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ABSTRACT. This paper is devoted to the study of semi-linear parabolic equations whose principal term is fractional, *i.e.* is integral and eventually singular. A typical example is the fractional Laplace operator. This work sheds light on the fact that, if the initial datum is not bounded, assumptions on the non-linearity are closely related to its behaviour at infinity. The sublinear and superlinear cases are first treated by classical techniques. We next present a third original case: if the associated first order Hamilton-Jacobi equation is such that perturbations propagate at finite speed, then the semi-linear parabolic equation somehow keeps memory of this property. By using such a result, locally bounded initial data that are merely integrable at infinity can be handled. Next, regularity of the solution is proved. Eventually, strong convergence of gradients as the fractional term disappears is proved for strictly convex non-linearity.

Keywords: semi-linear equation, Lévy operator, fractional Laplacian, unbounded data, unbounded solutions, viscosity solution, finite-infinite propagation speed, regularity, non-local vanishing viscosity method, convergence of gradients

Mathematics Subject Classification: 35B65, 35D05, 35D25, 35B05, 35K65, 35S30

1   Introduction

The present paper is concerned with the following semi-linear parabolic equation

$$\begin{aligned}
\partial_t u + H(t,x,u,\nabla u) + g[u] &= 0 &\quad \text{in } Q_T := (0,T) \times \mathbb{R}^N, &\qquad (1.1)\\
u(0,.) &= u_0 &\quad \text{on } \mathbb{R}^N &\qquad (1.2)
\end{aligned}$$

where ∇u denotes the gradient with respect to (wrt for short) x, H is a continuous first order non-linearity (or Hamiltonian) and $g[u]$ denotes a Lévy operator. An important example of Lévy operator is the fractional Laplacian, *i.e.* the pseudodifferential operator defined by the symbol $|\xi|^\lambda$, $0 < \lambda < 2$: for all Schwartz function φ

$$g[\varphi](x) := \mathcal{F}^{-1}\left(|\cdot|^\lambda \mathcal{F}\varphi \right)$$

where $\mathcal{F}$ denotes the Fourier transform. In general, Lévy operators $g[\varphi]$ have the following integral form

$$g[\varphi](x) = -\int_{\mathbb{R}^N \setminus \{0\}} \left(\varphi(x+z) - \varphi(x) - \frac{\nabla\varphi(x)\cdot z}{1+|z|^2} \right) d\mu(z) \qquad (1.3)$$

where μ denotes the Lévy measure, *i.e.* a non-negative Radon measure such that $\int \min(1,|z|^2)d\mu(z) < +\infty$. It is convenient to write

$$\mu = \mu_b + \mu_s \qquad (1.4)$$

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where μ_b is bounded and supported outside of the unit ball B, and μ_s is eventually singular ($\int |z|^2 \mu_s(dz) < +\infty$) and supported in B. In the case of the fractional Laplacian, the derivative of μ wrt the Lebesgue measure is $c_N(\lambda)|z|^{-N-\lambda}$, $c_N(\lambda) > 0$. Hence, the first order Hamilton-Jacobi equation is perturbed by a singular integral term. We refer to (1.1) as a fractional semi-linear parabolic equation.

By many ways, Lévy operators generalize diffusion operators. They are indeed the infinitesimal generators of Lévy (jump) processes and Brownian motions make part of this large class of stochastic processes. In particular, they satisfy maximum principles [16]. They appear in many models from mathematical finance [15], biology [29, 31], combustion theory [13], continuum mechanics [3, 20], fluid mechanics [32, 14, 24], physical science [35] *etc.* We can explain their extensive use in models by two fundamental facts. First, their distribution functions decay as a power law (instead of exponentially fast). Secondly, Lévy operators can appear when dimension is reduced; for instance, in [3], the geometric motion of a curve of defaults in a crystal is obtained by considering the Dirichlet to Neumann operator associated with the elliptic linear elasticity equation.

The study of fractional non-linear equations have attracted a lot of attention since more than thirty years and the literature is now rapidly growing as far as fractional non-linear elliptic and parabolic equations are concerned. A relevant example comes from the study of stochastic control of jump processes. Equation (1.1) can be interpreted as the Bellman equation of such an optimal control problem if there is no control on the jumps. Maximum principle techniques are used for instance in [9, 23] in order to construct solutions in Sobolev spaces. Soner [34] first used the weak notion of viscosity solution to deal with more general non-linear equations. Indeed, viscosity solution theory provides a good framework and there is a important literature about it; see for instance [33, 4, 28, 6, 7] and references therein. A second class of non-linear equations is the one composed with fractional scalar conservation laws [10, 11, 1, 2]. We would like also to mention the books by Garroni and Menaldi [21, 22] about linear equations.

The aim of this paper is to study Equation (1.1) in full details. In particular, existence, uniqueness and regularity of solutions are investigated. Because of several applications we have in mind, one of our goal is to be able to deal with *unbounded* initial data u_0 (See (1.2)). For instance, if Equation (1.1) is seen as the Bellman equation associated with a control problem involving jump processes, a natural initial condition is $u_0(x) = \max(0, x - K)$.

In [25], such a Cauchy problem is studied with $u_0 \in L^\infty$ in the case of the fractional Laplacian with $\lambda > 1$. Regularizing effects are exhibited: with bounded Lipschitz continuous initial data, the solution turns out to be C^2 in space en C^1 in time. We will see that these results are easily extended to the case of sublinear initial data, *i.e.* functions u_0 for which there exists $C > 0$ such that for all $x \in \mathbb{R}^N$

$$|u_0(x)| \le C(1 + |x|).$$

Computations next suggest that even superlinear initial conditions can be handled. Precisely, if u_0 satisfies for all $x \in \mathbb{R}^N$

$$|u_0(x)| \le C(1 + |x|^{\lambda'}) \tag{1.5}$$

for a suitable $\lambda' > 1$ and constant $C > 0$ (we say that such a function is superlinear), assumptions on the non-linearity can be modified in order to prove a comparison principle; roughly speaking, the dependence in x of H is relaxed and the one in p is strengthened. See the introduction of Section 3 for a more detailed discussion about assumptions on H. Jakobsen and Karlsen [28] developed a general theory for second order parabolic non-linear integro-differential equations. In particular, they establish comparison results and continuous dependence estimates. Because of the dependence of H on the Hessian of u, their arguments are more involved. In our case, classical techniques work with minor modifications and more general assumptions (a modulus of continuity appear in Assumptions (B2) and (B3) instead of a Lipschitz constant; (B2) can be local in x and u).

As far as the superlinear case is concerned, Condition (1.5) ensures in particular that the following uniform integrability condition is satisfied by the continuous function u_0:

$$\int \left(\sup_{B_1(z)} u_0 \right) \mu_b(dz) < +\infty \tag{1.6}$$

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where μ_b is the bounded part of μ (it is introduced in (1.4)). The third class of unbounded initial data u_0 we can treat corresponds to assuming that u_0 satisfy the uniform integrability condition (1.6). If assumptions of the superlinear case are strenghtened, existence and (strong) uniqueness can be proved for (1.1). Moreover, a local L^∞ bound is derived. We next make assumptions on the non-linearity ensuring that the associated first order Hamilton-Jacobi equation (with no integral operator) propagates perturbations of u_0 with finite speed. This means that the solution at a point (t,x) only depends on the value of the initial condition on a ball centered at x and of radius proportional to t [30, 26]. Following an original idea of the first author [1], we show that, for initial condition satisfying (1.6), the solutions of (1.1) keep memory of this finite speed property, even if perturbations now propagate with infinite speed. Precisely, we prove that if u and v are solutions of (1.1) associated with initial data u_0 and v_0, then

$$u(t,x) - v(t,x) \leq K(t,\cdot) * \sup_{|y-\cdot|\leq C_2 t} (u_0(y) - v_0(y))(x).$$

Following [1], such a property is referred to as a finite-infinite propagation speed property. Let us make an important remark: this last result only relies on the fact that the perturbation operator has a positive Kernel (or Green function) and it can also be applied to any Lévy operator, including the Laplacian. In this latter case, we recover (more or less) classical results where $(1+|z|^{N+\lambda})^{-1}$ is replaced with $\exp(-|z|^2)$.

We previously mentioned that the fractional Laplacian has a regularizing effect for $\lambda > 1$. The results we obtain here in the sublinear framework are in the spirit of those of [25] and the proofs rely on them. They are obtained in the case of a Lipschitz continuous initial datum u_0. As far as superlinear initial conditions are concerned, we give two different results: the first one asserts the C^1 regularity in space of the solutions for superlinear initial conditions that are locally Lipschitz continuous and for a non-linearity whose Lipschitz constant in p can depend linearly on $|x|$. If this assumption is strengthened, then $C^{1,1}$ in space solutions are obtained even for initial conditions that are not locally Lipschitz continuous.

We finally consider the case where the non-local part of Equation (1.1) vanishes. Precisely, we consider

$$\partial_t u^\varepsilon + H(t,x,u^\varepsilon,\nabla u^\varepsilon) + \varepsilon g[u^\varepsilon] = 0 \quad \text{in } Q_T, \tag{1.7}$$
$$u^\varepsilon(0,.) = u_0 \quad \text{on } \mathbb{R}^N \tag{1.8}$$

with u_0 (locally) Lipschitz continuous and let ε go to 0. The theory of viscosity solution ensures that the regular solution u^ε converges as $\varepsilon \to 0$ locally uniformly towards the (locally) Lipschitz continuous solution u^0 of (1.7) with $\varepsilon = 0$. Hence, the sequence $\{\nabla u^\varepsilon\}_{\varepsilon>0}$ converges towards ∇u^0 in the distribution sense but it is not clear if strong convergence in L^p_{loc}, $p \geq 1$, can be obtained. We prove it in the case where the non-linearity is strictly convex with respect to the gradient variable by using Young measure techniques. Let us mention that the result about the convergence of gradients remains true if the non-local operator is replaced with the Laplacian. Strong convergence of sequences of gradients appear in [18] in the context of non-linear elliptic equations in the divergence form. To the best of our knowledge, there are very few results in the literature as far as Hamilton-Jacobi equations are concerned and we did not find any references. To finish with, we shed some light on the fact that the proof only relies on local gradient bounds and the strict convexity of H in p.

Let us mention that we decided to present most of our results with the classical fractional Laplacian instead of a general Lévy operator to avoid blinding technicality. If general Lévy operators are at stake, most of the results can be adapted if assumptions are properly translated. For instance, Condition $\lambda' \in (0,\lambda)$ in the sub- and superlinear cases should be replaced with

$$\int |z|^{\lambda'} d\mu_b(z) < +\infty$$

and the results of Subsections 3.1 and 3.2 remain true under this assumption. As far as regularity results of Section 4 are concerned, the techniques of [25] we use there only rely on the fact that the Kernel of the fractional Laplacian satisfies

$$\sup_{t\in[0,T]} t^{\frac{1}{\lambda}} \int_{\mathbb{R}^N} (1+|x|^{\lambda'})|\nabla K(t,x)|dx < +\infty$$

for all $\lambda' \in [0, \lambda)$. As far as the integrable case is concerned, strong uniqueness result and existence for instance rely on the construction of barrier functions (see Lemma 2.5.2). We are convinced that such a construction could be done for a large class of Lévy operators, if not for all of them. See also [7] for further discussion.

The paper is organized as follows. In Section 2, norms and function spaces that are used throughout the paper are introduced. Next, the definition of a viscosity solution for (1.1) is given and technical results related to this notion of weak solutions are stated. We also recall in this section some useful properties of the kernel associated with the non-local operator. Section 3 is devoted to existence and uniqueness results and gradient estimates in the three frameworks described above. In Section 4, we investigate regularity issues in the sublinear and the superlinear cases. Next, a non-local vanishing viscosity method is considered in the last section; in particular, gradients are proved to converge strongly in L^p_{loc}, $p \geq 1$, in the case where the non-linearity is strictly convex with respect to the gradient variable. Eventually, we gathered in Appendices several (more or less) classical proofs for the reader's convenience; a list of the different assumptions used throughout this paper is also given just before references.

Notation. We let $|\cdot|$ denote the Euclidian norm, $B_R(x)$ denote the open ball of $\mathbb{R}^N$ centered at x of radius R and B_R denote the ball $B_R(0)$ and B denotes the unit ball B_1. For any real r, we let $r^+ := \min\{r, 0\}$ denote the positive part of r and $r^- := (-r)^+$ its negative part. A modulus (of continuity) is a non-decreasing function $m : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\lim_{r \to 0} m(r) = 0$ and $m(r + s) \leq m(r) + m(s)$ for every $r, s \geq 0$.

For a Lévy measure μ with a density M with respect to the Lebesgue measure, μ_b denotes the non-negative bounded measure whose density is $\mathbf{1}_{B^c} M$. The space of integrable function on $\mathbb{R}^N$ wrt μ_b is denoted by $L^1(\mathbb{R}^N, \mu_b)$. If now μ_s denotes $\mu - \mu_b$, one can check that (1.4) holds true.

2 Preliminaries

This section is devoted to definitions and technical tools that are used throughout the paper. We first introduce notation for norms and function spaces (Subsection 2.1), we next recall the definition of viscosity solutions for fractional semi-linear parabolic equations (Subsection 2.2). In Subsection 2.3 are gathered technical results related to stability issues. Subsection 2.4 is devoted to the properties of the Kernel K (or Green function) associated with the Lévy operator $g[\cdot]$ used in the proofs of the main results. The last subsection defines the notion of uniform integrability related to Condition (1.6) and gives some basic properties.

Let us mention that we assume throughout the paper that the function $H : [0, T] \times \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ is continuous and that $\lambda \in (0, 2)$.

2.1 Norms and function spaces

Data. Consider functions $u_0 : \mathbb{R}^N \to \mathbb{R}$. The space of continuous (resp. uniformly continuous) functions u_0 is denoted by $C(\mathbb{R}^N)$ (resp. $UC(\mathbb{R}^N)$). For $u_0 \in C(\mathbb{R}^N)$, $\lambda' \in (0, \lambda)$ and $\beta \in (1, \lambda)$ (if $\lambda > 1$), define the norms and semi-norm

$$
\begin{aligned}
\|u_0\|_{0,\lambda'} &:= \sup_{x \in \mathbb{R}^N} \frac{|u_0(x)|}{1 + |x|^{\lambda'}}, \\
[u_0]_{1,\beta} &:= \sup_{x,y \in \mathbb{R}^N,\, x \neq y} \frac{|u_0(x) - u_0(y)|}{(1 + |x|^{\beta-1} + |y|^{\beta-1})|x - y|}, \\
\|u_0\|_{1,\beta} &:= \|u_0\|_{0,\beta} + [u_0]_{1,\beta}
\end{aligned}
$$

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and the associated function spaces

$$
\begin{aligned}
C^0_{\lambda'}(\mathbb{R}^N) &:= \left\{ u_0 \in C(\mathbb{R}^N) : \|u_0\|_{0,\lambda'} < +\infty \right\}, \\
C^{0,1}_{\beta}(\mathbb{R}^N) &:= \left\{ u_0 \in C(\mathbb{R}^N) : \|u_0\|_{1,\beta} < +\infty \right\}.
\end{aligned}
$$

In discussions, $C^0_{\lambda'}(\mathbb{R}^N)$ and $C^{0,1}_{\beta}(\mathbb{R}^N)$ are respectively denoted $C^0_{\lambda'}$ and $C^{0,1}_{\beta}$ for short.

Solutions. Consider functions $u : \overline{Q}_T \to \mathbb{R}$ where $\overline{Q}_T := [0,T] \times \mathbb{R}^N$. Define the norms

$$
\|u\|_{0,\lambda'} := \sup_{t \in [0,T]} \|u(t,\cdot)\|_{0,\lambda'} \quad \text{and} \quad \|u\|_{1,\beta} := \sup_{t \in [0,T]} \|u(t,\cdot)\|_{1,\beta}
$$

and the associated function spaces

$$
\begin{aligned}
C^0_{\lambda'}(\overline{Q}_T) &:= \left\{ u \in C(\overline{Q}_T) : \|u\|_{0,\lambda'} < +\infty \right\}, \\
C^{0,1}_{\beta}(\overline{Q}_T) &:= \left\{ u \in C(\overline{Q}_T) : \|u\|_{1,\beta} < +\infty \right\}.
\end{aligned}
$$

In discussions, $C^0_{\lambda'}(\overline{Q}_T)$ and $C^{0,1}_{\beta}(\overline{Q}_T)$ are respectively denoted $C^0_{\lambda'}$ and $C^{0,1}_{\beta}$ for short.

2.2   Viscosity solutions for (1.1)

Let us introduce the notion of viscosity solution to (1.1). Even if this weak notion of solution is nowaways well known, we would like to recall some basic facts about it. First, a weak notion of differentiability is obtained by "touching" (from below or above) a semi-continuous function u with smooth test-functions ϕ. Secondly, it is very easy to pass to the limit in non-linear terms with this notion of solution [8]; in other words, good stability results are available. Thirdly, it is possible (and we will do so later) to prove strong uniqueness results; they are obtained under the form of comparison principles. To finish with, existence of solutions is generally proved by using Perron's method [27]. For a general introduction, the interested reader is referred to the survey [17] or to the book [5] for instance.

We now turn to the definition of viscosity solution for Equation (1.1). Following [25], we will use an appropriate notion of parabolic sub- and supergradients to define viscosity solutions of first order integro-differential equations.

Definition 2.2.1 (Parabolic supergradient)**.** Let $u : \overline{Q}_T \to \mathbb{R}$ be upper semicontinuous (usc for short). Then, $(a,p) \in \mathbb{R} \times \mathbb{R}^N$ is a *(parabolic) supergradient* of u at $(t,x) \in \overline{Q}_T$ if there exist $r > 0$ and $\sigma > 0$ such that for all $s \in [0,T]$ and all $y \in B_r(x)$:

$$
u(s,y) \le u(t,x) + a(s-t) + p \cdot (y-x) + \sigma(|y-x|^2) + o(|s-t|), \tag{2.1}
$$

where $\frac{o(l)}{l} \to 0$ as $l \to 0^+$.

In the following, $\partial^P u(t,x)$ denotes the set of all supergradients of u at (t,x) and it is referred to as the superdifferential of u at (t,x). If u is lower semicontinuous (lsc for short), we then define (parabolic) subgradients and subdifferentials by $\partial_P u(t,x) := -\partial^P(-u)(t,x)$.

We can now introduce the notion of viscosity solution to (1.1) in the spirit of [7]. We do it for the following general class of functions

$$
\mathcal{C}^{+(-)} := \left\{ u : \overline{Q}_T \to \mathbb{R} \text{ s.t. } \exists \Phi \in C^2(\mathbb{R}^N) \cap L^1(\mathbb{R}^N, \mu_b), \ \forall (t,x) \in \overline{Q}_T, \ u^{+(-)}(t,x) \le \Phi(x) \right\}
$$

where μ_b appears in (1.4) (see also the end of the Introduction). We will consider solutions that lie in $\mathcal{C} := \mathcal{C}^+ \cap \mathcal{C}^-$.

Let us mention that we will consider different subspaces of $\mathcal{C}$: sublinear functions, superlinear ones and uniformly integrable ones (see below for a definition).

