Existence of densities for jumping S.D.E.s

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Abstract

We consider a jumping Markov process $\{X_t^x\}_{t\geq 0}$. We study the absolute continuity of the law of X_t^x for $t > 0$. We first consider, as Bichteler-Jacod [2] and Bichteler-Gravereaux-Jacod [1], the case where the rate of jump is constant. We state some results in the spirit of those of [2, 1], with rather weaker assumptions and simpler proofs, not relying on the use of stochastic calculus of variations. We finally obtain the absolute continuity of the law of X_t^x in the case where the rate of jump depends on the spatial variable, and this last result seems to be new.

Key words : Stochastic differential equations, Jump processes, Absolute continuity.

MSC 2000 : 60H10, 60J75.

1 Introduction

Consider a d-dimensional Markov process with jumps $\{X_t^x\}_{t\geq 0}$, starting from $x \in \mathbb{R}^d$, with generator $\mathcal{L}$, defined for $\phi : \mathbb{R}^d \mapsto \mathbb{R}$ sufficiently smooth and $y \in \mathbb{R}^d$, by

$$\mathcal{L}\phi(y) = b(y).\nabla\phi(y) + \int_{\mathbb{R}^n} \gamma(y)\varphi(z)\left[\phi(y + h(y, z)) - \phi(y)\right] dz, \tag{1.1}$$

with possibly an additional diffusion term, and the integral part written in a (more general) *compensated* form. Here $n \in \mathbb{N}$ is fixed, and the functions $\gamma : \mathbb{R}^d \mapsto \mathbb{R}$ and $\varphi : \mathbb{R}^n \mapsto \mathbb{R}$ are nonnegative.

We aim to investigate the absolute continuity of the law of X_t^x with respect to the Lebesgue measure on $\mathbb{R}^d$, for $t > 0$. We will sometimes allow the presence of a Brownian part, but we will actually not use the regularizing effect of the Brownian motion.

Assume for a moment that $d = n = 1$. Roughly speaking, the law of X_t^x is expected to have a density as soon as $t > 0$, if for all $y \in \mathbb{R}$, $\gamma(y)\int_{\mathbb{R}} \varphi(z)dz = \infty$ and if $h(y, z)$ is not too much constant in z (for example $h(y, .)$ of class C^1 with a nonzero derivative almost everywhere). Indeed, in such a case, X^x has infinitely many jumps immediately after $t = 0$. Furthermore, the jumps are of the shape $X_t^x = X_{t-}^x + h(X_{t-}^x, Z)$, with Z a random variable independent of X_{t-}^x, with law $\varphi(z)dz$. This produces absolute continuity for X_t^x, if h is sufficiently non-constant in z.
This simple idea is not so easy to handle rigorously, since X^x has infinitely many jumps, and since $\varphi(z)dz$ is not a probability measure (because $\int_{\mathbb{R}^n} \varphi(z)dz = \infty$). To our knowledge, all the known results are based on the use of *stochastic calculus of variations*, i.e. on a sort of *differential calculus* with respect to the stochastic variable ω. The first results in this direction were obtained by Bismut [3]. Important results are due to Bichteler-Jacod [2], and then Bichteler-Gravereaux-Jacod [1]. We refer to Graham-Méléard [7] and Fournier-Giet [6] for applications to physical integro-differential equations such as the Boltzmann and the coagulation-fragmentation equations. See also Picard [11] and Denis [4] for alternative methods in the much more complicated case where the intensity measure of N is singular.

All the previously cited works concern the case where the *rate of jump* $\gamma(x)$ is constant. The case where γ is non constant is much more delicate. The main reason for this is that in such a case, the map $x \mapsto X_t^x$ cannot be regular (and even continuous). Indeed, if $\gamma(x) < \gamma(y)$, and if $\int_{\mathbb{R}^n} \varphi(z)dz = \infty$, then it is clear that for all small $t > 0$, X^y jumps infinitely more often than X^x before t. The only available result with γ not constant seems to be that of [5], of which the assumptions are very restrictive: monotonicity (in x) is assumed for h and γ.

First, we would like to give some results in the spirit of Bichteler-Jacod [2] and Bichteler-Gravereaux-Jacod [1], with simpler proofs. We will in particular not use the stochastic calculus of variations. We thus consider in Section 2 the case where γ is constant. In Subsection 2.1, we state and prove a first result under a strong nondegeneracy assumption on h, which is satisfying only in the case where $d = 1$. It relies on assumptions which ensure that one jump is sufficient to produce absolute continuity for the law of X_t^x. The proof is elementary, and our result follows the line of Theorem 2.5 in Bichteler-Jacod [2], but our assumptions are rather weaker. In Subsection 2.2, we study a much more complicated case, where a finite number of jumps are required to ensure the absolute continuity of the law of X_t^x. Our result is inspired by that of Bichteler-Gravereaux-Jacod [1] Theorem 2.14.
Although the results of Subsection 2.1 are contained in those of Subsection 2.2, we begin with Subsection 2.1 for the sake of clarity: the result and its proof are much simpler.
We will finally obtain a result in the case where γ is not constant in Section 3. This last result seems to be new, and improves consequently those of [5].

Our methods allow to improve slightly the results of [1, 2] concerning the existence of a density when γ is constant, and to obtain a result when γ depends on the variable position. Let us however recall that the smoothness of the density was studied in [1], which does not seem to be possible with our method.

In the whole paper, we denote the collection of Borelian subsets of $\mathbb{R}^d$ with vanishing Lebesgue measure by

$$\mathcal{A} = \left\{ A \in \mathcal{B}(\mathbb{R}^d);\ \int_A dx = 0 \right\}. \tag{1.2}$$

2 The case of a constant rate of jump

Consider the following d-dimensional S.D.E., for some $d \in \mathbb{N}$, starting from $x \in \mathbb{R}^d$:

$$X_t^x = x + \int_0^t b(X_s^x)ds + \int_0^t \int_{\mathbb{R}^n} h(X_{s-}^x, z)\tilde{N}(ds, dz) + \int_0^t \sigma(X_s^x)dB_s, \tag{2.1}$$

where

Assumption (I)**:** $N(ds, dz)$ is a Poisson measure on $[0, \infty) \times \mathbb{R}^n$, for some $n \in \mathbb{N}$, with intensity measure $\nu(ds, dz) = ds\varphi(z)dz$. The function $\varphi : \mathbb{R}^n \mapsto \mathbb{R}_+$ is supposed to be measurable. We denote by $\tilde{N} = N - \nu$ the associated *compensated* Poisson measure. The $\mathbb{R}^m$-valued (for some $m \in \mathbb{N}$) Brownian motion $\{B_t\}_{t \geq 0}$ is supposed to be independent of N.

In this case, the generator of the Markov process X^x is given, for any $\phi \in C_b^2(\mathbb{R}^d)$ and $y \in \mathbb{R}^d$, by

$$\mathcal{L}\phi(y) = b(y).\nabla\phi(y) + \frac{1}{2}\sum_{i,j=1}^d (\sigma\sigma^*)_{i,j}(x)\partial_i\partial_j\phi(x) + \int_{\mathbb{R}^n} \left[\phi(y + h(y,z)) - \phi(y) - h(y,z).\nabla\phi(y)\right]\varphi(z)dz. \tag{2.2}$$

We assume the following hypothesis, $\mathcal{M}_{d\times m}$ standing for the set of $d \times m$ matrices with real entries.

Assumption $(H1)$**:** The functions $b : \mathbb{R}^d \mapsto \mathbb{R}^d$ and $\sigma : \mathbb{R}^d \mapsto \mathcal{M}_{d\times m}$ are of class C^2 and have at most linear growth. The function $h : \mathbb{R}^d \times \mathbb{R}^n \mapsto \mathbb{R}^d$ is measurable. For each $z \in \mathbb{R}^n$, $x \mapsto h(x, z)$ is of class C^2 on $\mathbb{R}^d$. There exists $\eta \in L^2(\mathbb{R}^n, \varphi(z)dz)$ and a continuous function $\zeta : \mathbb{R}^d \mapsto \mathbb{R}$ such that for all $x \in \mathbb{R}^d, z \in \mathbb{R}^n$, $|h(x, z)| \leq (1 + |x|)\eta(z)$, while $|h'_x(x, z)| + |h''_{xx}(x, z)| \leq \zeta(x)\eta(z)$.

Then it is well-known that the following result holds.

