omega_{i_{i_1} \ldots i_{i_s} \ldots i_{i_t} \ldots i_{i_p}} \omega_{j_{i_1} \ldots i_{i_s} \ldots i_{i_t} \ldots i_{i_p}}
\]

where the indices with ˆ are omitted. Then

\[
|B^+ (\omega)|^2 \\
= \frac{1}{p!} \sum_{1 \leq i_1, \ldots, i_p \leq m} \langle B_{i s}, B_{j t} \rangle \omega_{i_{i_1} \ldots i_{i_s} \ldots i_{i_t} \ldots i_{i_p}} \omega_{j_{i_1} \ldots i_{i_s} \ldots i_{i_t} \ldots i_{i_p}} \\
+ \frac{1}{p!} \sum_{1 \leq i_1, \ldots, i_p \leq m} (-1)^{s+t} \langle B_{i s}, B_{j t} \rangle \omega_{i_{i_1} \ldots i_{i_s} \ldots i_{i_t} \ldots i_{i_p}} \omega_{j_{i_1} \ldots i_{i_s} \ldots i_{i_t} \ldots i_{i_p}} \\
= \frac{1}{(p-1)!} \sum_{1 \leq i_1, \ldots, i_{p-1} \leq m} \langle B_{ik}, B_{jk} \rangle \omega_{i_{i_1} \ldots i_{i_{p-1}} i_{i_{p-1}}} \omega_{j_{i_1} \ldots i_{i_{p-1}} i_{i_{p-1}}} \\
- \frac{1}{p!} \sum_{1 \leq i_1, \ldots, i_p \leq m} \langle B_{i s}, B_{j t} \rangle \omega_{i_{i_1} \ldots i_{i_s} \ldots i_{i_t} \ldots i_{i_p}} \omega_{j_{i_1} \ldots i_{i_s} \ldots i_{i_t} \ldots i_{i_p}} \\
- \frac{1}{p!} \sum_{1 \leq i_1, \ldots, i_p \leq m} \langle B_{i s}, B_{j t} \rangle \omega_{i_{i_1} \ldots i_{i_s} \ldots i_{i_t} \ldots i_{i_p}} \omega_{j_{i_1} \ldots i_{i_s} \ldots i_{i_t} \ldots i_{i_p}} \
\]
\[
\begin{align*}
&= \sum_{i,j,k} \langle B_{ik}, B_{jk} \rangle \langle i(e_i) \omega, i(e_j) \omega \rangle - \frac{1}{(p-2)!} \sum_{i_1, \ldots, i_{p-2} \leq m} \langle B_{ik}, B_{jl} \rangle \omega_{l_1 \ldots i_{p-2}} \omega_{j_1 \ldots i_{p-2}} \\
&= \sum_{i,j,k} \langle B_{ik}, B_{jk} \rangle \langle i(e_i) \omega, i(e_j) \omega \rangle + \frac{1}{(p-2)!} \sum_{i_1, \ldots, i_{p-2} \leq m} \langle B_{ik}, B_{jl} \rangle \omega_{i_1 \ldots i_{p-2}} \omega_{j_1 \ldots i_{p-2}} \\
&= \sum_{i,j,k} \langle B_{ik}, B_{jk} \rangle \langle i(e_i) \omega, i(e_j) \omega \rangle + \sum_{i,j,k,l} \langle B_{ik}, B_{jl} \rangle \langle i(e_j \wedge e_i) \omega, i(e_l \wedge e_k) \omega \rangle
\end{align*}
\]

Note that if \( M \) is Kaehler and \( \omega \) is the Kaehler form then \(|B^+(\omega)|^2 = |B^+|^2\), where \( B^+ \) is the holomorphic part of \( B \) (i.e. \( B^+(X, Y) = \frac{1}{2}(B(X, Y) + B(JX, JY)) \) where \( \omega(X, Y) = \langle JX, Y \rangle \).

Now, combining the above relation with (12), we obtain

\[
|B^+(\omega)|^2 = m \sum_{i,j} \langle B_{ij}, H \rangle \langle i(e_i) \omega, i(e_j) \omega \rangle + \sum_{i,j} (\mathcal{R}_\phi)_{ij} \langle i(e_i) \omega, i(e_j) \omega \rangle \\
- \frac{1}{2} \sum_{i,j,k,l} \mathcal{R}_{ijkl} \langle i(e_j \wedge e_i) \omega, i(e_l \wedge e_k) \omega \rangle
\]

where the last term is zero if \( p = 1 \). Putting \( X^{i_1 \ldots i_p} = \sum_{i \leq m} \omega_{i_1 \ldots i_{p-1}} e_i \) and \( \theta^{i_1 \ldots i_{p-2}} = \frac{1}{2} \sum_{1 \leq i,j \leq m} \omega_{i_1 \ldots i_{p-2}} e_i^* \wedge e_j^* \), we have