We now can give the definition of a viscosity solution of (1.1).

Definition 2.2.2 (Viscosity solutions for (1.1))**.** 1. A usc function $u \in \mathcal{C}^+$ is a *viscosity subsolution* of (1.1) if for all $(t,x) \in Q_T$ and all $(a,p) \in \partial^P u(t,x)$,

$$a + H(t,x,u(t,x),p) + g[u](t,x,p) \leq 0, \tag{2.2}$$

where $g[u](t,x,p) := -\int_{\mathbb{R}^N \setminus \{0\}} \left(u(t,x+z) - u(t,x) - \frac{p \cdot z}{1+|z|^2} \right) d\mu(z)$.

2. A lsc function $u \in \mathcal{C}^-$ is a *viscosity supersolution* of (1.1) if for all $(t,x) \in Q_T$ and all $(a,p) \in \partial_P u(t,x)$,

$$a + H(t,x,u(t,x),p) + g[u](t,x,p) \geq 0.$$

3. A continuous function $u \in \mathcal{C}$ is a *viscosity solution* of (1.1) if it is both a viscosity sub- and supersolution.

Remarks 2.1. 1. Notice that the integral term $g[u](t,x,p)$ makes sense since its integrand is μ-quasi-integrable, thanks to (2.1) and to the integrability condition on u at infinity. Actually, it is μ-integrable because of (2.2). Such an idea already appears in [25]. See also [7] for a complete discussion.

2. If $\lambda \in (0,1)$, then we obtain an equivalent definition by replacing the notion of (parabolic) supergradients with the one of first order Fréchet supergradients.

2.3 Stability results

This subsection is devoted to classical and technical results related with stability issues. It contains two technical lemmata that are used in the proofs of comparison principles. It also contains a discontinuous stability result that permits to pass to the limit in the equation in a very weak sense. Sketches of the proofs of these results are given in Appendix A. The reader can easily skip this technical subsection since it is not necessary in the understanding of the statements of the main results.

When proving comparison principles, we must show that subsolutions u lie below supersolutions v. In order to do so, we consider $M = \sup(u - v)$ and, in order to get viscosity inequalities, the number of variables is doubled and this doubling is penalized. Lemmata 2.3.1 and 2.3.2 are related to the doubling of the time variable.

In order to state and use these lemmata, we need to introduce the closure of sub- and superdifferentials.

Definition 2.3.1 (Limiting subgradients)**.** For u usc, $\overline{\partial}^P u(t,x)$ denotes the set of $(a,p) \in \mathbb{R} \times \mathbb{R}^N$ such that there exist $r, \sigma > 0$ and a sequence $((t_n, x_n, a_n, p_n, r_n, \sigma_n))_{n \in \mathbb{N}}$ such that $(a_n, p_n) \in \partial^P u(t_n, x_n)$, with (2.1) holding true with the constants $r_n, \sigma_n > 0$, and such that $(t_n, x_n, u(t_n, x_n), a_n, p_n, r_n, \sigma_n) \to (t,x,u(t,x),a,p,r,\sigma)$ as $n \to +\infty$.

We then define $\overline{\partial}_P u(t,x) := -\overline{\partial}^P(-u)(t,x)$ for u lsc.

Lemma 2.3.1. *If $u \in \mathcal{C}^+$ is a viscosity subsolution of (1.1), then (2.2) holds true for all $(t,x) \in (0,T] \times \mathbb{R}^N$ and all $(a,p) \in \overline{\partial}^P u(t,x)$.*

Remark 2.1. Remark that, passing to the limit in the equation, we conclude that the integrand of the integral term $g[u](t,x,p)$ is μ-quasi-integrable with $g[u](t,x,p) < +\infty$. Nevertheless, it could occur that $g[u](t,x,p) = -\infty$; in that case, (2.2) holds true in $\mathbb{R} \cup \{-\infty\}$.

Lemma 2.3.2. *Let $u \in \mathcal{C}^+$ and $v \in \mathcal{C}^-$ be, respectively, a sub- and a supersolution of Equation (1.1). Let $\phi : [0,T] \times \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ be smooth and such that the supremum*

$$M := \sup_{[0,T] \times \mathbb{R}^N \times \mathbb{R}^N} (u(t,x) - v(t,y) - \phi(t,x,y))$$

is achieved at some point $(\bar{t}, \bar{x}, \bar{y})$. *There then exist* $a, b \in \mathbb{R}$ *such that*

$$\left(a, \nabla_x \phi(\bar{t}, \bar{x}, \bar{y})\right) \in \overline{\partial}^P u(\bar{t}, \bar{x}),$$

$$\left(b, -\nabla_y \phi(\bar{t}, \bar{x}, \bar{y})\right) \in \overline{\partial}_P v(\bar{t}, \bar{y})$$

and $a - b = \partial_t \phi(\bar{t}, \bar{x}, \bar{y})$. *Moreover,*

$$g[v]\left(\bar{t}, \bar{y}, \nabla_y \phi(\bar{t}, \bar{x}, \bar{y})\right) - g[u]\left(\bar{t}, \bar{x}, \nabla_x \phi(\bar{t}, \bar{x}, \bar{y})\right)$$
$$\leq \int_{\mathbb{R}^N} \left(\phi(\bar{t}, \bar{x} + z, \bar{y} + z) - \phi(\bar{t}, \bar{x}, \bar{y}) - \frac{(\nabla_x + \nabla_y)\phi(\bar{t}, \bar{x}, \bar{y}) \cdot z}{1 + |z|^2}\right) d\mu(z). \quad (2.3)$$

Remark 2.2. Following the previous remark, (2.3) holds true in $\mathbb{R} \cup \{+\infty\}$. But, this lemma will be used with ϕ such that the last integral term of (2.3) is finite; hence, so will be the integral terms $g[v]\left(\bar{t}, \bar{y}, \nabla_y \phi(\bar{t}, \bar{x}, \bar{y})\right)$ and $g[u]\left(\bar{t}, \bar{x}, \nabla_x \phi(\bar{t}, \bar{x}, \bar{y})\right)$.

We now turn to the discontinuous stability result. When dealing with non-linear equations, it is crucial to be able to pass to the limit in the non-linear terms in an appropriate sense. It turns out that it is very easy to do so when working with viscosity solutions. Indeed, Barles and Perthame [8] remarked that the so-called relaxed lower/upper limits of a family of super-/subsolutions is still a super-/subsolution (see below for a precise definition). It is also classical to prove that the supremum of a family of subsolutions is a subsolution of (1.1). This result is useful to prove existence by the classical Perron's method.

Let us make precise the definition of relaxed semi-limits. Let us recall that if $\mathcal{O}$ is a nonempty subset of $\mathbb{R}^M$, then the relaxed upper limit of a sequence of functions $u_n : \mathcal{O} \to \mathbb{R}$ is defined as follows

$$\limsup^* u_n(x) = \limsup_{n \to +\infty, \mathcal{O} \ni y \to x} u_n(y).$$

The relaxed lower limit is defined by $\liminf_* u_n := -\limsup^*(-u_n)$. We then have the following result.

Proposition 2.1 (Discontinuous stability). *Let* $(u_n)_n$ *be a sequence of subsolution of* (1.1) *such that there exists* $\Phi \in C^2(\mathbb{R}^N) \cap L^1(\mathbb{R}^N, \mu_b)$ *with*

$$\forall n \in \mathbb{N}, \ \forall (t, x) \in \overline{Q}_T, \ u_n(t, x) \leq \Phi(x).$$

Then $\limsup^* u_n$ *is a viscosity subsolution of* (1.1). *Moreover,*

$$(\limsup^* u_n)(0, .) \leq \limsup^*(u_n(0, .)) \quad (2.4)$$

where the relaxed upper semi-limit of the left-hand side is computed wrt time and space while the one of the right-hand side is only computed wrt the space variable.

Remarks 2.2.   1. An analogous result for supersolution can be easily stated and proved. Hence one can pass to the limit in (1.1) wrt the local uniform convergence of sequences of solutions.

   2. Inequality (2.4) implies that if u_0 is an initial condition, then the notion of subsolution of (1.1)-(1.2) (the initial condition being replaced with $u(0, .) \leq u_0$) is stable by passing to the upper semi-limit. Such a property will be used to prove the convergence of a sequence of solutions under only L_{loc}^∞ estimates, by the so-called technique of the relaxed semi-limits (see [8]).

2.4   The kernel associated with the non-local operator

The semi-group generated by g is formally given by the convolution with the kernel defined for $t > 0$ by $K(t, \cdot) := \mathcal{F}^{-1}(e^{-t|\cdot|^\lambda})$. It is well-known that K satisfies the following properties (see [19] and the item

1) of the remarks below):

$$K(1, \cdot) \in C_b^\infty(\mathbb{R}^N) \cap W^{\infty,1}(\mathbb{R}^N) \text{ is positive and } \int_{\mathbb{R}^N} K(1, \cdot) = 1,$$

$$K(t, x) = t^{-N/\lambda} K(1, t^{-1/\lambda} x),$$

$$K(t, \cdot) * K(s, \cdot) = K(t + s, \cdot).$$

To deal with unbounded solutions, we need the following estimates

$$\int_0^T \int_{\mathbb{R}^N} (1 + |x|^{\lambda'}) K(t, x) dx \quad < \quad +\infty, \tag{2.5}$$

$$\mathcal{K}_{\lambda'} := \sup_{t \in [0,T]} t^{\frac{1}{\lambda}} \int_{\mathbb{R}^N} (1 + |x|^{\lambda'}) |\nabla K(t, x)| dx \quad < \quad +\infty, \tag{2.6}$$

for all $\lambda' \in [0, \lambda)$. These estimates follow immediately from the homogeneity property and the lemma below.

Lemma 2.4.1 ([12]). *Consider $\lambda \in (0, 2)$. There exists $C_\lambda > 0$ such that, as $|x| \to +\infty$, we have:* $K(1, x) \sim C_\lambda |x|^{-N-\lambda}$, $|\nabla K(1, x)| = \mathcal{O}\left(|x|^{-N-\lambda-1}\right)$ *and* $|g[K(1, \cdot)](x)| = \mathcal{O}\left(|x|^{-N-2\lambda}\right)$.

Remarks 2.3. 1. In [19], it is only proved that K is non-negative. But in [2], it is proved that it is radially decreasing. Combining this result with the ones of Lemma 2.4.1 permits to conclude that K cannot vanish.

  2. Actually, one have results similar to the ones of Lemma 2.4.1 for the derivatives of all order.

2.5 Uniformly integrable functions

In this subsection, we give the definition of uniform integrability of a function (with respect to μ_b) and give some basic properties.

Definition 2.5.1 (Uniformly integrable function). Let $u_0 : \mathbb{R}^N \to \mathbb{R}$ be locally bounded. We say that u_0 is μ_b-*uniformly integrable* if there exists $R > 0$ such that

$$\int_{\mathbb{R}^N} \sup_{B_R(z)} |u_0| \, d\mu_b(z) < +\infty . \tag{2.7}$$

Remark 2.3. We will use this notion for locally bounded functions $u : \overline{Q}_T \to \mathbb{R}$. In this case, we investigate either $\sup_{t \in [0,T]} |u(t, \cdot)|$ is μ_b-uniformly integrable or not. If yes, we simply say that u is μ_b-uniformly integrable in space uniformly wrt time.

Here is a result that makes precise the structure of such functions.

Lemma 2.5.1. *If u_0 is μ_b-uniformly integrable, then (2.7) holds true for all $R > 0$.*

Proof. Let $R > r > 0$. For $z \in \mathbb{R}^N$, there exists $y_z \in B_R(z)$ such that $B_{\frac{r}{2}}(y_z) \subset B_R(z)$ and

$$\sup_{B_R(z)} |u_0| = \sup_{B_{\frac{r}{2}}(y_z)} |u_0|.$$

For $y \in B_{\frac{r}{2}}(y_z)$, we have $B_{\frac{r}{2}}(y_z) \subset B_r(y)$ and $\frac{1}{1+|z|^{N+\lambda}} \leq \frac{C(R,N,\lambda)}{1+|y|^{N+\lambda}}$; hence,

$$\frac{\sup_{B_R(z)} |u_0|}{1 + |z|^{N+\lambda}} \leq C(R, N, \lambda) \frac{\sup_{B_r(y)} |u_0|}{1 + |y|^{N+\lambda}}$$

and integrating wrt $y \in B_{\frac{r}{2}}(y_z)$ yields:

$$
\begin{aligned}
|B_{\frac{r}{2}}| \, \frac{\sup_{B_R(z)} |u_0|}{1 + |z|^{N+\lambda}} &\leq C(R, N, \lambda) \int_{B_{\frac{r}{2}}(y_z)} \frac{\sup_{B_r(y)} |u_0|}{1 + |y|^{N+\lambda}} \, dy, \\
&\leq C(R, N, \lambda) \int_{B_R(z)} \frac{\sup_{B_r(y)} |u_0|}{1 + |y|^{N+\lambda}} \, dy.
\end{aligned}
$$

Let us now integrate wrt $z \in \mathbb{R}^N$.

$$
\begin{aligned}
|B_{\frac{r}{2}}| \int_{\mathbb{R}^N} \sup_{B_R(z)} |u_0| \, \frac{dz}{1 + |z|^{N+\lambda}} &\leq C(R, N, \lambda) \int_{\mathbb{R}^N} \int_{B_R(z)} \frac{\sup_{B_r(y)} |u_0|}{1 + |y|^{N+\lambda}} \, dy dz, \\
&= C(R, N, \lambda) |B_R| \int_{\mathbb{R}^N} \sup_{B_r(z)} |u_0| \, \frac{dz}{1 + |z|^{N+\lambda}};
\end{aligned}
$$

the last line is obtained by integrating first wrt z. Since a locally bounded function is integrable wrt μ_b if and only if it is integrable wrt to $\frac{dz}{1+|z|^{N+\lambda}}$, the rest of the proof of the lemma is obvious. ∎

Let us now construct barrier functions that allow to develop a viscosity solution theory.

Lemma 2.5.2 (Construction of barriers). *Let $u_0 : \mathbb{R}^N \to \mathbb{R}^+$ be a usc and μ_b-uniformly integrable. Then, there exists $\Phi[u_0] \in C^\infty(\mathbb{R}^N)$ positive, μ_b-uniformly integrable and such that*

$$
\forall x \in \mathbb{R}^N, \ u_0(x) \ \leq \ \Phi[u_0](x), \tag{2.8}
$$

$$
\lim_{|x| \to +\infty} \frac{u_0(x)}{\Phi[u_0](x)} \ = \ 0, \tag{2.9}
$$

$$
|\nabla \Phi[u_0]| + |g[\Phi[u_0]]| \ \leq \ C_{N,\lambda} \Phi[u_0], \tag{2.10}
$$

for some positive constant $C_{N,\lambda}$ that only depends on N and λ.

Remarks 2.4.   1. We will use this result for usc functions $u : \overline{Q}_T \to \mathbb{R}^+$ that are μ_b-uniformly integrable in space uniformly wrt time. In this case, we apply it to $\sup_{t \in [0,T]} u(t, \cdot)$ and $\Phi[u]$ simply denotes the barrier function $\Phi\left[\sup_{t \in [0,T]} u(t, \cdot)\right]$.

   2. In view of the proof below, one can check that if $u \leq v$ then $\Phi[u] \leq \Phi[v]$.

Proof of Lemma 2.5.2. Define the positive bounded Borel measure $\nu := \left(\sup_{B_1(z)} u_0\right) d\mu_b(z)$. Consider an increasing sequence $r_n \to +\infty$ such that $r_{n+1} \geq r_n + 1/2$ and

$$
\nu(B_{r_n}^c) \leq \frac{1}{n^3} \, \nu(\mathbb{R}^N).
$$

For $r \geq 0$, define

$$
f(r) := \mathbf{1}_{[0,r_1+1/2)} + \sum_{n \geq 1} n \, \mathbf{1}_{[r_n+1/2, r_{n+1}+1/2)}.
$$

Notice that $f \geq 1$, it is non-decreasing and such that $\lim_{r \to +\infty} f(r) = +\infty$. Define now for $x \in \mathbb{R}^N$,

$$
\Phi(x) := f(|x|) \left(\sup_{B_{\frac{1}{2}}(x)} u_0\right).
$$

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Notice that Φ is μ_b-uniformly integrable; indeed,

$$
\begin{aligned}
\int_{\mathbb{R}^N} \left(\sup_{B_{\frac{1}{2}}(z)} \Phi \right) d\mu_b(z) &\leq \int_{\mathbb{R}^N} f(|z| + 1/2) \left(\sup_{B_1(z)} u_0 \right) d\mu_b(z), \\
&= \int_{\mathbb{R}^N} f(|z| + 1/2) \, d\nu(z), \\
&= \nu(B_{r_1}) + \sum_{n \geq 1} n \, \nu(B_{r_{n+1}} \setminus B_{r_n}), \\
&\leq \nu(B_{r_1}) + \sum_{n \geq 1} n \, \nu(B_{r_n}^c), \\
&\leq \nu(B_{r_1}) + \sum_{n \geq 1} \frac{n}{n^3} \, \nu(\mathbb{R}^N) < +\infty.
\end{aligned}
$$

Write now $\widetilde{\Phi} := K(1, \cdot) * \Phi$. This convolution product is finite, thanks to Lemma 2.4.1, which shows that the kernel of g and the density of μ behaves the same way at infinity (up to a positive multiplicative constant). Moreover, the asymptotic properties of the derivatives of K and the theorem of derivation under the integral sign ensure that $\widetilde{\Phi} \in C^\infty(\mathbb{R}^N)$ with $\nabla \widetilde{\Phi} = (\nabla K(1, \cdot)) * \Phi$ and $g[\widetilde{\Phi}] = g[K(1, \cdot)] * \Phi$; notice that $g[\widetilde{\Phi}]$ is well-defined since we will prove below that $\widetilde{\Phi}$ is μ_b-uniformly integrable thus *a fortiori* μ_b-integrable. We have proved in particular (2.10), since adding to the positivity of Φ, Lemma 2.4.1 also implies that $|\nabla K(1, \cdot)| + |g[K(1, \cdot)]| \leq C_{N,\lambda} K(1, \cdot)$.