Proposition 2.1 *Assume (I) and (H1). Consider the natural filtration $\{\mathcal{F}_t\}_{t\geq 0}$ associated with the Poisson measure N and the Brownian motion B. Then, for any $x \in \mathbb{R}^d$, there exists a unique càdlàg $\{\mathcal{F}_t\}_{t\geq 0}$-adapted process $\{X_t^x\}_{t\geq 0}$ solution to (2.1) such that for all $x \in \mathbb{R}^d$, all $T \in [0,\infty)$,*

$$E\left[\sup_{s\in[0,T]} |X_s^x|^2\right] < \infty. \tag{2.3}$$

The process $\{X_t^x\}_{t\geq 0, x\in\mathbb{R}^d}$ furthermore satisfies the strong Markov property.

See Ikeda-Watanabe [8] for the case of globally Lipschitz coefficients. A standard localization procedure allows to obtain Theorem 2.1.

2.1 Absolute continuity using one jump

We first give some assumptions, statements, and examples. The proof is handled in a second part.

2.1.1 Statements

We first introduce some assumptions. Here I_d stands for the unit $d \times d$ matrix, while x_0 is a given point of $\mathbb{R}^d$.

Assumption $(H2)$: For all $x \in \mathbb{R}^d$, all $z \in \mathbb{R}^n$, $\det(I_d + h'_x(x,z)) \neq 0$.

Assumption $(H3)(x_0)$: There exists $\epsilon > 0$ such that for all $x \in B(x_0,\epsilon)$, there exists a subset $O(x) \subset \mathbb{R}^n$ such that, (recall (1.2)),

$$\int_{O(x)} \varphi(z)dz = \infty, \text{ and for all } A \in \mathcal{A}, \quad \int_{O(x)} \mathbf{1}_{\{h(x,z)\in A\}}\varphi(z)dz = 0, \tag{2.4}$$

and such that the map $(x,z) \mapsto \mathbf{1}_{\{z\in O(x)\}}$ is measurable on $B(x_0,\epsilon) \times \mathbb{R}^n$.

The main results of this section are the following.

Theorem 2.2 *Let $x_0 \in \mathbb{R}^d$ be fixed. Assume (I), (H_1), (H_2) and $(H_3)(x_0)$. Consider the unique solution $\{X_t^{x_0}\}_{t\geq 0}$ to (2.1). Then the law of $X_t^{x_0}$ has a density with respect to the Lebesgue measure on $\mathbb{R}^d$ as soon as $t > 0$.*

In the case where $(H3)(x)$ holds for all $x \in \mathbb{R}^d$, we can omit assumption $(H2)$.

Corollary 2.3 *Let $x_0 \in \mathbb{R}^d$ be fixed. Assume (I), (H_1) and that $(H_3)(x)$ holds for all $x \in \mathbb{R}^d$. Consider the unique solution $\{X_t^{x_0}\}_{t\geq 0}$ to (2.1). Then the law of $X_t^{x_0}$ has a density with respect to the Lebesgue measure on $\mathbb{R}^d$ as soon as $t > 0$.*

We do not state a result concerning the regularizing effect of the Brownian part of (2.1), since it seems reasonable that standard techniques of Malliavin calculus (see e.g. Nualart, [10]) may allow to prove that under $(H1), (H2)$ and if $\sigma\sigma^*(x_0)$ is invertible, then the law of $X_t^{x_0}$ has a density with respect to the Lebesgue measure on $\mathbb{R}^d$ as soon as $t > 0$.

These results relax subsequently the assumptions of [2], especially concerning the regularity (in z) and boundness conditions on h. Let us comment on our hypotheses. The second condition in (2.4) means that the image measure of $\mathbf{1}_{\{z\in O(x)\}}\varphi(z)dz$ (where dz stands for the Lebesgue measure on $\mathbb{R}^n$) by the map $z \mapsto h(x,z)$ has a density with respect to the Lebesgue measure on $\mathbb{R}^d$, for each $x \in B(x_0, \epsilon)$.

Proposition 2.4 *Assume that $n = d$ and that there exists an open subset O of $\mathbb{R}^n$ such that $(x,z) \mapsto h(x,z)$ is of class C^1 on $\mathbb{R} \times O$. If for all $x \in \mathbb{R}^d$, $\int_O \mathbf{1}_{\{\det h'_z(x,z)\neq 0\}}\varphi(z)dz = \infty$, then $(H3)(x_0)$ holds for any x_0.*

Indeed, it suffices to note that, since $n = d$, $h'_z(x,z)$ is a $d \times d$ matrix for each $x \in \mathbb{R}^d$, each $z \in O$. Choosing $\epsilon = 1$ and $O(x) = \{z \in O,\ \det h'_z(x,z) \neq 0\}$ allows us to conclude, noting that, due to the local inverse Theorem, the map $z \mapsto h(x,z)$ is a local C^1-diffeomorphism on O.

Assumptions $(H1)$ and $(H3)(x_0)$ are quite natural. Note that $(H2)$ is not only a technical condition, as this example shows.

Example 2.5 *Assume that $n = d = 1$, that $\varphi \equiv 1$, that b, σ satisfy $(H1)$ with $b(0) = \sigma(0) = 0$, and that $h(x,z) = -x\mathbf{1}_{\{|z|\leq 1\}} + (x/|z|)\mathbf{1}_{\{|z|>1\}}$. Then (I) and $(H1)$ are satisfied, while $(H3)(x)$ holds for all $x \neq 0$, but $(H2)$ fails. One can prove that in such a case, $P[X_t^{x_0} = 0] > 0$ for all $t > 0$, and thus the law of $X_t^{x_0}$ is not absolutely continuous. Indeed, it is clear that, if $T_1 = \inf\{t \geq 0;\ \int_0^t \int_{\mathbb{R}} \mathbf{1}_{\{|z|\leq 1\}} N(ds,dz) \geq 1\}$, then $T_1 < \infty$ a.s. (its law is exponential with parameter 2) and $X_{T_1} = X_{T_1-} + (-X_{T_1-}) = 0$. Since furthermore $b(0) = \sigma(0) = 0$ and $h(0,.) = 0$, an uniqueness argument and the strong Markov property show that $X_{T_1+t} = 0$ a.s. for all $t \geq 0$. Hence $P[X_t^{x_0} = 0] > 0$ for all $t > 0$.*

2.1.2 Proof

We now turn to the proof of Theorem 2.2. We first proceed to a localization procedure.

Lemma 2.6 *To prove Theorem 2.2 and Corollary 2.3, we may assume the additional condition $(H4)$ below.*

Assumption $(H4)$**:** The functions $b, b', b'', \sigma, \sigma', \sigma''$ are bounded. There exists $\tilde{\eta} \in L^2(\mathbb{R}^n, \varphi(z)dz)$ such that for all $x \in \mathbb{R}^d$, $z \in \mathbb{R}^n$, $|h(x,z)| + |h'_x(x,z)| + |h''_{xx}(x,z)| \leq \tilde{\eta}(z)$.

Proof We study the case of Theorem 2.2. Let $x_0 \in \mathbb{R}^d$ be fixed. Assume that Theorem 2.2 holds under (I), (H_1), (H_2), $(H_3)(x_0)$, $(H4)$, and consider some functions b, σ, h satisfying only (I), (H_1), (H_2), $(H_3)(x_0)$. For each $l \geq 1$, consider some functions b_l, σ_l, h_l satisfying (I), (H_1), (H_2), $(H_3)(x_0)$, $(H4)$ and such that for all $|x| \leq l$, all $z \in \mathbb{R}^n$, $b_l(x) = b(x)$, $\sigma_l(x) = \sigma(x)$, and $h_l(x,z) = h(x,z)$. Denote by $\{X_t^{x_0,l}\}_{t\geq 0}$ the solution to (2.1) with h, σ, b replaced by h_l, σ_l, b_l. Then, by assumption, the law of $X_t^{x_0,l}$ has a density if $t > 0$. Next, denote by $\tau_l = \inf\{t \geq 0,\ |X_t^{x_0}| \geq l\}$. It is clear from (2.3) that τ_l increases a.s. to infinity as l tends to infinity. Furthermore a uniqueness argument yields that a.s. $X_t^{x_0} = X_t^{x_0,l}$ for all $t \leq \tau_l$. Hence, for any $A \in \mathcal{A}$, any $t > 0$, by the Lebesgue Theorem,

$$P[X_t^{x_0} \in A] = \lim_{l\to\infty} P[X_t^{x_0} \in A, t < \tau_l] = \lim_{l\to\infty} P[X_t^{x_0,l} \in A, t < \tau_l] \leq \lim_{l\to\infty} P[X_t^{x_0,l} \in A] = 0, \tag{2.5}$$

since the law of $X_t^{x_0,l}$ has a density for each $l \geq 1$. This implies that the law of $X_t^{x_0}$ has a density. □

We now gather some known results about the *flow* $\{X_t^x\}_{t\geq 0, x\in\mathbb{R}^d}$.

