Uniqueness and stability in an inverse problem for the Schrödinger equation

Lucie Baudouin, Jean-Pierre Puel

UNIQUENESS AND STABILITY IN AN INVERSE PROBLEM FOR THE SCHRÖDINGER EQUATION.

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ABSTRACT: We study the Schrödinger equation $iy' + \Delta y + qy = 0$ in $\Omega \times (0,T)$ with Dirichlet boundary data $y|_{\partial\Omega\times(0,T)}$ and real valued initial condition $y|_{\Omega\times\{0\}}$ and we consider the inverse problem of determining the potential $q(x),\ x \in \Omega$ when $\frac{\partial y}{\partial \nu}|_{\Gamma_0\times(0,T)}$ is given. Here Ω is an open bounded domain of $\mathbb{R}^N$, Γ_0 is an open subset of $\partial\Omega$ satisfying a suitable geometrical condition and $T > 0$. More precisely, from a global Carleman estimate we prove a stability inequality between $\|p - q\|$ and $\left\|\frac{\partial y(q)}{\partial \nu} - \frac{\partial y(p)}{\partial \nu}\right\|$ with appropriate norms.

Keywords: Inverse problem, Schrödinger equation, Dirichlet boundary conditions.
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1 Introduction

Let $N \in \mathbb{N}$, $T > 0$ and let $\Omega \subset \mathbb{R}^N$ be a bounded domain with C^2-boundary $\partial\Omega$. Let Γ_0 be an open subset of $\partial\Omega$.

Throughout this paper, we use the following notations :

$$\nabla v = \left(\frac{\partial v}{\partial x_1}, \dots, \frac{\partial v}{\partial x_N}\right),\ \ D^2 v = \left(\frac{\partial^2 v}{\partial x_i \partial x_j}\right)_{1\le i,j\le N},$$

$$\Delta v = \sum_{i=1}^{N} \frac{\partial^2 v}{\partial x_i^2}\ ,\ \ v' = \frac{\partial v}{\partial t}\ \ and\ \ v'' = \frac{\partial^2 v}{\partial t^2}\ ,$$

$\nu \in \mathbb{R}^N$ denotes the unit outward normal vector to $\partial\Omega$,

$\dfrac{\partial v}{\partial \nu} = \nabla v.\nu$ is the normal derivative.

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We consider the Schrödinger equation :

$$\left\{ \begin{array}{l} iy'(x,t) + \Delta y(x,t) + q(x)y(x,t) = 0,\ x \in \Omega,\ t \in (0,T) \\ y(x,t) = h(x,t),\ x \in \partial\Omega,\ t \in (0,T) \\ y(x,0) = y_0(x),\ x \in \Omega. \end{array} \right. \tag{1}$$

This paper treats two kinds of inverse problems which can be stated as follows.

Non linear inverse Problem : Is it possible to retrieve the potential $q = q(x),\ x \in \Omega$ from measurement of the normal derivative

$$\left.\frac{\partial y}{\partial \nu}\right|_{\Gamma_0 \times (0,T)}$$

where y is the solution to (1)?

In this direction, we will answer to two more precise problems.

Uniqueness : Under geometrical conditions on Γ_0, does the equality $\dfrac{\partial y(q)}{\partial \nu} = \dfrac{\partial y(p)}{\partial \nu}$ on $\Gamma_0 \times (0,T)$ imply $q = p$ on Ω ?

Stability : Under geometrical conditions on Γ_0, is it possible to estimate $\|q-p\|_{L^2(\Omega)}$ or better, a stronger norm of $(p-q)$, by a suitable norm of

$$\left(\frac{\partial y(q)}{\partial \nu} - \frac{\partial y(p)}{\partial \nu}\right)|_{\Gamma_0 \times (0,T)} \quad ?$$

Indeed, we will only give a local answer about the determination of q. We will first work on a linearized version of the problem and consider the following Schrödinger equation :

$$\left\{ \begin{array}{l} iu'(x,t) + \Delta u(x,t) + q(x)u(x,t) = f(x)R(x,t),\ x \in \Omega,\ t \in (0,T) \\ u(x,t) = 0,\ x \in \partial\Omega,\ t \in (0,T) \\ u(x,0) = 0,\ x \in \Omega \end{array} \right. \tag{2}$$

Linear inverse problem : Is it possible to determine $f(x),\ x \in \Omega$ from the knowledge of the normal derivative

$$\left.\frac{\partial u}{\partial \nu}\right|_{\Gamma_0 \times (0,T)}$$

where R and q are given and u is the solution to (2)?

In the case of the wave equation, the uniqueness result for the linear inverse problem has been proved by M.V. KLIBANOV in [7] and a stability result of M. YAMAMOTO, deriving from it, can be read in [13].

Here we set $y = y(q)$ the weak solution to (1) and $u = u(f)$ the one to (2). If we formally linearize equation (1) around a non stationary solution, we obtain equation (2). In fact, we notice here that if we set $f = q - p$, $u = y(p) - y(q)$ and $R = y(q)$ on $\Omega \times (0, T)$, we obtain (2) after substraction of (1) with potential q from (1) with potential p and linearization.

In our inverse problem, we have to determine a coefficient of a lower order term in a Schrödinger equation from a single time dependent observation of Neumann data on a part Γ_0 of the boundary. On the other hand, there is another formulation for stationnary inverse problems knowing the Dirichlet to Neumann map and the relation between the two problems is not really clear. In this latter direction, results are given in [2] for the stationnary Schrödinger equation which appear to be similar to ours.

Assuming that $q \in L^\infty$ is a given function, we are concerned with the stability around q. That is to say q and $y(q)$ are known while p is unknown. Later in section 5, we will give a meaning to equation (1) when $y_0 \in L^2(\Omega)$ and $h \in L^2(\partial\Omega \times (0, T))$. Of course, additional assumptions will be required on y_0 and h in order to obtain our main result which states as follows.

Theorem 1. *Let $\mathcal{U}$ be a bounded subset of $L^\infty(\Omega)$, $q \in L^\infty(\Omega)$ and y be a solution of equation* (1).
We assume

$$\exists\, x_0 \in \mathbb{R}^N \setminus \overline{\Omega} \textit{ such that } \Gamma_0 \supset \{x \in \partial\Omega; (x - x_0).\nu(x) \geq 0\},$$

$$y(q) \in W^{1,2}(0, T, L^\infty(\Omega)),$$

$$y_0 \ \textit{ is real valued and } \ |y_0| \geq r_0 > 0, \ ae \ in \ \overline{\Omega}.$$

There exists a constant $C = C(\Omega, T, \Gamma_0, \|q\|_{L^\infty(\Omega)}, y_0, h, \mathcal{U}) > 0$ such that if

$$\frac{\partial y(q)}{\partial \nu} - \frac{\partial y(p)}{\partial \nu} \in H^1(0, T; L^2(\Gamma_0))$$

then $\forall\, p \in \mathcal{U}$,

$$\|q - p\|_{L^2(\Omega)} \leq C \left\| \frac{\partial y(q)}{\partial \nu} - \frac{\partial y(p)}{\partial \nu} \right\|_{H^1(0,T;L^2(\Gamma_0))}. \tag{3}$$

Remarks : 1) We have the same result if y_0 takes its values in $i\mathbb{R}$.
2) If we consider equation (1) on $(-T, T)$ and with y_0 taking its values in $\mathbb{C}$, then, under the formalism of Theorem 1, we can prove the estimate

$$\|q - p\|_{L^2(\Omega)} \leq C \left\| \frac{\partial y(q)}{\partial \nu} - \frac{\partial y(p)}{\partial \nu} \right\|_{H^1(-T,T;L^2(\Gamma_0))}.$$

A regularity result in the linear case, obtained in section 4, implies that the right hand side of inequality (3) is finite under some additional regularity on $y(q)$.

Corollary 2. *Let $\mathcal{U}$ be a bounded subset of $L^\infty(\Omega)$ and $q \in L^\infty(\Omega)$.*
We assume :

$$\exists\, x_0 \in \mathbb{R}^N \setminus \overline{\Omega} \text{ such that } \Gamma_0 \supset \{x \in \partial\Omega; (x - x_0).\nu(x) \geq 0\},$$

$$y(q) \in W^{1,2}(0, T, W^{1,\infty}(\Omega)),$$

$$y_0 \text{ is real valued and } \ |y_0| \geq r_0 > 0, \ ae \ in \ \overline{\Omega}.$$

Then there exists a constant $C = C(\Omega, T, \Gamma_0, \|q\|_{L^\infty}, y_0, h, \mathcal{U}) > 0$ *such that* $\forall\, p \in \mathcal{U}$ *verifying* $q - p \in H_0^1(\Omega)$*,*

$$C^{-1}\|q - p\|_{L^2(\Omega)} \leq \left\| \frac{\partial y(q)}{\partial \nu} - \frac{\partial y(p)}{\partial \nu} \right\|_{H^1(0,T;L^2(\Gamma_0))} \leq C\|q - p\|_{H_0^1(\Omega)}. \tag{4}$$

The condition on $y(q)$ requires sufficient smoothness on q, y_0 and h and compatibility conditions for y_0 and h on $\partial\Omega \times \{0\}$. In particular, $|h(x,0)| \geq r_0 > 0, \ x \in \partial\Omega$ must be satisfied since $|y_0(x)| \geq r_0 > 0, \ ae$ in $\overline{\Omega}$ and $h(0)$ has to be real valued.