Let us now prove that $\widetilde{\Phi}$ is μ_b-uniformly integrable. We have:

$$
\sup_{B_1(\cdot)} \widetilde{\Phi} \leq K(1, \cdot) * \sup_{B_1(\cdot)} \Phi;
$$

indeed, for $x \in \mathbb{R}^N$,

$$
\begin{aligned}
\sup_{B_1(x)} \widetilde{\Phi} &= \sup_{z \in B_1(x)} \int_{\mathbb{R}^N} K(1, y) \Phi(z - y) dy, \\
&\leq \int_{\mathbb{R}^N} K(1, y) \sup_{z \in B_1(x)} \Phi(z - y) dy, \\
&= \int_{\mathbb{R}^N} K(1, y) \sup_{z' \in B_1(x - y)} \Phi(z') dy, \\
&= K(1, \cdot) * \left(\sup_{B_1(\cdot)} \Phi \right) (x).
\end{aligned}
$$

Remark that Lemma 2.4.1 implies that $\mu_b \leq C K(1, \cdot)$. Moreover, $K(1, \cdot)$ is even and $K(1, \cdot) \star K(1, \cdot) = K(2, \cdot)$. Hence, it follows that

$$
\begin{aligned}
\int_{\mathbb{R}^N} \sup_{B_1(z)} \widetilde{\Phi} \, d\mu_b(z) &\leq C \int_{\mathbb{R}^N} \sup_{B_1(z)} \widetilde{\Phi} \, K(1, -z) dz \\
&= C \, K(1, \cdot) * \sup_{B_1(\cdot)} \widetilde{\Phi} \, (0) \\
&\leq C \, K(1, \cdot) * K(1, \cdot) * \sup_{B_1(\cdot)} \Phi \, (0) \\
&= C \, K(2, \cdot) * \sup_{B_1(\cdot)} \Phi \, (0) \\
&\leq \widetilde{C} \int_{\mathbb{R}^N} \sup_{B_1(z)} \Phi \, d\mu_b(z);
\end{aligned}
$$

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the last line is obtained by using the homogeneity property of K which shows that $K(2, \cdot)$ behaves as $K(1, \cdot)$ and *a fortiori* as the density of μ at infinity (up to a positive multiplicative constant). Recalling now that Φ is μ_b-uniformly integrable, we deduce that so is $\widetilde{\Phi}$.

What is left to prove is (2.8) and (2.9). We have

$$\widetilde{\Phi}(x) \geq \int_{B_{\frac{1}{2}}(x) \cap \{|z| \geq |x|\}} K(1, x - z) \Phi(z) dz \geq \varepsilon \left(\inf_{z: |z - x| \leq 1/2, |z| \geq |x|} \Phi(z) \right)$$

where $\varepsilon := \inf_{z \in E_x} K(1, -z)$ with $E_x := B_{\frac{1}{2}}(x) \cap \{|z| \geq |x|\} - x$. Since $K(1, \cdot)$ is positive and radial, ε is a positive constant that does not depend on x. On using the definition of Φ, we get

$$\widetilde{\Phi}(x) \geq \varepsilon \inf_{z: |z - x| \leq 1/2, |z| \geq |x|} \left(f(|z|) \sup_{B_{\frac{1}{2}}(z)} u_0 \right) \geq \varepsilon \, f(|x|) \, u_0(x).$$

One can check that $\Phi[u_0] := \frac{1}{\varepsilon} \widetilde{\Phi}$ satisfies all the desired properties. ∎

To finish with, let us give an example of non-trivial uniformly integrable function.

Example 2.5.1. Define $e_1 := (1, 0, \ldots, 0) \in \mathbb{R}^N$ and let x_1 denote the variable on the e_1-axis of $\mathbb{R}^N$. Consider the function defined by:

$$u_0(x) := \sum_n \mathbf{1}_{B_1(n^2 e_1)}(x) \, |x_1|^{N-1+\lambda}.$$

Then u_0 is uniformly integrable wrt μ_b, but does clearly not belong to C_λ^0. Moreover, if $N \geq 2$, then u_0 does also not satisfy the following integrability condition:

$$\int \sup_{B_{|z|}} |u_0| \, d\mu_b(z) < +\infty. \tag{2.11}$$

3   Existence and uniqueness of unbounded solutions for (1.1)

In this section, we construct a unique unbounded solution of (1.1),(1.2) for unbounded initial datum u_0. In order to do so, different assumptions on the behaviour of u_0 at infinity are made: we will distinguish the cases of sublinear data, superlinear ones and "uniformly" μ_b- integrable data. Precisely, we assume that u_0 satisfies one of the following condition

$$|u_0(x)| \leq C(1 + |x|), \quad |u_0(x)| \leq C(1 + |x|^{\lambda'}) \quad \text{or} \quad \int_{\mathbb{R}^N} \sup_{B_R(z)} |u_0(z)| \, d\mu_b(z) < +\infty$$

for some $C > 0$, $\lambda' > 1$ and $R > 0$. We use the weak notion of viscosity solution in order to get uniqueness for instance. As mentioned in the introduction, the assumptions on the non-linearity H vary from one case to the other.

Before treating each case successively, we would like first to discuss the different sets of assumptions we need. To make it clear, let us assume that H is locally Lispchitz continuous, even if this assumption can be relaxed a bit (compare (A2) and (A2'), (B2) and (B2')). As far as the behaviour wrt u is concerned, we classically assume that the Lipschitz constant of H is bounded from below. The way H depends on (x, p) differs from the sublinear case to the superlinear one. If comparison principles are at stake, the Lipschitz constant of H in x has to be sublinear in p. But in the superlinear case, this constant can be local in x which is not the case in the sublinear case (compare (A2) and (B2)). Similar differences about the behaviour in p appear (compare (A3) and (B3)). These differences can be understood by looking at the proof arguments: in the sublinear case, the penalization in space parameter vanishes before the doubling variable penalization one and it is the contrary in the superlinear case. In the latter case, in order to get local bounds for instance, the behaviour of the coefficients must be prescribed (see (B2') and (B4)). If finite propagation speed is expected for the pure Hamilton-Jacobi equation, the Hamiltonian H has to be assumed globally Lipschitz continuous in p.

3.1 The sublinear case

We first consider uniformly continuous initial data. This implies that we expect solutions to be sublinear at infinity *i.e.* to lie in C_1^0. Naturally, for the range of exponent $\lambda \in (0, 1]$, we consider data and solutions that lie more precisely in $C_{\lambda'}^0$ with $\lambda' \in (0, \lambda)$. As far as the non-linearity is concerned, we assume that:

(A1) There exists $\nu \in \mathbb{R}$ such that for all $x \in \mathbb{R}^N$, $u, v \in \mathbb{R}$, $u \leq v$, $p \in \mathbb{R}^N$, $t \in [0, T]$,

$$H(t, x, v, p) - H(t, x, u, p) \geq \nu(v - u).$$

(A2) There exists a modulus of continuity m such that for all $x, y \in \mathbb{R}^N$, $u \in \mathbb{R}$, $p \in \mathbb{R}^N$, $t \in [0, T]$,

$$|H(t, x, u, p) - H(t, y, u, p)| \leq m((1 + |p|)|x - y|).$$

(A3) For all $R > 0$, there exists a modulus of continuity m_R such that for all $x \in \mathbb{R}^N$, $u \in \mathbb{R}$, $p, q \in B_R$, $t \in [0, T]$,

$$|H(t, x, u, p) - H(t, x, u, q)| \leq m_R(|p - q|).$$

Remark 3.1. It is classical that through a change of unknown function in the equation (namely, $U(t, x) := e^{-(\min\{\nu, 0\}+1)t} u(t, x)$), (A1) reduces to the following assumption:

(A1') For all $x \in \mathbb{R}^N$, $u, v \in \mathbb{R}$, $u \leq v$, $p \in \mathbb{R}^N$, $t \in [0, T]$, $H(t, x, v, p) - H(t, x, u, p) \geq v - u$.

For the sake of simplicity, the results of this section are proved under (A1') without changing explicitly the unknown function.

We obtain first a comparison principle. Its proof is given in Appendix A.3.

Theorem 3.1 (Comparison principle). *Assume (A1)-(A3). Let $\lambda' \in (0, 1] \cap (0, \lambda)$ and $u_0 \in UC(\mathbb{R}^N) \cap C_{\lambda'}^0(\mathbb{R}^N)$. Let u and v be, respectively, a sub- and a supersolution of (1.1) such that*

$$\sup_{[0,T] \times \mathbb{R}^N} \frac{u^+(t, x)}{1 + |x|^{\lambda'}} < +\infty \quad and \quad \sup_{[0,T] \times \mathbb{R}^N} \frac{v^-(t, x)}{1 + |x|^{\lambda'}} < +\infty .$$

Then $u \leq v$ whenever $u(0, .) \leq u_0 \leq v(0, .)$.

We now focus on existence issue and provide a L_{loc}^∞ bound on the solution. We need to prescribe the behaviour at infinity of the source terms of the equations for the range of exponent $\lambda \in (0, 1]$ (see Remark 3.2 below).

(A4) There exists a constant C_0 such that for all $x \in \mathbb{R}^N$,

$$\sup_{[0,T]} |H(t, x, 0, 0)| \leq C_0(1 + |x|^{\lambda'}).$$

Theorem 3.2 (Existence and L_{loc}^∞ estimate). *Let $\lambda' \in (0, 1] \cap (0, \lambda)$. Assume (A1)-(A4). Let $u_0 \in UC(\mathbb{R}^N) \cap C_{\lambda'}^0(\mathbb{R}^N)$. There then exists a (unique) viscosity solution $u \in C_{\lambda'}^0(\overline{Q}_T)$ of (1.1)-(1.2). Moreover, there exists a constant M only depending on $\|u_0\|_{0,\lambda'}$ and C_0 from (A4) such that $\|u\|_{0,\lambda'} \leq M$, i.e. for all $t \in (0, T) \times \mathbb{R}^N$*

$$|u(t, x)| \leq M(1 + |x|^{\lambda'}) .$$

Remark 3.2. If $\lambda \in (1, 2)$, then (A2) implies (A4).

In order to get a gradient estimate, Assumption (A2) is strenghtened.

(A2') There exists a constant $C_1 \geq 0$ such that for all $x, y \in \mathbb{R}^N$, $u \in \mathbb{R}$, $p \in \mathbb{R}^N$, $t \in [0, T]$,

$$|H(t, x, u, p) - H(t, y, u, p)| \leq C_1(1 + |p|)|x - y|.$$

Theorem 3.3 (Gradient estimate). *Assume (A1),(A2'),(A3). Let $\lambda' \in (0,1] \cap (0,\lambda)$ and $u_0 \in C^0_{\lambda'}(\mathbb{R}^N)$ be Lipschitz continuous. Then, if $u \in C^0_{\lambda'}(\overline{Q}_T)$ is a viscosity solution of (1.1)-(1.2), then u is Lipschitz continuous wrt the space variable; moreover there exists a constant M, only depending on $\|\nabla u_0\|_\infty$ and C_1 from (A2'), such that $\|\nabla u\|_\infty \leq M$.*

The proofs of the existence and the gradient estimate can easily be adapted from [25] and are left to the reader.

3.2 The superlinear case

We now turn to the superlinear case (1.5). In order to deal with $\lambda > 1$ and initial conditions that are not sublinear anymore, the assumptions on the non-linearity H need to be adapted, in particular its regularity wrt the x and p variables. But even in the case $\lambda \leq 1$, we exhibit different conditions on the non-linearity H that ensures the well-posedness of the problem.

(B2) For all $R > 0$, there exists a modulus of continuity m_R such that for all $x, y \in B_R$, $u \in [-R, R]$, $p \in \mathbb{R}^N$, $t \in [0,T]$,
$$|H(t, x, u, p) - H(t, y, u, p)| \leq m_R((1 + |p|)|x - y|).$$

(B3) There exists a modulus of continuity m such that for all $x \in \mathbb{R}^N$, $u \in \mathbb{R}$, $p, q \in \mathbb{R}^N$, $t \in [0,T]$,
$$|H(t, x, u, p) - H(t, x, u, q)| \leq m((1 + |x|)|p - q|).$$

In this framework, the comparison principle holds true for sub- and supersolutions with (eventually) superlinear growth.

Theorem 3.4 (Comparison principle). *Assume (A1),(B2),(B3). Let $\lambda' \in (0,\lambda)$. Let u and v be, respectively, a sub- and a supersolution of (1.1) such that*
$$\sup_{[0,T]\times\mathbb{R}^N} \frac{u(t,x)}{1 + |x|^{\lambda'}} < +\infty \quad and \quad \inf_{[0,T]\times\mathbb{R}^N} \frac{v(t,x)}{1 + |x|^{\lambda'}} > -\infty.$$

Then $u \leq v$ whenever $u(0,.) \leq v(0,.)$.

Proof. Let $M := \sup_{[0,T]\times\mathbb{R}^N}(u - v)$. We must prove that $M \leq 0$. Let us assume the contrary and let us exhibit a contradiction. Let $\lambda'' \in (\lambda', \lambda)$ and γ be a non-negative constant that will be appropriately chosen later. Consider two parameters $\varepsilon, \eta > 0$ and define

$$M_{\varepsilon,\eta} = \sup_{[0,T]\times\mathbb{R}^N\times\mathbb{R}^N} \left(u(t,x) - v(t,y) - \frac{|x-y|^2}{2\varepsilon} - \eta e^{\gamma t}\frac{(1+|x|^2)^{\frac{\lambda''}{2}}}{\lambda''} \right).$$

Since for all $\eta > 0$:
$$\lim_{|(x,y)| \to +\infty} \frac{|x-y|^2 + \eta(1+|x|^2)^{\frac{\lambda''}{2}}}{(1+|x|+|y|)^{\lambda'}} = +\infty, \tag{3.1}$$

there exists $(\bar{t}, \bar{x}, \bar{y}) \in [0,T] \times \mathbb{R}^N \times \mathbb{R}^N$ where the supremum is attained. There exists $\eta_0 > 0$ such that for all $\varepsilon > 0$, $M_{\varepsilon,\eta_0} \geq M/2 > 0$. Moreover,

$$\lim_{\varepsilon \to 0} \sup_{\mathbb{R}^N\times\mathbb{R}^N} \left(u(0,x) - v(0,y) - \frac{|x-y|^2}{2\varepsilon} - \eta_0\frac{(1+|x|^2)^{\frac{\lambda''}{2}}}{\lambda''} \right)$$

$$= \sup_{\mathbb{R}^N} \left(u(0,x) - v(0,x) - \eta_0\frac{(1+|x|^2)^{\frac{\lambda''}{2}}}{\lambda''} \right) \leq 0$$

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and it follows that $\bar{t} \neq 0$ for ε small enough. By Lemma 2.3.2, there exist

$$\left(a, p + \eta_0 e^{\gamma \bar{t}}(1 + |\bar{x}|^2)^{\frac{\lambda''-2}{2}} \, \bar{x}\right) \in \overline{\partial}^P u(\bar{t}, \bar{x}),$$

$$(b, p) \in \overline{\partial}_P v(\bar{t}, \bar{y})$$

with $a - b = \gamma \eta_0 e^{\gamma \bar{t}} \frac{(1+|\bar{x}|^2)^{\frac{\lambda''}{2}}}{\lambda''}$, $p = \frac{\bar{x}-\bar{y}}{\varepsilon}$ and

$$g[v](\bar{t}, \bar{y}, p) - g[u]\left(\bar{t}, \bar{x}, p + \eta_0 e^{\gamma \bar{t}}(1 + |\bar{x}|^2)^{\frac{\lambda''-2}{2}} \, \bar{x}\right) \leq -\frac{\eta_0 e^{\gamma \bar{t}}}{\lambda''} g\left[(1 + |\cdot|^2)^{\frac{\lambda''}{2}}\right](\bar{x}).$$

Subtracting the corresponding viscosity inequalities and using (A1') (see Remark 3.1), we get

$$M/2 + \gamma \eta_0 \frac{e^{\gamma \bar{t}}(1 + |\bar{x}|^2)^{\frac{\lambda''}{2}}}{\lambda''} \leq H(\bar{t}, \bar{y}, u(\bar{t}, \bar{x}), p)$$

$$- H\left(\bar{t}, \bar{x}, u(\bar{t}, \bar{x}), p + \eta_0 e^{\gamma \bar{t}}(1 + |\bar{x}|^2)^{\frac{\lambda''-2}{2}} \, \bar{x}\right) - \frac{\eta_0 e^{\gamma \bar{t}}}{\lambda''} g\left[(1 + |\cdot|^2)^{\frac{\lambda''}{2}}\right](\bar{x}).$$

By (3.1), there exists $R \geq 0$ that depends on η_0 but not on $0 < \varepsilon \leq 1$ such that $\bar{x}, \bar{y} \in B_R$ and $u(\bar{t}, \bar{x}) \in [-R, R]$. By (B2) and (B3), we get

$$M/2 + \gamma \eta_0 \frac{e^{\gamma \bar{t}}(1 + |\bar{x}|^2)^{\frac{\lambda''}{2}}}{\lambda''} \leq m_R\left(|\bar{x} - \bar{y}| + \frac{|\bar{x} - \bar{y}|^2}{\varepsilon}\right)$$

$$+ m\left(2\eta_0 e^{\gamma \bar{t}}(1 + |\bar{x}|^2)^{\frac{\lambda''}{2}}\right) - \frac{\eta_0 e^{\gamma \bar{t}}}{\lambda''} g\left[(1 + |\cdot|^2)^{\frac{\lambda''}{2}}\right](\bar{x}).$$

By Lemma B.0.1 in Appendix B, the function $g[(1 + |\cdot|^2)^{\frac{\lambda''}{2}}]$ is bounded. If C denotes its L^∞ norm and $\gamma = C + 2\lambda''K$ (where K is an arbitrary positive real number), then

$$M/2 \quad \leq \quad m_R\left(|\bar{x} - \bar{y}| + \frac{|\bar{x} - \bar{y}|^2}{\varepsilon}\right) + m\left(2\eta_0 e^{\gamma \bar{t}}(1 + |\bar{x}|^2)^{\frac{\lambda''}{2}}\right) - 2K\eta_0 e^{\gamma \bar{t}}(1 + |\bar{x}|^2)^{\frac{\lambda''}{2}},$$

$$\leq \quad m_R\left(|\bar{x} - \bar{y}| + \frac{|\bar{x} - \bar{y}|^2}{\varepsilon}\right) + \sup_{r>0}(m(r) - Kr). \tag{3.2}$$

Classical arguments allow to prove that $\lim_{\varepsilon \to 0} \frac{|\bar{x}-\bar{y}|^2}{\varepsilon} = 0$, since η_0 is a fixed positive number. The limit as $\varepsilon \to 0$ thus yields $M/2 \leq \sup_{r>0}(m(r) - Kr)$. Taking the infimum wrt $K > 0$ gives the contradiction $0 < M/2 \leq 0$. The proof of Theorem 3.4 is now complete. ∎

Theorem 3.5 (Existence and L^∞_{loc} estimate). *Let $\lambda' \in (0, \lambda)$. Assume (A1),(B2),(B3),(A4). Let $u_0 \in C^0_{\lambda'}(\mathbb{R}^N)$. There then exists a (unique) viscosity solution $u \in C^0_{\lambda'}(\overline{Q}_T)$ of (1.1)-(1.2). Moreover, $\|u\|_{0,\lambda'} \leq M$, i.e. for all $(t, x) \in (0, T) \times \mathbb{R}^N$*

$$|u(t, x)| \leq M(1 + |x|^{\lambda'}).$$

where M is some constant that only depends on $\|u_0\|_{0,\lambda'}$, C_0 from (A4) and m from (B3).

We do not give an explicit proof of this result, since it is similar and easier than the one we will give for the existence of solutions with integrable initial data (see next subsection).

We now turn to gradient estimates. To get some, we need to make precise the way the derivative of the non-linearity H wrt x depends on $|x|$ (see R in (B2)) and to strengthen a little bit (B3).