Lemma 2.7 *Assume* (I), $(H1)$, $(H4)$. *Consider the flow of solutions* $\{X_t^x\}_{t\geq 0, x\in\mathbb{R}^d}$ *to (2.1). Then a.s., the map* $x \mapsto X_t^x$ *is of class* C^1 *on* $\mathbb{R}^d$ *for each* $t \geq 0$. *If furthermore* $(H2)$ *holds, then a.s., for all* $t \geq 0$, *all* $x \in \mathbb{R}^d$, $\det \frac{\partial}{\partial x} X_t^x \neq 0$.

Proof It is well-known (see Protter [13] Theorems 39 and 40 Section 7 for very similar results) that under $(I), (H1), (H4)$, the map $x \mapsto X_t^x$ is a.s. of class C^1 on $\mathbb{R}^d$ for each $t \geq 0$, and that one may differentiate (2.1) with respect to x:

$$\frac{\partial}{\partial x} X_t^x = I_d + \int_0^t b'(X_s^x)\frac{\partial}{\partial x} X_s^x ds + \int_0^t \int_{\mathbb{R}^n} h'_x(X_{s-}^x, z)\frac{\partial}{\partial x} X_{s-}^x \tilde{N}(ds,dz) + \int_0^t \sigma'(X_s^x)\frac{\partial}{\partial x} X_s^x dB_s. \quad (2.6)$$

Then, following the ideas of Jacod ([9], Theorem 1 and Corollary page 443), we deduce an explicit expression for $V_t^x = \det \frac{\partial}{\partial x} X_t^x$ in terms of Doléans-Dade exponentials (a continuity argument shows that this explicit expression holds a.s. simultaneously for all $x \in \mathbb{R}^d$). We thus obtain, still using the results of [9] simultaneously for all $x \in \mathbb{R}^d$, that a.s., $\det \frac{\partial}{\partial x} X_t^x \neq 0$ for all $x \in \mathbb{R}^d$ and all $t < \tau = \inf_{x\in\mathbb{R}^d} T^x$, where

$$T^x = \inf\{t \geq 0; \int_0^t \int_{\mathbb{R}^n} \mathbf{1}_{\{\det(I_d + h'_x(X_{s-}^x, z))=0\}} N(ds,dz) \geq 1\}. \quad (2.7)$$

But $(H2)$ ensures that $\tau = \infty$ a.s. $\square$

We may now prove Theorem 2.2.

Proof of Theorem 2.2 Due to Lemma 2.6, we suppose the additional condition $(H4)$. We consider $x_0 \in \mathbb{R}^d$ and $t > 0$ fixed.

Step 1: Due to $(H3)(x_0)$, we may build, for each $x \in B(x_0, \epsilon)$, an increasing sequence $\{O_p(x)\}_{p\geq 1}$ of subsets of $\mathbb{R}^n$ such that

$$\cup_{p\geq 1} O_p(x) = O(x) \text{ and } \forall\, p \geq 1,\ \int_{O_p(x)} \varphi(z)dz = p, \quad (2.8)$$

in such a way that for each $p \geq 1$, the map $(x,z) \mapsto \mathbf{1}_{\{z \in O_p(x)\}}$ is measurable on $B(x_0, \epsilon) \times \mathbb{R}^n$.

We also consider the stopping time

$$\tau = \inf\{s \geq 0;\ |X_s^{x_0} - x_0| \geq \epsilon\} > 0 \text{ a.s.} \tag{2.9}$$

The positivity of τ comes from the fact that X^{x_0} is a.s. right-continuous and starts from x_0.

We finally consider the stopping time, for $p \geq 1$,

$$S_p = \inf\left\{s \geq 0;\ \int_0^s \int_{\mathbb{R}^n} \mathbf{1}_{\left\{z \in O_p\left(X_{(u\wedge\tau)-}^{x_0}\right)\right\}} N(du, dz) \geq 1\right\}, \tag{2.10}$$

and the associated *mark* $Z_p \in \mathbb{R}^n$, uniquely defined by $N(\{S_p\} \times \{Z_p\}) = 1$.

Due to (2.8), and to the fact that $X_{(u\wedge\tau)-}^{x_0}$ always belongs to $B(x_0, \epsilon)$, one may prove that

(i) $p \mapsto S_p$ is a.s. nonincreasing,

(ii) $\lim_{p\to\infty} S_p = 0$ a.s.,

(iii) conditionally to $\mathcal{F}_{S_p-}$, the law of Z_p is given by $\frac{1}{p}\varphi(z)\mathbf{1}_{\left\{z \in O_p\left(X_{(S_p\wedge\tau)-}^{x_0}\right)\right\}} dz$, where

$$\mathcal{F}_{S_p-} = \sigma\left\{B \cap \{S_p > s\};\ s \geq 0,\ B \in \mathcal{F}_s\right\}. \tag{2.11}$$

Indeed, (i) is obvious by construction, since $p \mapsto O_p(x)$ is increasing for each $x \in \mathbb{R}^d$. Next, an easy computation shows that the compensator of the (random) point measure $N^p(ds, dz) = \mathbf{1}_{\{z \in O_p(X_{(s\wedge\tau)-}^{x_0})\}} N(ds, dz)$ is given by $pds \times p^{-1}\mathbf{1}_{\{z \in O_p(X_{(s\wedge\tau)-}^{x_0})\}}\varphi(z)dz$. Since for each $x \in \mathbb{R}^d$, $\int_{\mathbb{R}^n} p^{-1}\mathbf{1}_{\{z \in O_p(x)\}}\varphi(z)dz = 1$, we deduce that the *rate* of jump of N^p is constant and equal to p, so that S_p, which is its first instant of jump, has an exponential distribution with parameter p. This and (i) ensure (ii). We also obtain (iii) as a consequence of the shape of the compensator of N^p.

Step 2: We now prove that conditionally to $\sigma(S_p)$, the law of $X_{S_p}^{x_0}$ has a density with respect to the Lebesgue measure on $\mathbb{R}^d$, on the set $\Omega_p^0 = \{\tau \geq S_p\}$. Since S_p is $\mathcal{F}_{S_p-}$-measurable, it suffices to prove that for any $A \in \mathcal{A}$, a.s., $P[\Omega_p^0, X_{S_p}^{x_0} \in A \mid \mathcal{F}_{S_p-}] = 0$. But, using the notations of Step 1, a.s., $X_{S_p}^{x_0} = X_{S_p-}^{x_0} + h[X_{S_p-}^{x_0}, Z_p]$ on Ω_p^0. Furthermore, we know that on Ω_p^0, $X_{S_p-}^{x_0} \in B(x_0, \epsilon)$ a.s. Thus, using

Step 1 (see (iii)), since $\{\tau \geq S_p\}$ and $X^{x_0}_{S_p-}$ are $\mathcal{F}_{S_p-}$-measurable,

$$\begin{aligned} P[\Omega^0_p, X^{x_0}_{S_p} \in A \mid \mathcal{F}_{S_p-}] &= \mathbf{1}_{\Omega^p_0} P\left[X^{x_0}_{S_p-} + h[X^{x_0}_{S_p-}, Z_p] \in A \mid \mathcal{F}_{S_p-}\right] \\ &= \mathbf{1}_{\left\{\tau \geq S_p, X^{x_0}_{S_p-} \in B(x_0,\epsilon)\right\}} \int_{\mathbb{R}^n} \mathbf{1}_{\left\{h\left[X^{x_0}_{S_p-}, z\right] \in A - X^{x_0}_{S_p-}\right\}} \frac{1}{p} \varphi(z) \mathbf{1}_{\left\{z \in O_p\left(X^{x_0}_{S_p-}\right)\right\}} dz = 0 \end{aligned} \tag{2.12}$$

due to $(H3)(x_0)$ (use (2.4) with $x = X^{x_0}_{S_p-}$), since for any $y \in \mathbb{R}^d$, $A - y = \{x - y, x \in A\}$ belongs to $\mathcal{A}$.