\[
\sum_{i,j} (\mathcal{R}_\phi)_{ij} \langle i(e_i) \omega, i(e_j) \omega \rangle = \frac{1}{(p-1)!} \sum_{i_1, \ldots, i_{p-1}} \mathcal{R}_\phi(X^{i_1 \ldots i_p}, X^{i_1 \ldots i_p}) \\
\leq \frac{(m-1)}{(p-1)!} K(\phi(x)) \sum_{i_1, \ldots, i_{p-1}} |X^{i_1 \ldots i_p}|^2 = p(m-1)K(\phi(x))|\omega|^2
\]

and

\[
\frac{1}{2} \sum_{i,j,k,l} \mathcal{R}_{ijkl} \langle i(e_j \wedge e_i) \omega, i(e_l \wedge e_k) \omega \rangle = \frac{2}{(p-2)!} \sum_{i_1, \ldots, i_{p-2}} \bar{p}(\theta^{i_1 \ldots i_{p-2}}, \theta^{i_1 \ldots i_{p-2}}) \\
\geq \frac{2}{(p-2)!} \bar{p}(\phi(x)) \sum_{i_1, \ldots, i_{p-2}} |\theta^{i_1 \ldots i_{p-2}}|^2 \\
= \frac{1}{(p-2)!} \bar{p}(\phi(x)) \sum_{i_1, \ldots, i_p} \omega^{i_1 \ldots i_p} = p(p-1) \bar{p}(\phi(x))|\omega|^2
\]
consequently, for all $p \in \{1, \ldots, m\}$, it follows from (13) and the fact that $H = 0$

$$|B^+(\omega)|^2(x) \leq p(m-1)\mathcal{K}^1(\phi(x))|\omega|^2 - p(p-1)\overline{\partial}^p(\phi(x))|\omega|^2$$

From this we conclude that if $(p-1)\overline{\partial}^p(x) > (m-1)\mathcal{K}^1(x)$ holds for any $x \in N$, then there is no minimal immersion from $(M^m, g)$ into $(N^n, h)$. \hfill \Box

**Remark 3.3:** In the theorem 3.1, the nonpositivity of the curvature operator of $(N^n, h)$ is not required. In ([2]) Corlette proved (theorem 3.1) a similar result for harmonic maps $\phi$ from $(M^m, g)$ to $(N^n, h)$ (in fact he assumes that $\phi$ is a twisted harmonic map which is not necessary) by assuming only the nonpositivity of the curvature operator of $(N^n, h)$ without pinching condition. Indeed, under this hypothesis he proved that if $(M^m, g)$ is compact and has a parallel $p$-form then $\nabla^*(d\phi \wedge \omega) = 0$ (here $\nabla$ is the pullback of the Levi-Civita connection of $TN$). Note that Corlette doesn’t obtain a result of non existence. Moreover if $\phi$ is a minimal isometric immersion, then $\phi$ is a harmonic map and the theorem of Corlette can be proved without the compacity of $M$ and a short computation shows that $\nabla^*(d\phi \wedge \omega) = -B^+(\omega)$.

**Proof of theorem 3.2:** From the relation (13) and the inequality (15) of the previous proof, we deduce that if $\phi$ is a minimal immersion, we have

$$|B^+(\omega)|^2 \leq \sum_{i,j} (\overline{\mathcal{R}}_{\phi})_{ij} \langle i(e_i)\omega, i(e_j)\omega \rangle - p(p-1)\overline{\partial}^p(\phi(x))|\omega|^2$$

Since $M$ is locally oriented, we can define locally the Hodge operator $\star$ and the $m-p$-form $\star \omega$. But $|B^+(\star \omega)|^2$ is independent on the choice of the orientability and is consequently globally defined. Then we have

$$|B^+(\omega)|^2 + |B^+(\star \omega)|^2 \leq \sum_{i,j} (\overline{\mathcal{R}}_{\phi})_{ij} (\langle i(e_i)\omega, i(e_j)\omega \rangle + \langle i(e_i) \star \omega, i(e_j) \star \omega \rangle)$$

$$- (p(p-1) + (m-p)(m-p-1))\overline{\partial}^p(\phi(x))|\omega|^2 \tag{16}$$

Now using the properties about inner and exterior products recalled in the first section, we have

$$\sum_{i,j} ((\overline{\mathcal{R}}_{\phi})_{ij} \langle i(e_i)\omega, i(e_j)\omega \rangle + (\overline{\mathcal{R}}_{\phi})_{ij} \langle i(e_i) \star \omega, i(e_j) \star \omega \rangle) =$$

$$\sum_{i,j} ((\overline{\mathcal{R}}_{\phi})_{ij} \langle i(e_i)\omega, i(e_j)\omega \rangle + (\overline{\mathcal{R}}_{\phi})_{ij} \langle e^*_i \wedge \omega, e^*_j \wedge \omega \rangle) =$$

$$\sum_{i,j} ((\overline{\mathcal{R}}_{\phi})_{ij} \langle i(e_i)\omega, i(e_j)\omega \rangle + (\overline{\mathcal{R}}_{\phi})_{ij} \langle i(e_j)(e^*_i \wedge \omega), \omega \rangle) =$$