(B2') There exists a constant C_1 such that for all $x, y \in \mathbb{R}^N$, $u \in \mathbb{R}$, $p \in \mathbb{R}^N$, $t \in [0, T]$,

$$|H(t, x, u, p) - H(t, y, u, p)| \leq C_1(1 + |x|^{\lambda'-1} + |y|^{\lambda'-1} + |p|)|x - y|.$$

(B3') There exists a constant C_2 such that for all $x, y \in \mathbb{R}^N$, $u \in \mathbb{R}$, $p, q \in \mathbb{R}^N$, $t \in [0, T]$,

$$|H(t, x, u, p) - H(t, x, u, q)| \leq C_2(1 + |x|)|p - q|.$$

Theorem 3.6 (Gradient estimate). *Let $\lambda \in (1, 2)$ and $\lambda' \in (1, \lambda)$. Assume (A1),(B2'),(B3'). Let $u_0 \in C^{0,1}_{\lambda'}(\mathbb{R}^N)$. If $u \in C^0_{\lambda'}(\overline{Q}_T)$ is a viscosity solution of (1.1)-(1.2), then $u \in C^{0,1}_{\lambda'}(\overline{Q}_T)$ and there exists a constant M, only depending on $\|u_0\|_{1,\lambda'}$, C_1 from (B2') and C_2 from (B3'), such that $\|u\|_{1,\lambda'} \leq M$.*

Proof. Consider a parameter $\varepsilon > 0$ and define $u^\varepsilon(t, x) := \sup_{y \in \mathbb{R}^N} \left(u(t, y) - e^{\gamma t} \frac{|x-y|^2}{2\varepsilon} \right)$ with

$$\gamma = \max \left\{ 2C_1 + 1, 4C_2(\lambda' - 1) + \| g[(1 + |\cdot|^2)^{\lambda'-1}] \|_\infty + 1 \right\} \tag{3.3}$$

where C_1 and C_2 appear in (B2') and (B3'). It is easy to see that u^ε is continuous and such that $\sup_{[0,T]\times\mathbb{R}^N} \frac{u^\varepsilon(t,x)}{1+|x|^{\lambda'}} < +\infty$. Let us prove that u^ε is a viscosity subsolution of the following equation:

$$\partial_t u^\varepsilon + H(t, x, u^\varepsilon, \nabla u^\varepsilon) + g[u^\varepsilon] \leq C\varepsilon \frac{(1+|x|^2)^{\lambda'-1}}{2(\lambda'-1)} + C\varepsilon^{\frac{\lambda'}{2-\lambda'}}, \tag{3.4}$$

where C is some constant that only depends on C_1 and λ'. Let $t \in (0, T]$, $x \in \mathbb{R}^N$ and $(a, p) \in \partial^P u^\varepsilon(t, x)$. It is well-known that there exists a point $\overline{y}$ such that $u^\varepsilon(t, x) = u(t, \overline{y}) - e^{\gamma t} \frac{|x-\overline{y}|^2}{2\varepsilon}$, $\left(a + \gamma e^{\gamma t} \frac{|x-\overline{y}|^2}{2\varepsilon}, p \right) \in \partial^P u(t, \overline{y})$ and $p = e^{\gamma t}(\frac{x-\overline{y}}{\varepsilon})$. Moreover, using the fact that

$$u^\varepsilon(t, x + z) - u^\varepsilon(t, x) - \frac{p.z}{1+|z|^2} \geq u(t, \overline{y} + z) - u(t, \overline{y}) - \frac{p.z}{1+|z|^2},$$

we get $g[u^\varepsilon](t, x, p) \leq g[u](t, \overline{y}, p)$. The viscosity inequality corresponding to the solution u of (1.1) at $(t, \overline{y})$ and Assumptions (A1') (see Remark 3.1) and (B2') then imply that

$$a + H(t, x, u^\varepsilon(t, x), p) + g[u^\varepsilon](t, x, p) \leq$$
$$C_1 \left((1 + |x|^{\lambda'-1} + |\overline{y}|^{\lambda'-1}) + e^{\gamma t} \frac{|x-\overline{y}|}{\varepsilon} \right) |x - \overline{y}| - \gamma e^{\gamma t} \frac{|x-\overline{y}|^2}{2\varepsilon}.$$

By (3.3), the right-hand side of this inequality is bounded from above by

$$C_1(1 + 2|x|^{\lambda'-1})|x - \overline{y}| + C_1|x - \overline{y}|^{\lambda'} + C_1 e^{\gamma t} \frac{|x-\overline{y}|^2}{\varepsilon} - \gamma e^{\gamma t} \frac{|x-\overline{y}|^2}{2\varepsilon}$$
$$\leq \sup_{r>0} \left(C_1(1 + 2|x|^{\lambda'-1})r + C_1 r^{\lambda'} - \frac{r^2}{2\varepsilon} \right) \leq C \frac{(1+|x|^2)^{\lambda'-1}}{2(\lambda'-1)}\varepsilon + C\varepsilon^{\frac{\lambda'}{2-\lambda'}}$$

where C is some constant that only depends on C_1 and λ'. This achieves the proof of (3.4).

Let us prove that for all $C' \geq C$,

$$w(t, x) := u^\varepsilon(t, x) - C' e^{\gamma t} \varepsilon \frac{(1+|x|^2)^{\lambda'-1}}{2(\lambda'-1)} - C\varepsilon^{\frac{\lambda'}{2-\lambda'}} t$$

is a subsolution of (1.1). Let $t \in (0, T]$, $x \in \mathbb{R}^N$ and $(a, p) \in \partial^P w(t, x)$. Then,

$$\left(a + \gamma C' e^{\gamma t} \varepsilon \frac{(1+|x|^2)^{\lambda'-1}}{2(\lambda'-1)} + C\varepsilon^{\frac{\lambda'}{2-\lambda'}}, p + C' e^{\gamma t} \varepsilon (1 + |x|^2)^{\lambda'-2} x \right) \in \partial^P u^\varepsilon(t, x)$$

and

$$g[w](t, x, p) = g[u^\varepsilon]\left(t, x, p + C' e^{\gamma t}\varepsilon(1 + |x|^2)^{\lambda'-2}x \right) - C' e^{\gamma t}\varepsilon \frac{1}{2(\lambda'-1)} g\left[(1 + |\cdot|^2)^{\lambda'-1} \right](x).$$

The viscosity inequality corresponding to the subsolution u^ε of (3.4) and Assumptions (A1') and (B3') then imply that

$$a + H(t, x, w(t, x), p) + g[w](t, x, p) \leq 2C_2 C' e^{\gamma t} \varepsilon (1 + |x|^2)^{\lambda' - 1}$$
$$+ C \frac{(1 + |x|^2)^{\lambda' - 1}}{2(\lambda' - 1)} \varepsilon - C' e^{\gamma t} \varepsilon \frac{1}{2(\lambda' - 1)} g \left[(1 + |\cdot|^2)^{\lambda' - 1} \right](x) - \gamma C' e^{\gamma t} \varepsilon \frac{(1 + |x|^2)^{\lambda' - 1}}{2(\lambda' - 1)}.$$

By (3.3), the right-hand side of this inequality is non-positive for $C' \geq C$ and we conclude that w is a subsolution of (1.1).

Clearly $\sup_{\overline{Q}_T} \frac{w(t,x)}{1 + |x|^{\lambda'}} < +\infty$ and by the comparison principle we get

$$w(t, x) \leq u(t, x) + \sup_{\mathbb{R}^N} (w(0, x) - u_0(x)) \tag{3.5}$$

for all $t \in [0, T]$ and all $x \in \mathbb{R}^N$. Simple computations show that

$$\sup_{\mathbb{R}^N} (w(0, x) - u_0(x)) = \sup_{\mathbb{R}^N \times \mathbb{R}^N} \left(u_0(y) - u_0(x) - \frac{|x - y|^2}{2\varepsilon} - C' \varepsilon \frac{(1 + |x|^2)^{\lambda' - 1}}{2(\lambda' - 1)} \right)$$

$$\leq \sup_{\mathbb{R}^N \times \mathbb{R}^N} \left(\|u_0\|_{1, \lambda'} (1 + |x|^{\lambda' - 1} + |y|^{\lambda' - 1}) |x - y| - \frac{|x - y|^2}{2\varepsilon} - C' \varepsilon \frac{(1 + |x|^2)^{\lambda' - 1}}{2(\lambda' - 1)} \right)$$

$$\leq \sup_{\mathbb{R}^N \times \mathbb{R}^N} \left(\|u_0\|_{1, \lambda'} (1 + 2|x|^{\lambda' - 1}) |x - y| + \|u_0\|_{1, \lambda'} |x - y|^{\lambda'} - \frac{|x - y|^2}{2\varepsilon} - C' \varepsilon \frac{(1 + |x|^2)^{\lambda' - 1}}{2(\lambda' - 1)} \right)$$

$$\leq \sup_{x \in \mathbb{R}^N} \left(\sup_{r > 0} \left(\|u_0\|_{1, \lambda'} (1 + 2|x|^{\lambda' - 1}) r + \|u_0\|_{1, \lambda'} r^{\lambda'} - \frac{r^2}{2\varepsilon} \right) - C' \varepsilon \frac{(1 + |x|^2)^{\lambda' - 1}}{2(\lambda' - 1)} \right)$$

$$\leq \sup_{x \in \mathbb{R}^N} \left(C'' \varepsilon \frac{(1 + |x|^2)^{\lambda' - 1}}{2(\lambda' - 1)} + C'' \varepsilon^{\frac{\lambda'}{2 - \lambda'}} - C' \varepsilon \frac{(1 + |x|^2)^{\lambda' - 1}}{2(\lambda' - 1)} \right),$$

where C'' only depends on $\|u_0\|_{1, \lambda'}$ and λ'. If we take $C' \geq C''$ sufficiently large then

$$\sup_{\mathbb{R}^N} (w(0, x) - u_0(x)) \leq C'' \varepsilon^{\frac{\lambda'}{2 - \lambda'}}.$$

Inequality (3.5) then implies that for all $t \in [0, T]$, $x, y \in \mathbb{R}^N$ and $\varepsilon > 0$,

$$u(t, y) \leq u(t, x) + e^{\gamma t} \frac{|x - y|^2}{2\varepsilon} + C' e^{\gamma t} \varepsilon \frac{(1 + |x|^2)^{\lambda' - 1}}{2(\lambda' - 1)} + \varepsilon^{\frac{\lambda'}{2 - \lambda'}} (C'' + Ct);$$

it follows that

$$|u(t, x) - u(t, y)| \leq e^{\gamma t} \frac{|x - y|^2}{2\varepsilon} + C' e^{\gamma t} \varepsilon \frac{(1 + |x|^2)^{\lambda' - 1}}{2(\lambda' - 1)} + \varepsilon^{\frac{\lambda'}{2 - \lambda'}} (C'' + Ct)$$

and taking $\varepsilon = \frac{|x - y|}{(1 + |x|^{\lambda' - 1})}$ implies the result. ∎

3.3 The uniformly integrable framework

We present in this subsection a viscosity solution theory in an uniformly integrable framework (see the definition in Section 2). As we shall see, this theory seems to be related with a property of the speed of propagation for (1.1). It is as if the equation keeps memory of the finite propagation speed of the associated pure first order Hamilton-Jacobi equation (*i.e.* (1.1) without the integral term).

Let us begin by proving a strong uniqueness result. To do this, Assumption $(B3)$ is strenghtened as follows:

(C3) There exists a modulus of continuity such that for all $x, y \in \mathbb{R}^N$, $p, q \in \mathbb{R}^N$, $u \in \mathbb{R}$, $t \in [0, T]$,

$$|H(t, x, u, p) - H(t, x, u, q)| \leq m(|p - q|).$$

Theorem 3.7 (Comparison principle). *Assume (A1),(B2),(C3). Let u and v be, respectively, a sub- and a supersolution of* (1.1) *such that u^+ and v^- are μ_b-uniformly integrable in space uniformly wrt time. Then $u \leq v$ whenever $u(0, .) \leq v(0, .)$.*

Proof. Let us assume as previously that $M := \sup_{[0,T] \times \mathbb{R}^N}(u - v) > 0$ and let us exhibit a contradiction. We need to construct barrier functions in order to perturb this supremum. We do so by applying Lemma 2.5.2. Precisely, define $\Phi := \Phi[u^+] + \Phi[v^-]$ and

$$M_{\varepsilon,\eta} = \sup_{[0,T] \times \mathbb{R}^N \times \mathbb{R}^N} \left(u(t, x) - v(t, y) - \frac{|x - y|^2}{2\varepsilon} - \eta e^{\gamma t}(\Phi(x) + \Phi(y)) \right).$$

By Property (2.9) of barrier functions, we claim that there exists $(\bar{t}, \bar{x}, \bar{y}) \in [0, T] \times \mathbb{R}^N \times \mathbb{R}^N$ where the supremum is attained. Arguing as in the proof of Theorem 3.4, we infer that $M_{\varepsilon,\eta_0} \geq M/2$ and $\bar{t} \neq 0$ for some $\eta_0 > 0$ and all ε small enough. Lemma 2.3.2 yields the existence of

$$\begin{aligned}
\left(a, p + \eta_0 e^{\gamma \bar{t}} \nabla \Phi(\bar{x}) \right) &\in \overline{\partial}^P u(\bar{t}, \bar{x}), \\
\left(b, p - \eta_0 e^{\gamma \bar{t}} \nabla \Phi(\bar{y}) \right) &\in \overline{\partial}_P v(\bar{t}, \bar{y})
\end{aligned}$$

with $a - b = \gamma \eta_0 e^{\gamma \bar{t}} (\Phi(\bar{x}) + \Phi(\bar{y}))$, $p = \frac{\bar{x} - \bar{y}}{\varepsilon}$ and

$$g[v] \left(\bar{t}, \bar{y}, p - \eta_0 e^{\gamma \bar{t}} \nabla \Phi(\bar{y}) \right) - g[u] \left(\bar{t}, \bar{x}, p + \eta_0 e^{\gamma \bar{t}} \nabla \Phi(\bar{x}) \right) \leq -\eta_0 e^{\gamma \bar{t}} \left(g[\Phi](\bar{x}) + g[\Phi](\bar{y}) \right).$$

The viscosity inequalities and (A1') now give

$$\begin{aligned}
M/2 + \gamma \eta_0 e^{\gamma \bar{t}} (\Phi(\bar{x}) + \Phi(\bar{y})) \leq{}& H\left(\bar{t}, \bar{y}, u(\bar{t}, \bar{x}), p - \eta_0 e^{\gamma \bar{t}} \nabla \Phi(\bar{y}) \right) \\
&- H\left(\bar{t}, \bar{x}, u(\bar{t}, \bar{x}), p + \eta_0 e^{\gamma \bar{t}} \nabla \Phi(\bar{x}) \right) - \eta_0 e^{\gamma \bar{t}} \left(g[\Phi](\bar{x}) + g[\Phi](\bar{y}) \right).
\end{aligned}$$

Using again Property (2.9) of barrier functions, we claim that $\bar{x}, \bar{y} \in B_R$ and $u(\bar{t}, \bar{x}) \in [-R, R]$ for some $R \geq 0$ which does not depend on $0 < \varepsilon \leq 1$ (but does depend on η_0). By (B2) and (C3), we get

$$\begin{aligned}
M/2 + \gamma \eta_0 e^{\gamma \bar{t}} (\Phi(\bar{x}) + \Phi(\bar{y})) \leq{}& m_R \left(|\bar{x} - \bar{y}| + \frac{|\bar{x} - \bar{y}|^2}{\varepsilon} \right) \\
&+ m\left(\eta_0 e^{\gamma \bar{t}} (|\nabla \Phi(\bar{x})| + |\nabla \Phi(\bar{y})|) \right) - \eta_0 e^{\gamma \bar{t}} \left(g[\Phi](\bar{x}) + g[\Phi](\bar{y}) \right).
\end{aligned}$$

We next use the key property on the barrier function, (2.10), to get

$$|\nabla \Phi| + g[\Phi] \leq |\nabla \Phi[u^+]| + g[\Phi[u^+]] + |\nabla \Phi[v^-]| + g[\Phi[v^-]] \leq C_{N,\lambda} (\Phi[u^+] + \Phi[v^-]) = C_{N,\lambda} \Phi.$$

Combining the two previous inequalities yield

$$\begin{aligned}
M/2 + \gamma \eta_0 e^{\gamma \bar{t}} (\Phi(\bar{x}) + \Phi(\bar{y})) \leq{}& m_R \left(|\bar{x} - \bar{y}| + \frac{|\bar{x} - \bar{y}|^2}{\varepsilon} \right) \\
&+ m\left(C_{N,\lambda} \, \eta_0 e^{\gamma \bar{t}} (\Phi(\bar{x}) + \Phi(\bar{y})) \right) + C_{N,\lambda} \, \eta_0 e^{\gamma \bar{t}} (\Phi(\bar{x}) + \Phi(\bar{y}))
\end{aligned}$$

and, on taking $\gamma = C_{N,\lambda}(K + 1)$, we deduce that

$$M/2 \leq m_R \left(|\bar{x} - \bar{y}| + \frac{|\bar{x} - \bar{y}|^2}{\varepsilon} \right) + \sup_{r > 0} (m(r) - Kr);$$

taking now successively the limit as $\varepsilon \to 0$ and the infimum wrt $K > 0$ gives the desired contradiction $0 < M/2 \leq 0$. The proof is complete. $\blacksquare$

To get existence result, we naturally need a uniform integrability condition on source terms

(C4) The function $(t, x) \mapsto H(t, x, 0, 0)$ is μ_b-uniformly integrable in space uniformly wrt time.

Theorem 3.8 (Existence and L_{loc}^∞ estimate). *Assume (A1),(B2),(C3),(C4). Let u_0 be a continuous and μ_b-uniformly integrable. There then exists a (unique) continuous and μ_b-uniformly integrable in space uniformly wrt time viscosity solution u of* (1.1)-(1.2).

Proof. Let us first approximate the initial condition u_0 and the Hamiltonian H in order to apply the result of [25]. Define $u_n^0 := u_0 \times \chi_n$ where $\chi_n : \mathbb{R}^N \to [0,1]$ equals 1 in B_n and 0 outside B_{n+1}. We also define $H_n := T_n(H)$ where $T_n(r) := \max(-n, \min(n, r))$ is a truncature function. The $W^{1,\infty}$ theory of [25] ensures the existence of a bounded continuous viscosity solution u_n associated with these data. Assume for a while that we have some L_{loc}^∞ estimate on u_n. Proposition 2.1 implies that the upper (resp. lower) semi-limit of u_n is a subsolution (resp. supersolution) of (1.1)-(1.2). The comparison principle then yields $\limsup^* u_n \leq \liminf_* u_n$; since the other inequality always holds true, we have: $\limsup^* u_n = \liminf_* u_n$. If u denotes this function, we see that u is both usc and lsc and both a viscosity sub- and supersolution.