The first inequality of Corollary 2 in (4) shows the stability of the nonlinear inverse problem and gives uniqueness while the second inequality gives the continuous dependance of the normal derivative of the solution with respect to the potential.

Many of the results we can refer to concern the wave equation. They are related to the same kind of inverse problems of determining a potential, some of them ([10], [13]) with a Dirichlet boundary data and a Neumann measurement and others with a Neumann boundary data and a Dirichlet measurement ([5], [6]). These references are all based upon local or global Carleman estimates. Nevertheless, in our approach, as in [6] for example, in order to prove Theorem 1, we do not use any of the compactness-uniqueness arguments which are required in [13] for the same kind of situation. Indeed, our present proof is based upon a global Carleman estimate (Proposition 3) which leads to the result in a direct way.

Up to our knowledge, the result of determination of a time independent potential in Schrödinger equation from a single time dependent measurement on a suitable part of the boundary is new. Let us notice that in the different context of Cauchy problem, V. ISAKOV in [4] uses local Carleman estimates for the Schrödinger equation to prove uniqueness of the solution.

This paper is organized as follows :

We first establish a global Carleman estimate for a Schrödinger equation with a potential (Section 2). This estimate leads us to show a theorem describing uniqueness and stability of the linear inverse problem (Section 3). The idea is inspired by O. Yu. IMANUVILOV and M. YAMAMOTO [6].

Then, after recalling some classical properties of regularity concerning our equations we prove a two sided inequality in the linear case (Section 4).

In section 5, we complete the proof of Theorem 1 and Corollary 2 from the results obtained for the linear problem. In section 6, under additional hypotheses, we finally improve the result of Theorem 1 by showing stability for a stonger norm of $(p-q)$, using there an observability estimate proved from the same Carleman estimate.

2 A global Carleman estimate

In this step, we will show a global Carleman estimate concerning a function $v = v(x,t)$ equals to zero on $\partial\Omega \times (-T,T)$ and solution of a Schrödinger equation with a bounded potential.

First, we assume that it is possible to find a regular and positive weight function $\psi = \psi(x)$ defined on $\mathbb{R}^N$ and pseudo- convex with respect to the Schrödinger operator. Indeed, we will suppose that ψ verifies the following properties.

- $\psi \in C^4(\mathbb{R}^N)$,
- $\psi(x) \geq 0,\ \forall x \in \Omega,$
- $|\nabla\psi(x)| \geq \beta > 0,\ \forall x \in \Omega,$
- $\exists\,\Lambda_1 > 0,\ \exists\,\varepsilon > 0$ *such that* $\forall\,\xi \in \mathbb{R}^N,\ \forall\,\lambda > \Lambda_1,$ $$\lambda|\nabla\psi.\,\xi|^2 + D^2\psi\left(\xi,\overline{\xi}\right) \geq \varepsilon\,|\xi|^2 \tag{5}$$
- $\nabla\psi.\nu < 0,\ \forall x \in \partial\Omega \setminus \Gamma_0.$

A classical answer to the problem of choosing a weight ψ and a geometrical condition upon Γ_0 is the following :

$$\begin{array}{c} x_0 \in \mathbb{R}^N \setminus \overline{\Omega} \\ \psi = |x-x_0|^2 \\ \Gamma_0 \supset \{x \in \partial\Omega,\ (x-x_0).\nu(x) \geq 0\} \end{array} \tag{6}$$

In this case all the required conditions are satisfied.

Then, for $s > 0$ and $\lambda > 0$ we define on $\Omega \times (-T,T)$ the functions θ and φ by

$$\theta(x,t) = \frac{e^{\lambda\psi(x)}}{(T-t)(T+t)} \quad \text{and} \quad \varphi(x,t) = \frac{\alpha - e^{\lambda\psi(x)}}{(T-t)(T+t)}$$

where $\alpha > \|e^{\lambda\psi}\|_{L^\infty(\Omega)}$. We also set $Lv = iv' + \Delta v + qv$.

Proposition 3 (Carleman Estimate)**.** *Let* $q \in L^\infty(\Omega)$, $\|q\|_{L^\infty} \leq m$ *and let* ψ, θ *and* φ *satisfy the above conditions. There exists* $\Lambda_0 > 0$, $s_0 > 0$ *and a constant* $M = M(\Omega, T, \Gamma_0, \beta, \varepsilon, m, \Lambda_0, s_0) > 0$ *such that* *for all $\lambda > \Lambda_0$ and for all $s > s_0$,*

$$\begin{aligned}
& s\lambda \int_{-T}^{T}\int_{\Omega} |\nabla v|^2 e^{-2s\varphi}\, dxdt + s^3\lambda^4 \int_{-T}^{T}\int_{\Omega} |v|^2 e^{-2s\varphi}\, dxdt \\
+ \quad & \int_{-T}^{T}\int_{\Omega} \left(|\widetilde{P}_1 v|^2 + |\widetilde{P}_2 v|^2\right) e^{-2s\varphi}\, dxdt \qquad (7) \\
\leq \quad & M \int_{-T}^{T}\int_{\Omega} |Lv|^2 e^{-2s\varphi}\, dxdt + Ms\lambda \int_{-T}^{T}\int_{\Gamma_0} \theta \left|\frac{\partial v}{\partial \nu}\right|^2 e^{-2s\varphi} \nabla\psi.\nu \, d\sigma dt.
\end{aligned}$$

for all v satisfying

$$\left|\begin{array}{l} Lv \in L^2(\Omega \times (-T,T)), \\ v \in L^2(-T,T; H_0^1(\Omega)), \\ \dfrac{\partial v}{\partial \nu} \in L^2(-T,T; L^2(\Gamma_0)), \end{array}\right.$$

where $\widetilde{P}_1$ and $\widetilde{P}_2$ will be defined later by (8) *and* (9).

Proof :
We can refer to [1] for the general method. The main idea consists in setting $v = e^{s\varphi} w$ and calculating

$$Pw = e^{-s\varphi} L(e^{s\varphi} w).$$

Thus we have

$$Pw = iw' + is\varphi' w + \Delta w + 2s\nabla\varphi.\nabla w + sw\Delta\varphi + s^2|\nabla\varphi|^2 w + qw,$$

and we set

$$P_1 w + P_2 w = Pw - qw$$

where

$$\begin{aligned} P_1 w &= iw' + \Delta w + s^2 |\nabla\varphi|^2 w, \\ P_2 w &= is\varphi' w + 2s\nabla\varphi.\nabla w + s\Delta\varphi w. \end{aligned}$$

We just have represented $Pw - qw$ as the sum of adjoint (P_1) and skew-adjoint (P_2) operators. Then,

$$\begin{aligned} \int_{-T}^{T}\int_{\Omega} |Pw - qw|^2\, dxdt \quad = \quad & \int_{-T}^{T}\int_{\Omega} |P_1 w|^2\, dxdt + \int_{-T}^{T}\int_{\Omega} |P_2 w|^2\, dxdt \\ & +2Re \int_{-T}^{T}\int_{\Omega} P_1 w \overline{P_2 w}\, dxdt, \end{aligned}$$

where $\overline{z}$ is the conjugate of z and $Re(z)$ its real part.
As $v \in L^2(-T,T; H_0^1(\Omega))$ and $v' \in L^2(-T,T; H^{-1}(\Omega))$ (because $Lv \in L^2(\Omega \times (-T,T))$), we have $v \in C([-T,T]; L^2(\Omega))$ and $w \in C([-T,T]; L^2(\Omega))$ with $w(x, \pm T) = 0$.