Step 3: We may now deduce that for any $p \geq 1$, the law of $X^{x_0}_t$ has a density on the set $\Omega^1_p = \{S_p \leq \tau \wedge t\}$. We deduce from Step 2 that on $\Omega^1_p \subset \Omega^0_p$ the law of $(S_p, X^{x_0}_{S_p})$ is of the shape $\nu_p(ds) f_p(s,x) dx$. Hence, for any $A \in \mathcal{A}$, using the strong Markov property, we obtain, conditionning with respect to $\mathcal{F}_{S_p}$,

$$P[\Omega^1_p, X^{x_0}_t \in A] = E\left[\mathbf{1}_{\Omega^1_p} E\left\{\int_0^t \nu_p(ds) \int_{\mathbb{R}^d} f_p(s,x) dx \mathbf{1}_{\{X^x_{t-s} \in A\}}\right\}\right]. \tag{2.13}$$

It thus suffices to show that a.s., for any $s < t$ fixed,

$$\int_{\mathbb{R}^d} f_p(s,x) dx \mathbf{1}_{\{X^x_{t-s} \in A\}} = 0. \tag{2.14}$$

Of course, it suffices to check that a.s., for $s < t$ fixed,

$$\int_{\mathbb{R}^d} dx \mathbf{1}_{\{X^x_{t-s} \in A\}} = 0. \tag{2.15}$$

But this is immediate from Lemma 2.7, using that the Jacobian of the map $x \mapsto X^x_{t-s}$ does (a.s.) never vanish and that A is Lebesgue-nul: one may find, due to the local inverse Theorem, a countable family of open subsets R_i of $\mathbb{R}^d$, on which $x \mapsto X^x_{t-s}$ is a C^1 diffeomorphism, and such that $\mathbb{R}^d = \cup_{i=1}^\infty R_i$. The conclusion follows, performing the substitution $x \mapsto y = X^x_{t-s}$ on each R_i. This allows us to conclude that $P[\Omega^p_1, X^{x_0}_t \in A] = 0$.

Step 4: The conclusion readily follows: due to Step 1 (see (2.9) and (ii)), $\mathbf{1}_{\Omega^1_p}$ goes a.s. to 1 as p tends to infinity. We thus infer from the Lebesgue Theorem that for any $A \in \mathcal{A}$,

$$P[X^{x_0}_t \in A] = \lim_{p \to \infty} P[\Omega^1_p, X^{x_0}_t \in A] = 0, \tag{2.16}$$

thanks to Step 3. Thus the law of $X^{x_0}_t$ has a density with respect to the Lebesgue measure on $\mathbb{R}^d$. □

We finally show how to relax assumption $(H2)$ when $(H3)(x)$ holds everywhere.

Proof of Corollary 2.3 Due to Lemma 2.6, we may suppose the additional assumption $(H4)$. We consider $\delta_0 > 0$ such that for any $M \in \mathcal{M}_{d\times d}$ satisfying $|M| \le \delta_0$, $\det(I_d + M) \ge 1/2$. We then split $\mathbb{R}^n$ into $O_F \cup O_I$, with

$$O_F = \{z \in \mathbb{R}^n,\ \tilde{\eta}(z) \ge \delta_0\}, \quad O_I = \{z \in \mathbb{R}^n,\ \tilde{\eta}(z) < \delta_0\}. \tag{2.17}$$

Since $\tilde{\eta} \in L^2(\mathbb{R}^n, \varphi(z)dz)$, we deduce that $\lambda_F := \int_{O_F} \varphi(z)dz \le \delta_0^{-2} \int_{\mathbb{R}^n} \tilde{\eta}^2(z)\varphi(z)dz < \infty$. We next consider the solution $\{Y_t^x\}_{t\ge 0}$ to the S.D.E.

$$Y_t^x = x + \int_0^t b(Y_s^x)ds + \int_0^t \int_{\mathbb{R}^n} h(Y_{s-}^x, z)\mathbf{1}_{\{z\in O_I\}}\tilde{N}(ds,dz) - \int_0^t \int_{O_F} h(Y_{s-}^x, z)\varphi(z)dz + \int_0^t \sigma(Y_s^x)dB_s. \tag{2.18}$$

Clearly, this S.D.E. satisfies (I), $(H1)$, (H_2), and $(H3)(x)$ for all x, so that due to Theorem 2.2, the law of Y_t^x has a density for each $t > 0$, each $x \in \mathbb{R}^d$. The solution $X_t^{x_0}$ to (2.1) may now be realized in the following way (see Ikeda-Watanabe [8] for details): consider a standard Poisson process with intensity λ_F and with instants of jump $0 = T_0 < T_1 < T_2 < ...$, a family of i.i.d. $\mathbb{R}^n$-valued random variables $(Z_i)_{i\ge 1}$ with law $\lambda_F^{-1}\varphi(z)\mathbf{1}_{\{z\in O_F\}}dz$, and a family of i.i.d. solutions $(\{Y_t^{i,x}\}_{x\in\mathbb{R}^d, t\ge 0})_{i\ge 1}$ to (2.18), all these random objects being independent. Set

$$X_0^{x_0} = x_0, \quad \forall\, i \ge 0,\ X_{T_{i+1}}^{x_0} = Y_{T_{i+1}-T_i}^{i, X_{T_i}^{x_0}} + h(Y_{T_{i+1}-T_i}^{i, X_{T_i}^{x_0}}, Z_i) \text{ and } \forall\, t \ge 0,\ X_t^{x_0} = \sum_{i\ge 0} Y_{t-T_i}^{i, X_{T_i}^{x_0}} \mathbf{1}_{\{t\in[T_i, T_{i+1})\}}. \tag{2.19}$$

Then $\{X_t^{x_0}\}_{t\ge 0}$ is solution (in law) to (2.1). To conclude, notice that for any $t > 0$, one has $t \notin \cup_i\{T_i\}$ a.s., so that for any $A \in \mathcal{A}$,

$$P[X_t^{x_0} \in A] \;=\; \sum_{i\ge 0} P\left[Y_{t-T_i}^{i, X_{T_i}^{x_0}} \in A,\ t \in (T_i, T_{i+1})\right] \le \sum_{i\ge 0} P\left[Y_{t-T_i}^{i, X_{T_i}} \in A,\ t > T_i\right] = 0. \tag{2.20}$$

The last equality comes from the facts that for each i, $\{Y_s^{i,x}\}_{s\ge 0, x\in\mathbb{R}^d}$ is independent of (T_i, X_{T_i}), and that the law of $Y_t^{i,x}$ has a density for each $t > 0$, each $x \in \mathbb{R}^d$. $\square$

2.2 Absolute continuity using a finite number of jumps

We now would like to investigate the case where one jump is not sufficient to produce a density for the law of X_t^x. For example, assume that $d = 2$, and that $X_t^x = (X_t^{x,1}, X_t^{x,2})$ has sufficiently many jumps which produce a density for $X_t^{x,1}$, sufficiently many jumps which produce a density for $X_t^{x,2}$, and that these two kinds of jump are independent enough. Then the result of Theorem 2.2 might still hold, even if $(H3)$ fails. In other words, we would like to prove a result in the spirit of Bichteler-Gravereaux-Jacod [1] Theorem 2.14. We keep in mind assumptions (I), $(H1)$, $(H2)$ and $(H4)$ defined in the previous subsection.

We first give our result, which relies on a general (but quite untracktable) non-degeneracy assumption. Then we turn to the proof, and we conclude the section with examples of applications.

2.2.1 Statements

We invite the reader to have a look at Assumption $(H6)$ (stated in Subsection 2.2.3), which is a tractable version of $(H5)$ below ($(H6)$ is however less general). We first of all introduce some notation.

Notation 2.8 *Assume* (I) *and* $(H1)$.

(i) For $x = (x_i) \in \mathbb{R}^d$, *we set* $|x| = (\sum_i x_i^2)^{1/2}$, *and for* $M = (M_{ij})_{i,j} \in \mathcal{M}_{d\times d}$ *we set* $|M| = \sum_{i,j} |M_{ij}|$.

(ii) For $x \in \mathbb{R}^d$ *and* $\epsilon > 0$, *we consider the set*

$$O_{x,\epsilon} = \{z \in \mathbb{R}^n,\ |h(x,z)| \le \epsilon\}. \tag{2.21}$$

(iii) For $x \in \mathbb{R}^d$, $\eta > 0$ *and* $\epsilon > 0$, *we consider the following set of regular functions:*

$$\mathcal{D}_{x,\eta,\epsilon} = \left\{ \psi \in C^1(B(x,\eta) \mapsto \mathbb{R}^d),\ \sup_{y\in B(x,\eta)} [|\psi(y) - y| + |\psi'(y) - I_d|] \le \epsilon \right\}. \tag{2.22}$$

Note that for any ψ *in* $\mathcal{D}_{y,\eta,\epsilon}$, $\psi'(y)$ *is a* $d\times d$ *matrix for each* $y \in B(x,\epsilon)$, *and that one has* $\psi(B(x,\eta)) \subset B(x, \eta+\epsilon)$.