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\[ \sum_{i,j} (\overline{R}_\phi)_{ij} i(e_i) \omega, i(e_j) \omega) + \sum_i (\overline{R}_\phi)_{ii} |\omega|^2 - \sum_{i,j} (\overline{R}_\phi)_{ij} (e_i^* \wedge i(e_j) \omega, \omega) = \sum_i (\overline{R}_\phi)_{ii} |\omega|^2 \]  

(17)

A straightforward computation gives

\[ \sum_i (\overline{R}_\phi)_{ii} \leq \text{Scal}(\phi(x)) - (n-m)(n+m-1)\overline{K}^0(\phi(x)) \]

and from (16) and (17) we deduce that for all \( x \in M \)

\[ 0 \leq \text{Scal}(\phi(x)) - \left( (n-m)(n+m-1)\overline{K}^0 + (p(p-1) + (m-p)(m-p-1))\overline{p}^0 \right)(\phi(x)) \]

Consequently if

\[ \text{Scal}(x) < (n-m)(n+m-1)\overline{K}^0(x) + (p(p-1) + (m-p)(m-p-1))\overline{p}^0(x) \]

for all \( x \in N \), there is no minimal immersion from \( (M^m, g) \) into \( (N^n, h) \). Using the inequalities (2), we can easily see that the above condition implies

\[ \left( \frac{m(m-1)}{p(p-1) + (m-p)(m-p-1)} \right) \overline{K}^0(x) < \overline{p}^0(x) \leq \overline{K}^0(x) \]

for all \( x \in N \). And the Theorem 3.2 is of interest only for \( \overline{K}^0 < 0 \). \( \square \)

To finish this section, we study the particular case where the ambient space is the complex hyperbolic space \( \mathbb{CH}^n(c) \) with constant holomorphic curvature equal to \( c \) \((c < 0)\). Let us recall that in this case the curvature tensor of \( \mathbb{CH}^n(c) \) has the expression

\[ \overline{R}(X, Y, Z, W) = \frac{c}{4} \left( \langle X \wedge Y, Z \wedge W \rangle + \langle X \wedge Y, JZ \wedge JW \rangle + 2 \langle X, JY \rangle \langle Z, JW \rangle \right) \]  

(18)

for all \( X, Y, Z, W \in \Gamma(T\mathbb{CH}^n(c)) \). Here \( J \) denotes the complex structure of \( \mathbb{CH}^n(c) \). For any isometric immersion \( \phi \) of a Riemannian manifold \( (M^m, g) \) into \( \mathbb{CH}^n(c) \), we define the \((1,1)\)-tensor \( J_\phi \) on \( M \) by \( J_\phi X = \sum_{i \leq m} \langle Jd\phi(X), d\phi(e_i) \rangle e_i, \forall X \in \Gamma(TM) \) and for all orthonormal frame \((e_i)_{1 \leq i \leq m} \). Recall that the immersion \( \phi \) is said to be totally real if \( J_\phi \equiv 0 \).

For this kind of immersions we have the

**Theorem 3.3** Let \((M^m, g)\) be an \( m \)-dimensional Riemannian manifold admitting a non-trivial parallel \( p \)-form \((p \geq 1)\). Then there is no minimal totally real immersion of \((M^m, g)\) into \( \mathbb{CH}^n(c) \).
Proof: Let $\phi$ be a minimal immersion of $(M^m, g)$ into $\mathbb{H}^n$ and assume that $M$ has a non trivial parallel $p$-form $\omega$. Then from (13) we have

$$|B^+(\omega)|^2 = \sum_{i,j} (\overline{R}_{ij})_{ij} \langle i(e_i)\omega, i(e_j)\omega \rangle - \frac{1}{2} \sum_{i,j,k,l} \overline{R}_{ijkl} \langle i(e_j \wedge e_i)\omega, i(e_l \wedge e_k)\omega \rangle$$

Now the conclusion follows from a straightforward computation. Using (18) the above equality becomes

$$|B^+(\omega)|^2 = \frac{c}{4} \left( p(m-p)|\omega|^2 + 3 \sum_{i,j \leq m} \langle J_\phi e_i, J_\phi e_j \rangle \langle i(e_i)\omega, i(e_j)\omega \rangle - \sum_{i,j \leq m} \langle i(e_j \wedge e_i)\omega, i(e_j \wedge J_\phi e_i)\rangle \right)$$

where $(e_i)_{1 \leq i \leq m}$ is a local orthonormal frame.