It now remains to derive a L_{loc}^∞ estimate on u_n. To do this, define $\Phi := \Phi[|u_0|] + \Phi[|H(\cdot, \cdot, 0, 0)|]$. In view of Remark 2.4 following Lemma 2.5.2, we claim that $\Phi \geq \Phi[|u_n^0|] + \Phi[|H_n(\cdot, \cdot, 0, 0)|]$. Hence, we have

$$|u_n^0| + \sup_{[0,T]} |H_n(t, \cdot, 0, 0)| \leq \Phi,$$

$$|\nabla \Phi| + |g[\Phi]| \leq C_{N,\lambda} \Phi.$$

Let us look for a supersolution of (1.1) of the form $w(t, x) := e^{\gamma t} \Phi(x) + t \sup_{r>0} (m(r) - Kr)$. We first choose K sufficiently large so that $\sup_{r>0} (m(r) - Kr)$ is finite. By using (A1') and (C3), simple computations show that we have

$$\partial_t w(t, x) + H_n(t, x, w, \nabla w) + g[w] \geq \gamma e^{\gamma t} \Phi(x) + \sup_{r>0} (m(r) - Kr)$$
$$+ H_n(t, x, 0, 0) - m \left(e^{\gamma t} |\nabla \Phi(x)| \right) + e^{\gamma t} g[\Phi](x).$$

By the property on the barrier function, we see that

$$\partial_t w(t, x) + H_n(t, x, w, \nabla w) + g[w] \geq \gamma e^{\gamma t} \Phi(x) + \sup_{r>0} (m(r) - Kr)$$
$$- e^{\gamma t} \Phi(x) - m \left(C_{N,\lambda} \, e^{\gamma t} \Phi(x) \right) - C_{N,\lambda} \, e^{\gamma t} \Phi(x).$$

If we take $\gamma := 1 + C_{N,\lambda}(1 + K)$, we get

$$\partial_t w(t, x) + H_n(t, x, w, \nabla w) + g[w] \geq 0.$$

The comparison principle then implies that $u_n \leq w$. We argue similarly to get the other inequality and prove that $|u_n| \leq w$. This is the desired L_{loc}^∞ estimate and the proof is now complete. ∎

To get a Lipschitz continuity regularity result, we need to use the following assumptions

(C2) The Hamiltonian H is locally Lipschitz continuous wrt x and there are a constant C_1 and a locally bounded function $\Theta : \mathbb{R}^N \to \mathbb{R}^+$ such that Θ^2 is μ_b-uniformly integrable and for a.e. $x \in \mathbb{R}^N$, $u \in \mathbb{R}$, $p \in \mathbb{R}^N$, $t \in [0, T]$,

$$|\nabla_x H(t, x, u, p)| \leq \Theta(x) + C_1 |p|.$$

(C3') There exists a constant C_2 such that for all $x, y \in \mathbb{R}^N$, $u \in \mathbb{R}$, $p, q \in \mathbb{R}^N$, $t \in [0, T]$,

$$|H(t, x, u, p) - H(t, x, u, q)| \leq C_2 |p - q|.$$

Theorem 3.9 (Gradient estimate). *Assume (A1),(C2),(C3'). Let u_0 be locally Lipschitz continuous, μ_b-uniformly integrable and such that $|\nabla u_0|^2$ is μ_b-uniformly integrable. Then, if u is a continuous viscosity solution of (1.1)-(1.2) μ_b-uniformly integrable in space uniformly wrt time, then u is locally Lipschitz continuous wrt the space variable.*

Proof. Consider $\Phi := \Phi[|u|] + \Phi[|\nabla u_0|^2] + \Phi[\Theta^2]$ a barrier function given by Lemma 2.5.2. For $x, y \in \mathbb{R}^N$, define

$$\phi(x,y) := \int_0^1 \Phi((1-\tau)x + \tau y)d\tau + \Phi(x) + \Phi(y).$$

For $\varepsilon > 0$, define

$$M := \sup_{[0,T] \times \mathbb{R}^N \times \mathbb{R}^N} \left(u(t,x) - u(t,y) - e^{\gamma t}\frac{|x-y|^2}{2\varepsilon} - \varepsilon e^{\gamma t}\phi(x,y) \right).$$

Let us prove that for γ sufficiently large, this supremum is non-positive. Assume this is true. In this case, for every $x, y \in \mathbb{R}^N$ and $\varepsilon > 0$, we have

$$u(t,y) - u(t,x) \leq e^{\gamma t}\frac{|x-y|^2}{2\varepsilon} + \varepsilon e^{\gamma t}\phi(x,y).$$

It follows that

$$|u(t,x) - u(t,y)| \leq e^{\gamma t}\frac{|x-y|^2}{2\varepsilon} + \varepsilon e^{\gamma t}\phi(x,y)$$

and taking $\varepsilon = \frac{|x-y|}{\phi(x,y)}$ implies the result since ϕ is locally bounded.

It remains to prove that M is non-positive. Recall that M is achieved at some $(\bar{t}, \bar{x}, \bar{y}) \in [0,T] \times \mathbb{R}^N \times \mathbb{R}^N$, thanks to the definition of ϕ and to Property (2.9) of $\Phi[|u|]$. If $\bar{t} = 0$, then

$$M \leq \sup_{\mathbb{R}^N \times \mathbb{R}^N} \left(u_0(x) - u_0(y) - \frac{|x-y|^2}{2\varepsilon} - \varepsilon\phi(x,y) \right).$$

Using Property (2.8) of barrier function, we claim that Φ is continuous and such that $\Phi \geq |\nabla u_0|^2$; hence, it is obvious that for every $x, y \in \mathbb{R}^N$,

$$|u_0(x) - u_0(y)| \leq I(x,y)\,|x-y|,$$

where $I(x,y) := \int_0^1 \sqrt{\Phi((1-\tau)x + \tau y)}\,d\tau$. It follows that

$$M \leq \sup_{\mathbb{R}^N \times \mathbb{R}^N} \left(I(x,y)\,|x-y| - \frac{|x-y|^2}{2\varepsilon} - \varepsilon\phi(x,y) \right).$$

But, $I(x,y)\,|x-y| - \frac{|x-y|^2}{2\varepsilon} \leq \sup_{r>0}\left(I(x,y)\,r - \frac{r^2}{2\varepsilon} \right)$ and an easy computation shows that this supremum equals $\frac{1}{2}I^2(x,y)\varepsilon$. Since Jensen's inequality implies that

$$I^2(x,y) = \left(\int_0^1 \sqrt{\Phi((1-\tau)x + \tau y)}\,d\tau \right)^2 \leq \int_0^1 \Phi((1-\tau)x + \tau y)\,d\tau \leq \phi(x,y),$$

we conclude that $M \leq \sup_{\mathbb{R}^N \times \mathbb{R}^N}\left(\frac{1}{2}\varepsilon\phi(x,y) - \varepsilon\phi(x,y) \right) \leq 0$ in the case where $\bar{t} = 0$.

To prove the non-positivity of M in the other case, we assume the contrary and we seek a contradiction. In particular, we have $u(\bar{t}, \bar{x}) \geq u(\bar{t}, \bar{y})$. Moreover, recall that Lemma 2.3.2 yields the existence of

$$\left(a, p + \varepsilon e^{\gamma \bar{t}}\nabla_x\phi(\bar{x},\bar{y}) \right) \in \overline{\partial}^P u(\bar{t}, \bar{x}),$$
$$\left(b, p - \varepsilon e^{\gamma \bar{t}}\nabla_y\phi(\bar{x},\bar{y}) \right) \in \overline{\partial}_P v(\bar{t}, \bar{y})$$

with $a - b = \gamma e^{\gamma \bar{t}} \frac{|\bar{x} - \bar{y}|^2}{2\varepsilon} + \gamma \varepsilon e^{\gamma \bar{t}} \phi(\bar{x}, \bar{y})$, $p = e^{\gamma \bar{t}} (\frac{\bar{x} - \bar{y}}{\varepsilon})$ and

$$g[v] \left(\bar{t}, \bar{y}, p - \varepsilon e^{\gamma \bar{t}} \nabla_y \phi(\bar{x}, \bar{y}) \right) - g[u] \left(\bar{t}, \bar{x}, p + \varepsilon e^{\gamma \bar{t}} \nabla_x \phi(\bar{x}, \bar{y}) \right)$$
$$\leq \varepsilon e^{\gamma \bar{t}} \int_{\mathbb{R}^N} \left(\phi(\bar{x} + z, \bar{y} + z) - \phi(\bar{x}, \bar{y}) - \frac{(\nabla_x + \nabla_y)\phi(\bar{x}, \bar{y}) \cdot z}{1 + |z|^2} \right) d\mu(z) =: \varepsilon e^{\gamma \bar{t}} J.$$

By the definition of ϕ, simple computations show that the integral term I above equals

$$J = J(\bar{x}, \bar{y}) = \int_0^1 g[\Phi]((1 - \tau)\bar{x} + \tau\bar{y}) \, d\tau + g[\Phi](\bar{x}) + g[\Phi](\bar{y}).$$

Using Property (2.10) of barrier functions, we see that $|g[\Phi]| \leq C_{N,\lambda} \Phi$ and we obtain

$$J(\bar{x}, \bar{y}) \leq \int_0^1 C_{N,\lambda} \, \Phi((1 - \tau)\bar{x} + \tau\bar{y}) \, d\tau + C_{N,\lambda} \, \Phi(\bar{x}) + C_{N,\lambda} \, \Phi(\bar{y}) = C_{N,\lambda} \phi(\bar{x}, \bar{y}).$$

It follows that the non-local terms satisfy

$$g[v] \left(\bar{t}, \bar{y}, p - \varepsilon e^{\gamma \bar{t}} \nabla_y \phi(\bar{x}, \bar{y}) \right) - g[u] \left(\bar{t}, \bar{x}, p + \varepsilon e^{\gamma \bar{t}} \nabla_x \phi(\bar{x}, \bar{y}) \right) \leq C_{N,\lambda} \varepsilon e^{\gamma \bar{t}} \phi(\bar{x}, \bar{y}).$$

The viscosity inequalities and (A1') now give

$$\gamma e^{\gamma \bar{t}} \frac{|\bar{x} - \bar{y}|^2}{2\varepsilon} + \gamma \varepsilon e^{\gamma \bar{t}} \phi(\bar{x}, \bar{y}) \leq H \left(\bar{t}, \bar{y}, u(\bar{t}, \bar{x}), p - \varepsilon e^{\gamma \bar{t}} \nabla_y \phi(\bar{x}, \bar{y}) \right)$$
$$- H \left(\bar{t}, \bar{x}, u(\bar{t}, \bar{x}), p + \varepsilon e^{\gamma \bar{t}} \nabla_x \phi(\bar{x}, \bar{y}) \right) + C_{N,\lambda} \varepsilon e^{\gamma \bar{t}} \phi(\bar{x}, \bar{y}).$$

Using (C3'), we get

$$\gamma e^{\gamma \bar{t}} \frac{|\bar{x} - \bar{y}|^2}{2\varepsilon} + \gamma \varepsilon e^{\gamma \bar{t}} \phi(\bar{x}, \bar{y}) \leq H \left(\bar{t}, \bar{y}, u(\bar{t}, \bar{x}), p \right) - H \left(\bar{t}, \bar{x}, u(\bar{t}, \bar{x}), p \right)$$
$$+ C_2 \varepsilon e^{\gamma \bar{t}} |(\nabla_x + \nabla_y)\phi(\bar{x}, \bar{y})| + C_{N,\lambda} \varepsilon e^{\gamma \bar{t}} \phi(\bar{x}, \bar{y}).$$

Using that $|(\nabla_x + \nabla_y)\phi(\bar{x}, \bar{y})| \leq C_{N,\lambda} \phi(\bar{x}, \bar{y})$, thanks again to Property (2.10) of barrier functions, we deduce that

$$\gamma e^{\gamma \bar{t}} \frac{|\bar{x} - \bar{y}|^2}{2\varepsilon} + \gamma \varepsilon e^{\gamma \bar{t}} \phi(\bar{x}, \bar{y}) \leq H \left(\bar{t}, \bar{y}, u(\bar{t}, \bar{x}), p \right) - H \left(\bar{t}, \bar{x}, u(\bar{t}, \bar{x}), p \right) + (1 + C_2) C_{N,\lambda} \varepsilon e^{\gamma \bar{t}} \phi(\bar{x}, \bar{y}).$$

Since $\Phi \geq \Theta^2$, Assumption (C2) implies that

$$\gamma e^{\gamma \bar{t}} \frac{|\bar{x} - \bar{y}|^2}{2\varepsilon} + \gamma \varepsilon e^{\gamma \bar{t}} \phi(\bar{x}, \bar{y}) \leq (I(\bar{x}, \bar{y}) + C_1 |p|) |\bar{x} - \bar{y}| + (1 + C_2) C_{N,\lambda} \varepsilon e^{\gamma \bar{t}} \phi(\bar{x}, \bar{y}),$$

where we recall that $I(x, y) := \int_0^1 \sqrt{\Phi((1 - \tau)x + \tau y)} \, d\tau$. Using the fact that $I \leq \sqrt{\phi}$, we get

$$\gamma e^{\gamma \bar{t}} \frac{|\bar{x} - \bar{y}|^2}{2\varepsilon} + \gamma \varepsilon e^{\gamma \bar{t}} \phi(\bar{x}, \bar{y}) \leq \sqrt{\phi(\bar{x}, \bar{y})} \, |\bar{x} - \bar{y}| + C_1 e^{\gamma \bar{t}} \frac{|\bar{x} - \bar{y}|^2}{\varepsilon} + (1 + C_2) C_{N,\lambda} \varepsilon e^{\gamma \bar{t}} \phi(\bar{x}, \bar{y}).$$

Rearranging terms, we get

$$(\gamma - (1 + C_2) C_{N,\lambda}) \varepsilon e^{\gamma \bar{t}} \phi(\bar{x}, \bar{y}) \leq \sqrt{\phi(\bar{x}, \bar{y})} \, |\bar{x} - \bar{y}| + (C_1 - \gamma/2) e^{\gamma \bar{t}} \frac{|\bar{x} - \bar{y}|^2}{\varepsilon}.$$

Take now $\gamma := \max\{2(C_1 + 1), (1 + C_2) C_{N,\lambda} + 1\}$. We get

$$\varepsilon \phi(\bar{x}, \bar{y}) \leq \sqrt{\phi(\bar{x}, \bar{y})} \, |\bar{x} - \bar{y}| - \frac{|\bar{x} - \bar{y}|^2}{\varepsilon} \leq \frac{1}{2} \varepsilon \phi(\bar{x}, \bar{y})$$

which yields the desired contradiction since ϕ is positive. We then have proved that $M \leq 0$. $\blacksquare$

We end this section by studying how (1.1) keeps memory of the fact that the pure first order Hamilton-Jacobi equation propagates perturbations of the initial conditions with finite speed. It is well-known that the key assumption to get such a property is (C3'). In order to simplify the proof of the result below, we shall assume in addition that

(C1) H does not depend on u.

We then have the following result.

Theorem 3.10 (Finite-infinite propagation speed). *Assume (C1),(B2),(C3'). Let u and v be, respectively, a sub- and super-solution of (1.1) such that u and v are μ_b-uniformly integrable in space uniformly wrt time. Then for all $t \in (0,T]$ and all $x \in \mathbb{R}^N$,*

$$u(t,x) - v(t,x) \leq K(t,\cdot) * \sup_{B_{C_2 t(\cdot)}} (u(0,\cdot) - v(0,\cdot))\ (x), \tag{3.6}$$

where C_2 is the constant in (C3').

Proof. Notice first that the right-hand side of (3.6) is well-defined, thanks to Lemma 2.5.1 and to the uniform integrability of $u^+(0,\cdot)$ and $v^-(0,\cdot)$ wrt μ_b. Now, we let the reader check that $w = u - v$ is a viscosity subsolution of the equation

$$\partial_t w + g[w] = C_2 |\nabla w|. \tag{3.7}$$

By the Lax-Oleinik formula, we know that $\alpha(t,x) := \sup_{B_{C_2 t}(x)}(u(0,\cdot) - v(0,\cdot))$ is the unique continuous viscosity solution of

$$\partial_t w = C_2 |\nabla w| \tag{3.8}$$

such that $w(0,\cdot) = u(0,\cdot) - v(0,\cdot)$. Consider a sequence $(w_n^0)_n$ of $W^{1,\infty}$ initial conditions that converges locally uniformly to $w(0,\cdot)$. Define $\alpha_n(t,x) := \sup_{B_{C_2 t}(x)} w_n^0$. By stability, $\alpha_n \to \alpha$ locally uniformly as $n \to +\infty$. Moreover, we know that α_n is Lipschitz continuous and thus satisfies (3.8) almost everywhere. Let $(\theta_m)_m$ and $(\rho_m)_m$ be mollifiers, in time and space respectively. Assume that $\mathrm{supp}(\theta_m) \subset (-\infty,0)$. Define

$$\alpha_{n,m}(t,x) = \alpha_n *_{t,x} (\rho_m \theta_m)(t,x) := \int_0^{+\infty} \alpha_n(s,.) * \rho_m(x)\ \theta(t-s)ds.$$

The function $\alpha_{n,m}$ is smooth and

$$\partial_t \alpha_{n,m} = \partial_t(\alpha_n *_{t,x} (\rho_m \theta_m)) = (\partial_t \alpha_n) *_{t,x} (\rho_m \theta_m) = C_2 |\nabla \alpha_n| *_{t,x} (\rho_m \theta_m) \geq C_2 |\nabla \alpha_{n,m}|.$$

This means that $\alpha_{n,m}$ is a supersolution of (3.8). Using now that $\partial_t K + g[K] = 0$, we obtain

$$\partial_t(K(t,\cdot) * \alpha_{n,m}(t,.))(x) = -g[K(t,\cdot) * \alpha_{n,m}(t,.)](x) + K(t,\cdot) * (\partial_t \alpha_{n,m}(t,.))(x),$$
$$\geq -g[K(t,\cdot) * \alpha_{n,m}(t,.)](x) + C_2 K(t,\cdot) * |\nabla \alpha_{n,m}(t,.)|(x),$$
$$\geq -g[K(t,\cdot) * \alpha_{n,m}(t,.)](x) + C_2 |\nabla(K(t,\cdot) * \alpha_{n,m}(t,.))(x)|.$$

Hence $K(t,\cdot) * \alpha_{n,m}(t,.)(x)$ is a smooth supersolution of (3.7); moreover, it converges locally uniformly to $K(t,\cdot) * \alpha_n(t,.)(x)$ as $n \to +\infty$. By stability, $K(t,\cdot) * \alpha_n(t,.)(x)$ is a supersolution of (3.7). Write next

$$K(t,.) * \alpha_n(t,.) = \int_{\mathbb{R}^N} K(1,y)\alpha_n(t, x - t^{1/\lambda}y)dy$$

and conclude that the limit of this function as t goes to 0^+ exists and equals $\alpha_n(0,.) = w_n^0$. Taking now the limit as $n \to +\infty$, we see that $K(t,\cdot) * \alpha(t,.)(x)$ is a supersolution of (3.7) that satisfies $\alpha(0,.) = w(0,\cdot)$. The comparison principle then yields $w(t,x) \leq K(t,\cdot) * \alpha(t,.)(x)$, that is to say, (3.6) holds true. ∎

4 Regularizing effect

In this section, we focus on the range of exponents $\lambda \in (1,2)$ and we prove that the unbounded continuous (viscosity) solution of (1.1) we constructed in the previous section is in fact $C^{1,1}$ with respect to x and Lipschitz continuous with respect to t.

