

Viscous potential free-surface flows in a fluid layer of finite depth

Denys Dutykh, Frédéric Dias

► **To cite this version:**

Denys Dutykh, Frédéric Dias. Viscous potential free-surface flows in a fluid layer of finite depth. Comptes Rendus Mathématique, Elsevier Masson, 2007, 345 (2), pp.113-118. 10.1016/j.crma.2007.06.007 . hal-00145315v3

HAL Id: hal-00145315

<https://hal.archives-ouvertes.fr/hal-00145315v3>

Submitted on 2 Aug 2007

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Viscous potential free-surface flows in a fluid layer of finite depth

Denys Dutykh, Frédéric Dias

CMLA, ENS Cachan, CNRS, PRES UniverSud, 61, avenue du Président Wilson, 94230 Cachan cedex, France

Abstract

It is shown how to model weakly dissipative free-surface flows using the classical potential flow approach. The Helmholtz–Leray decomposition is applied to the linearized 3D Navier–Stokes equations. The governing equations are treated using Fourier–Laplace transforms. We show how to express the vortical component of the velocity only in terms of the potential and free-surface elevation. A new predominant nonlocal viscous term is derived in the bottom kinematic boundary condition. The resulting formulation is simple and does not involve any correction procedure as in previous viscous potential flow theories [7]. Corresponding long wave model equations are derived.

Résumé

Écoulements viscopotentiels avec surface libre en profondeur finie. Nous montrons comment modéliser les écoulements à surface libre faiblement dissipatifs en utilisant une approche potentielle. La décomposition de Helmholtz–Leray est appliquée aux équations de Navier–Stokes linéarisées. Le problème est étudié au moyen de la transformée de Fourier–Laplace. Nous montrons comment exprimer la partie rotationnelle de la vitesse en fonction du potentiel des vitesses et de l’élévation de la surface libre. Un nouveau terme nonlocal prépondérant apparaît dans la condition cinématique au fond. La formulation finale est simple et ne requiert pas de corrections ultérieures comme dans [7]. Un modèle d’ondes longues est obtenu à partir de ces nouvelles équations.

Key words: potential flow, free-surface flow, viscosity, dissipation, water waves, wave damping

Mots-clés : écoulement potentiel, écoulement avec surface libre, viscosité, dissipation, vagues, atténuation de vague

Version française abrégée

Récemment, Dias et al. [3] ont donné une nouvelle formulation du problème des vagues en profondeur infinie en présence de faible dissipation. L’idée principale est de partir des équations de Navier–Stokes linéarisées (1) où ν représente la viscosité cinématique, \mathbf{v} le vecteur vitesse, \mathbf{g} l’accélération due à la

Email addresses: Denys.Dutykh@cmla.ens-cachan.fr (Denys Dutykh), dias@cmla.ens-cachan.fr (Frédéric Dias).

URLs: www.cmla.ens-cachan.fr/~dutykh (Denys Dutykh), www.cmla.ens-cachan.fr/~dias (Frédéric Dias).

gravité, p la pression, ρ la densité, t le temps, et d'utiliser la décomposition de Helmholtz–Leray (2) pour la vitesse. Ensuite l'on exprime la composante rotationnelle $\boldsymbol{\psi}$ de la vitesse en fonction de la composante potentielle ϕ et de la déformation η de la surface libre. Nous présentons ici une extension au cas d'une couche de fluide de profondeur finie. Dans ce cas-là il faut tenir compte des frottements au fond.

En utilisant la transformée de Fourier–Laplace, on écrit la solution générale pour ϕ et $\boldsymbol{\psi}$. Ensuite l'on utilise la condition dynamique sur la surface libre. La nullité des deux composantes tangentielles de la contrainte donne les équations (3) et (4), qui peuvent être recombinaées pour donner l'équation (5). Cette même équation, combinée avec la condition cinématique sur la surface libre (6), donne un lien entre η et $\boldsymbol{\psi}$ (7) et un autre lien entre η et ϕ (8). Le remplacement de la composante rotationnelle dans la condition cinématique à la surface libre donne la première équation modifiée (9). De même l'équilibre de la contrainte normale au niveau de la surface libre donne l'équation de Bernoulli modifiée (10).