We will first look for lower bounds for

$$Re \int_{-T}^{T} \int_{\Omega} P_1 w \overline{P_2 w} \, dxdt,$$

reminding that

$$\begin{array}{rcl} P_1 w & = & iw' + \Delta w + s^2 |\nabla \varphi|^2 w, \\ \overline{P_2 w} & = & -is\varphi' \overline{w} + 2s \nabla \varphi . \nabla \overline{w} + s \Delta \varphi \overline{w}. \end{array}$$

We multiply each term of $P_1 w$ by each term of $\overline{P_2 w}$. The properties of w and some integrations by parts allow to write the following equalities.

$$I_{11} = Re \int_{-T}^{T} \int_{\Omega} iw'(-is\varphi' \overline{w}) \, dxdt = -\frac{s}{2} \int_{-T}^{T} \int_{\Omega} \varphi'' |w|^2 \, dxdt.$$

Writing $Im(z)$ for the imaginary part of $z \in \mathbb{C}$, we have $Im(z) - Im(\overline{z}) = 2Im(z)$ and taking $z = 2s\lambda \displaystyle\int_{-T}^{T} \int_{\Omega} \theta \nabla \psi \nabla \overline{w} w' \, dxdt$, we show that :

$$\begin{array}{rcl} I_{12} & = & Re \displaystyle\int_{-T}^{T} \int_{\Omega} iw'(2s \nabla \varphi . \nabla \overline{w}) \, dxdt \\ & = & s\lambda \, Im \displaystyle\int_{-T}^{T} \int_{\Omega} \theta(\Delta \psi + \lambda |\nabla \psi|^2) w \overline{w'} \, dxdt \\ & & - \, s\lambda \, Im \displaystyle\int_{-T}^{T} \int_{\Omega} \theta' w \nabla \psi . \nabla \overline{w} \, dxdt. \end{array}$$

Moreover, since $Im \, z = -Im \, \overline{z}$, then :

$$I_{13} = Re \int_{-T}^{T} \int_{\Omega} iw'(s \Delta \varphi \overline{w}) \, dxdt = -s\lambda \, Im \int_{-T}^{T} \int_{\Omega} \theta(\Delta \psi + \lambda |\nabla \psi|^2) w \overline{w}' dxdt$$

and since $Im \displaystyle\int_{-T}^{T} \int_{\Omega} \varphi' \nabla \overline{w} . \nabla w \, dxdt = 0$,

$$I_{21} = Re \int_{-T}^{T} \int_{\Omega} \Delta w(-is\varphi' \overline{w}) \, dxdt = s\lambda \, Im \int_{-T}^{T} \int_{\Omega} \theta' \overline{w} \nabla \psi . \nabla w \, dxdt.$$

The next inequality uses the fact that $\nabla w = \dfrac{\partial w}{\partial \nu} . \, \nu$ on $\partial \Omega \times (0,T)$ because $w = 0$ on $\partial \Omega \times (0,T)$:

$$
\begin{aligned}
I_{22} &= Re \int_{-T}^{T}\int_{\Omega} \Delta w (2s\nabla\varphi.\nabla\overline{w})\, dxdt \\
&= -\, s\lambda \int_{-T}^{T}\int_{\Omega} \theta(\Delta\psi + \lambda|\nabla\psi|^2)|\nabla w|^2\, dxdt \\
&\quad -\, s\lambda \int_{-T}^{T}\int_{\partial\Omega} \theta \left|\frac{\partial w}{\partial\nu}\right|^2 \nabla\psi.\nu\, d\sigma dt \\
&\quad +\, 2s\lambda^2 \int_{-T}^{T}\int_{\Omega} \theta|\nabla\psi.\nabla w|^2\, dxdt \\
&\quad +\, 2s\lambda\, Re \int_{-T}^{T}\int_{\Omega} \theta \sum_{i,j=1}^{n} \frac{\partial\psi}{\partial x_i\partial x_j}\frac{\partial w}{\partial x_i}\frac{\partial \overline{w}}{\partial x_j}\, dxdt.
\end{aligned}
$$

Using integrations by parts we obtain :

$$
\begin{aligned}
I_{23} &= Re \int_{-T}^{T}\int_{\Omega} \Delta w (s\Delta\varphi\overline{w})\, dxdt \\
&= s\lambda \int_{-T}^{T}\int_{\Omega} \theta(\Delta\psi + \lambda|\nabla\psi|^2)|\nabla w|^2\, dxdt \\
&\quad -\, \frac{s\lambda}{2} \int_{-T}^{T}\int_{\Omega} \theta\Delta^2\psi|w|^2\, dxdt \\
&\quad -\, \frac{s\lambda^2}{2} \int_{-T}^{T}\int_{\Omega} \theta(|\Delta\psi|^2 + 2\nabla\psi.\nabla(\Delta\psi) + \Delta(|\nabla\psi|^2))|w|^2\, dxdt \\
&\quad -\, s\lambda^3 \int_{-T}^{T}\int_{\Omega} \theta(|\nabla\psi|^2\Delta\psi + \nabla\psi.\nabla(|\nabla\psi|^2))|w|^2\, dxdt \\
&\quad -\, \frac{s\lambda^4}{2} \int_{-T}^{T}\int_{\Omega} \theta|\nabla\psi|^4|w|^2\, dxdt,
\end{aligned}
$$

and we obviously have

$$
\begin{aligned}
I_{31} &= Re \int_{-T}^{T}\int_{\Omega} s^2|\nabla\varphi|^2 w (is\varphi'\overline{w})\, dxdt = 0, \\
I_{32} &= Re \int_{-T}^{T}\int_{\Omega} s^2|\nabla\varphi|^2 w (2s\nabla\varphi.\nabla\overline{w})\, dxdt \\
&= s^3\lambda^3 \int_{-T}^{T}\int_{\Omega} \theta^3(|\nabla\psi|^2\Delta\psi + \nabla\psi.\nabla(|\nabla\psi|^2))|w|^2\, dxdt \\
&\quad +\, 3s^3\lambda^4 \int_{-T}^{T}\int_{\Omega} \theta^3|\nabla\psi|^4|w|^2\, dxdt,
\end{aligned}
$$

and

$$
\begin{aligned}
I_{33} &= Re \int_{-T}^{T}\int_{\Omega} s^2|\nabla\varphi|^2 w(s\Delta\varphi\overline{w})\,dxdt \\
&= -s^3\lambda^3 \int_{-T}^{T}\int_{\Omega} \theta^3|\nabla\psi|^2\Delta\psi|w|^2\,dxdt - s^3\lambda^4 \int_{-T}^{T}\int_{\Omega} \theta^3|\nabla\psi|^4|w|^2\,dxdt.
\end{aligned}
$$

These four last equalities explain why we required $\psi \in C^4(\mathbb{R}^n)$.

Thereafter, we obtain :

$$
\begin{aligned}
& Re \int_{-T}^{T}\int_{\Omega} P_1 w\overline{P_2 w}\,dxdt = \\
- & \frac{s}{2}\int_{-T}^{T}\int_{\Omega} \varphi''|w|^2\,dxdt - 2s\lambda\, Im \int_{-T}^{T}\int_{\Omega} \theta' w\nabla\psi.\nabla\overline{w}\,dxdt \\
- & s\lambda \int_{-T}^{T}\int_{\partial\Omega} \theta\left|\frac{\partial w}{\partial\nu}\right|^2 \nabla\psi.\nu\,d\sigma dt + 2s\lambda^2 \int_{-T}^{T}\int_{\Omega} \theta|\nabla\psi.\nabla w|^2\,dxdt \\
+ & 2s\lambda\, Re \int_{-T}^{T}\int_{\Omega} \theta D^2\psi(\nabla w,\nabla\overline{w})\,dxdt - \frac{s\lambda}{2}\int_{-T}^{T}\int_{\Omega} \theta\Delta^2\psi|w|^2\,dxdt \\
- & \frac{s\lambda^2}{2}\int_{-T}^{T}\int_{\Omega} \theta(|\Delta\psi|^2 + 2\nabla\psi.\nabla(\Delta\psi) + \Delta(|\nabla\psi|^2))|w|^2\,dxdt \\
- & s\lambda^3 \int_{-T}^{T}\int_{\Omega} \theta(|\nabla\psi|^2\Delta\psi + \nabla\psi.\nabla(|\nabla\psi|^2))|w|^2\,dxdt \\
- & \frac{s\lambda^4}{2}\int_{-T}^{T}\int_{\Omega} \theta|\nabla\psi|^4|w|^2\,dxdt + s^3\lambda^3 \int_{-T}^{T}\int_{\Omega} \theta^3\nabla\psi.\nabla(|\nabla\psi|^2)|w|^2\,dxdt \\
+ & 2s^3\lambda^4 \int_{-T}^{T}\int_{\Omega} \theta^3|\nabla\psi|^4|w|^2\,dxdt.
\end{aligned}
$$

We call X_1 the terms which are neglectible within respect to
$s\lambda^2 \int_{-T}^{T}\int_{\Omega} \theta|\nabla\psi.\nabla w|^2\,dxdt$ or $s^3\lambda^4 \int_{-T}^{T}\int_{\Omega} \theta^3|\nabla\psi|^4|w|^2\,dxdt.$
Then :

$$
\begin{aligned}
X_1 &= -\frac{s}{2}\int_{-T}^{T}\int_{\Omega} \varphi''|w|^2\,dxdt - 2s\lambda\, Im \int_{-T}^{T}\int_{\Omega} \theta' w\nabla\psi.\nabla\overline{w}\,dxdt \\
&- \frac{s\lambda}{2}\int_{-T}^{T}\int_{\Omega} \theta\Delta^2\psi|w|^2\,dxdt \\
&- \frac{s\lambda^2}{2}\int_{-T}^{T}\int_{\Omega} \theta(|\Delta\psi|^2 + 2\nabla\psi.\nabla(\Delta\psi) + \Delta(|\nabla\psi|^2))|w|^2\,dxdt \\
&- s\lambda^3 \int_{-T}^{T}\int_{\Omega} \theta(|\nabla\psi|^2\Delta\psi + \nabla\psi.\nabla(|\nabla\psi|^2))|w|^2\,dxdt \\
&- \frac{s\lambda^4}{2}\int_{-T}^{T}\int_{\Omega} \theta|\nabla\psi|^4|w|^2\,dxdt + s^3\lambda^3 \int_{-T}^{T}\int_{\Omega} \theta^3\nabla\psi.\nabla(|\nabla\psi|^2)|w|^2\,dxdt.
\end{aligned}
$$