(iv) We consider the function $g : \mathbb{R}^d \times \mathbb{R}^n \mapsto \mathbb{R}^d$ *defined by* $g(x,z) = x + h(x,z)$.

(v) Consider $x_0 \in \mathbb{R}^d$, $\epsilon > 0$ *and* $\alpha \in \mathbb{N}$ *to be fixed. For* $\psi_1, ..., \psi_\alpha \in \mathcal{D}_{x_0,2\alpha\epsilon,\epsilon}$, *we define the functions*

$g_i^{x_0,\psi_1,...,\psi_i}$ *for* $i \in \{1,...,\alpha\}$ *recursively, by*

$$\begin{aligned} g_1^{x_0,\psi_1} &: O_{\psi_1(x_0),\epsilon} \subset \mathbb{R}^n \mapsto B(x_0,2\epsilon) \subset \mathbb{R}^d, \\ g_1^{x_0,\psi_1}(z_1) &= g[\psi_1(x_0),z_1], \end{aligned} \tag{2.23}$$

and, for $i \in \{1,...,\alpha-1\}$,

$$\begin{aligned} g_{i+1}^{x_0,\psi_1,...,\psi_{i+1}}(z_1,...,z_i,.) &: O_{\psi_{i+1}(g_i^{x_0,\psi_1,...,\psi_i}(z_1,...,z_i)),\epsilon} \subset \mathbb{R}^n \mapsto B(x_0,2(i+1)\epsilon) \subset \mathbb{R}^d, \\ g_{i+1}^{x_0,\psi_1,...,\psi_{i+1}}(z_1,...,z_i,z_{i+1}) &= g[\psi_{i+1}(g_i^{x_0,\psi_1,...,\psi_i}(z_1,...,z_i)),z_{i+1}]. \end{aligned} \tag{2.24}$$

Note that (2.23) and (2.24) always make sense, since ψ_i belongs to $\mathcal{D}_{x_0,2\alpha\epsilon,\epsilon}$ and due to point (ii). For $x_0 \in \mathbb{R}^d$ to be fixed, our non-degeneracy assumption is the following.

Assumption $(H5)(x_0)$**:** There exists $\epsilon > 0$ and $\alpha \in \mathbb{N}$ such that for any $\psi_1,...,\psi_\alpha$ in $\mathcal{D}_{x_0,2\alpha\epsilon,\epsilon}$, the following conditions hold: there exists $O_1(\psi_1) \subset O_{\psi_1(x_0),\epsilon}$ such that

$$\int_{O_1(\psi_1)} \varphi(z)dz = \infty, \tag{2.25}$$

and such that for all $z_1 \in O_1(\psi_1)$, there exists $O_2(\psi_1,\psi_2;z_1) \subset O_{\psi_2[g_1^{x_0,\psi_1}(z_1)],\epsilon}$ such that

$$\int_{O_2(\psi_1;\psi_2,z_1)} \varphi(z)dz = \infty, \quad ..., \tag{2.26}$$

and such that for all $z_{\alpha-1} \in O_{\alpha-1}(\psi_1,...,\psi_{\alpha-1};z_1,...,z_{\alpha-2})$, there exists

$O_\alpha(\psi_1,...,\psi_\alpha;z_1,...,z_{\alpha-1}) \subset O_{\psi_\alpha[g_{\alpha-1}^{x_0,\psi_1,...,\psi_{\alpha-1}}(z_1,...,z_{\alpha-1})],\epsilon}$ such that

$$\int_{O_\alpha(\psi_1,...,\psi_\alpha;z_1,...,z_{\alpha-1})} \varphi(z)dz = \infty, \tag{2.27}$$

and such that for all negligible Lebesgue subset $A \in \mathcal{A}$ (recall (1.2)),

$$\int_{O_1(\psi_1)} dz_1 \int_{O_2(\psi_1,\psi_2;z_1)} dz_2 ... \int_{O_\alpha(\psi_1,...,\psi_\alpha;z_1,...,z_{\alpha-1})} dz_\alpha \mathbf{1}_{\left\{g_\alpha^{x_0,\psi_1,...,\psi_\alpha}(z_1,...,z_\alpha) \in A\right\}} = 0. \tag{2.28}$$

We finally require that for all $i \in \{1,...,\alpha\}$, the map

$$\psi_1,...,\psi_i,z_1,...,z_i \mapsto \mathbf{1}_{\{z_i \in O_i(\psi_1,...,\psi_i;z_1,...,z_{i-1})\}} \tag{2.29}$$

is measurable with respect to all its variables.

We then have the following results, which generalize Theorem 2.2 and Corollary 2.3.

Theorem 2.9 *Let $x_0 \in \mathbb{R}^d$ be fixed. Assume (I), (H_1), (H_2) and $(H_5)(x_0)$. Consider the unique solution $\{X_t^{x_0}\}_{t\geq 0}$ to (2.1). Then the law of $X_t^{x_0}$ has a density with respect to the Lebesgue measure on $\mathbb{R}^d$ as soon as $t > 0$.*

Corollary 2.10 *Let $x_0 \in \mathbb{R}^d$ be fixed. Assume (I), (H_1), and that $(H_5)(x)$ holds for all $x \in \mathbb{R}^d$. Consider the unique solution $\{X_t^{x_0}\}_{t\geq 0}$ to (2.1). Then the law of $X_t^{x_0}$ has a density with respect to the Lebesgue measure on $\mathbb{R}^d$ as soon as $t > 0$.*

The idea of Theorem 2.9 is relatively simple, since assumption $(H5)(x_0)$ almost contains its proof (see Subsection 2.2.2 below). Note that α stands for the *maximum number of jumps* necessary to produce a density for the law of X_t^x.

2.2.2 Proof

We first of all remark that we may assume the additionnal assumption $(H4)$ as before: copy line by line the proof of Lemma 2.6. By the same way, the proof of Corollary 2.10 is the same as that of Corollary 2.3, using of course Theorem 2.9 instead of Theorem 2.2. We will need the following result concerning the flow $\{X_t^x\}_{t\geq 0, x\in\mathbb{R}^d}$.

Lemma 2.11 *Assume (I), $(H1)$ and $(H4)$. Consider the flow of solutions $\{X_t^x\}_{t\geq 0, x\in\mathbb{R}^d}$ to (2.1). Then for any $x_0 \in \mathbb{R}^d$,*

$$\lim_{t\to 0} E\left[\sup_{s\in[0,t]}\sup_{x\in B(x_0,1)}\left\{|X_s^x - x|^2 + \left|\frac{\partial X_s^x}{\partial x} - I_d\right|^2\right\}\right] = 0. \tag{2.30}$$

Proof First, a standard computation using (2.1), (2.6) and $(H4)$ shows that there exists a constant $C > 0$ such that for all $x \in \mathbb{R}^d$, all $t \in [0,1]$,

$$E\left[\sup_{s\in[0,t]}\left\{|X_s^x - x|^2 + \left|\frac{\partial X_s^x}{\partial x} - I_d\right|^2\right\}\right] \leq Ct. \tag{2.31}$$

Another standard computation using (2.1), (2.6) and $(H4)$ shows that there exists a constant $C > 0$ such that for all $x, y \in \mathbb{R}^d$,

$$E\left[\sup_{s\in[0,1]}\left\{|X_s^x - X_s^y|^2 + \left|\frac{\partial X_s^y}{\partial x} - \frac{\partial X_s^x}{\partial x}\right|^2\right\}\right] \le C|x-y|^2. \tag{2.32}$$

We deduce from an easy adaptation of the Kolmogorov criterion of continuity, see e.g. Revuz-Yor [12] p 25, that there exists $C > 0$ such that

$$\text{if } Z = \sup_{s\in[0,1]}\sup_{x,y\in B(x_0,1),x\neq y}\left\{\frac{|X_s^x - X_s^y|^2}{|x-y|^{1/2}} + \frac{\left|\frac{\partial X_s^y}{\partial x} - \frac{\partial X_s^x}{\partial x}\right|^2}{|x-y|^{1/2}}\right\}, \quad E[Z] < \infty. \tag{2.33}$$