La seconde partie de la note consiste à incorporer une correction de type couche limite dans la condition cinématique au fond. L'idée est d'introduire l'épaisseur de la couche limite proportionnelle à $\delta = \sqrt{\nu}$, la coordonnée de couche limite $\zeta = (z + h)/\delta$, puis d'effectuer un développement du potentiel et de la vitesse rotationnelle en fonction du petit paramètre δ . On obtient ensuite une hiérarchie de problèmes à résoudre. Ce qui nous intéresse, c'est la correction à apporter à la condition cinématique au fond (13). Après quelques calculs, on obtient la correction présentée dans l'équation (14). Finalement nous proposons le nouveau système d'équations (15)–(18) pour l'étude des vagues avec faible viscosité en profondeur finie. Notons que les termes nonlinéaires ont été ajoutés de façon heuristique. A partir de ce système, nous pouvons dériver le système de Boussinesq faiblement dissipatif donné par les équations (19)–(20).

1. Introduction

The effects of viscosity on gravity waves have been addressed since the end of the nineteenth century in the context of the linearized Navier–Stokes (NS) equations. It is well-known that Lamb [9] studied this question in the case of oscillatory waves on deep water. What is less known is that Boussinesq studied this effect as well [1]. In this particular case they both showed that

$$\frac{\partial \alpha}{\partial t} = -2\nu k^2 \alpha,$$

where α denotes the wave amplitude, ν the kinematic viscosity of the fluid and k the wavenumber of the decaying wave. This equation leads to the classical law for viscous decay, namely $\alpha(t) = \alpha_0 e^{-2\nu k^2 t}$.

In this work we keep the features of undamped free-surface flows while adding dissipative effects. The classical theory of viscous potential flows [7] is based on pressure and boundary conditions corrections due to the presence of viscous stresses. We present here a novel approach.

Currently, potential flows with ad-hoc dissipative terms are used for example in direct numerical simulations of weak turbulence of gravity waves [5,6,11]. There have also been several attempts to introduce dissipative effects into long wave modelling [10,4,2].

The present article is a direct continuation of the recent study [3]. In that work the authors considered two-dimensional (2D) periodic waves in infinite depth, while in the present study we remove these two hypotheses and all the computations are done in 3D. This point is important since the vorticity structure is more complicated in 3D. In other words we consider a general wavetrain on the free surface of a fluid layer of finite depth. As a result we obtain a qualitatively different formulation which contains a nonlocal term in the bottom kinematic condition. The inclusion of this term is natural since it represents the correction to potential flow due to the presence of a boundary layer. Moreover, this term is predominant since its magnitude scales with $\mathcal{O}(\sqrt{\nu})$, while other terms in the free-surface boundary conditions are of order $\mathcal{O}(\nu)$. Other researchers have obtained nonlocal corrections but they differ from ours [8].

2. Derivation

Consider the linearized 3D incompressible NS equations describing free-surface flows in a fluid layer of uniform depth h :

$$\frac{\partial \mathbf{v}}{\partial t} = -\frac{1}{\rho} \nabla p + \nu \Delta \mathbf{v} + \mathbf{g}, \quad \nabla \cdot \mathbf{v} = 0, \quad (1)$$

with \mathbf{v} the velocity vector, p the pressure, ρ the fluid density and \mathbf{g} the acceleration due to gravity. We represent $\mathbf{v} = (u, v, w)$ in the form of the Helmholtz–Leray decomposition:

$$\mathbf{v} = \nabla \phi + \nabla \times \boldsymbol{\psi}, \quad \boldsymbol{\psi} = (\psi_1, \psi_2, \psi_3). \quad (2)$$

After substitution of the decomposition (2) into (1), one notices that the equations are verified provided that the functions ϕ and $\boldsymbol{\psi}$ satisfy the following equations:

$$\Delta \phi = 0, \quad \phi_t + \frac{p - p_0}{\rho} + gz = 0, \quad \frac{\partial \boldsymbol{\psi}}{\partial t} = \nu \Delta \boldsymbol{\psi}.$$

Next we discuss the boundary conditions. We assume that the velocity field satisfies the conventional no-slip condition at the bottom $\mathbf{v}|_{z=-h} = \mathbf{0}$, while at the free surface we have the usual kinematic condition $\eta_t = w$ and three dynamic conditions $[\underline{\boldsymbol{\sigma}} \cdot \mathbf{n}] = \mathbf{0}$, where $\underline{\boldsymbol{\sigma}}$ is the stress tensor, $[f]$ denotes the jump of a function f across the free surface, and the normal vector \mathbf{n} equals $(0, 0, 1)$ due to linearization.