Now, we can notice that :

$$
\begin{aligned}
&1) \quad s\lambda\, Im \int_{-T}^{T}\int_{\Omega} \theta' w \nabla\psi.\nabla\overline{w}\, dxdt \leq s\lambda \int_{-T}^{T}\int_{\Omega} (\theta')^{\frac{1}{2}} |\nabla\psi.\nabla w|^2\, dxdt \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad + s\lambda \int_{-T}^{T}\int_{\Omega} (\theta')^{\frac{3}{2}} |w|^2\, dxdt, \\
&2) \quad \alpha\; is\; such\; that\; \varphi > 0 \; on\; \Omega \times (-T,T), \\
&3) \quad |\theta| \leq C\theta^3,\; |\theta'| \leq C\theta^2 \; and\; |\varphi''| \leq C\theta^3 \; on\; (-T,T)\times\Omega,\; C = C(T) > 0.
\end{aligned}
$$

Then,

$$
\begin{aligned}
|X_1| \quad \leq \quad & Cs\lambda \int_{-T}^{T}\int_{\Omega} \theta |\nabla\psi.\nabla w|^2\, dxdt + Cs\lambda^4 \int_{-T}^{T}\int_{\Omega} \theta |w|^2\, dxdt \\
& + Cs^3\lambda^3 \int_{-T}^{T}\int_{\Omega} \theta^3 |w|^2\, dxdt.
\end{aligned}
$$

We can also write :

$$
\begin{aligned}
Re \int_{-T}^{T}\int_{\Omega} P_1 w \overline{P_2 w}\, dxdt \quad \geq \quad & X_1 + 2s\lambda^2 \int_{-T}^{T}\int_{\Omega} \theta |\nabla\psi.\nabla w|^2\, dxdt \\
& + 2s\lambda\, Re \int_{-T}^{T}\int_{\Omega} \theta D^2\psi(\nabla w, \nabla\overline{w})\, dxdt \\
& + 2s^3\lambda^4 \int_{-T}^{T}\int_{\Omega} \theta^3 |\nabla\psi|^4 |w|^2\, dxdt \\
& - s\lambda \int_{-T}^{T}\int_{\partial\Omega} \theta \left|\frac{\partial w}{\partial\nu}\right|^2 \nabla\psi.\nu\, d\sigma dt.
\end{aligned}
$$

Moreover,

$$
\int_{-T}^{T}\int_{\Omega} |Pw - qw|^2\, dxdt \leq 2 \int_{-T}^{T}\int_{\Omega} |Pw|^2\, dxdt + 2 \int_{-T}^{T}\int_{\Omega} q^2 |w|^2\, dxdt.
$$

Therefore, from these two last inequalities and if we impose

$$
|\nabla\psi(x)| \geq \beta > 0, \;\; \forall x \in \Omega,
$$

we obtain :

$$
\begin{aligned}
& 4s\lambda^2 \int_{-T}^{T}\int_{\Omega} \theta |\nabla\psi.\nabla w|^2\, dxdt + 4s\lambda\, Re \int_{-T}^{T}\int_{\Omega} \theta D^2\psi(\nabla w, \nabla\overline{w})\, dxdt \\
& + 4s^3\lambda^4\beta^4 \int_{-T}^{T}\int_{\Omega} \theta^3 |w|^2\, dxdt + \int_{-T}^{T}\int_{\Omega} |P_1 w|^2 + |P_2 w|^2\, dxdt \\
\leq \quad & 2\,|X_1| + 2 \int_{-T}^{T}\int_{\Omega} |Pw|^2\, dxdt + 2 \int_{-T}^{T}\int_{\Omega} q^2 |w|^2\, dxdt \\
& + 2s\lambda \int_{-T}^{T}\int_{\partial\Omega} \theta \left|\frac{\partial w}{\partial\nu}\right|^2 \nabla\psi.\nu\, d\sigma dt.
\end{aligned}
$$

Hence, it is clear that if we take $\lambda > \Lambda_2$ and $s > s_0$ large enough, then $\displaystyle\int_{-T}^{T}\int_\Omega q^2|w|^2\,dxdt$ and all the terms of X_1 will be absorbed by the two dominating terms of the left hand side. Then, we see there exists $M_1 > 0$ depending on $\Omega, T, m, \beta, \Lambda_2, s_0$ and independant of s and λ such that

$$\begin{aligned}
& s\lambda^2\int_{-T}^{T}\int_\Omega \theta|\nabla\psi.\nabla w|^2\,dxdt + s\lambda\,Re\int_{-T}^{T}\int_\Omega \theta D^2\psi(\nabla w,\nabla\overline{w})\,dxdt \\
+ \quad & s^3\lambda^4\int_{-T}^{T}\int_\Omega \theta^3|w|^2\,dxdt + \int_{-T}^{T}\int_\Omega (|P_1w|^2+|P_2w|^2)\,dxdt \\
\leq \quad & M_1\int_{-T}^{T}\int_\Omega |Pw|^2\,dxdt + M_1\,s\lambda\int_{-T}^{T}\int_{\partial\Omega}\theta\left|\frac{\partial w}{\partial\nu}\right|^2\nabla\psi.\nu\,d\sigma dt.
\end{aligned}$$

At this step, applying condition (5) on ψ, we obtain that $\forall\ \lambda > \Lambda_0$, where $\Lambda_0 = \max(\Lambda_2, \Lambda_1)$,

$$\begin{aligned}
& \varepsilon\, s\lambda\int_{-T}^{T}\int_\Omega \theta|\nabla w|^2\,dxdt + s^3\lambda^4\beta^4\int_{-T}^{T}\int_\Omega \theta^3|w|^2\,dxdt \\
+ \quad & \int_{-T}^{T}\int_\Omega |P_1w|^2\,dxdt + \int_{-T}^{T}\int_\Omega |P_2w|^2\,dxdt \\
\leq \quad & M_1\int_{-T}^{T}\int_\Omega |Pw|^2\,dxdt + M_1\,s\lambda\int_{-T}^{T}\int_{\partial\Omega}\theta\left|\frac{\partial w}{\partial\nu}\right|^2\nabla\psi.\nu\,d\sigma dt.
\end{aligned}$$

Let us remark that $\theta > 0$ on $(-T,T)\times\Omega$ and $\nabla\psi.\nu < 0$ on $\partial\Omega\setminus\Gamma_0$. Then, by modifying the constant M_1 into $M_2 = M_2(\Omega, T, m, \beta, \varepsilon, \Lambda_0, s_0) > 0$ we obtain :

$$\begin{aligned}
& s\lambda\int_{-T}^{T}\int_\Omega |\nabla w|^2\,dxdt + s^3\lambda^4\int_{-T}^{T}\int_\Omega |w|^2\,dxdt \\
& + \int_{-T}^{T}\int_\Omega (|P_1w|^2+|P_2w|^2)\,dxdt \\
& \leq M_2\int_{-T}^{T}\int_\Omega |Pw|^2\,dxdt + M_2\,s\lambda\int_{-T}^{T}\int_{\Gamma_0}\theta\left|\frac{\partial w}{\partial\nu}\right|^2\nabla\psi.\nu\,d\sigma dt.
\end{aligned}$$

We can now rewrite our inequality with v instead of w. We have

$$\begin{aligned}
& |v|^2e^{-2s\varphi} = |w|^2, \\
& e^{-2s\varphi}|\nabla v|^2 = |\nabla w + s\nabla\varphi w|^2 \leq 2|\nabla w|^2 + 2s^2|\nabla\varphi|^2|w|^2, \\
& \left|\frac{\partial v}{\partial\nu}\right|^2 e^{-2s\varphi} = \left|\frac{\partial w}{\partial\nu}\right|^2 \text{ on } \partial\Omega, \\
& Pw = e^{-s\varphi}L(v),
\end{aligned}$$

and defining

$$\widetilde{P}_1v = e^{s\varphi}P_1w, \tag{8}$$
$$\widetilde{P}_2v = e^{s\varphi}P_2w, \tag{9}$$

we finally show (7) : $\exists M = M(\Omega, T, \Gamma_0, \beta, \varepsilon, m, \Lambda_0, s_0)$ such that $\forall\ s > s_0$, $\forall\ \lambda > \Lambda_0$,

$$\begin{aligned} & s\lambda \int_{-T}^{T} \int_\Omega |\nabla v|^2 e^{-2s\varphi}\, dxdt + s^3\lambda^4 \int_{-T}^{T} \int_\Omega |v|^2 e^{-2s\varphi}\, dxdt \\ + & \int_{-T}^{T} \int_\Omega \left(|\widetilde{P}_1 v|^2 + |\widetilde{P}_2 v|^2 \right) e^{-2s\varphi}\, dxdt \\ \leq & \ M \int_{-T}^{T} \int_\Omega |Lv|^2 e^{-2s\varphi}\, dxdt + Ms\lambda \int_{-T}^{T} \int_{\Gamma_0} \theta \left| \frac{\partial v}{\partial \nu} \right|^2 e^{-2s\varphi} \nabla\psi.\nu\, d\sigma dt. \end{aligned}$$