4 Geometry of submanifolds with $b_p(M) \neq 0$

Let $(M^m, g)$ and $(N^n, h)$ be two Riemannian manifolds and assume that $(M^m, g)$ is compact. We use the same notations as in the previous sections. Moreover in this section, $k(x)$ denotes the smallest eigenvalue at $x$ of the Ricci curvature of $(M^m, g)$ and we put $k_0 = \min_M k(x)$. On the other hand, if $K^1$ is bounded above, we will set $K^1_{\text{max}} = \max_N K^1$ and $p^1_{\text{max}} = \max_N p^1$. The first result is the following theorem

**Theorem 4.1** Let $(M^m, g)$ be a compact Riemannian manifold of dimension $m \geq 2$ so that $b_p(M) \neq 0$ for some $p \geq 1$. Then for any isometric immersion $\phi$ from $(M^m, g)$ into an $n$-dimensional Riemannian manifold $(N^n, h)$, there exists at least a point $x \in M$ so that

$$\frac{m}{\sqrt{p}} \frac{p-1}{p} B(x)|H(x)| \geq k(x) - \left( \frac{p-1}{p} \right) (m-1)K^1 + p^1(\phi(x))$$

(19)

The following corollary is an immediate consequence of this theorem.

**Corollary 4.1** Let $(M^m, g)$ be a compact Riemannian manifold of dimension $m \geq 2$ minimally immersed in an $n$-dimensional Riemannian manifold $(N^n, h)$ ($n > m$). If $p^1$ is bounded above and if for an integer $p$ so that $1 \leq p \leq m/2$, we have
\[
k_0 > \left(\frac{p-1}{p}\right) \left((m-1)\overline{K}_{max} + \overline{\rho}_{max}\right)
\]
then \(b_q(M) = 0\) for \(q \in \{1, \ldots, p\}\).

Remark 4.1: The inequality (19) is an equality at each point for the standard embedding of \(S^p(\sqrt{p/m}) \times S^{k_1}(\sqrt{k_1/m}) \times \ldots \times S^{k_q}(\sqrt{k_q/m})\) into \(S^{m+r}\) where \(p + k_1 + \ldots + k_q = m\) with \(k_i \geq p\) \((1 \leq i \leq q)\). For the case \(p = 2\), (19) is also an equality at each point for the standard embedding from \(\mathbb{C}P^q\) with holomorphic curvature \(2q/(q+1)\) into \(S^{q^2+2q}\).

This theorem 4.1 is a generalization of a result obtained by El Soufi (see theorem 3.1 of [5]) in the particular case \(p = 2\).

To prove the theorem 4.1 we need the following proposition which gives an estimate of the term \(\langle R^p(\omega), \omega \rangle\) for any \(p\)-form \(\omega\).

Proposition 4.1 Let \((M^m, g)\) be an \(m\)-dimensional compact Riemannian manifold isometrically immersed in an \(n\)-dimensional Riemannian manifold \((N^n, h)\). Then for any \(p \in \{1, \ldots, m\}\) and for any \(p\)-form \(\omega\) of \(M\), we have for all \(x \in M\)

\[
\langle R^p(\omega), \omega \rangle \geq p \left(pk(x) - (p-1) \left((m-1)\overline{K} + \overline{\rho}\right) (\phi(x)) - m \left(\frac{p-1}{\sqrt{p}}\right) |H(x)||B(x)| \right) |\omega|^2
\]

Before proving this proposition, we introduce the following \(p\)-tensor associated to any \(p\)-form \(\omega\) of \(M\) and the isometric immersion \(\phi\)

\[
B^-(\omega) = \frac{1}{(p-2)!} \sum_{i, i_1, \ldots, i_{p-2} \leq m} ((i(e_i \wedge e_{i_1} \wedge \ldots \wedge e_{i_{p-2}}) \omega) \vee B(e_i, \ldots) \otimes (e_{i_1}^* \wedge \ldots \wedge e_{i_{p-2}}^*)
\]

where \((e_i)_{1 \leq i \leq m}\) is an orthonormal frame at a point \(x \in M\) and \(\vee\) denotes the symmetric product defined in the preliminaries (see (4)). It will be convenient to choose the norm \(|B^-(\omega)|\) so that

\[
|B^-(\omega)|^2 = \frac{1}{(p-2)!} \sum_{j, k, i_1, \ldots, i_{p-2} \leq m} |B^-(\omega)_{jk, i_1, \ldots, i_{p-2}}|^2
\]