4.1 The sublinear case

In order to get $C^{1,1}$ solution for (1.1), we need to strenghen assumptions. In particular, we need the following one.

(A3') For all $R > 0$, there exists a constant C_R such that for all $x \in \mathbb{R}^N$, $u, v \in \mathbb{R}$, $p, q \in B_R$, $t \in [0, T]$,

$$|H(t, x, u, p) - H(t, x, v, q)| \leq C_R(|u - v| + |p - q|).$$

Theorem 4.1. *Let $\lambda \in (1,2)$. Assume (A1),(A2'),(A3') and consider a Lipschitz continuous function u_0. Then the (unique) viscosity solution of (1.1)-(1.2) is $C^{1,1}$ wrt the space variable and locally Lipschitz continuous wrt the time variable on $(0, T] \times \mathbb{R}^N$.*

Proof. Let us regularize the non-linearity and the initial data in order to ensure, thanks to the results of [25], existence and uniqueness of C^2 in space and C^1 in time bounded solution of (1.1). Let us consider $u_n^0 \in W^{1,\infty}(\mathbb{R}^N)$ which converges locally uniformly to u_0 on $\mathbb{R}^N$ by satisfying:

$$\|u_n^0\|_{0,1} + \|\nabla u_n^0\|_\infty \leq C$$

for some constant C. Consider a sequence of smooth non-linearities H_n which converges locally uniformly to H in $[0, T] \times \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N$ by satisfying the assumptions of Theorem 4.1 (with same constants). Let u_n be the viscosity solution associated to H_n and u_n^0. The estimates of Theorems 3.2 and 3.3 imply that there exists a constant $M > 0$ such that:

$$\|u_n\|_{0,1} + \|\nabla u_n\|_\infty \leq M. \tag{4.1}$$

This gives us in particular an L_{loc}^∞ estimate; hence, we can argue as in the proof of Theorem 3.5 to establish the local uniform convergence on $\overline{Q}_T$ of u_n toward the unique $C_{\lambda'}^0$ viscosity solution u of (1.1)-(1.2). It thus rests to derive a $C^{1,1}$ in space estimate on u_n.

In order to use the results of [25], we need to consider H_n that satisfies in addition:

$$\sup_{\overline{Q}_T} |H_n(t, x, 0, 0)| < +\infty,$$

$$\forall R > 0, \quad \sup_{[0,T] \times \mathbb{R}^N \times [-R,R] \times B_R} \left(|\partial_u H_n| + |\nabla_p H_n| + |D_{p,x}^2 H_n| + |\nabla_p \partial_u H_n| + |D_p^2 H_n|\right) < +\infty.$$

Then, $u_n \in C_b(\overline{Q}_T) \cap L^\infty([0,T]; W^{1,\infty}(\mathbb{R}^N))$ is C^2 in space and C^1 in time on $(0, T] \times \mathbb{R}^N$ with

$$\sup_{[0,T]} \|t^{1/\lambda} D^2 u_n(t, .)\|_\infty < +\infty. \tag{4.2}$$

Moreover, u_n satisfies the Duhamel's Formula: for all $t > 0$ and all $x \in \mathbb{R}^N$,

$$u_n(t, x) = K(t, \cdot) * u_n^0(x) - \int_0^t K(t - s, .) * H_n(s, ., u_n(s, .), \nabla u_n(s, .))(x) ds. \tag{4.3}$$

By (2.6) and the properties of u_n and H_n, the theorem of derivation under the integral sign implies that

$$D^2 u_n(t, x) = \nabla K(t, \cdot) * \nabla u_n^0(x) - \int_0^t \nabla K(t - s, .) * \nabla \left(H_n(s, ., u_n(s, .), \nabla u_n(s, .))\right)(x) ds. \tag{4.4}$$

Compute next

$$\nabla\left(H_n(s,.,u_n(s,.),\nabla u_n(s,.))\right) \;=\; \nabla_x H_n(s,.,u_n(s,.),\nabla u_n(s,.))$$
$$+\partial_u H_n(s,.,u_n(s,.),\nabla u_n(s,.))\nabla u_n(s,.)$$
$$+D^2 u_n(s,.)\nabla_p H_n(s,.,u_n(s,.),\nabla u_n(s,.)).$$

By (4.1), (A2') and (A3'), this implies:

$$\|\nabla\left(H_n(s,.,u_n(s,.),\nabla u_n(s,.))\right)(x)\|_\infty \le C_1(1+M) + C_M\left(M + \|D^2 u_n(s,.)\|_\infty\right).$$

By (2.6), we get:

$$\|\nabla K(t-s,.)*\nabla\left(H_n(s,.,u_n(s,.),\nabla u_n(s,.))\right)\|_\infty$$
$$\le (t-s)^{-1/\lambda}\,\mathcal{K}_0[C_1(1+M)+C_M M] + (t-s)^{-1/\lambda}s^{-1/\lambda}\,\mathcal{K}_0 C_M\|s^{1/\lambda}D^2 u_n(s,.)\|_\infty.$$

Since $\|\nabla K(t,\cdot)*\nabla u_n^0\|_\infty \le t^{-1/\lambda}\mathcal{K}_0 M$, we deduce from (4.4) that

$$\|t^{1/\lambda}D^2 u_n(t,.)\|_\infty \;\le\; \mathcal{K}_0 M + t^{1/\lambda}\int_0^t (t-s)^{-1/\lambda}ds\,\mathcal{K}_0[C_1(1+M)+C_M M]$$
$$+t^{1/\lambda}\int_0^t (t-s)^{-1/\lambda}s^{-1/\lambda}ds\,\mathcal{K}_0 C_M\sup_{\tau\in[0,T]}\|\tau^{1/\lambda}D^2 u_n(\tau,.)\|_\infty,$$
$$=\; \mathcal{K}_0 M + \frac{\lambda}{1-\lambda}\,t\,\mathcal{K}_0[C_1(1+M)+C_M M]$$
$$+I_\lambda t^{\frac{\lambda-1}{\lambda}}\mathcal{K}_0 C_M\sup_{\tau\in[0,T]}\|\tau^{1/\lambda}D^2 u_n(\tau,.)\|_\infty$$

where $I_\lambda = \int_0^1 (1-s)^{-1/\lambda}s^{-1/\lambda}ds$. Let $T_1\in(0,T]$ be such that $I_\lambda T_1^{\frac{\lambda-1}{\lambda}}\mathcal{K}_0 C_M \le 1/2$. We have proved that

$$\sup_{[0,T_1]}\|t^{1/\lambda}D^2 u_n(t,.)\|_\infty \le 2\left(\mathcal{K}_0 M + \frac{\lambda}{1-\lambda}\,T_1\,\mathcal{K}_0[C_1(1+M)+C_M M]\right) =: C(M,\lambda,\mathcal{K}_0).$$

Taking any T_0 as initial time, we can argue similarly to prove that for all $n\in\mathbb{N}$ and all $T_0\in[0,T-T_1]$,

$$\sup_{[T_0,T_0+T_1]}\|(t-T_0)^{1/\lambda}D^2 u_n(t,.)\|_\infty \le C(M,\lambda,\mathcal{K}_0).$$

Eventually, we get for all $t\in[0,T]$ and $x\in\mathbb{R}^N$:

$$|D^2 u_n(t,x)| \le t^{-1/\lambda}\tilde{C}(M,\lambda,\mathcal{K}_0). \tag{4.5}$$

This gives the desired $C^{1,1}$ estimate in space. The equation $\partial_t u_n = -H_n(t,x,u_n,\nabla u_n)-g[u_n]$ now allows to derive a Lipschitz continuous in time estimate on u_n. The proof is complete. ∎

Corollary 4.1 (Duhamel's formula). *The unique viscosity solution of* (1.1)-(1.2) *satisfies:*

$$u(t,x) = K(t,\cdot)*u_0(x) - \int_0^t K(t-s,.)*H(s,.,u(s,.),\nabla u(s,.))(x)ds.$$

Proof. We need to check that we can pass to the limit in (4.3). This reduces to get pointwise convergence of gradients. To prove this, remark that we have the $C^{1,1}$ estimate (4.5) and $\|\nabla u_n\|_\infty \le M$. Hence, we have compactness. The fact that u_n converges locally uniformly towards u tells us that the limit of any subsequence of ∇u_n has to be ∇u. Indeed, it coincides with the distribution limit. ∎

4.2 The superlinear case

To get the C^1 and/or $C^{1,1}$ regularity in the superlinear case, we need to strengthen Assumption (A1).

(B1) There exists a constant C_3 such that for all $x \in \mathbb{R}^N$, $u \in \mathbb{R}$, $p \in \mathbb{R}^N$, $t \in [0, T]$,

$$|H(t, x, u, p) - H(t, x, v, p)| \le C_3|u - v|.$$

It turns out that the assumptions that permit to get existence of a viscosity solution in the superlinear case (see the assumptions of Theorem 3.5) only permit to prove C^1 regularity in space; moreover, initial conditions have to be locally Lipschitz; precisely, they have to lie in $C^{0,1}_{\lambda'}$. If now one considers the assumptions that permit to get existence in the case of finite-infinite speed of propagation (Theorem 3.10), solutions can be proved to be $C^{1,1}$. Moreover, initial conditions need not being assumed to be locally Lipschitz continuous; precisely, they have to lie in $C^0_{\lambda'}$. Let us first state both results before turning to their proofs.

Theorem 4.2. *Let $\lambda \in (1, 2)$ and $\lambda' \in (1, \lambda)$. Assume (B1),(B2'),(B3'),(A4). Let $u_0 \in C^{0,1}_{\lambda'}(\mathbb{R}^N)$. Then the (unique) viscosity solution of (1.1)-(1.2) is C^1 wrt the space variable on $(0, T] \times \mathbb{R}^N$.*

Remark 4.1. Notice that (B2') and the continuity of $H(t, 0, 0, 0)$ wrt $t \in [0, T]$ already imply (A4).

Remark 4.2. A Duhamel's formula holds true in this case too (see Corollary 4.1) under the assumptions of Theorem 4.2.

Theorem 4.3. *Let $\lambda \in (1, 2)$ and $\lambda' \in (1, \lambda)$. Assume (B1),(B2'),(C3'),(A4). Let $u_0 \in C^0_{\lambda'}(\mathbb{R}^N)$. Then the (unique) viscosity solution of (1.1)-(1.2) is $C^{1,1}$ wrt the space variable and locally Lipschitz continuous wrt the time variable in $(0, T] \times \mathbb{R}^N$.*

Proof of Theorem 4.2. Following the proof of Theorem 4.1, we first regularize the non-linearity and the initial data in such a way that $\|u_n^0\|_{1,\lambda'}$ remains bounded. We have a $C^{0,1}_{\lambda'}$ bound on $(u_n)_n$ given by Theorems 3.5 and 3.6. Let M denote this bound. By the theorem of derivation under the integral sign and Duhamel's Formula (4.3),

$$\nabla u_n(t, x) = \nabla K(t, \cdot) * u_n^0(x) - \int_0^t \nabla K(t - s, .) * H_n(s, x, u_n(s, .), \nabla u_n(s, .))(x)ds. \tag{4.6}$$

Define $\mathcal{H}_n(s, x) := H_n(s, x, u_n(s, x), \nabla u_n(s, x))\mathbf{1}_{(0,T)}(s)$ and $\mathcal{K}(s, x) := \nabla K(s, x)\mathbf{1}_{(0,T)}(s)$. The last term of the right-hand side of (4.6) is equal to $\mathcal{H}_n * \mathcal{K}$ where the convolution is computed wrt (t, x). Assumptions (A4),(B1) and (B3') imply:

$$\|\mathcal{H}_n\|_{0,\lambda'} \le C_0 + 4C_3M.$$

By (2.6), $\mathcal{K}$ is integrable with respect to the measure $(1 + |z|^{\lambda'})d\tau dz$. Hence, the continuity of the translations in $L^1((1 + |z|^{\lambda'})d\tau dz)$ implies that $(\mathcal{H}_n * \mathcal{K})_n$ is equicontinuous in $\overline{Q}_T$; indeed, we can write

$$
\begin{aligned}
|\mathcal{H}_n * \mathcal{K}(t, x) - \mathcal{H}_n * \mathcal{K}(s, y)| &\le \|\mathcal{H}_n\|_{0,\lambda'} \int_\mathbb{R} \int_{\mathbb{R}^N} |\mathcal{K}(\tau + (t - s), z + (x - y)) - \mathcal{K}(\tau, z)| \, (1 + |z|^{\lambda'})d\tau dz, \\
&\le (C_0 + 4C_3M) \, \omega(|t - s| + |x - y|),
\end{aligned}
$$

where ω is a modulus of continuity. Using now the bound on $\|u_n^0\|_{0,\lambda'}$, it is immediate to see that the sequence $(\nabla K(t, \cdot) * u_n^0)_n$ is equicontinuous in $(0, T] \times \mathbb{R}^N$; hence, so is $(\nabla u_n)_n$ and the local uniform limit u of u_n is C^1 in space in $(0, T] \times \mathbb{R}^N$. The proof of Theorem 4.2 is complete. ∎

Proof of Theorem 4.3. Once again, we regularize the initial conditions as in Theorem 4.1; in particular, the corresponding solution of (1.1) remains bounded in $C^0_{\lambda'}$, and M denotes its bound. Let us derive local gradient estimate on ∇u_n with the help of (4.6). By (A4),(B1) and (C3),

$$\|H_n(s, ., u_n(s, .), \nabla u_n(s, .))\|_{0,\lambda'} \le C_0 + C_3M + C_2\|\nabla u_n(s, .)\|_{0,\lambda'}.$$

By (2.6), we get

$$\|\nabla K(t-s,.) * H_n(s,.,u_n(s,.),\nabla u_n(s,.))\|_{0,\lambda'} \le 2(t-s)^{-1/\lambda}\left(\mathcal{K}_0 + \mathcal{K}_{\lambda'}\right)(C_0 + C_3 M)$$
$$+ 2(t-s)^{-1/\lambda}s^{-1/\lambda}\left(\mathcal{K}_0 + \mathcal{K}_{\lambda'}\right)C_2\|s^{1/\lambda}\nabla u_n(s,.)\|_{0,\lambda'}.$$

Since $\|\nabla K(t,\cdot) * u_n^0\|_{0,\lambda'} \le 2t^{-1/\lambda}\left(\mathcal{K}_0 + \mathcal{K}_{\lambda'}\right)M$, we deduce from (4.6) that

$$\|t^{1/\lambda}\nabla u_n(t,.)\|_{0,\lambda'} \le 2\left(\mathcal{K}_0 + \mathcal{K}_{\lambda'}\right)M + 2\frac{\lambda}{1-\lambda}\,t\,\left(\mathcal{K}_0 + \mathcal{K}_{\lambda'}\right)(C_0 + C_3 M)$$
$$+ 2I_\lambda t^{\frac{\lambda-1}{\lambda}}\left(\mathcal{K}_0 + \mathcal{K}_{\lambda'}\right)C_2 \sup_{[0,T]} \|s^{1/\lambda}\nabla u_n(s,.)\|_{0,\lambda'}.$$

Let $T_1 \in [0,T]$ be such that $2I_\lambda T_1^{\frac{\lambda-1}{\lambda}}\left(\mathcal{K}_0 + \mathcal{K}_{\lambda'}\right)C_2 \le 1/2$. We have proved that

$$\sup_{[0,T_1]} \|t^{1/\lambda}\nabla u_n(t,.)\|_{0,\lambda'} \le 4\left(\left(\mathcal{K}_0 + \mathcal{K}_{\lambda'}\right)M + \frac{\lambda}{1-\lambda}\,T_1\,\left(\mathcal{K}_0 + \mathcal{K}_{\lambda'}\right)(C_0 + C_3 M)\right).$$

Taking any T_0 as initial time, we can argue similarly to prove that for all $n \in \mathbb{N}$ and all $T_0 \in [0,T-T_1]$,

$$\sup_{[T_0,T_0+T_1]} \|(t-T_0)^{1/\lambda}\nabla u_n(t,.)\|_{0,\lambda'} \le 4\left(\left(\mathcal{K}_0 + \mathcal{K}_{\lambda'}\right)M + \frac{\lambda}{1-\lambda}\,T_1\,\left(\mathcal{K}_0 + \mathcal{K}_{\lambda'}\right)(C_0 + C_3 M)\right).$$

The local uniform limit u of u_n is then locally Lipschitz continuous wrt x in $(0,T] \times \mathbb{R}^N$. This process can be iterated with the formula (4.4) to prove the $C^{1,1}$ regularity in space and *a fortiori* the local Lipschitz continuity regularity in time. The proof of Theorem 4.3 is now complete. ∎

Let us end this section with a remark on regularity results in the uniformly integrable framework.

Remark 4.1. The key property on the kernel used in the preceding proof is Estimate (2.6). By the homogeneity property of K, this property still holds true for $(1 + |x|^{\lambda'})$ replaced by $\Phi(|x|)$, with $\Phi : \mathbb{R}^+ \to \mathbb{R}^+$ non-decreasing and such that $\int \Phi(|z|)d\mu_b(z) < +\infty$. This suggests that our techniques can be adapted to initial condition satisfying (2.11).

5 Convergence of gradients as "viscosity" vanishes

In this section, we prove a result about the sequence of the gradients of the solution u^ε of (1.7)-(1.8) under the following assumption

(D) The non-linearity H is strictly convex wrt p.

Theorem 5.1. *Let $\lambda \in (1,2)$ and $\lambda' \in (1,\lambda)$. Let $u_0 \in C_{\lambda'}^{0,1}(\mathbb{R}^N)$ (resp. let u_0 be Lipschitz continuous). Assume (D) and the assumptions of Theorem 4.2 (resp. of Theorem 4.1). Then the unbounded regular solution u^ε of (1.7)-(1.8) converges locally uniformly to the solution u^0 of (1.7) with $\varepsilon = 0$ and the sequence of gradients satisfies: for all $p \in [1,+\infty)$,*

$$\nabla u^\varepsilon(t,x) \underset{\varepsilon\to 0}{\to} \nabla u^0(t,x) \qquad in \quad L_{loc}^p(\overline{Q}_T).$$

Proof. To avoid technicalities with Young measures, we only do the proof with $H(t,x,u,p) = H(p)$ and let the reader check that the following ideas can be adapted in the general case.