Hence the end of the proof of Proposition 3. $\square$

Remark : Under the conditions upon ψ, we can notice that $\theta e^{-2s\varphi}$ and $\nabla\psi.\nu$ are bounded on $(-T, T) \times \Gamma_0$ and replace

$$\int_{-T}^{T} \int_{\Gamma_0} \theta \left| \frac{\partial v}{\partial \nu} \right|^2 e^{-2s\varphi} \nabla\psi.\nu\, d\sigma dt \ \text{ by } \ C \int_{-T}^{T} \int_{\Gamma_0} \left| \frac{\partial v}{\partial \nu} \right|^2 d\sigma dt$$

We knew, from counter-examples that a geometrical condition was necessary and the choice of Γ_0 given in (6) is very usual. Indeed, E. MACHTYNGIER [9], R. TRIGGIANI, P.- F. YAO and I. LASIECKA in references [8], [11] and [12] used that kind of open set of the boundary. In [9], E. MACHTYNGIER shows an observability inequality which estimates initial data by boundary Neumann data for a Schrödinger equation without a potential on $\Gamma_0 = \{x \in \partial\Omega; (x - x_0).\nu(x) \geq 0\}$ using a multiplier identity and Holmgren's uniqueness theorem. Moreover, observability inequalities are technically related to our inverse problem (see [13] and section 6). The reference [11] is based on Carleman estimates, that R. TRIGGIANI proved for a more general kind of coupled Schrödinger equation and applied it to exact controllability. However, both of them are not directly applicable to obtain the result we are expecting.

3 Stability in the linear case

We first consider the linear inverse problem and give the following result.

Theorem 4. *Let $q \in L^\infty(\Omega)$ and u be a solution of equation* (2).
We assume that

$$\begin{gathered} \Gamma_0 \ \textit{satisfies} \ (6), \\ R \in W^{1,2}(0, T, L^\infty(\Omega)), \\ R(0) \ \textit{is real valued and} \ |R(x, 0)| \geq r_0 > 0, \ \textit{ae in} \ \overline{\Omega}. \end{gathered}$$

There exists a constant $C = C(\Omega, T, \|q\|_{L^\infty(\Omega)}, R) > 0$ such that if

$$\frac{\partial u}{\partial \nu} \in H^1(0, T; L^2(\Gamma_0)),$$

then,

$$\|f\|_{L^2(\Omega)} \leq C \left\| \frac{\partial u}{\partial \nu} \right\|_{H^1(0,T;L^2(\Gamma_0))}. \tag{10}$$

Proof :

As we need to estimate $\frac{\partial u}{\partial \nu}$ in $H^1(0,T;L^2(\Gamma_0))$ norm, we work on the equation satisfied by $v = u'$:

$$\begin{cases} iv'(x,t) + \Delta v(x,t) + q(x)v(x,t) = f(x)R'(x,t), \ x \in \Omega, \ t \in (0,T) \\ v(x,t) = 0, \ x \in \partial\Omega, \ t \in (0,T) \\ v(x,0) = -if(x)R(x,0), \ x \in \Omega \end{cases} \tag{11}$$

The Carleman inequality we just obtained is the key of the proof. We extend the function v on $\Omega \times (-T,T)$ by the formula $v(x,t) = -\overline{v}(x,-t)$ for every $(x,t) \in \Omega \times (-T,0)$. Since $R(0)$ and f are real valued, $v \in C([-T,T];H_0^1(\Omega))$ and $\frac{\partial v}{\partial \nu} \in L^2((-T,T) \times \Gamma)$. We also extend R on $\Omega \times (-T,T)$ by the formula $R(x,t) = \overline{R}(x,-t)$ for every $(x,t) \in \Omega \times (-T,0)$ and if we denote the extention of R' by the same notation, then $R' \in L^2(-T,T;W^{1,\infty}(\Omega))$. Thus, v satisfies the same equation (11), set in $(-T,T)$.

We set $w = e^{-s\varphi}v$, $P_1 w = iw' + \Delta w + s^2|\nabla\varphi|^2 w$ and $e^{-s\varphi}\widetilde{P}_1 v = P_1 w$ as in section 2 and we define :

$$I = Im \int_{-T}^{0} \int_{\Omega} \widetilde{P}_1 v \, e^{-2s\varphi} \overline{v} \, dxdt.$$

On the one hand,

$$\begin{aligned} I &= Im \int_{-T}^{0} \int_{\Omega} P_1 w \overline{w} \, dxdt \\ &= Im \int_{-T}^{0} \int_{\Omega} \left(iw' + \Delta w + s^2|\nabla\varphi|^2 w \right) \overline{w} \, dxdt \\ &= Re \int_{-T}^{0} \int_{\Omega} w' \overline{w} \, dxdt - Im \int_{-T}^{0} \int_{\Omega} \left(|\nabla w|^2 - s^2 |\nabla\varphi|^2 |w|^2 \right) \, dxdt \\ &= \frac{1}{2} \int_{-T}^{0} \int_{\Omega} \left(|w|^2 \right)' \, dxdt \\ &= \frac{1}{2} \int_{\Omega} |w(x,0)|^2 \, dx \\ &= \frac{1}{2} \int_{\Omega} |f(x)|^2 |R(x,0)|^2 e^{-2s\varphi(x,0)} \, dx \end{aligned}$$

On the other hand, Cauchy-Schwarz inequality and Carleman estimate stated as in

Proposition 3 in section 2, give :

$$\begin{aligned} I &\le \left(\int_{-T}^{T}\int_\Omega |\widetilde{P}_1 v|^2 e^{-2s\varphi}\,dxdt\right)^{\frac12}\left(\int_{-T}^{T}\int_\Omega |v|^2 e^{-2s\varphi}\,dxdt\right)^{\frac12} \\ &\le s^{-\frac32}\left(M\int_{-T}^{T}\int_\Omega |fR'|^2 e^{-2s\varphi}\,dxdt + Ms\int_{-T}^{T}\int_{\Gamma_0}\theta\left|\frac{\partial v}{\partial\nu}\right|^2 e^{-2s\varphi}\nabla\psi.\nu\,d\sigma dt\right) \end{aligned}$$

Then, $\varphi(x,t) = \dfrac{\alpha - e^{\lambda\psi(x)}}{(T-t)(T+t)}$ is such that $e^{-2s\varphi(x,t)} \le e^{-2s\varphi(x,0)}$ for all $x \in \Omega$ and $t \in (-T,T)$ and it is easy to see that under the conditions satisfied by ψ, $\theta e^{-2s\varphi}$ and $\nabla\psi.\nu$ are bounded on $(-T,T)\times\Gamma_0$. Therefore

$$I \le s^{-\frac32}\left(M\int_{-T}^{T}\int_\Omega |fR'|^2 e^{-2s\varphi(x,0)}\,dxdt + Ms\int_{-T}^{T}\int_{\Gamma_0}\left|\frac{\partial v}{\partial\nu}\right|^2 d\sigma dt\right).$$

Moreover, using the definition of the extensions of v and R', we easily get

$$I \le s^{-\frac32}\left(M\int_{0}^{T}\int_\Omega |fR'|^2 e^{-2s\varphi(x,0)}\,dxdt + Ms\int_{0}^{T}\int_{\Gamma_0}\left|\frac{\partial v}{\partial\nu}\right|^2 d\sigma dt\right).$$

From $R \in W^{1,2}(0,T,L^\infty(\Omega))$ and $|R(x,0)| \ge r_0 > 0$ $ae\ in\ \overline{\Omega}$, we deduce that :

$$\exists\, g_0 \in L^2(0,T),\ |R'(x,t)| \le g_0(t)|R(x,0)|,\ \forall x \in \Omega,\ t \in (0,T).$$

Hence we have :

$$\begin{aligned} \int_\Omega |f|^2|R(0)|^2 e^{-2s\varphi(0)}\,dx &\le Ms^{-\frac32}\int_0^T\int_\Omega |f|^2|g_0|^2|R(0)|^2 e^{-2s\varphi(0)}\,dxdt \\ &+ Ms^{-\frac12}\int_0^T\int_{\Gamma_0}\left|\frac{\partial v}{\partial\nu}\right|^2 d\sigma dt. \end{aligned}$$