Using Fourier–Laplace transforms, which we denote by $\mathcal{L}_{\mathcal{F}} \equiv \mathcal{L} \circ \mathcal{F}$, $f(\mathbf{x}, t) \xrightarrow{\mathcal{L}_{\mathcal{F}}} \hat{f}(\mathbf{k}, s)$, $\mathbf{k} = (k_x, k_y)$ we can determine the structure of the unknown functions ϕ , $\boldsymbol{\psi}$ in the transform space. We assume that all the functions involved in the present computation satisfy the necessary regularity requirements and have sufficient decay at infinity so that the integral transforms can be applied. The solution for ϕ is obtained from the transformed continuity equation $\Delta \phi = 0 \xrightarrow{\mathcal{L}_{\mathcal{F}}} \hat{\phi}_{zz} - |\mathbf{k}|^2 \hat{\phi} = 0$ and $\boldsymbol{\psi}$ from the corresponding transformed equation $\boldsymbol{\psi}_t = \nu \Delta \boldsymbol{\psi} \xrightarrow{\mathcal{L}_{\mathcal{F}}} s \hat{\boldsymbol{\psi}} = \nu (\hat{\boldsymbol{\psi}}_{zz} - |\mathbf{k}|^2 \hat{\boldsymbol{\psi}})$:

$$\hat{\phi} = \hat{\phi}_0^+(\mathbf{k}, s) e^{|\mathbf{k}|z} + \hat{\phi}_0^-(\mathbf{k}, s) e^{-|\mathbf{k}|z}, \quad \hat{\psi}_i = \hat{\psi}_{i0}(\mathbf{k}, s) (e^{|\mathbf{k}|z} + C_i(\mathbf{k}, s) e^{-|\mathbf{k}|z}),$$

where $m^2 := |\mathbf{k}|^2 + s/\nu$ and $\hat{\phi}_0^+$, $\hat{\phi}_0^-$, $\hat{\psi}_0$, $\mathbf{C} := (C_1, C_2, C_3)$ are unknown functions of the transform parameters (\mathbf{k}, s) , determined by the initial and appropriate boundary conditions.

There are three dynamic conditions on the free surface. Let us use first those related to the tangential stresses (the third one will be used later), where $\mu = \rho\nu$:

$$\sigma_{xz} = \mu \left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} \right) = 0, \quad \sigma_{yz} = \mu \left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} \right) = 0, \quad \text{at } z = 0.$$

Substituting decomposition (2) into these two identities yields

$$2 \frac{\partial^2 \phi}{\partial x \partial z} + \frac{\partial^2 \psi_2}{\partial x^2} - \frac{\partial^2 \psi_1}{\partial x \partial y} + \frac{\partial^2 \psi_3}{\partial y \partial z} - \frac{\partial^2 \psi_2}{\partial z^2} = 0, \quad z = 0, \quad (3)$$

$$2 \frac{\partial^2 \phi}{\partial y \partial z} + \frac{\partial^2 \psi_2}{\partial x \partial y} - \frac{\partial^2 \psi_1}{\partial y^2} + \frac{\partial^2 \psi_1}{\partial z^2} - \frac{\partial^2 \psi_3}{\partial x \partial z} = 0, \quad z = 0. \quad (4)$$

The next step consists in taking the Fourier–Laplace transform to these relations. We do not give here the explicit expressions since this operation is straightforward. The combination $(-ik_x)(3) + (-ik_y)(4)$ gives the important relation

$$ik_y \hat{\psi}_{10}(1 + C_1) - ik_x \hat{\psi}_{20}(1 + C_2) = -\frac{2|\mathbf{k}|^3}{m^2 + |\mathbf{k}|^2} (\hat{\phi}_0^+ - \hat{\phi}_0^-). \quad (5)$$

Let us turn to the free-surface kinematic condition $\frac{\partial \eta}{\partial t} = w \equiv \frac{\partial \phi}{\partial z} + \frac{\partial \psi_2}{\partial x} - \frac{\partial \psi_1}{\partial y}$, $z = 0$. In transform space it becomes

$$s\hat{\eta} = |\mathbf{k}|(\hat{\varphi}_0^+ - \hat{\varphi}_0^-) + ik_y \hat{\psi}_{10}(1 + C_1) - ik_x \hat{\psi}_{20}(1 + C_2). \quad (6)$$

Equations (5) and (6) can be rewritten as

$$\frac{|\mathbf{k}|(\hat{\varphi}_0^+ - \hat{\varphi}_0^-)}{\nu(m^2 + |\mathbf{k}|^2)} = \hat{\eta}, \quad (7)$$

$$ik_y \hat{\psi}_{10}(1 + C_1) - ik_x \hat{\psi}_{20}(1 + C_2) = -2\nu |\mathbf{k}|^2 \hat{\eta}. \quad (8)$$