To conclude, consider, for each $n \in \mathbb{N}$, some points $\{x_i\}_{i\in\{1,\dots,n^d\}}$ of $B(x_0,1)$ such that for any $x \in B(x_0,1)$, $\min_i |x - x_i| \le 1/n$. We then obtain, using (2.31) and (2.33), for some C not depending on n,

$$\begin{aligned} &E\left[\sup_{s\in[0,t]}\sup_{x\in B(x_0,1)}\left\{|X_s^x - x|^2 + \left|\frac{\partial X_s^x}{\partial x} - I_d\right|^2\right\}\right] \\ &\le E\left[\sum_{i=1}^{n^d}\sup_{s\in[0,t]}\left\{|X_s^{x_i} - x_i|^2 + \left|\frac{\partial X_s^{x_i}}{\partial x} - I_d\right|^2\right\} + n^{-1/2}Z\right] \le C[n^d t + n^{-1/2}] \end{aligned} \tag{2.34}$$

One easily concludes, choosing n to be equal to the integer part of $t^{-1/2d}$. □

Proof of Theorem 2.9 As said before, we may assume the additional condition $(H4)$. Let $t > 0$ and $x_0 \in \mathbb{R}^d$ be fixed. We assume that $(H5)(x_0)$ is satisfied, for some $\epsilon > 0$. We suppose ϵ smaller than $1/2\alpha$, where $\alpha \in \mathbb{N}$ was defined in $(H5)(x_0)$. This ensures that $B(x_0, 2\alpha\epsilon) \subset B(x_0, 1)$. We denote, for $x \in \mathbb{R}^d$ and $s \ge 0$, by $\{X_r^{x,s}\}_{r\ge s}$ the process defined by

$$X_r^{x,s} = x + \int_s^r b(X_u^{x,s})du + \int_s^r\int_{\mathbb{R}^n} h(X_{u-}^{x,s}, z)\tilde{N}(du,dz) + \int_s^r \sigma(X_u^{x,s})dB_u, \tag{2.35}$$

We consider, for $0 \le s \le r$, the (stochastic) maps $\psi_{s,r}$ and $\psi_{s,r-}$ from $\mathbb{R}^d$ into itself, defined by

$$\psi_{s,r}(x) = X_r^{x,s}, \quad \psi_{s,r-}(x) = X_{r-}^{x,s}. \tag{2.36}$$

Step 1: We first build recursively some well-chosen jump times and marks. Due to $(H5)(x_0)$, we may build, for any $\psi_1 \in \mathcal{D}_{x_0,2\alpha\epsilon,\epsilon}$, an increasing sequence $\{O_1^p(\psi_1)\}_{p\ge1}$ of subsets of $\mathbb{R}^n$ such that

$$\cup_{p\ge1}O_1^p(\psi_1) = O_1(\psi_1), \text{ and } \forall\, p \ge 1, \int_{O_1^p(\psi_1)}\varphi(z)dz = p. \tag{2.37}$$

We also consider the stopping times

$$\begin{aligned}\tau_1 &= \inf\{s \geq 0;\ \psi_{0,s} \notin \mathcal{D}_{x_0,2\alpha\epsilon,\epsilon}\}, \\ S_1^p &= \inf\left\{s \geq 0;\ \int_0^s \int_{\mathbb{R}^n} \mathbf{1}_{\{z \in O_1^p(\psi_{0,(u\wedge\tau_1)-})\}} N(du,dz) \geq 1\right\},\end{aligned} \tag{2.38}$$

and we denote by Z_1^p the associated *mark*, uniquely defined by $N(\{S_1^p\} \times \{Z_1^p\}) = 1$.

We next define recursively O_i^p, τ_i, S_i^p, and Z_i^p for $i \in \{1, ..., \alpha\}$, in the following way. Let $i \in \{1, ..., \alpha - 1\}$ be fixed. Due to $(H5)(x_0)$, we may build, for any $\psi_1, ..., \psi_{i+1} \in \mathcal{D}_{x_0,2\alpha\epsilon,\epsilon}$, any $z_1 \in O_1^p(\psi_1)$, ..., $z_i \in O_i^p(\psi_1, ..., \psi_i; z_1, ..., z_{i-1})$, an increasing sequence $\{O_{i+1}^p(\psi_1, ..., \psi_{i+1}; z_1, ..., z_i)\}_{p\geq 1}$ of subsets of $\mathbb{R}^n$ such that

$$\begin{aligned}\cup_{p\geq 1} O_{i+1}^p(\psi_1, ..., \psi_{i+1}; z_1, ..., z_i) = O_{i+1}(\psi_1, ..., \psi_{i+1}; z_1, ..., z_i), \\ \text{and } \forall\, p \geq 1,\ \int_{O_{i+1}^p(\psi_1,...,\psi_{i+1};z_1,...,z_i)} \varphi(z)dz = p.\end{aligned} \tag{2.39}$$

We also consider the stopping times

$$\tau_{i+1} = \inf\left\{s > S_i^p;\ \psi_{S_i^p,s} \notin \mathcal{D}_{x_0,2\alpha\epsilon,\epsilon}\right\}, \tag{2.40}$$

$$S_{i+1}^p = \inf\left\{s > S_i^p;\ \int_{S_i^p}^s \int_{\mathbb{R}^n} \mathbf{1}_{\{z \in O_{i+1}^p(\psi_{0,S_1^p-}, \psi_{S_1^p,S_2^p-}, ..., \psi_{S_i^p,(u\wedge\tau_{i+1})-}; Z_1^p, ..., Z_i^p)\}} N(du,dz) \geq 1\right\},$$

and we denote by Z_{i+1}^p the associated *mark*, uniquely defined by $N(\{S_{i+1}^p\} \times \{Z_{i+1}^p\}) = 1$.

Step 2: We remark that

$$\text{for} \quad \Omega_p^0 = \cap_{i\in\{1,...,\alpha\}} \{\tau_i \geq S_i^p\}, \quad \lim_{p\to\infty} P[\Omega_p^0] = 1, \tag{2.41}$$

$$\text{and for} \quad \Omega_p^1 = \Omega_p^0 \cap \{S_\alpha^p \leq t\}, \quad \lim_{p\to\infty} P[\Omega_p^1] = 1. \tag{2.42}$$

Noting that $(S_{i+1}^p - S_i^p)_{i\in\{1,...,\alpha\}}$ is a family of independent exponentially distributed random variables with parameter p, (2.41) follows from Lemma 2.11 and the strong Markov property. Using finally that $t > 0$ while S_α^p follows a Gamma distribution with parameters α and p allows to obtain (2.42).

Step 3: We now show that the law of $X^{x_0}_{S^\alpha_p}$ conditionally to S^α_p has a density with respect to the Lebesgue measure on $\mathbb{R}^d$ on the set Ω^0_p. First note that on Ω^0_p,

$$X^{x_0}_{S^p_\alpha} = g_\alpha^{x_0,\psi_{0,S^p_1-},\dots,\psi_{S^p_{\alpha-1},S^p_\alpha-}}(Z^p_1,\dots,Z^p_\alpha). \tag{2.43}$$

Indeed, recalling Notation 2.8, one can check that $X^{x_0}_{S^p_1-} = \psi_{0,S^p_1-}(x_0)$, so that

$$X^{x_0}_{S^p_1} = \psi_{0,S^p_1-}(x_0) + h\left[\psi_{0,S^p_1-}(x_0), Z^p_1\right] = g_1^{x_0,\psi_{0,S^p_1-}}\left(Z^p_1\right). \tag{2.44}$$

Thus $X^{x_0}_{S^p_2-} = \psi_{S^p_1,S^p_2-}\left(g_1^{x_0,\psi_{0,S^p_1-}}(Z^p_1)\right)$, so that

$$X^{x_0}_{S^p_2} = \psi_{S^p_1,S^p_2-}\left(g_1^{x_0,\psi_{0,S^p_1-}}(Z^p_1)\right) + h\left[\psi_{S^p_1,S^p_2-}\left(g_1^{x_0,\psi_{0,S^p_1-}}(Z^p_1)\right), Z^p_2\right] = g_2^{x_0,\psi_{0,S^p_1-},\psi_{S^p_1,S^p_2-}}\left(Z^p_1, Z^p_2\right), \tag{2.45}$$

and so on...