But $g_0 \in L^2(0,T) \Rightarrow \int_0^T |g_0(t)|^2\,dt \le K < +\infty$ and so we write

$$\left[\int_\Omega |f(x)|^2|R(x,0)|^2 e^{-2s\varphi(x,0)}\,dx\right]\left(1 - \frac{MK}{s^{\frac32}}\right) \le Ms^{-\frac12}\int_0^T\int_{\Gamma_0}\left|\frac{\partial v}{\partial\nu}\right|^2 d\sigma dt.$$

Then, if s is large enough, ($s > (MK)^{\frac23}$), we see that there exist a constant $C = C(M,s) > 0$ such that :

$$\int_\Omega |f(x)|^2|R(x,0)|^2 e^{-2s\varphi(x,0)}\,dx \le C\int_0^T\int_{\Gamma_0}\left|\frac{\partial v}{\partial\nu}\right|^2 d\sigma dt.$$

Since $|R(x,0)| \ge r_0 > 0,\ ae\ in\ \overline{\Omega}$ and $e^{-2s\varphi(x,0)} \ge e^{-2s\frac{\alpha-1}{T^2}} > 0,\ \forall x \in \Omega$, we obtain

$$\int_\Omega |f(x)|^2\,dx \le C\int_0^T\int_{\Gamma_0}\left|\frac{\partial v}{\partial\nu}\right|^2 d\sigma dt,$$

and it is (10) :

$$\|f\|_{L^2(\Omega)} \leq C \left\| \frac{\partial u}{\partial \nu} \right\|_{H^1(0,T;L^2(\Gamma_0))}.$$

Therefore, Theorem 4 has been proved. $\square$

Remark : if we replace the assumption "$R(0)$ *is real valued*" by the following "$R(0)$ *takes its values in* $i\mathbb{R}$", using the same idea, but with a different extension of v and R, we will be able to prove the same result.

Corollary 5. *Let u be the solution of equation* (2) *and* Γ_0 *given by (6).*
We assume :

$$q \in L^\infty(\Omega), \ R \in W^{1,2}(0,T,W^{1,\infty}(\Omega)),$$

$$R(0) \textit{ is real valued and } |R(x,0)| \geq r_0 > 0 \; ae \; in \; \overline{\Omega}.$$

Then, there exists a constant $C = C(\Omega, T, \|q\|_{L^\infty(\Omega)}, R) > 0$ such that for all $f \in H_0^1(\Omega)$:

$$C^{-1}\|f\|_{L^2(\Omega)} \leq \left\| \frac{\partial u}{\partial \nu} \right\|_{H^1(0,T;L^2(\Gamma_0))} \leq C\|f\|_{H_0^1(\Omega)}. \tag{12}$$

The proof will be given in the following section.

Remark : It follows from Theorem 4 that

$$\left(\frac{\partial u}{\partial \nu} = 0 \text{ on } (0,T) \times \Gamma_0 \right) \Rightarrow (f = 0 \text{ on } \Omega)$$

and it corresponds to the uniqueness result, for the linear case, proposed in section 1. In the non linear situation, we easily show uniqueness by choosing $f = q - p$, $u = y(p) - y(q)$ and $R = y(p)$ on $\Omega \times (0,T)$.
This result can be writen in the following way :

Theorem 6. *Let $p \in L^\infty(\Omega)$ and $q \in L^\infty(\Omega)$. We assume that $y(p)$ or $y(q)$ belongs to $W^{1,2}(0,T,W^{1,\infty}(\Omega))$, y_0 is real valued and $|y_0(x)| \geq r_0 > 0$ almost everywhere in $\overline{\Omega}$.*

$$If \ \frac{\partial y(q)}{\partial \nu} = \frac{\partial y(p)}{\partial \nu} \ on \ (0,T) \times \Gamma_0, \ \ then \ \ q = p \ on \ \Omega.$$

4 Existence and regularity properties

The estimates we will need to prove Corollary 5 can be summed up by the following lemmas.

Lemma 7. *Let us consider*

$$\begin{cases} iy'(x,t) + \Delta y(x,t) + q(x)y(x,t) = g(x,t), \ x \in \Omega, \ t \in (0,T) \\ y(x,t) = 0, \ x \in \partial\Omega, \ t \in (0,T) \\ y(x,0) = y_0(x), \ x \in \Omega \end{cases} \tag{13}$$

where $q \in L^\infty(\Omega)$, $y_0 \in H_0^1(\Omega)$ *and* $g \in X$, *with* $X = L^1(0,T,H_0^1(\Omega))$ *or* $X = W^{1,1}(0,T,L^2(\Omega))$. *This equation admits a unique weak solution* $y \in C([0,T],H_0^1(\Omega))$ *such that the mapping* $(g,y_0) \to \frac{\partial y}{\partial \nu}$ *is linear and continuous from* $X \times H_0^1(\Omega)$ *to* $L^2(\Gamma \times (0,T))$ *and* $\exists\, C = C(\Omega, T, \|q\|_{L^\infty(\Omega)}) > 0$ *such that :*

$$\forall\, t \in (0,T),\ \ \|y(t)\|_{H_0^1(\Omega)} \le C\left(\|y_0\|_{H_0^1(\Omega)} + \|g\|_X\right) \tag{14}$$

$$\left\|\frac{\partial y}{\partial \nu}\right\|_{L^2(\Gamma\times(0,T))} \le C\left(\|y_0\|_{H_0^1(\Omega)} + \|g\|_X\right) \tag{15}$$

Proof :
Concerning estimate (14), we can refer to [3]. It is a classical result which can be formally obtained by two manipulations using Gronwall inequality. In a first time we have to work on "$Im\displaystyle\int_\Omega (13).\overline{y}\,dx$" and show that if $y_0 \in L^2(\Omega)$ and $g \in L^1(0,T,L^2(\Omega))$ then (13) admits a unique solution $y \in C([0,T],L^2(\Omega))$ such that $\forall\, t \in (0,T)$,

$$\|y(t)\|_{L^2(\Omega)} \le C\left(\|y_0\|_{L^2(\Omega)} + \|g\|_{L^1(0,T,L^2(\Omega))}\right).$$

Then, in a second time, working on "$Re\displaystyle\int_\Omega (13).\overline{y'}\,dx$", we manage to obtain (14) in both of the two cases for space X.

Estimate (15) can be deduced from this other result :

Lemma 8. *Let* $\gamma = \gamma(x,t) \in C^2(\overline{\Omega\times(0,T)},\mathbb{R}^n)$. *Under the same hypothesis as in the preceding lemma, the following multipliers identity holds for every weak solution of* (13) *with initial data* $y_0 \in H_0^1(\Omega)$ *and* $g \in X$ *:*

$$\begin{aligned}
&\int_0^T\int_\Gamma \gamma.\nu\left|\frac{\partial y}{\partial \nu}\right|^2 d\sigma dt = Im\int_\Omega y\gamma.\nabla\overline{y}\,dx\Big|_0^T \\
&+Re\int_0^T\int_\Omega y\nabla(\text{div}\gamma).\nabla\overline{y}\,dxdt + 2Re\int_0^T\int_\Omega\sum_{i,j=1}^n \frac{\partial\gamma_j}{\partial x_i}\frac{\partial y}{\partial x_i}\frac{\partial\overline{y}}{\partial x_j}\,dxdt \\
&-2Re\int_0^T\int_\Omega qy\gamma.\nabla\overline{y}\,dxdt - Re\int_0^T\int_\Omega q|y|^2\,\text{div}\gamma\,dxdt \\
&+2Re\int_0^T\int_\Omega g\gamma.\nabla\overline{y}\,dxdt + Re\int_0^T\int_\Omega g\overline{y}\,\text{div}\gamma\,dxdt.
\end{aligned}$$

We first obtain this identity for very regular data $g \in W^{1,1}(0,T,\mathcal{D}(\Omega))$ and $y_0 \in \mathcal{D}(\Omega)$ by calculating

$$''\ Re\int_0^T\int_\Omega (13).(\gamma.\nabla\overline{y} + \frac{1}{2}\overline{y}\,\text{div}\gamma)\,dxdt\ ''.$$

and the result holds by integration by parts. At this step, reference [9] gives a similar result for the Schrödinger equation without potential. Then, by density, the estimate holds true for every solution of (13) with initial data $y_0 \in H_0^1(\Omega)$ and $g \in X$.