Such a tensor has been introduced for the first time in [8] where the second fundamental form is replaced by the Hessian of a function. First note that if \(\omega\) is a volume form at a point \(p\) of \(M\) where we have define the Hodge operator \(\star\) so that \(\omega = \star 1\), we deduce from (20) shown in the proof of the proposition 4.1, that
\[
\frac{1}{2n}|B^-(v_g)|^2 = \frac{n-1}{n} \sum_{ijk \leq n} \langle B_{ik}, B_{jk} \rangle \langle i(e_i) \ast 1, i(e_j) \ast 1 \rangle \\
- \frac{1}{n} \sum_{ijkl \leq n} \langle B_{ij}, B_{kl} \rangle \langle i(e_i \wedge e_i) \ast 1, i(e_k \wedge e_j) \ast 1 \rangle \\
= \frac{n-1}{n} \sum_{ijk \leq n} \langle B_{ik}, B_{jk} \rangle \langle e_i^*, e_j^* \rangle - \frac{1}{n} \sum_{ijkl \leq n} \langle B_{ij}, B_{kl} \rangle \langle e_i^* \wedge e_j^*, e_k^* \wedge e_j^* \rangle \\
= |B|^2 - n|H|^2 = |B - H \otimes g|^2
\]

In other words we obtain the square of the umbilicity tensor. Moreover if \( \alpha \) denotes the Kaehler form of a Kaehlerian manifold, a straightforward calculation gives

\[
|B^-(\alpha)|^2 = 4|B^-|^2
\]

where \( B^- \) is the anti-holomorphic part of \( B \) (i.e. \( B^-(X, Y) = \frac{1}{2}(B(X, Y) - B(JX, JY)) \)) where \( \alpha(X, Y) = \langle JX, Y \rangle \).

On the other hand we will see in the proof of theorem 4.1 that \( B^-(\omega) \) is vanishing identically for the standard embedding of \( S^p(\sqrt{p/m}) \times S^{k_1}(\sqrt{k_1/m}) \times \cdots \times S^{k_q}(\sqrt{k_q/m}) \) into \( S^{m+r} \) where \( p + k_1 + \cdots + k_q = m \) with \( k_i \geq p \) (1 \( \leq i \leq q \).

**PROOF OF THE PROPOSITION 4.1:** For \( p = 1 \) and from the relation (6), the equality of the proposition is obvious. Suppose now that \( p \geq 2 \). Let \( x \in M \) and let \( \langle e_i \rangle_{1 \leq i \leq m} \) be a local orthonormal frame in a neighborhood of \( x \). We have

\[
\frac{(p-2)!}{2}|B^-(\omega)|^2 = \frac{1}{2} \sum_{i,j,i_1,\ldots,i_{p-2}} |B^-(\omega)|_{i,j,i_1,\ldots,i_{p-2}}^2 = \\
= \frac{1}{2} \sum_{i,j,k,l,i_1,\ldots,i_{p-2}} \langle (i(e_i \wedge e_i \wedge \ldots \wedge e_{i_{p-2}}) \omega \wedge B(e_i, \ldots) \rangle_{kl}, (i(e_j \wedge e_i \wedge \ldots \wedge e_{i_{p-2}}) \omega \wedge B(e_j, \ldots) \rangle_{kl} \\
= \sum_{i,j,k,l,i_1,\ldots,i_{p-2}} \langle i(e_i \wedge e_i \wedge \ldots \wedge e_{i_{p-2}}) \omega \rangle_k \langle i(e_j \wedge e_i \wedge \ldots \wedge e_{i_{p-2}}) \omega \rangle_k \langle B_{dl}, B_{jl} \rangle \\
+ \sum_{i,j,k,l,i_1,\ldots,i_{p-2}} \langle i(e_i \wedge e_i \wedge \ldots \wedge e_{i_{p-2}}) \omega \rangle_k \langle i(e_j \wedge e_i \wedge \ldots \wedge e_{i_{p-2}}) \omega \rangle_l \langle B_{dl}, B_{jk} \rangle \\
= \sum_{i,j,k,l,i_1,\ldots,i_{p-2}} \langle B_{dl}, B_{jl} \rangle \omega_{i_k l_{i_1} \ldots i_{p-2}} \omega_{j_k l_{i_1} \ldots i_{p-2}} + \sum_{i,j,k,l,i_1,\ldots,i_{p-2}} \langle B_{dl}, B_{jk} \rangle \omega_{i_k l_{i_1} \ldots i_{p-2}} \omega_{j_k l_{i_1} \ldots i_{p-2}} \\
= (p-1)! \sum_{i,j,k} \langle B_{ik}, B_{jk} \rangle \langle i(e_i) \omega, i(e_j) \omega \rangle - (p-2)! \sum_{i,j,k,l} \langle B_{ij}, B_{kl} \rangle \langle i(e_i \wedge e_i) \omega, i(e_k \wedge e_j) \omega \rangle
\]