The kernel associated with $\varepsilon g[\cdot]$ is $K_\varepsilon(t,x) = K(\varepsilon t,x)$. We then have

$$u^\varepsilon(t,x) = K(\varepsilon t,.) * u_0(x) + \int_0^t K(\varepsilon(t-s),\cdot) * H(\nabla u^\varepsilon(s,\cdot))ds. \tag{5.1}$$

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Consider the Young measure ν associated with the locally bounded family $\{\nabla u^\varepsilon\}_{\varepsilon>0}$ with $\varepsilon \to 0$. Precisely, its disintegration is a family of probabilities $\{\nu_{t,x}\}_{(t,x)\in Q_T}$ such that for all $F \in C(\mathbb{R}^N)$ and all $\phi \in C_c(\overline{Q}_T)$,

$$\int_0^T \int_{\mathbb{R}^N} F(\nabla u^\varepsilon(t,x))\phi(t,x)\,dtdx \to \int_0^T \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} F(\xi)\phi(t,x)\,d\nu_{t,x}(\xi)dtdx \quad \text{as} \quad \varepsilon \to 0 \qquad (5.2)$$

(up to a subsequence). Indeed, Prokhorov's Theorem gives us the existence of a Young measure ν_R on $\mathbb{R}^N$ indexed by $(0,T) \times B_R$ for all $R > 0$ associated to the bounded sequence $(\mathbf{1}_{(0,T)\times B_R} \nabla u^\varepsilon)_{\varepsilon>0}$. An argument of diagonalization (with $R_n = n$ for instance) then implies the existence of the measure ν indexed by Q_T that coincides with ν_R on each $(0,T) \times B_R$. Notice finally, that by the local bound we have on ∇u^ε (see Theorems 3.3 and 3.6 respectively), (5.2) holds true for F not necessarily bounded on $\mathbb{R}^N$ and φ with compact support on $\overline{Q}_T$; indeed, $\mathrm{supp}(\nu_{t,x}) \subset \{|\xi| \leq \sup_{\varepsilon>0}|\nabla u^\varepsilon(t,x)|\}$. This implies that

$$\int_0^t \int_{\mathbb{R}^N} F(\nabla u^\varepsilon(s,x))ds\,\phi(x)dx \to \int_0^t \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} F(\xi)\,d\nu_{s,x}(\xi)ds\,\phi(x)dx \quad \text{as} \quad \varepsilon \to 0.$$

We will use it with $F = H$. The previous convergence result implies that

$$\int_0^t H(\nabla u^\varepsilon(s,x))ds \to \int_0^t \int_{\mathbb{R}^N} H(\xi)\,d\nu_{s,x}(\xi)ds \quad \text{in} \quad \mathcal{D}'(\mathbb{R}^N).$$

We try to prove that for almost every (a.e. for short) $(t,x) \in Q_T$, $\nu_{t,x}$ is a Dirac mass centered at $\nabla u^0(t,x)$. Classical results about Young measures will thus imply the convergence in $L^p_{\mathrm{loc}}(\overline{Q}_T)$ for all $p \geq 1$.

We claim that, as $\varepsilon \to 0$,

$$\int_0^t K(\varepsilon(t-s),\cdot) * H(\nabla u^\varepsilon(s,\cdot))(x)ds \to \int_0^t \int_{\mathbb{R}^N} H(\xi)\,d\nu_{s,x}(\xi)ds \quad \text{in} \quad \mathcal{D}'(\mathbb{R}^N). \qquad (5.3)$$

Indeed, thanks to the evenness of K wrt x, we have:

$$\int_0^t \int_{\mathbb{R}^N} K(\varepsilon(t-s),\cdot) * H(\nabla u^\varepsilon(s,\cdot))(x)\,\varphi(x)dxds$$

$$= \int_0^t \int_{\mathbb{R}^N} K(\varepsilon(t-s),\cdot) * \varphi\,(x)\,H(\nabla u^\varepsilon(s,x))dxds.$$

Recalling that $\int_{\mathbb{R}^N} K(t,\cdot) = 1$, the homogeneity property of K implies from the equality above that

$$\left|\int_0^t \int_{\mathbb{R}^N} K(\varepsilon(t-s),\cdot) * H(\nabla u^\varepsilon(s,.))(x)\,\varphi(x)dxds - \int_0^t \int_{\mathbb{R}^N} H(\nabla u^\varepsilon(s,x))\,\varphi(x)dxds\right|, \qquad (5.4)$$

$$\leq \int_0^t \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} K(\varepsilon(t-s),z)\,|\varphi(x-z)-\varphi(x)|\,|H(\nabla u^\varepsilon(s,x))|dzdxds,$$

$$\leq \int_0^t \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} K(t-s,z)\,|\varphi(x-\varepsilon^{1/\lambda}z)-\varphi(x)|\,|H(\nabla u^\varepsilon(s,x))|dzdxds,$$

$$\leq \varepsilon^{1/\lambda}\|\nabla\varphi\|_{L^1(\mathbb{R}^N)} \int_0^t \int_{\mathbb{R}^N} K(t-s,z)|z| \sup_{1\geq\varepsilon>0} \sup_{|x|\leq R_\varphi+\varepsilon^{1/\lambda}|z|} |H(\nabla u^\varepsilon(s,x))|dzds$$

where $\mathrm{supp}(\varphi) \subset B_{R_\varphi}$. By (2.5), the growth conditions on H and the local bound we have on ∇u^ε, we deduce that (5.4) goes to 0 as $\varepsilon \to 0$.

Since u^ε converges towards u^0 locally uniformly, we can pass to the limit in Duhamel's formula (see Corollary 4.1 and Remark 4.2) in the distribution sense as $\varepsilon \to 0$. We thus obtain by using (5.3) that:

$$u^0(t,x) = u_0(x) + \int_0^t \int_{\mathbb{R}^N} H(\xi)d\nu_{s,x}(\xi) \quad \text{a.e.} \quad x. \qquad (5.5)$$

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Since u^0 is locally Lipschitz continuous wrt (t, x), we also have an integral (mild) formulation:

$$u^0(t, x) = u_0(x) + \int_0^t H(\nabla u^0(s, x))ds \quad \text{a.e.} \quad x. \tag{5.6}$$

Combining (5.5) and (5.6), we obtain:

$$\int_0^t \int_{\mathbb{R}^N} H(\xi)d\nu_{s,x}(\xi)ds = \int_0^t H(\nabla u^0(s, x))ds \quad \text{a.e.} \quad x.$$

Both sides are Lipschitz continuous wrt t, and we obtain for a.e. (t, x):

$$\int_{\mathbb{R}^N} H(\xi)d\nu_{t,x}(\xi) = H(\nabla u^0(t, x)).$$

Now use the facts that $\int_{\mathbb{R}^N} \xi d\nu_{t,x}(\xi) = \nabla u^0(t, x)$ and that H is strictly convex to conclude that $\nu_{t,x} = \delta_{\nabla u^0(t,x)}$. The proof is now complete. ∎

A Sketches of proofs of classical results

A.1 Proofs of Lemmata 2.3.1 and 2.3.2

Proof of Lemma 2.3.1. Let $(a, p) \in \overline{\partial}^P u(t, x)$. Following [5], it is quite classical to pass to the limit in the Hamiltonian; hence, we only need to check that we can pass to the limit in the integral term. To do so, let $f_n(z)$ denote the function $u(t_n, x_n + z) - u(t_n, x_n) - \frac{p_n \cdot z}{1 + |z|^2}$ where t_n, x_n and p_n appear in the definition of the closure of superdifferential. Let Φ be the barrier function associated to $u \in \mathcal{C}^+$. By the definition of a supergradient, we have:

$$f_n(z) \leq \Phi(x_n + z)\, \mathbf{1}_{B_{r/2}^c}(z) + g(z)$$

for n large enough, where g is continuous and μ-integrable. For $R > 2|x|$, define:

$$\omega(R) := \int_{B_R^c} \Phi(x_n + z)d\mu(z) + \int_{B_R^c} g(z)d\mu(z),$$

where

$$
\begin{aligned}
\int_{B_R^c} \Phi(x_n + z)d\mu(z) &= c_N(\lambda) \int_{B_R^c} \Phi(z')\frac{dz'}{|z' - x_n|^{1+\lambda}}, \\
&\leq 2^{N+\lambda}c_N(\lambda) \int_{B_R^c} \Phi(z')\frac{dz'}{|z'|^{1+\lambda}},
\end{aligned}
$$

since $|z' - x_n| \geq |z'| - |x_n| \geq \frac{1}{2}|z'|$. By the integrability property of Φ, we see that $\lim_{R \to +\infty} \omega(R) = 0$. Moreover, Fatou's lemma implies:

$$\limsup_{n \to +\infty} \int_{\mathbb{R}^N \setminus \{0\}} f_n(z)d\mu(z) \leq \int_{B_R \setminus \{0\}} \limsup_{n \to +\infty} f_n(z)d\mu(z) + \omega(R).$$

The positive part of $\limsup_{n \to +\infty} f_n(z)$ is μ-integrable since it is bounded above by $\Phi(x + z)\, \mathbf{1}_{B_r^c} + g(z)$; hence, we can use again Fatou's lemma to pass to the limit as $R \to +\infty$ and deduce that

$$\limsup_{n \to +\infty} \int_{\mathbb{R}^N \setminus \{0\}} f_n(z)d\mu(z) \leq \int_{\mathbb{R}^N \setminus \{0\}} \limsup_{n \to +\infty} f_n(z)d\mu(z).$$

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The limit equation then becomes:

$$a + H(t, x, u(t, x), p) \leq \int_{\mathbb{R}^N \setminus \{0\}} \limsup_{n \to +\infty} f_n(z) d\mu(z). \tag{A.1}$$

The right-hand side of (A.1) is finite, which implies that $\limsup_{n \to +\infty} f_n(z)$ is μ-integrable. Finally, the upper-semicontinuity of u and the fact that $u_n(t_n, x_n) \to u(t, x)$ imply:

$$\limsup_{n \to +\infty} f_n(z) \leq u(t, x + z) - u(t, x) - \frac{p \cdot z}{1 + |z|^2} =: f(z);$$

hence, the negative part of $f(z)$ is μ-integrable and (A.1) gives:

$$a + H(t, x, u(t, x), p) \leq \int_{\mathbb{R}^N \setminus \{0\}} f(z) d\mu(z).$$

The proof of Lemma 2.3.1 is complete. ∎

Proof of Lemma 2.3.2. To prove the existence of a and b, we let the reader check that we can slightly modify ϕ, without changing its first order partial derivatives at $(\bar{t}, \bar{x}, \bar{y})$, in such a way that M achieves a strict maximum at $(\bar{t}, \bar{x}, \bar{y})$ and ϕ is positive and such that

$$\limsup_{|(x,y)| \to +\infty} \sup_{t \in [0,T]} \frac{u(t, x) - v(t, y)}{\phi(t, x, y)} \leq 0. \tag{A.2}$$

Consider a parameter $\delta > 0$ and define

$$M_\delta := \sup_{[0,T] \times [0,T] \times \mathbb{R}^N \times \mathbb{R}^N} \left(u(t, x) - v(s, y) - \frac{|s - t|^2}{2\delta} - \phi(t, x, y) \right).$$

This supremum is achieved at some point $(t_\delta, s_\delta, x_\delta, y_\delta)$. By (A.2), the set $\{(t_\delta, s_\delta, x_\delta, y_\delta) : \delta > 0\}$ is relatively compact and we know that $(t_\delta, s_\delta, x_\delta, y_\delta) \to (\bar{t}, \bar{t}, \bar{x}, \bar{y})$ as $\delta \to 0$. Moreover,

$$(a_\delta, \nabla_x \phi(t_\delta, x_\delta, y_\delta)) \in \partial^p u(t_\delta, x_\delta),$$
$$(b_\delta, -\nabla_y \phi(t_\delta, x_\delta, y_\delta)) \in \partial_p v(s_\delta, y_\delta),$$

where $b_\delta = \frac{t_\delta - s_\delta}{\delta}$ and $a_\delta = b_\delta + \partial_t \phi(t_\delta, x_\delta, y_\delta)$. By writing the viscosity inequality for u, we get

$$a_\delta \leq -H \left(t_\delta, u(t_\delta, x_\delta), \nabla_x \phi(t_\delta, x_\delta, y_\delta) \right) - g[u] \left(t_\delta, x_\delta, \nabla_x \phi(t_\delta, x_\delta, y_\delta) \right).$$

One can see that the constants in the definition of the supergradient $(a_\delta, \nabla_x \phi(t_\delta, x_\delta, y_\delta))$ can be chosen as follows: $\sigma_\delta := \sup_{|t-t_\delta|+|x-x_\delta|+|y-y_\delta| \leq 1} |D_x^2 \phi(t, x, y)|$ and $r = 1$. According to the growth condition on u, this gives us an upper bound on $-g[u] (t_\delta, x_\delta, \nabla_x \phi(t_\delta, x_\delta, y_\delta))$ which does not depend on small δ. The viscosity inequality above then gives us an upper bound on a_δ independently on small δ. Similar arguments allow to prove that b_δ is bounded from below independently on small δ. Since $a_\delta - b_\delta = \partial_t \phi(t_\delta, x_\delta, y_\delta)$, we deduce that a_δ and b_δ are bounded independently on small δ. There then exit $a, b \in \mathbb{R}$ such that $(a_\delta, b_\delta) \to (a, b)$ as $\delta \to 0$ (up to a subsequence). It is now straightforward that:

$$\left(a, \nabla_x \phi(\bar{t}, \bar{x}, \bar{y}) \right) \in \overline{\partial}^P u(\bar{t}, \bar{x}),$$

$$\left(b, -\nabla_y \phi(\bar{t}, \bar{x}, \bar{y}) \right) \in \overline{\partial}_P v(\bar{t}, \bar{y})$$

and $a - b = \partial_t \phi(\bar{t}, \bar{x}, \bar{y})$.

It remains to prove that (2.3) holds true. Writing that $M = u(\bar{t},\bar{x}) - v(\bar{t},\bar{y}) - \phi(\bar{t},\bar{x},\bar{y})$, we get

$$u(\bar{t},\bar{x}+z) - u(\bar{t},\bar{x}) - \frac{\nabla_x\phi(\bar{t},\bar{x},\bar{y})\cdot z}{1+|z|^2} - v(\bar{t},\bar{y}+z) + v(\bar{t},\bar{y}) + \frac{\nabla_y\phi(\bar{t},\bar{x},\bar{y})\cdot z}{1+|z|^2}$$
$$\leq \phi(\bar{t},\bar{x}+z,\bar{y}+z) - \phi(\bar{t},\bar{x},\bar{y}) - \frac{(\nabla_x+\nabla_y)\phi(\bar{t},\bar{x},\bar{y})\cdot z}{1+|z|^2}$$

for all $z \in \mathbb{R}^N$. We have seen in the preceding proof that the negative part of the left-hand side of this inequality is μ-integrable; hence, so do the negative part of the right-hand side and integrating with respect to $d\mu(z)$ completes the proof of Lemma 2.3.2. ■

A.2   Main ideas of the proof of the discontinuous stability

Sketch of the proof of Proposition 2.1. Following the preceding proof of Lemma 2.3.1, one can easily adapt the ideas of [5] to prove that $u := \limsup^* u_n$ is a viscosity subsolution of (1.1) with generalized initial condition:

$$\min\{\partial_t u(0,.) + H(0,x,u(0,.),\nabla u(0,.)) + g[u(0,.)], \ u(0,.) - u_0\} \leq 0, \tag{A.3}$$

where $u_0 := \limsup^*(u_n(0,.))$; the main difficulty is to handle the non-local term and this is done by using Fatou's lemma.

Let us prove (2.4). For $\lambda' \in (0,\lambda)$, $\varepsilon > 0$ and $C > 0$, define

$$\chi(t,y) = u(t,y) - \frac{\phi_\varepsilon(y-x)}{\varepsilon} - \Phi(y) - Ct,$$

where $\phi_\varepsilon(z) := \frac{(\varepsilon+|z|^2)^{\frac{\lambda'}{2}}}{\lambda'}$. Using some ideas of the proof of Lemma 2.5.2, we claim that Φ can be slightly modified in such way that Φ stays positive, C^2 and μ_b-integrable and satisfies in addition

$$\limsup_{|x|\to+\infty} \frac{\sup_{[0,T]} u(t,x)}{\Phi(x)} \leq 0;$$

hence, if $x \in \mathbb{R}^N$ is fixed, χ achieves its maximum at some $(\bar{t},\bar{y})$. We claim that $(\bar{t},\bar{y}) \to (0,x)$ as $\varepsilon \to 0$ and $C \to +\infty$. Let us fix ε. On denoting

$$p_C := \frac{(\bar{y}-x)\,\nabla\phi_\varepsilon(\bar{y}-x)}{\varepsilon} + \nabla\Phi(\bar{y}),$$

the optimilaty conditions satisfied by the global maximum $(\bar{t},\bar{y})$ imply that $(C,p_C) \in \partial^P u(\bar{t},\bar{y})$ and

$$g[u](\bar{t},\bar{y},p_C) \geq g\left[\frac{\phi_\varepsilon(\cdot-x)}{\varepsilon}+\Phi\right](\bar{y}) =: I_C.$$

By the smoothness of ϕ_ε and Φ, we can see that $\limsup_{C\to+\infty}(|p_C|+|I_C|) < +\infty$; hence, if C is sufficiently large, then

$$C + H(\bar{t},\bar{y},u(\bar{t},\bar{y}),p_C) + I_C > 0.$$

It follows that

$$C + H(\bar{t},\bar{y},u(\bar{t},\bar{y}),p_C) + g[u](\bar{t},\bar{y},p_C) > 0$$

and that necessarily $\bar{t} = 0$. Moreover, since $\chi(0,\bar{y}) \geq \chi(0,x)$, we have $u(0,\bar{y}) \geq u(0,x) + \Phi(\bar{y}) - \Phi(x)$ and (A.3) implies that $u(0,x) \leq u_0(\bar{y}) + \Phi(x) - \Phi(\bar{y})$. Since u_0 is usc as upper semi-limit, we can pass to the limit as $\varepsilon \to 0$ and $C \to +\infty$ to deduce that $u(0,x) \leq u_0(x)$. The proof of Proposition 2.1 is now complete. ■

A.3 Proof of the comparison principle in the sublinear case

Proof of Theorem 3.1. The proof proceeds in two steps. We first prove that there exists a constant $K > 0$ such that

$$\sup_{[0,T] \times \mathbb{R}^N \times \mathbb{R}^N} (u(t,x) - v(t,y) - K|x-y|) < +\infty \tag{A.4}$$

and then prove that $u \leq v$.