We finally choose $\gamma = \gamma(x) \in C^2(\overline{\Omega \times (0,T)}, \mathbb{R}^n)$ such that $\gamma = \nu$ on the C^2-boundary $\partial\Omega$. Then $\gamma.\nu = 1$ on $(0,T) \times \partial\Omega$ and applying estimate (14) with Lemma 8, we manage to obtain estimate (15). $\square$

Proof of Corollary 5 :
Since $f \in H_0^1(\Omega)$ and $R \in W^{1,2}(0,T,W^{1,\infty}(\Omega))$, we have $fR' \in L^1(0,T,H_0^1(\Omega))$ and $fR(0) \in H_0^1(\Omega)$. Thereafter, we know that equation (11) has a solution $v \in C([0,T];H_0^1(\Omega))$ and it also implies $\dfrac{\partial v}{\partial \nu} \in L^2((0,T) \times \Gamma)$. Of course, the left inequality in (12) derives from Theorem 4. Besides, from Lemma 7 :

$$\begin{aligned}
\left\|\frac{\partial v}{\partial \nu}\right\|_{L^2((0,T)\times\Gamma_0)} &\leq \left\|\frac{\partial v}{\partial \nu}\right\|_{L^2((0,T)\times\partial\Omega)} \\
&\leq C\left(\|fR(0)\|_{H_0^1(\Omega)} + \|fR'\|_{L^1(0,T,H_0^1(\Omega))}\right) \\
&\leq C\|f\|_{H_0^1(\Omega)}
\end{aligned}$$

This proves the right hand side of inequality (12) and the proof of Corollary 5 is complete. $\square$

5 Proof of Theorem 1 and Corollary 2

We would like first to give a meaning to equation (1) we are studying.

Lemma 9. *Let $q \in L^\infty(\Omega)$, $y_0 \in L^2(\Omega)$ and $h \in L^2(\partial\Omega \times (0,T))$. Then, there exists a unique solution*

$$y \in C([0,T], H^{-1}(\Omega)) \cap H^{-1}(0,T,L^2(\Omega)),$$

defined by transposition, of the problem (1) :

$$\begin{cases} iy'(x,t) + \Delta y(x,t) + q(x)y(x,t) = 0, \ x \in \Omega, \ t \in (0,T) \\ y(x,t) = h(x,t), \ x \in \partial\Omega, \ t \in (0,T) \\ y(x,0) = y_0(x), \ x \in \Omega. \end{cases}$$

Proof :
We define the adjoint system

$$\begin{cases} i\varphi' + \Delta\varphi + q\varphi = g, & \text{in } \Omega \times (0,T) \\ \varphi = 0, & \text{on } \partial\Omega \times (0,T) \\ \varphi(T) = 0, & \text{in } \Omega \end{cases}$$

It is well-known that we have the following regularity properties about this Schrödinger equation :

a) If $g \in L^1(0,T,H_0^1(\Omega))$ then $\varphi \in C([0,T],H_0^1(\Omega))$ and $\dfrac{\partial\varphi}{\partial\nu} \in L^2(\Gamma\times(0,T))$ (Lemma 7).

b) If $g \in H_0^1(0,T,L^2(\Omega)) \hookrightarrow W^{1,1}(0,T,L^2(\Omega))$ then $\varphi \in C([0,T],H^2(\Omega))$ and $\dfrac{\partial\varphi}{\partial\nu} \in C([0,T],H^{\frac{1}{2}}(\partial\Omega))$.
Indeed, we first have $\varphi \in C([0,T],L^2(\Omega))$. The study of the equation satisfied by φ' gives $\varphi \in C^1([0,T],L^2(\Omega))$ and that leads to $\Delta\varphi = g - q\varphi - i\varphi' \in C([0,T],L^2(\Omega))$. Then $\varphi \in C([0,T],H^2(\Omega))$ and its normal derivative is in $C([0,T],H^{\frac{1}{2}}(\partial\Omega))$.

We say that y is a solution of (1) in the transposition sense if and only if it is possible, for every g, to give a meaning to

$$\int_0^T\int_\Omega \overline{g}y\,dxdt = i\int_\Omega y_0\overline{\varphi}(0)\,dx + \int_0^T\int_{\partial\Omega} h\frac{\partial\overline{\varphi}}{\partial\nu}\,d\sigma dt.$$

We take $g \in \mathcal{D}(0,T,\mathcal{D}(\Omega))$. By density in $L^1(0,T,H_0^1(\Omega))$ and $H_0^1(0,T,L^2(\Omega))$ we are able to define $y \in L^\infty(0,T,H^{-1}(\Omega)) \cap H^{-1}(0,T,L^2(\Omega))$ with $q \in L^\infty(\Omega)$, $y_0 \in L^2(\Omega)$ and $h \in L^2(\partial\Omega\times(0,T))$.

We refer to [9] for the transposition method concerning the Schrödinger equation and the way to prove that we finally obtain $y \in C([0,T],H^{-1}(\Omega)) \cap H^{-1}(0,T,L^2(\Omega))$. Nevertheless, we would like to underline that the important point here is that we have a potential $q \in L^\infty(\Omega)$ and we have to give a meaning to qy (indeed, we proved that $qy \in H^{-1}(0,T;L^2(\Omega))$, since $y \in H^{-1}(0,T;L^2(\Omega))$). Let us also notice that the regularity we obtain in y implies $\dfrac{\partial y}{\partial\nu} \in H^{-2}(0,T;H^{-\frac{3}{2}}(\Gamma))$. □

Thereafter, we define $u = y(p) - y(q)$, which verifies :

$$\begin{cases} iu' + \Delta u + pu = (q-p)y(q), & \text{in } \Omega\times(0,T) \\ u = 0, & \text{on } \partial\Omega\times(0,T) \\ u(0) = 0, & \text{in } \Omega. \end{cases} \tag{16}$$

The key of our proof is that in the linear case, all the constants depend on the L^∞ norm of the potential. Then, since $p \in \mathcal{U}$, where $\mathcal{U}$ is bounded in L^∞, we are in fact, with (16) in a situation similar to the linear case (2).

Proof of Theorem 1 :
We have $y(q) \in W^{1,2}(0,T,L^\infty(\Omega))$ and we know that

$$W^{1,2}(0,T,L^\infty(\Omega)) \subset C([0,T],W^{1,\infty}(\Omega))$$

then we have $y(x,0) = y_0 \in L^\infty(\Omega)$. Thus, hypothesis $|y_0(x)| \geq r_0 > 0,\ ae\ in\ \overline{\Omega}$ makes sense and we can apply the result of Theorem 4, which leads to :

$$\|q-p\|_{L^2(\Omega)} \leq C\ \left\|\frac{\partial u}{\partial\nu}\right\|_{H^1(0,T;L^2(\Gamma_0))}.$$

And the proof of Theorem 1 is complete. $\square$

Proof of Corollary 2 :
We assume that $y(q) \in W^{1,2}(0,T,W^{1,\infty}(\Omega))$ and $q-p \in H_0^1(\Omega)$. Then, it comes $(q-p)y(q) \in W^{1,2}(0,T,H_0^1(\Omega))$ and there exists a unique solution $u \in C([0,T],H_0^1(\Omega))$ to (16).

We are in the same situation as in the linear case (2) and we can apply the result of Corollary 5 which leads to :

$$C^{-1}\|q-p\|_{L^2(\Omega)} \leq \left\|\frac{\partial u}{\partial \nu}\right\|_{H^1(0,T;L^2(\Gamma_0))} \leq C\|q-p\|_{H_0^1(\Omega)}.$$

It means that there exists a constant $C = C(\Omega,T,\Gamma_0,\|q\|_{L^\infty},y_0,h,\mathcal{U}) > 0$ such that for all $p \in \mathcal{U}$ satisfying $q-p \in H_0^1(\Omega)$,

$$C^{-1}\|q-p\|_{L^2(\Omega)} \leq \left\|\frac{\partial y(q)}{\partial \nu} - \frac{\partial y(p)}{\partial \nu}\right\|_{H^1(0,T;L^2(\Gamma_0))} \leq C\|q-p\|_{H_0^1(\Omega)},$$

and the proof is complete. $\square$

Remark : The importance of Carleman estimate being global has to be underlined. It is also well shown, in reference [6], how a global Carleman estimate leads really faster to a conclusion in a non linear situation for the wave equation. We can refer to [13] for a situation using only a local estimate with a weaker result. Indeed, to prove a stability inequality in a non linear situation from the knowledge of the linear case, an observability inequality and a compactness-uniqueness argument are required. However, to improve our results, we will precisely use an observability estimate.

6 Improvement of the symetry of the two-sided estimates

It is known that for the wave equation, a symetric two sided estimate can be shown, for instance in [13]. The result obtained for the Schrödinger equation in Corollary 2 is not symetric in terms of the norms of $(p-q)$. We will here improve the result of Theorem 1 under slightly stronger regularity hypothesis on $y(q)$.

Proposition 10 (Observability Estimate)**.** *We assume $q \in L^\infty$, $z_0 \in H_0^1(\Omega)$ and Γ_0 is given by* (6)*. If z is the weak solution of*

$$\left\{\begin{array}{l} iz'(x,t)+\Delta z(x,t)+q(x)z(x,t)=0,\ x\in\Omega,\ t\in(0,T) \\ z(x,t)=0,\ x\in\partial\Omega,\ t\in(0,T) \\ z(x,0)=z_0(x),\ x\in\Omega \end{array}\right. \tag{17}$$

then, there exists a constant $C = C(\Omega,T,\Gamma_0,\|q\|_{L^\infty}) > 0$ such that

$$\|z_0\|_{H_0^1(\Omega)} \leq C\left\|\frac{\partial z}{\partial \nu}\right\|_{L^2(0,T;L^2(\Gamma_0))}. \tag{18}$$

Proof :
It is a well-known consequence of the Carleman estimate (Proposition 3).
Let $0 < T_0 < T_1 < T$. First of all, since all the needed conditions are satisfied (from Lemma 7), we can apply Proposition 3 to the weak solution z of (17).