Finally, we have proved that
\[
\frac{1}{2} |B^{-}(\omega)|^2 = (p - 1) \sum_{i,j,k} \langle B_{ik}, B_{jk} \rangle \langle i(e_i)\omega, i(e_j)\omega \rangle - \sum_{i,j,k,l} \langle B_{ik}, B_{jl} \rangle \langle i(e_i \wedge e_j)\omega, i(e_k \wedge e_l)\omega \rangle
\]

(20)

Now, combining this with the relations (10) and (11) and using the expression of \( \langle R_p(\omega), \omega \rangle \) (see (6)), we get

\[
\frac{1}{2} |B^{-}(\omega)|^2 = \langle R_p(\omega), \omega \rangle - p \sum_{i,j} \text{Ric}_{ij} \langle i(e_i)\omega, i(e_j)\omega \rangle + (p - 1) \sum_{i,j} \langle R_{i,j} \rangle^2 (x) \left| \omega \right|^2 + m(p - 1) \sum_{i,j} \langle B_{ij}, H \rangle \langle i(e_i)\omega, i(e_j)\omega \rangle
\]

(21)

From the hypotheses on the curvature of \( N \) and by techniques already used in the proof of the theorem 3.1 (see (14) and (15)) we deduce that

\[
\frac{1}{2} |B^{-}(\omega)|^2 \leq \langle R_p(\omega), \omega \rangle - p \sum_{i,j} \text{Ric}_{ij} \langle i(e_i)\omega, i(e_j)\omega \rangle + p(p - 1) \left( (m - 1)\overline{K}^1 + \overline{p}^1(\omega) \right) |\omega|^2 + m(p - 1) \sum_{i,j} \langle B_{ij}, H \rangle \langle i(e_i)\omega, i(e_j)\omega \rangle
\]

(22)

Now, let us estimate the last term. For this, assume that at the point \( x \in M \), \( (e_i)_{1 \leq i \leq m} \) diagonalizes the symmetric tensor \( \langle B(X, Y), H \rangle \). We have

\[
\sum_{i,j} \langle B_{ij}, H \rangle \langle i(e_i)\omega, i(e_j)\omega \rangle = \frac{1}{(p - 1)!} \sum_{i_1, \ldots, i_{p-1}} \langle B_{i_{i_1}, H} \rangle \omega_{i_{i_1} \ldots i_{p-1}}^2
\]

\[
= \frac{1}{p!} \sum_{i, i_{1}, \ldots, i_{p-1}} \left( \langle B_{i_{i_1}, H} \rangle + \langle B_{i_{i_1}i_2}, H \rangle + \ldots + \langle B_{i_{i_{p-1}i_{p-1}}, H} \rangle \right) \omega_{i_{i_1} \ldots i_{p-1}}^2
\]

\[
\leq \frac{1}{p!} \sum_{i, i_{1}, \ldots, i_{p-1}} \left( |B_{i_{i_1}}| + |B_{i_{i_1}i_2}| + \ldots + |B_{i_{i_{p-1}i_{p-1}}}| \right) |H| \omega_{i_{i_1} \ldots i_{p-1}}^2
\]

\[
\leq \frac{\sqrt{p}}{p!} \sum_{i, i_{1}, \ldots, i_{p-1}} |B| |H| \omega_{i_{i_1} \ldots i_{p-1}}^2
\]

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Finally we have proved
\[\sum_{i,j} \langle B_{ij}, H \rangle \langle i(e_i)\omega, i(e_j)\omega \rangle \leq \sqrt{p} \|B\| \|H\| \|\omega\|^2 \] (23)

Since \(Ric \geq kg\), we have \(\sum_{i,j} Ric_{ij} \langle i(e_i)\omega, i(e_j)\omega \rangle \geq pk\|\omega\|^2\), and we deduce from this, (22) and (23), the inequality of the proposition 4.1.

The proof of the theorem 4.1 is now an immediate consequence of the proposition 4.1.

**Proof of theorem 4.1:** Since \(b_p(M) \neq 0\), there exists a nontrivial \(p\)-form \(\omega\) so that \(\Delta \omega = 0\). And from the Weitzenböck formula (5), we deduce
\[\int_M \langle R_p(\omega), \omega \rangle \leq 0 \] (24)

Now, applying the estimate of the proposition 4.1 we get the desired inequality.

We can show a similar result to the theorem 4.1, with the scalar curvature \(Scal\) of \((M^m, g)\) instead of the Ricci curvature.