Using (8) one can replace the rotational part in the kinematic free-surface condition:

$$\eta_t = \phi_z + \mathcal{L}_{\mathcal{F}}^{-1}[-2\nu |\mathbf{k}|^2 \hat{\eta}] = \phi_z + 2\nu \Delta \eta. \quad (9)$$

In order to account for the presence of viscous stresses, we have to modify the dynamic free-surface condition as well. This is done using the balance of normal stresses at the free surface:

$$\sigma_{zz} = 0 \text{ at } z = 0 \Rightarrow p - p_0 = 2\rho\nu \frac{\partial w}{\partial z} \equiv 2\rho\nu \left(\frac{\partial^2 \phi}{\partial z^2} + \frac{\partial^2 \psi_2}{\partial x \partial z} - \frac{\partial^2 \psi_1}{\partial y \partial z} \right).$$

Using (8) one can show that $\frac{\partial^2 \psi_2}{\partial x \partial z} - \frac{\partial^2 \psi_1}{\partial y \partial z} = \mathcal{O}(\nu^{\frac{1}{2}})$, so Bernoulli's equation becomes

$$\phi_t + g\eta + 2\nu\phi_{zz} + \mathcal{O}(\nu^{\frac{3}{2}}) = 0. \quad (10)$$

Since we only consider weak dissipation ($\nu \sim 10^{-6} - 10^{-3} \text{ m}^2/\text{s}$), we neglect terms of order $o(\nu)$.

The second step in our derivation consists in introducing a boundary layer correction at the bottom. Obviously, this was not done in the previous study [3], since the derivation dealt with the infinite depth case. In order to include this modification, we consider a semi-infinite fluid layer as it is usually done in boundary layer theory. The fluid occupies the domain $z > -h$. In this derivation we use the pure Leray decomposition of the velocity field $\mathbf{v} = \nabla\phi + \mathbf{u}$ together with the divergence-free constraint $\nabla \cdot \mathbf{u} = 0$. Expecting that the rotational part \mathbf{u} varies rapidly in a distance $\delta = \sqrt{\nu}$,¹ we introduce the boundary-layer coordinate $\zeta \equiv (z + h)/\delta$, so that $\mathbf{u} = \mathbf{u}(\mathbf{x}, \zeta) = (\mathbf{u}_x, u_z)(\mathbf{x}, \zeta)$. The solid boundary is given by $\zeta = 0$, and the potential part of the flow is not subject to this change of variables. With the new scaling, the divergence-free condition becomes

$$\frac{\partial u_z}{\partial \zeta} + \delta \nabla_{\mathbf{x}} \cdot \mathbf{u}_x = 0. \quad (11)$$

As done in [10], we expand the unknown functions in powers of the small parameter δ :

$$\phi = \phi_0(\mathbf{x}, z, t) + \delta\phi_1(\mathbf{x}, z, t) + \dots, \quad \mathbf{u} = \mathbf{q}_0(\mathbf{x}, \zeta, t) + \delta\mathbf{q}_1(\mathbf{x}, \zeta, t) + \dots$$

Substituting the expansion for \mathbf{u} into (11) gives the following relations:

$$\delta^0: \quad \frac{\partial q_{0z}}{\partial \zeta} = 0, \quad \delta^1: \quad \frac{\partial q_{1z}}{\partial \zeta} = -\nabla_{\mathbf{x}} \cdot \mathbf{q}_{0x}, \quad (12)$$

where \mathbf{q}_{0x} denotes the first two components of the vector \mathbf{q}_0 corresponding to the horizontal coordinates \mathbf{x} . Recall that we would like to determine the correction to the bottom boundary condition

$$\phi_z = -u_z|_{\zeta=0} = -(q_{0z} + \delta q_{1z})|_{\zeta=0} + o(\delta). \quad (13)$$

So we only need to compute q_{0z} and q_{1z} at the bottom $\zeta = 0$.

¹ Of course we should nondimensionalize all quantities in order to define small numbers. One would find that δ is in fact equivalent to $\sqrt{Re^{-1}} \times L$, where Re is the Reynolds number and L a typical length.