Consider the σ-field

$$\mathcal{G}_p = \sigma\left\{\psi_{0,S^p_1-},\dots,\psi_{S^p_{\alpha-1},S^p_\alpha-}, S^p_1,\dots,S^p_\alpha\right\}. \tag{2.46}$$

We next claim that our construction leads to the conclusion that on Ω^p_0, the law of $(Z^p_1,\dots,Z^p_\alpha)$ conditionally to $\mathcal{G}_p$ is given by

$$\begin{aligned}&\frac{1}{p}\mathbf{1}_{\{z_1\in O^p_1(\psi_{0,S^p_1-})\}}\varphi(z_1)dz_1 \times \frac{1}{p}\mathbf{1}_{\{z_2\in O^p_2(\psi_{0,S^p_1-},\psi_{S^p_1,S^p_2-},z_1)\}}\varphi(z_2)dz_2\\ &\qquad\times\dots\times\frac{1}{p}\mathbf{1}_{\{z_\alpha\in O^p_\alpha(\psi_{0,S^p_1-},\dots,\psi_{S^p_{\alpha-1},S^p_\alpha-};z_1,\dots,z_{\alpha-1})\}}\varphi(z_\alpha)dz_\alpha.\end{aligned} \tag{2.47}$$

Hence, for any Lebesgue-null set $A\in\mathcal{A}$, since Ω^p_0 is $\mathcal{G}_p$-measurable,

$$\begin{aligned}P[\Omega^0_p, X^{x_0}_{S^p_\alpha}\in A\mid\mathcal{G}_p] &= \mathbf{1}_{\Omega^0_p}P\left\{g_\alpha^{x_0,\psi_{0,S^p_1-},\dots,\psi_{S^p_{\alpha-1},S^p_\alpha-}}(Z^p_1,\dots,Z^p_\alpha)\in A\,\middle|\,\mathcal{G}_p\right\}\\ &= \mathbf{1}_{\Omega^0_p}\frac{1}{p^\alpha}\int_{O^p_1(\psi_{0,S^p_1-})}\varphi(z_1)dz_1\dots\int_{O^p_\alpha(\psi_{0,S^p_1-},\dots,\psi_{S^p_{\alpha-1},S^p_\alpha-};z_1,\dots,z_{\alpha-1})\}}\varphi(z_\alpha)dz_\alpha\\ &\qquad\mathbf{1}_{\left\{g_\alpha^{x_0,\psi_{0,S^p_1-},\dots,\psi_{S^p_{\alpha-1},S^p_\alpha-}}(z_1,\dots,z_\alpha)\in A\right\}} = 0 \quad \text{a.s.},\end{aligned} \tag{2.48}$$

where the last equality comes from $(H5)(x_0)$ (see (2.28)). Since S^α_p is $\mathcal{G}_p$-measurable, this concludes the Step.

Step 4: We deduce from Step 3 that the law of $(S^p_\alpha, X^{x_0}_{S^\alpha_p})$ is of the shape $\nu_p(ds)f_p(s,x)dx$. One thus may conclude exactly as in the proof of Theorem 2.2, Steps 3 and 4, using S^p_α instead of S_p: we first prove that for each $p \geq 1$, the law of $X^{x_0}_t$ has a density on the set Ω^p_1 (see the proof of Theorem 2.2, Step 3), and then we let p tend to infinity (see the proof of Theorem 2.2, Step 4). □

2.2.3 Applications

The aim of this subsection is to give examples of functions h satisfying $(H5)$. In the whole subsection, we will denote, for $k \in \mathbb{N}$, for $(A_i)_{i\in\{1,\dots,k\}}$ a collection of $d \times l_i$ matrices, by $(A_1 \; ; \; \dots \; ; \; A_k)$ the corresponding $d \times (l_1 + \dots + l_k)$ matrix. We first show that $(H5)$ generalizes $(H3)$

Lemma 2.12 *Assume (I) and $(H1)$. Suppose that for some $x_0 \in \mathbb{R}^d$, $(H3)(x_0)$ is satisfied. Then $(H5)(x_0)$ is satisfied.*

Proof We know from $(H3)(x_0)$ that there exists $\epsilon_0 > 0$ such that for all $x \in B(x_0, \epsilon_0)$, there exists $O(x)$ such that (2.4) is fullfilled. Then $(H5)(x_0)$ is satisfied with $\alpha = 1$ and $\epsilon = \epsilon_0/3$. Indeed, it suffices to choose, for each $\psi_1 \in \mathcal{D}_{x_0,2\epsilon,\epsilon}$, $O_1(\psi_1) = O(\psi_1(x_0)) \cap O_{\psi_1(x_0),\epsilon}$. Then (2.4) and $(H1)$ ensure that (2.25) holds, while (2.28) follows from (2.4), since $g^{x_0,\psi_1}_1(z_1) = \psi_1(x_0) + h[\psi_1(x_0), z_1]$, with $\psi_1(x_0) \in B(x_0, 3\epsilon) = B(x_0, \epsilon_0)$. □

We next show that $(H5)$ is satisfied under some conditions in the spirit of those of Bichteler-Gravereaux-Jacod [1] Theorem 2.14, at least in the case where $\sigma \equiv 0$ (recall that mixed non-degeneracy conditions concerning both σ and h are supposed in [1], which seems to be very difficult to obtain not using Malliavin calculus concerning at least the Brownian part). We of course rewrite (and generalize) the assumptions of [1] in terms of our notations.

Proposition 2.13 *Assume (I) and $(H1)$. Suppose that $(H6)$ below is satisfied. Then $(H5)(x_0)$ is satisfied for any $x_0 \in \mathbb{R}^d$.*

Assumption $(H6)$**:** There exists $\alpha \in \mathbb{N}$, and some disjoint open subsets $B_1, \dots, B_\alpha$ of $\mathbb{R}^n$ such that:

(i) $(x,z) \mapsto h(x,z)$ is of class C^1 on $\mathbb{R}^d \times B_i$ for all $i \in \{1, \dots, \alpha\}$;

(ii) for all $x \in \mathbb{R}^d$, there exists $B_1(x) \subset B_1$, ..., $B_\alpha(x) \subset B_\alpha$ such that for all $i \in \{1, ..., \alpha\}$,

$$\int_{B_i(x)} \varphi(z)dz = \infty, \tag{2.49}$$

and such that for all $z_1 \in B_1(x)$, ..., $z_\alpha \in B_\alpha(x)$, the $d \times d$ matrix

$$M(x, z_1, ..., z_\alpha) = \sum_{i=1}^{\alpha} [(I_d + h'_x(x, z_i))^{-1} h'_z(x, z_i)][(I_d + h'_x(x, z_i))^{-1} h'_z(x, z_i)]^* \tag{2.50}$$

is well-defined (that is $(I_d + h'_x(x, z_i))$ is invertible for all i) and non degenerated (that is $\det M \neq 0$). Here K^* stands for the transposed matrix of K.

(iii) For all $i \in \{1, ..., \alpha\}$, the map $(x, z) \mapsto \mathbf{1}_{\{z \in B_i(x)\}}$ is measurable.

This assumption is a *strict ellipticity* condition: at each point $x \in \mathbb{R}^d$, the vector space spanned by the *directions* of all possible jumps at x is $\mathbb{R}^d$. See the comments in [1] for more details.

Note that $M(x, z_1, ..., z_\alpha)$ is invertible if and only if the rank of the following $d \times n\alpha$ matrix is d:

$$\text{rank}\left((I_d + h'_x(x, z_1))^{-1} h'_z(x, z_1) \; ; \; (I_d + h'_x(x, z_2))^{-1} h'_z(x, z_2) \; ; \; ... \; ; \; (I_d + h'_x(x, z_\alpha))^{-1} h'_z(x, z_\alpha)\right) = d. \tag{2.51}$$

Proof Let $x_0 \in \mathbb{R}^d$ be fixed. We will prove that $(H5)(x_0)$ holds, with $\alpha = d$, and with $\epsilon > 0$ small enough, in order that for all $\psi \in \mathcal{D}_{x_0, 2\alpha\epsilon, \epsilon}$, all $x \in B(x_0, 2\alpha\epsilon)$, the $d \times d$ matrix $\psi'(x)$ is invertible. We consider $\psi_1, ..., \psi_d$ in $\mathcal{D}_{x_0, 2\alpha\epsilon, \epsilon}$ to be fixed.