**Theorem 4.2** Let \((M^m, g)\) be a compact Riemannian manifold of dimension \(m \geq 2\) so that \(b_p(M) \neq 0\) for a \(p \geq 1\). Then for any isometric immersion \(\phi\) from \((M^m, g)\) into an \(n\)-dimensional Riemannian manifold \((N^n, h)\), there exists at least a point \(x \in M\) so that
\[m \left( \frac{p - 1}{\sqrt{p}} + \frac{m - p - 1}{\sqrt{m - p}} \right) |B(x)||H(x)| \geq Scal(x) - (m - 2)((m - 1)K_1 + \bar{p}^1)(\phi(x)) \] (25)

We immediately deduce the following

**Corollary 4.2** Let \((M^m, g)\) be a compact Riemannian manifold of dimension \(m \geq 2\) minimally immersed into an \(n\)-dimensional Riemannian manifold \((N^n, h)\) \((n > m)\). If \(\bar{p}^1\) is bounded above and if
\[\min_M Scal > (m - 2) \left( (m - 1)K_1^\text{max} + \bar{p}_\text{max}^1 \right)\]
then for any \(p \in \{1, \ldots, m\}\), we have \(b_p(M) = 0\).

**Remark 4.2:** If \(p = m/2\), we can improve (25) to obtain
\[m(m - 2)|H(x)|^2 \geq Scal(x) - (m - 2)((m - 1)K_1^1 + \bar{p}^1)(\phi(x)) \] (26)
The theorem 4.2 and the inequality (26) was obtained by El Soufi (theorem 3.1 and theorem 3.2 of [5]) in the particular case where $m = 4$ and $p = 2$.

On the other hand, note that for $p \neq m/2$, the theorem 4.1 is not a consequence of the theorem 4.2.

For the same reasons as in the proof of theorem 3.2, we can choose locally an orientation on $M$ and define locally the Hodge operator $\star$. But for all $p$-form $\omega$ of $M$, the quantity $\langle R_{m-p}(\star \omega), \star \omega \rangle$ is globally defined.

The theorem 4.2 is a consequence of the following proposition

**Proposition 4.2** Let $(M^m, g)$ be an $m$-dimensional compact Riemannian manifold isometrically immersed in an $n$-dimensional Riemannian manifold $(N^n, h)$. Then for all $p \in \{1, ..., m-1\}$ and for all $p$-form $\omega$ on $(M^m, g)$, we have for all $x \in M$

$$\langle R_p(\omega), \omega \rangle + \frac{p}{m-p} \langle R_{m-p}(\star \omega), \star \omega \rangle \geq p \left( \text{Scal}(x) - (m-2) \left( (m-1)K_1 + p^1 \right)(\phi(x)) \right)$$

$$- m \left( \frac{p-1}{\sqrt{p}} + \frac{m-p-1}{\sqrt{m-p}} \right) |H(x)||B(x)| |\omega|^2$$

**Remark 4.3:** If $p = m/2$, we can improve this inequality (see the proof of the proposition 4.2) to obtain

$$\langle R_p(\omega), \omega \rangle + \frac{p}{m-p} \langle R_{m-p}(\star \omega), \star \omega \rangle \geq$$

$$\langle R_p(\omega), \omega \rangle + \frac{p}{m-p} \langle R_{m-p}(\star \omega), \star \omega \rangle \geq$$

$$p \left( \text{Scal}(x) - (m-2) \left( (m-1)K_1 + p^1 \right)(\phi(x)) - m(m-2)|H(x)||\omega|^2 \right)$$

And if for $p \geq 1$, $b_p(M) \neq 0$, we get (26).

**Proof of the Proposition 4.2:** From the inequality (22) we obtain that for all $p \geq 1$

$$\langle R_p(\omega), \omega \rangle \geq p \sum_{i,j} R_{ij} \langle i(e_i)\omega, i(e_j)\omega \rangle$$

$$- p(p-1) \left( (m-1)K_1 + p^1 \right)(\phi(x)) |\omega|^2$$

$$- m(p-1) \sum_{i,j} \langle B_{ij}, H \rangle \langle i(e_i)\omega, i(e_j)\omega \rangle$$