Using the same asymptotic considerations as above, we can write down the following sequence of problems:

$$\begin{aligned} \Delta\phi_0 = 0, \quad \frac{\partial\phi_0}{\partial z}\Big|_{z=-h} &= 0, & \frac{\partial\mathbf{q}_0}{\partial t} = \frac{\partial^2\mathbf{q}_0}{\partial\zeta^2}, \quad \mathbf{q}_0 = -\nabla\phi_0|_{\zeta=0}, \\ \Delta\phi_1 = 0, \quad \frac{\partial\phi_1}{\partial z}\Big|_{z=-h} &= -q_{1z}, & \frac{\partial\mathbf{q}_1}{\partial t} = \frac{\partial^2\mathbf{q}_1}{\partial\zeta^2}, \quad \mathbf{q}_1 = -\nabla\phi_1|_{\zeta=0}, \end{aligned}$$

together with the radiation condition $\mathbf{q} \rightarrow \mathbf{0}$ as $\zeta \rightarrow \infty$.

This sequence of linear problems can be solved using Fourier transforms. In Fourier space one finds immediately that $\hat{\phi}_0(t, z, \mathbf{k}) = \hat{\varphi}_0(t, \mathbf{k})(e^{|\mathbf{k}|z} + e^{-|\mathbf{k}|z})$. Since we know $\hat{\phi}_0$, we can determine the rotational component $\hat{\mathbf{q}}_0$.

Analytical solutions to the equation $\partial\hat{\mathbf{q}}_{0\mathbf{x}}/\partial t = \partial^2\hat{\mathbf{q}}_{0\mathbf{x}}/\partial\zeta^2$ are well-known. If we assume that initially the flow is potential and the boundary condition is $\hat{\mathbf{q}}_{0\mathbf{x}} = i\mathbf{k}\hat{\phi}_0(z = -h; \mathbf{k})$, the solution is

$$\hat{\mathbf{q}}_{0\mathbf{x}} = \frac{1}{2\sqrt{\pi}} \int_0^t \frac{\zeta}{(t-\tau)^{\frac{3}{2}}} e^{-\frac{\zeta^2}{4(t-\tau)}} i\mathbf{k}\hat{\phi}_0(\tau, z = -h, \mathbf{k}) d\tau.$$

Let us now integrate the second equation in (12) from 0 to ∞ , using the appropriate decay at infinity:

$$\hat{q}_{1z}|_{\zeta=0} = - \int_0^\infty i\mathbf{k} \cdot \hat{\mathbf{q}}_{0\mathbf{x}} d\zeta = \frac{1}{2\sqrt{\pi}} \int_0^\infty \int_0^t \frac{\zeta}{(t-\tau)^{\frac{3}{2}}} e^{-\frac{\zeta^2}{4(t-\tau)}} |\mathbf{k}|^2 \hat{\phi}_0(\tau, z = -h, \mathbf{k}) d\tau d\zeta.$$

One can interchange integral signs and evaluate the inner integral on ζ to obtain: $\hat{q}_{1z}|_{\zeta=0} = \frac{1}{\sqrt{\pi}} \int_0^t \frac{|\mathbf{k}|^2 \hat{\phi}_0(\tau, z = -h, \mathbf{k})}{\sqrt{t-\tau}} d\tau$.

Hence, the bottom boundary condition becomes, at order δ ,

$$\frac{\partial\phi}{\partial z}\Big|_{z=-h} = -\sqrt{\frac{\nu}{\pi}} \int_0^t \frac{\mathcal{F}^{-1}(|\mathbf{k}|^2 \hat{\phi}_0(-h, \mathbf{k}))}{\sqrt{t-\tau}} d\tau = \sqrt{\frac{\nu}{\pi}} \int_0^t \frac{\nabla_{\mathbf{x}}^2 \phi_0|_{z=-h}}{\sqrt{t-\tau}} d\tau = -\sqrt{\frac{\nu}{\pi}} \int_0^t \frac{\phi_{0zz}|_{z=-h}}{\sqrt{t-\tau}} d\tau. \quad (14)$$

One recognizes on the right-hand side a half-order integral operator. Summarizing the developments made above and generalizing our equations by including nonlinear terms (this is a conjecture at this stage), we obtain a new set of viscous potential free-surface flow equations:

$$\Delta\phi = 0, \quad (\mathbf{x}, z) \in \Omega = \mathbb{R}^2 \times [-h, \eta] \quad (15)$$

$$\eta_t + \nabla\eta \cdot \nabla\phi = \phi_z + 2\nu\Delta\eta, \quad z = \eta \quad (16)$$

$$\phi_t + \frac{1}{2}|\nabla\phi|^2 + g\eta = -2\nu\phi_{zz}, \quad z = \eta \quad (17)$$

$$\phi_z = -\sqrt{\frac{\nu}{\pi}} \int_0^t \frac{\phi_{zz}}{\sqrt{t-\tau}} d\tau, \quad z = -h. \quad (18)$$