Remark : Some changes are made. We work on $[0, T]$ and with

$$\theta(x,t) = \frac{e^{\lambda\psi(x)}}{T(T-t)}, \;\; \varphi(x,t) = \frac{\alpha - e^{\lambda\psi(x)}}{T(T-t)}.$$

Then we have :

$$\int_0^T \int_\Omega |\nabla z|^2 e^{-2s\varphi}\, dxdt + s^2\lambda^3 \int_0^T \int_\Omega |z|^2 e^{-2s\varphi}\, dxdt \leq M \int_0^T \int_{\Gamma_0} \left|\frac{\partial z}{\partial \nu}\right|^2 d\sigma dt$$

and since $e^{-2s\varphi} \geq c > 0$ on $[T_0, T_1] \times \Omega$, it means that :

$$\|z\|_{L^2(T_0,T_1;H_0^1(\Omega))} \leq C \left\|\frac{\partial z}{\partial \nu}\right\|_{L^2((0,T)\times\Gamma_0)}$$

Let $\chi \in C^\infty(0,T)$ be a function such that $0 \leq \chi(t) \leq 1,\ \forall t \in [0,T]$, $\chi = 1$ on $[0, T_0]$ and $\chi = 0$ on $[T_1, T]$. Then, $\chi' = 0$ on $[0, T_0] \cup [T_1, T]$ and we obtain

$$\|\chi' z\|_{L^2(0,T;H_0^1(\Omega))} \leq C \left\|\frac{\partial z}{\partial \nu}\right\|_{L^2((0,T)\times\Gamma_0)}$$

with $C = C(\chi)$. Moreover, $w = \chi z$ satisfies :

$$\begin{cases} iw' + \Delta w + qw = i\chi' z, \quad \text{in } \Omega \times (0,T) \\ w = 0, \quad \text{on } \partial\Omega \times (0,T) \\ w(T) = 0, \quad \text{in } \Omega. \end{cases}$$

Then, applying again Lemma 7 and since $\chi' z \in L^2(0,T;H_0^1(\Omega))$, we have

$$\|w(t)\|_{H_0^1(\Omega)} \leq C\, \|\chi' z\|_{L^2(0,T;H_0^1(\Omega))}, \quad \forall\, t \in [0,T].$$

Therefore, with $t = 0$ and recalling that $w(0) = z_0$, we obtain :

$$\|z_0\|_{H_0^1(\Omega)} \leq C\, \|\chi' z\|_{L^2(0,T;H_0^1(\Omega))} \leq C \left\|\frac{\partial z}{\partial \nu}\right\|_{L^2((0,T)\times\Gamma_0)},$$

what proves the inverse inequality (18). □

Thereafter, we manage to obtain better stability results about our inverse problem. The main idea is to use this observability estimate and the price to pay is to assume more regularity on the given function R and to obtain a result with non explicit constants that we had till now.

Theorem 11. *Let $q \in L^\infty(\Omega)$ and u be a solution of equation* (2).
Assume that

$$\Gamma_0 \text{ satisfies } (6),$$
$$R \in W^{1,2}(0,T,W^{1,\infty}(\Omega)) \cap W^{2,1}(0,T;L^\infty(\Omega)),$$
$$R(0) \text{ is real valued and } |R(x,0)| \geq r_0 > 0, \text{ ae in } \overline{\Omega}.$$

Then, there exists a constant $C = C(\Omega,T,\Gamma_0,\|q\|_{L^\infty(\Omega)},R) > 0$ such that for all $f \in H_0^1(\Omega)$,

$$C^{-1}\|f\|_{H_0^1(\Omega)} \leq \left\|\frac{\partial u}{\partial \nu}\right\|_{H^1(0,T;L^2(\Gamma_0))} \leq C\|f\|_{H_0^1(\Omega)}. \tag{19}$$

Proof :
We work on equation (11), satisfied by $v = u'$:

$$\begin{cases} iv'(x,t) + \Delta v(x,t) + q(x)v(x,t) = f(x)R'(x,t), \ x \in \Omega, \ t \in (0,T) \\ v(x,t) = 0, \ x \in \partial\Omega, \ t \in (0,T) \\ v(x,0) = -if(x)R(x,0), \ x \in \Omega \end{cases}$$

We introduce :

$$\begin{cases} iz' + \Delta z + qz = 0, & \text{in } \Omega \times (0,T) \\ z = 0, & \text{on } \partial\Omega \times (0,T) \\ z(0) = -ifR(0), & \text{in } \Omega \end{cases}$$

and

$$\begin{cases} i\varphi' + \Delta\varphi + q\varphi = fR', & \text{in } \Omega \times (0,T) \\ \varphi = 0, & \text{on } \partial\Omega \times (0,T) \\ \varphi(0) = 0, & \text{in } \Omega. \end{cases}$$

Then,

$$v(x,t) = z(x,t) + \varphi(x,t), \quad x \in \Omega, \ t \in (0,T).$$

On the one hand, as we have $R \in W^{2,1}(0,T,L^\infty(\Omega))$ and because of Lemma 7, we can write :

$$\left\|\frac{\partial \varphi}{\partial \nu}\right\|_{L^2((0,T)\times\Gamma_0)} \leq C\|fR'\|_{W^{1,1}(0,T;L^2(\Omega))} \leq C\|f\|_{L^2(\Omega)}.$$

On the other hand, since $R \in W^{1,2}(0,T,W^{1,\infty}(\Omega))$ the observability inequality gives

$$\|fR(0)\|_{H_0^1(\Omega)} \leq C\left\|\frac{\partial z}{\partial \nu}\right\|_{L^2((0,T)\times\Gamma_0)}.$$

Moreover, we have to notice that if $R(0) \in W^{1,\infty}(\Omega)$ and $|R(x,0)| \geq r_0 > 0$, then $\dfrac{1}{R(0)} \in W^{1,\infty}(\Omega)$ and it yields

$$\|f\|_{H_0^1(\Omega)} \leq C\|fR(0)\|_{H_0^1(\Omega)}.$$

We finally obtain :

$$\|f\|_{H_0^1(\Omega)} \leq C \left\| \frac{\partial v}{\partial \nu} \right\|_{L^2((0,T)\times\Gamma_0)} + C \|f\|_{L^2(\Omega)}.$$

Then, writting this estimate with u and applying Theorem 4 to take away the term $\|f\|_{L^2(\Omega)}$, we obtain the left hand side of (19).
To conclude, we can directly apply Lemma 7 to (11) since $f \in H_0^1(\Omega)$ and $R \in W^{1,2}(0,T,W^{1,\infty}(\Omega))$. We then obtain :

$$\begin{aligned} \left\| \frac{\partial v}{\partial \nu} \right\|_{L^2(0,T;L^2(\Gamma_0))} &\leq C\|fR(0)\|_{H_0^1(\Omega)} + C\|fR'\|_{L^1(0,T;H_0^1(\Omega))} \\ &\leq C \|f\|_{H_0^1(\Omega)}. \end{aligned}$$

Since $v = u'$, we actually know there exists a strictly positive constant C depending on $\Omega, T, \Gamma_0, \|q\|_{L^\infty(\Omega)}$ and R such that

$$C^{-1}\|f\|_{H_0^1(\Omega)} \leq \left\| \frac{\partial u}{\partial \nu} \right\|_{H^1(0,T;L^2(\Gamma_0))} \leq C\|f\|_{H_0^1(\Omega)}$$

and the proof of Theorem 11 is complete. □

As for the proof of Theorem 1 and Corollary 2 in Section 5, we can derive from Theorem 11 the following one.

Theorem 12. *Let $\mathcal{U}$ be a bounded subset of $L^\infty(\Omega)$ and $q \in L^\infty(\Omega)$. Assume that*

$$\Gamma_0 \textit{ satisfies } (6),$$
$$y(q) \in W^{1,2}(0,T,W^{1,\infty}(\Omega)) \cap W^{2,1}(0,T,L^\infty(\Omega)),$$
$$y_0 \textit{ is real valued and } |y_0| \geq r_0 > 0, \ ae \ in \ \overline{\Omega}.$$

Then, there exists a constant $C = C(\Omega, T, \Gamma_0, \|q\|_{L^\infty}, y_0, h, \mathcal{U}) > 0$ such that $\forall\ p \in \mathcal{U}\ verifying\ q - p \in H_0^1(\Omega)$,

$$C^{-1} \|p-q\|_{H_0^1(\Omega)} \leq \left\| \frac{\partial y(q)}{\partial \nu} - \frac{\partial y(p)}{\partial \nu} \right\|_{H^1(0,T;L^2(\Gamma_0))} \leq C \|p-q\|_{H_0^1(\Omega)}.$$

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