Since $\star \omega$ is a $(m-p)$-form we have also

$$\langle R_{m-p}(\star \omega), \star \omega \rangle \geq (m-p) \sum_{i,j} R_{ij} \langle i(e_i) \star \omega, i(e_j) \star \omega \rangle$$

$$- (m-p)(m-p-1) \left( (m-1)K_1 + p^1 \right)(\phi(x)) |\omega|^2$$

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Multiplying (29) by $p/(m - p)$, and summing the obtained inequality with (28), we find

\[
\langle R_p(\omega), \omega \rangle + \frac{p}{m - p} \langle R_{m-p}(\star \omega), \star \omega \rangle \geq
\]

\[
p \sum_{i,j} (\text{Ric}_{ij} \langle i(e_i) \omega, i(e_j) \omega \rangle + \text{Ric}_{ij} \langle i(e_i) \star \omega, i(e_j) \star \omega \rangle)
\]

\[- p(m - 2) \left( (m - 1)K^1 + \bar{p}^1 \right) \langle \phi(x) \rangle |\omega|^2
\]

\[- m(p - 1) \sum_{i,j} \langle B_{ij}, H \rangle \langle i(e_i) \omega, i(e_j) \omega \rangle - mp \left( \frac{m - p - 1}{m - p} \right) \sum_{i,j} \langle B_{ij}, H \rangle \langle i(e_i) \star \omega, i(e_j) \star \omega \rangle
\]

By computations which are similar to (17) we get

\[
\sum_{i,j} (\text{Ric}_{ij} \langle i(e_i) \omega, i(e_j) \omega \rangle + \text{Ric}_{ij} \langle i(e_i) \star \omega, i(e_j) \star \omega \rangle) = \text{Scal} |\omega|^2
\]

Thus

\[
\langle R_p(\omega), \omega \rangle + \frac{p}{m - p} \langle R_{m-p}(\star \omega), \star \omega \rangle \geq \text{pScal} |\omega|^2
\]

\[- p(m - 2) \left( (m - 1)K^1 + \bar{p}^1 \right) \langle \phi(x) \rangle |\omega|^2
\]

\[- m(p - 1) \sum_{i,j} \langle B_{ij}, H \rangle \langle i(e_i) \omega, i(e_j) \omega \rangle - mp \left( \frac{m - p - 1}{m - p} \right) \sum_{i,j} \langle B_{ij}, H \rangle \langle i(e_i) \star \omega, i(e_j) \star \omega \rangle
\]

( Note that if $p = m/2$, then $p - 1 = p \left( \frac{m - p - 1}{m - p} \right)$, and we show with the same arguments as in the proof of (30) that

\[
m(p - 1) \sum_{i,j} \langle B_{ij}, H \rangle \langle i(e_i) \omega, i(e_j) \omega \rangle + mp \left( \frac{m - p - 1}{m - p} \right) \sum_{i,j} \langle B_{ij}, H \rangle \langle i(e_i) \star \omega, i(e_j) \star \omega \rangle =
\]

\[p(m - 2)m |H|^2 |\omega|^2
\]

and reporting this in (31), we obtain the inequality (27) of the remark 3.3. )

From the estimate (23), we deduce
\[ m(p-1) \sum_{i,j} \langle B_{ij}, H \rangle \langle i(e_i) \omega, i(e_j) \omega \rangle \leq m(p-1) \sqrt{p} |H| |B| |\omega|^2 \]

and

\[ mp \left( \frac{m-p-1}{m-p} \right) \sum_{i,j} \langle B_{ij}, H \rangle \langle i(e_i) \star \omega, i(e_j) \star \omega \rangle \leq \frac{mp(m-p-1)}{\sqrt{m-p}} |H| |B| |\omega|^2 \]

now reporting these inequalities in (31), we find the inequality of the proposition.

The proof of the theorem is now immediate.

**Proof of Theorem 4.2:** Let \( \omega \) be a harmonic \( p \)-form. Since the Hodge operator commutes with the Laplacian, then \( \star \omega \) is a harmonic \( (m-p) \)-form, and we have

\[ \int_M \left( \langle R_p(\omega), \omega \rangle + \frac{p}{m-p} \langle R_{m-p}(\star \omega), \star \omega \rangle \right) \leq 0 \]

and the theorem follows from the proposition 4.2.

**Remark 4.4:** We can improve all the results of this section by considering the particular case of submanifolds of the complex projective space \( \mathbb{C}P^n(c) \) \((c > 0)\). We just need to compute in (21) the terms with the curvature tensor of the complex projective space. Then we obtain the same statements as previously by replacing \((m-1)K + p_1\) by \(c/8((m-1)+3 \| J_\phi \|^2 \) where for all \( x \in M, \| J_\phi \| (x) = \sup\{ \| J_\phi (X) \| / X \in T_x M \text{ and } |X| = 1 \} \). In particular, if \((M^m, g)\) is an \( m \)-dimensional compact Riemannian manifold minimally immersed in \( \mathbb{C}P^n(c) \) and if

\[ \min_M (\text{Scal}) > \frac{c(m-2)((m-1)+3 \max_M \| J_\phi \|^2)}{8} \]

then for any \( p \in \{1, \ldots, m\} \), we have \( b_p(M) = 0 \).

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**References**