Using this weakly damped potential flow formulation one can derive the following system of Boussinesq equations with horizontal velocity \mathbf{u}_h defined at the depth $z_\theta = -\theta h$, $0 \leq \theta \leq 1$:

$$\eta_t + \nabla \cdot ((h + \eta)\mathbf{u}_h) + h^3 \left(\frac{\theta^2}{2} - \theta + \frac{1}{3} \right) \nabla^2(\nabla \cdot \mathbf{u}_h) = 2\nu\Delta\eta + \sqrt{\frac{\nu}{\pi}} \int_0^t \frac{\nabla \cdot \mathbf{u}_h}{\sqrt{t-\tau}} d\tau, \quad (19)$$

$$\mathbf{u}_{ht} + \frac{1}{2}\nabla|\mathbf{u}_h|^2 + g\nabla\eta - h^2\theta \left(1 - \frac{\theta}{2} \right) \nabla(\nabla \cdot \mathbf{u}_{ht}) = 2\nu\Delta\mathbf{u}_h. \quad (20)$$

3. Conclusion

In the present paper we have shown how to express the rotational component of the velocity field in terms of the potential part of Helmholtz–Leray decomposition. This expression contains differential and integral operators. Obviously, this analysis is only linear. In future work we will try to extend the present derivation to the nonlinear case. A long wave approximation was derived from this new potential flow formulation.

It is interesting to note that dissipative terms of this form have been used to verify the theory of weak turbulence of surface gravity waves in deep water [5,6,11]. They were added without justification to model dissipation at small scales. Note that a good qualitative agreement was obtained between the Kolmogorov spectrum predicted by weak turbulence theory and the results of DNS. Hence, the present work can be considered as an attempt to justify the inclusion of these terms.

Our final remark concerns the nonlocal term in the kinematic bottom boundary condition. This term can be also considered as a boundary layer correction at the bottom. In modelling viscous effects this term plays the main role, since its magnitude is $\mathcal{O}(\sqrt{\nu})$. Of course, the numerical implementation of this term is another matter.

What is the value of ν to be taken in numerical simulations? There is surprisingly little published information of this subject. What is clear is that the molecular diffusion is too small to model true viscous damping and one should rather consider the eddy viscosity parameter.

Acknowledgments

The authors are grateful to Prof. A. I. Dyachenko and Dr. D. Bouche for helpful discussions.

References

- [1] J. Boussinesq. Lois de l’extinction de la houle en haute mer. *C. R. Acad. Sci. Paris*, 121:15–20, 1895.
- [2] M. Chen and O. Goubet. Long-time asymptotic behaviour of dissipative boussinesq systems. *Discrete and Continuous Dynamical Systems*, 17:61–80, 2007.
- [3] F. Dias, A.I. Dyachenko, and V.E. Zakharov. Theory of weakly damped free-surface flows: a new formulation based on potential flow solutions. *Submitted*, 2007.
- [4] D. Dutykh and F. Dias. Dissipative boussinesq equations. *C. R. Acad. Sci. Paris*, to appear, 2007.
- [5] A.I. Dyachenko, A.O. Korotkevich, and V.E. Zakharov. Weak turbulence of gravity waves. *JETP Lett.*, 77:546–550, 2003.
- [6] A.I. Dyachenko, A.O. Korotkevich, and V.E. Zakharov. Weak turbulent kolmogorov spectrum for surface gravity waves. *Phys. Rev. Lett.*, 92:134501, 2004.
- [7] D.D. Joseph and J. Wang. The dissipation approximation and viscous potential flow. *J. Fluid Mech.*, 505:365–377, 2004.
- [8] T. Kakutani and K. Matsuuchi. Effect of viscosity on long gravity waves. *J. Phys. Soc. Japan*, 39:237–246, 1975.
- [9] H. Lamb. *Hydrodynamics*. Cambridge University Press, 1932.
- [10] C.C. Mei. *The applied dynamics of ocean surface waves*. World Scientific, 1994.
- [11] V.E. Zakharov, A.O. Korotkevich, A.N. Pushkarev, and A.I. Dyachenko. Mesoscopic wave turbulence. *JETP Lett.*, 82:487–491, 2005.