Hence, let us prove (A.4). First we note that the uniform continuity of u_0 and (A2) imply that we can choose $K > 0$ such that

$$\sup_{\mathbb{R}^N \times \mathbb{R}^N} (u_0(x) - u_0(y) - K|x-y|) < +\infty, \tag{A.5}$$

$$\sup_{[0,T] \times \mathbb{R}^N \times \mathbb{R}^N} (H(t,y,u(t,x),0) - H(t,x,u(t,x),0) - K|x-y|) < +\infty. \tag{A.6}$$

Consider $(\beta_R)_{R \geq 1}$ a family of C^2 functions on $\mathbb{R}^N$ such that

$$\begin{cases} \text{(i)} \ \beta_R \geq 0, \\ \text{(ii)} \ \liminf_{|x| \to +\infty} \frac{\beta_R(x)}{|x|} \geq 2R_0, \\ \text{(iii)} \ \|\beta_R\|_{0,\lambda'} + \|\nabla \beta_R\|_\infty + \|D^2\beta_R\|_\infty \leq C \text{ for } R \geq 1, \\ \text{(iv)} \ \lim_{R \to +\infty} \beta_R(x) = 0 \text{ for } x \in \mathbb{R}^N, \end{cases} \tag{A.7}$$

where $R_0 = \max\left\{\sup_{[0,T] \times \mathbb{R}^N} \frac{u(t,x)}{1+|x|}, \inf_{[0,T] \times \mathbb{R}^N} \frac{v(t,x)}{1+|x|}, 0\right\}$ and C is some positive constant. In view of (A3), (A.6), (A.7)-(iii) and Lemma B.0.1, there exists a constant σ such that for all $R \geq 1$, $x,y \in \mathbb{R}^N$, $p \in B_{Ke^T}$, $t \in [0,T]$,

$$H(t,y,u(t,x),p+\nabla\beta_R(y)) - H(t,x,u(t,x),p-\nabla\beta_R(x)) - K|x-y| - g[\beta_R](x) - g[\beta_R](y) \leq \sigma. \tag{A.8}$$

Define

$$M_R := \sup_{[0,T] \times \mathbb{R}^N \times \mathbb{R}^N} \left(u(t,x) - v(t,y) - Ke^t(1+|x-y|^2)^{1/2} - \beta_R(x) - \beta_R(y) - \sigma t \right).$$

There exists $(\bar{t}, \bar{x}, \bar{y}) \in [0,T] \times \mathbb{R}^N \times \mathbb{R}^N$ where the supremum is attained. Now either (A.4) holds true or $\lim_{R \to +\infty} M_R = +\infty$. In the latter case, M_R is in particular greater than 0 and greater than the supremum in (A.5) for R large enough. It follows that $u(\bar{t}, \bar{x}) > v(\bar{t}, \bar{y})$ and $\bar{t} \neq 0$. By Lemma 2.3.2, there exist $a,b \in \mathbb{R}$ such that

$$(a, p + \nabla\beta_R(\bar{x})) \in \overline{\partial}^P u(\bar{t}, \bar{x}),$$
$$(b, p - \nabla\beta_R(\bar{y})) \in \overline{\partial}_P v(\bar{t}, \bar{y}),$$

with $a - b = Ke^{\bar{t}}(1+|\bar{x}-\bar{y}|^2)^{1/2} + \sigma$, $p = Ke^{\bar{t}}\frac{(\bar{x}-\bar{y})}{(1+|\bar{x}-\bar{y}|^2)^{1/2}}$ and

$$g[v](\bar{t}, \bar{y}, p - \nabla\beta_R(\bar{y})) - g[u](\bar{t}, \bar{x}, p + \nabla\beta_R(\bar{x})) \leq -g[\beta_R](\bar{x}) - g[\beta_R](\bar{y}).$$

Since u is a subsolution and v is a supersolution of (1.1), Lemma 2.3.1 gives

$$a + H(\bar{t}, \bar{x}, u(\bar{t}, \bar{x}), p + \nabla\beta_R(\bar{x})) + g[u](\bar{t}, \bar{x}, p + \nabla\beta_R(\bar{x})) \leq 0,$$
$$b + H(\bar{t}, \bar{y}, v(\bar{t}, \bar{y}), p - \nabla\beta_R(\bar{y})) + g[v](\bar{t}, \bar{x}, p - \nabla\beta_R(\bar{y})) \geq 0.$$

Subtracting the two inequalities, it comes

$$\sigma \leq H(\bar{t}, \bar{y}, v(\bar{t}, \bar{y}), p - \nabla\beta_R(\bar{y})) - H(\bar{t}, \bar{x}, u(\bar{t}, \bar{x}), p + \nabla\beta_R(\bar{x})) - K|\bar{x} - \bar{y}| - g[\beta_R](\bar{x}) - g[\beta_R](\bar{y}).$$

Since $u(\bar{t}, \bar{x}) > v(\bar{t}, \bar{y})$ and H is strictly non-decreasing wrt u (see (A1')), we find that

$$\sigma < H(\bar{t}, \bar{y}, u(\bar{t}, \bar{x}), p - \nabla\beta_R(\bar{y})) - H(\bar{t}, \bar{x}, u(\bar{t}, \bar{x}), p + \nabla\beta_R(\bar{x})) - K|\bar{x} - \bar{y}| - g[\beta_R](\bar{x}) - g[\beta_R](\bar{y}).$$

By (A.8), we get a contradiction and this implies that (A.4) holds true.

We this information in hand we repeat the above line of arguments to show that

$$M := \sup_{[0,T]\times\mathbb{R}^N\times\mathbb{R}^N} (u - v) \leq 0.$$

Let $\lambda'' \in (\lambda', \lambda)$ be fixed. Consider two parameters $\varepsilon, \eta > 0$ and define

$$M_{\varepsilon,\eta} := \sup_{[0,T]\times\mathbb{R}^N\times\mathbb{R}^N} \left(u(t,x) - v(t,y) - \frac{|x-y|^2}{2\varepsilon} - \eta\frac{(1+|x|^2)^{\frac{\lambda''}{2}}}{\lambda''} \right).$$

There exists $(\bar{t}, \bar{x}, \bar{y}) \in [0,T] \times \mathbb{R}^N \times \mathbb{R}^N$ where the supremum is attained. If $M > 0$ then for ε and η small enough, $M_{\varepsilon,\eta} \geq M/2 > 0$ and $\bar{t} \neq 0$; indeed,

$$M_{\varepsilon,\eta} \geq \sup_{[0,T]\times\mathbb{R}^N\times\mathbb{R}^N} \left(u(t,x) - v(t,x) - \eta\frac{(1+|x|^2)^{\frac{\lambda''}{2}}}{\lambda''} \right) \to M$$

as $\eta \to 0$ and

$$\sup_{\mathbb{R}^N\times\mathbb{R}^N} \left(u(0,x) - v(0,y) - \frac{|x-y|^2}{2\varepsilon} - \eta\frac{(1+|x|^2)^{\frac{\lambda''}{2}}}{\lambda''} \right) \leq \sup_{\mathbb{R}^N\times\mathbb{R}^N} \left(u_0(x) - u_0(y) - \frac{|x-y|^2}{2\varepsilon} \right) \to 0$$

as $\varepsilon \to 0$, thanks to the uniform continuity of u_0. It follows that $u(\bar{t}, \bar{x}) - v(\bar{t}, \bar{y}) \geq M/2$ and by (A.4):

$$\lim_{\varepsilon\to 0} \limsup_{\eta\to 0} \frac{|\bar{x} - \bar{y}|^2}{\varepsilon} = 0, \tag{A.9}$$

$$\frac{|\bar{x} - \bar{y}|^2}{4\varepsilon} + \eta\frac{(1+|\bar{x}|^2)^{\frac{\lambda''}{2}}}{\lambda''} \leq u(\bar{t}, \bar{x}) - v(\bar{t}, \bar{y}) - \frac{|\bar{x} - \bar{y}|^2}{4\varepsilon} \leq C(1+\varepsilon) \tag{A.10}$$

for some positive constant C. By Lemma 2.3.2, there exist

$$\left(a, p + \eta(1 + |\bar{x}|^2)^{\frac{\lambda''-2}{2}} \bar{x} \right) \in \overline{\partial}^P u(\bar{t}, \bar{x}),$$

$$(a, p) \in \overline{\partial}_P v(\bar{t}, \bar{y})$$

with $p = \frac{\bar{x} - \bar{y}}{\varepsilon}$ and

$$g[v](\bar{t}, \bar{y}, p) - g[u]\left(\bar{t}, \bar{x}, p + \eta(1 + |\bar{x}|^2)^{\frac{\lambda''-2}{2}} \bar{x} \right) \leq \frac{\eta}{\lambda''} g\left[(1 + |\cdot|^2)^{\frac{\lambda''}{2}} \right](\bar{x}).$$

Substracting the corresponding viscosity inequalities and using (A1') (see Remark 3.1), we get

$$M/2 \leq H(\bar{t}, \bar{y}, u(\bar{t}, \bar{x}), p) - H\left(\bar{t}, \bar{x}, u(\bar{t}, \bar{x}), p + \eta(1 + |\bar{x}|^2)^{\frac{\lambda''-2}{2}} \bar{x} \right) - \frac{\eta}{\lambda''} g\left[(1 + |\cdot|^2)^{\frac{\lambda''}{2}} \right](\bar{x}).$$

If $\eta \leq 1$, then (A.10) implies that $p + \eta(1 + |\bar{x}|^2)^{\frac{\lambda''-2}{2}} \bar{x} \in B_{R_\varepsilon}$ with

$$R_\varepsilon = \sqrt{\frac{C(1+\varepsilon)}{4\varepsilon}} + (\lambda'' C(1+\varepsilon))^{\frac{\lambda''-1}{\lambda''}}.$$

By (A3),

$$M/2 \le H(\bar{t}, \bar{y}, u(\bar{t}, \bar{x}), p) - H\left(\bar{t}, \bar{x}, u(\bar{t}, \bar{x}), p\right) + m_{R_\varepsilon}\left(\eta^{\frac{1}{\lambda''}}(\lambda'' C(1+\varepsilon))^{\frac{\lambda''-1}{\lambda''}}\right) - \frac{\eta}{\lambda''} g\left[(1+|\cdot|^2)^{\frac{\lambda''}{2}}\right](\bar{x}).$$

Assumption (A2) now gives

$$M/2 \le m\left(|\bar{x} - \bar{y}| + \frac{|\bar{x} - \bar{y}|^2}{\varepsilon}\right) + m_{R_\varepsilon}\left(\eta^{\frac{1}{\lambda''}}(\lambda'' C(1+\varepsilon))^{\frac{\lambda''-1}{\lambda''}}\right) - \frac{\eta}{\lambda''} g\left[(1+|\cdot|^2)^{\frac{\lambda''}{2}}\right](\bar{x}).$$

By Lemma B.0.1 and (A.9), we see that the right-hand side of the inequality above tends to 0 as $\eta \to 0$ and $\varepsilon \to 0$ successively. This gives us the desired contradiction $0 < M/2 \le 0$ and this completes the proof of Theorem 3.1. ∎

B A technical result

Define for $\beta \in (1, 2)$:

$$C_\beta^2(\mathbb{R}^N) := \left\{ f \in C_\beta^{0,1}(\mathbb{R}^N) \cap C_b^2(\mathbb{R}^N) : D^2 f(x) = \mathcal{O}\left(|x|^{\beta-2}\right) \text{ as } |x| \to +\infty \right\}.$$

Lemma B.0.1. *Let $\lambda \in (0, 2)$ and $\lambda'' \in (0, \lambda)$. Then:*

- *If $\lambda'' > 1$ then $g : C_{\lambda''}^2(\mathbb{R}^N) \to C_b(\mathbb{R}^N)$.*

- *If $\lambda'' \le 1$, then $g : C_b^2(\mathbb{R}^N) \to C_b(\mathbb{R}^N)$.*

Proof. Let $f \in C_b^2(\mathbb{R}^N)$ with $\lambda'' \in (0, 1]$. By a second order expansion, we get

$$|g[f]|(x) \le \sup_{B_r(x)} |D^2 f| \int_{B_r} |z|^2 d\mu(z) d\tau + \sup_{B_r(x)^c} |\nabla f| \int_{B_r^c} \frac{|z|}{1+|z|^2} \, d\mu(z) + \sup_{B_r(x)^c} |f| \int_{B_r^c} d\mu(z)$$

for all $r \ge 0$ and all $x \in \mathbb{R}^N$. The proof is complete in this case.

If now $\lambda'' \in (1, 2)$, write for $x \in \mathbb{R}^N \setminus B_1$

$$g[f](x) = -\int_{B_r}\left(f(x+z) - f(x) - \nabla f(x) \cdot \frac{z}{1+|z|^2}\right) d\mu(z) - \int_{\mathbb{R}^N \setminus B_r} (\dots) d\mu(z)$$

and choose $r = |x|/2$. Observe that

$$\sup_{z' \in B_{|x|/2}} (1 + |x + z'|)^{\lambda''-2} \le (1 + |x|/2)^{\lambda''-2}$$

so that

$$\begin{aligned}
|g[f](x)| \le{} & C(1+|x|/2)^{\lambda''-2} \int_{B_{|x|/2}} |z|^2 d\mu(z) + C(1+|x|)^{\lambda''-1} \int_{\mathbb{R}^N \setminus B_{|x|/2}} \frac{|z|}{1+|z|^2} d\mu(z) \\
& + C \int_{\mathbb{R}^N \setminus B_{|x|/2}} (1+|z|)^{\lambda''-1} \frac{|z|}{1+|z|^2} d\mu(z) + C \int_{\mathbb{R}^N \setminus B_{|x|/2}} |z|^{\lambda''} d\mu(z), \\
\le{} & C(1+|x|)^{\lambda''-2} J_2 + C(1+|x|)^{\lambda''-1} I_{-1} + C I_{\lambda''-2} + C I_{\lambda''}
\end{aligned}$$

where

$$J_2 = \int_{|z| \le |x|/2} |z|^2 \mu(dz) = C \int_0^{|x|/2} r^{1-\lambda} dr \le C(1 + |x|^{2-\lambda})$$

and for $\alpha = -1, \lambda'' - 2, \lambda''$ (hence $\alpha < \lambda$)

$$I_\alpha = \int_{\mathbb{R}^N \setminus B_{|x|/2}} |z|^\alpha d\mu(z) = C \int_{|x|/2}^{+\infty} r^\alpha r^{N-1} r^{-N-\lambda} dr = C \int_{|x|/2}^{+\infty} r^{\alpha-1-\lambda} dr = C|x|^{\alpha-\lambda}.$$

Hence, we obtain for $x \in \mathbb{R}^N \setminus B_1$

$$|g[f](x)| \leq C + C|x|^{\lambda''-\lambda} + C|x|^{\lambda''-2-\lambda} \leq C.$$

This completes the proof of the lemma. ∎

C Different sets of assumptions

H continuous, $\lambda \in (0,2)$ and $\lambda' \in (0,\lambda)$.

- THE SUBLINEAR CASE

  (A1) There exists $\nu \in \mathbb{R}$ such that for all $x \in \mathbb{R}^N$, $u, v \in \mathbb{R}$, $u \leq v$, $p \in \mathbb{R}^N$, $t \in [0,T]$,

  $$H(t,x,v,p) - H(t,x,u,p) \geq \nu(v-u).$$

  (A1') For all $x \in \mathbb{R}^N$, $u, v \in \mathbb{R}$, $u \leq v$, $p \in \mathbb{R}^N$, $t \in [0,T]$, $H(t,x,v,p) - H(t,x,u,p) \geq v - u$.

  (A2) There exists a modulus of continuity m such that for all $x, y \in \mathbb{R}^N$, $u \in \mathbb{R}$, $p \in \mathbb{R}^N$, $t \in [0,T]$,

  $$|H(t,x,u,p) - H(t,y,u,p)| \leq m((1+|p|)|x-y|).$$

  (A2') There exists a constant $C_1 \geq 0$ such that for all $x, y \in \mathbb{R}^N$, $u \in \mathbb{R}$, $p \in \mathbb{R}^N$, $t \in [0,T]$,

  $$|H(t,x,u,p) - H(t,y,u,p)| \leq C_1(1+|p|)|x-y|.$$

  (A3) For all $R > 0$, there exists a modulus of continuity m_R such that for all $x \in \mathbb{R}^N$, $u \in \mathbb{R}$, $p, q \in B_R$, $t \in [0,T]$,
  $$|H(t,x,u,p) - H(t,x,u,q)| \leq m_R(|p-q|).$$

  (A3') For all $R > 0$, there exists a constant C_R such that for all $x \in \mathbb{R}^N$, $u, v \in \mathbb{R}$, $p, q \in B_R$, $t \in [0,T]$,
  $$|H(t,x,u,p) - H(t,x,v,q)| \leq C_R(|u-v| + |p-q|).$$

  (A4) There exists a constant C_0 such that for all $x \in \mathbb{R}^N$,

  $$\sup_{[0,T]} |H(t,x,0,0)| \leq C_0(1+|x|^{\lambda'}).$$

- THE SUPERLINEAR CASE

  (B1) There exists a constant C_3 such that for all $x \in \mathbb{R}^N$, $u \in \mathbb{R}$, $p \in \mathbb{R}^N$, $t \in [0,T]$,

  $$|H(t,x,u,p) - H(t,x,v,p)| \leq C_3|u-v|.$$

  (B2) For all $R > 0$, there exists a modulus of continuity m_R such that for all $x, y \in B_R$, $u \in [-R,R]$, $p \in \mathbb{R}^N$, $t \in [0,T]$,

  $$|H(t,x,u,p) - H(t,y,u,p)| \leq m_R((1+|p|)|x-y|).$$

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(B2') There exists a constant C_1 such that for all $x, y \in \mathbb{R}^N$, $u \in \mathbb{R}$, $p \in \mathbb{R}^N$, $t \in [0, T]$,

$$|H(t, x, u, p) - H(t, y, u, p)| \leq C_1((1 + |x|^{\lambda'-1} + |y|^{\lambda'-1}) + |p|)|x - y|.$$

(B3) There exists a modulus of continuity m such that for all $x \in \mathbb{R}^N$, $u \in \mathbb{R}$, $p, q \in \mathbb{R}^N$, $t \in [0, T]$,

$$|H(t, x, u, p) - H(t, x, u, q)| \leq m((1 + |x|)|p - q|).$$

(B3') There exists a constant C_2 such that for all $x, y \in \mathbb{R}^N$, $u \in \mathbb{R}$, $p, q \in \mathbb{R}^N$, $t \in [0, T]$,

$$|H(t, x, u, p) - H(t, x, u, q)| \leq C_2(1 + |x|)|p - q|.$$

- THE UNIFORMLY INTEGRABLE CASE

  (C2) The Hamiltonian H is locally Lipschitz continuous wrt x and there are a constant C_1 and a locally bounded function $\Theta : \mathbb{R}^N \to \mathbb{R}^+$ such that $\int_{\mathbb{R}^N} \sup_{B_R(z)} \Theta^2 \, d\mu_b < +\infty$, for some $R > 0$, and for a.e. $x \in \mathbb{R}^N$, $u \in \mathbb{R}$, $p \in \mathbb{R}^N$, $t \in [0, T]$,

  $$|\nabla_x H(t, x, u, p)| \leq \Theta(x) + C_1|p|.$$

  (C3) There exists a modulus of continuity such that for all $x, y \in \mathbb{R}^N$, $p, q \in \mathbb{R}^N$, $u \in \mathbb{R}$, $t \in [0, T]$,

  $$|H(t, x, u, p) - H(t, x, u, q)| \leq m(|p - q|).$$

  (C3') There exists a constant C_2 such that for all $x, y \in \mathbb{R}^N$, $u \in \mathbb{R}$, $p, q \in \mathbb{R}^N$, $t \in [0, T]$,

  $$|H(t, x, u, p) - H(t, x, u, q)| \leq C_2|p - q|.$$

  (C4) There exists $R > 0$ such that

  $$\int_{\mathbb{R}^N} \sup_{[0,T] \times B_R(z)} |H(\cdot, \cdot, 0, 0)| \, d\mu_b < +\infty.$$

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  (D) The non-linearity H is strictly convex wrt p.

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