Step 1 For any $k \in \mathbb{N} \cup \{0\}$, any $j \in \{1, ..., d-1\}$, any $d \times k$ matrix M such that rank $M \geq j$, and any $x \in \mathbb{R}^d$, one may find, due to (2.51), an index $i(x, M) \in \{1, ..., \alpha\}$ such that, for any $z_i \in B_{i(x,M)}(x)$, the rank of the following $d \times (k + n)$ matrix is at least $j + 1$:

$$\text{rank}\left(M \; ; \; (I_d + h'_x(x, z_i))^{-1} h'_z(x, z_i)\right) \geq j + 1, \tag{2.52}$$

which implies that the rank of the following $d \times (k + n)$ matrix is also at least $j + 1$:

$$\text{rank}\left((I_d + h'_x(x, z_i))M \; ; \; h'_z(x, z_i)\right) \geq j + 1. \tag{2.53}$$

Step 2: Set $x_1 = \psi_1(x_0)$, $M_1 = 0$, and consider $O_1(\psi_1) = B_{i(x_1,M_1)}(x_1) \cap O_{x_1,\epsilon}$. Due to (2.49) and $(H1)$, we deduce that (2.25) holds. Furthermore, the map $z_1 \mapsto g_1^{x_0,\psi_1}(z_1)$ is of class C^1 on $O_1(\psi_1)$. Using Step 1, we obtain that for any $z_1 \in O_1(\psi_1)$, the rank of the following $d \times n$ matrix is at least 1:

$$\text{rank}\ [g_1^{x_0,\psi_1}]'(z_1) = \text{rank}\ [h'_z(\psi_1(x_0), z_1)] = \text{rank}\ \Big((I_d + h'_x(x_1, z_1))M_1\ ;\ h'_z(x_1, z_1)\Big) \geq 1. \tag{2.54}$$

We used here the explicit expression (2.23) of g_1. Next, we fix z_1 in $O_1(\psi_1)$, and we set $x_2 = \psi_2[g_1^{x_0,\psi_1}(z_1)]$, and $M_2 = \psi'_2(g_1^{x_0,\psi_1}(z_1))[g_1^{x_0,\psi_1}]'(z_1) \in \mathcal{M}_{d\times n}$. Note that $\text{rank}(M_2) \geq 1$ due to (2.54) and since $\psi'_2(g_1^{x_0,\psi_1}(z_1))$ is invertible. We choose

$$O_2(\psi_1, \psi_2; z_1) = B_{i(x_2,M_2)}(x_2) \cap O_{x_2,\epsilon}. \tag{2.55}$$

Then (2.49) and $(H1)$ ensure that (2.26) holds, while an immediate computation shows that for all $z_1 \in O_1(\psi_1)$, all $z_2 \in O_2(\psi_1, \psi_2; z_1)$, the rank of the following $d \times 2n$ matrix is at least 2, thanks to Step 1:

$$\text{rank}\ [g_2^{x_0,\psi_1,\psi_2}]'(z_1, z_2) = \text{rank}\,((I_d + h'_x(x_2, z_2))M_2\ ;\ h'_z(x_2, z_2)) \geq 2. \tag{2.56}$$

We used here the explicit expression (2.24) of g_2. By the same way, one may build recursively a sequence of sets $O_i(\psi_1, ..., \psi_i; z_1, ..., z_{i-1})$, for $i = 1, ..., d$, in such a way that for all $i \in \{1, ..., \alpha\}$

$$\int_{O_i(\psi_1,...,\psi_i;z_1,...,z_{i-1})} \varphi(z_i)dz_i = \infty, \tag{2.57}$$

and such that for any $(z_1, ..., z_d) \in \Delta_{x_0,\psi_1,...,\psi_d}$, where

$$\Delta_{x_0,\psi_1,...,\psi_d} = \{(z_1, ..., z_d),\ z_1 \in O_1(\psi_1), ..., z_d \in O_d(\psi_1, ..., \psi_d; z_1, ..., z_{d-1}))\}, \tag{2.58}$$

the rank of the following $d \times nd$ matrix is (at least) d:

$$\text{rank}\ [g_d^{x_0,\psi_1,...,\psi_d}]'(z_1, ..., z_d) = d. \tag{2.59}$$

This ensures that the map $z_1, ..., z_d \mapsto g_d^{x_0,\psi_1,...,\psi_d}(z_1, ..., z_d)$ is a submersion on $\Delta_{x_0,\psi_1,...,\psi_d}$, from which (2.28) follows immediately, for any $A \in \mathcal{A}$. $\square$

We now give an example of function h satisfying $(H6)$.

Example 2.14 *Assume that (I) holds with $d = 2$ and $n = 1$, and with $\varphi(z) = z^{-2}\mathbf{1}_{\{z \in (0,1)\}}$. Consider a C^2 function $\gamma : \mathbb{R} \mapsto (0,\infty)$ with at most polynomial growth, and set $h(x,z) = \gamma(x) \begin{pmatrix} z \\ z^2 \end{pmatrix}$. Then h clearly satisfies $(H1)$, and it satisfies $(H6)$, and thus $(H5)(x_0)$ for all $x_0 \in \mathbb{R}^2$. Note that h does not satisfies $(H3)(x_0)$, for any x_0.*

Remark that h is quite degenerated, since the state space of $\{X_t^{x_0}\}_{t \geq 0}$ is 2, while the image of $h(x,.)$ is a one-dimensional curve, for each $x \in \mathbb{R}^2$. However, X_t^x has a density for $t > 0$ because there are many possible *directions* of jumps: for each x, the vector space spanned by the directions $\{h'_z(x,z), z \in (0,1)\}$ is $\mathbb{R}^2$.

Proof Set $\alpha = 2$, and consider two disjoint open subsets B_1 and B_2 of $(0,1)$ such that $B_1 \cup B_2 = (0,1)$, and such that

$$\int_{B_1} \varphi(z)dz = \int_{B_2} \varphi(z)dz = \infty. \tag{2.60}$$

The function h is clearly of class C^1 on $\mathbb{R}^2 \times B_i$ for $i = 1,2$. Fix $x \in \mathbb{R}^2$, and consider $\epsilon(x) > 0$ such that $1 + \partial_{x_1}\gamma(x)z + \partial_{x_2}\gamma(x)z^2 > 0$ for all $z \in (0,\epsilon(x))$, and such that, $z \mapsto f(x,z) = (2z + \partial_{x_1}\gamma(x)z^2)/(1 - \partial_{x_2}\gamma(x)z^2)$ is injective on $(0,\epsilon(x))$. Setting $B_i(x) = B_i \cap (0,\epsilon(x))$, it is clear that (2.49) holds for all x. To show that $M(x,z_1,z_2)$ is non degenerated, it suffices to prove that for all $x \in \mathbb{R}^2$, all $z_1 \in B_1(x)$, $z_2 \in B_2(x)$, the two-dimensional vectors

$$v_1 = (I_d + h'_x(x,z_1))^{-1}h'_z(x,z_1) \text{ and } v_2 = (I_d + h'_x(x,z_2))^{-1}h'_z(x,z_2) \tag{2.61}$$

are well-defined and not colinear. But, for $i = 1,2$, we get, using the expression of h,

$$v_i = \frac{\gamma(x)}{1 + \partial_{x_1}\gamma(x)z_i + \partial_{x_2}\gamma(x)z_i^2} \begin{pmatrix} 1 - \partial_{x_2}\gamma(x)z_i^2 \\ 2z_i + \partial_{x_1}\gamma(x)z_i^2 \end{pmatrix}. \tag{2.62}$$

Thanks to our choice for $\epsilon(x)$ and since $z_1 \neq z_2$ (because $B_1 \cap B_2 = \emptyset$), while z_1, z_2 belong to $(0,\epsilon(x))$, we deduce that v_1 and v_2 are well-defined and not colinear, which concludes the proof. $\square$

We carry on with an *hypoelliptic case*, where the dependence of h in x plays an important role.

Example 2.15 *Assume that (I) holds with $d = 2$ and $n = 1$, and with $\varphi(z) = z^{-2}\mathbf{1}_{\{z\in(0,1)\}}$. Set $h(x,z) = \begin{pmatrix} z \\ x_1 z \end{pmatrix}$, where $x = (x_1, x_2)$. Then h clearly satisfies $(H1)$, and it satisfies $(H6)$, and thus $(H5)(x_0)$ for all $x_0 \in \mathbb{R}^2$. Note that h does not satisfies $(H3)(x_0)$, for any x_0.*

Remark that replacing here $h(x,z) = \begin{pmatrix} z \\ x_1 z \end{pmatrix}$ by $h(x,z) = \begin{pmatrix} z \\ z \end{pmatrix}$ would not work, since in such a case, if $b = \sigma = 0$, if $x_0 = 0$, the solution $X_t^{x_0}$ belongs a.s. to $\{(x_1, x_2) \in \mathbb{R}^2;\ x_1 = x_2\}$, and thus can not have a density. Thus the dependence of h in x plays an important role.

Proof Set $\alpha = 2$, and consider two disjoint open subsets B_1 and B_2 of $(0,1)$ such that $B_1 \cup B_2 = (0,1)$, and such that (2.60) holds. Set then $B_1(x) = B_1$ and $B_2(x) = B_2$ for all $x \in \mathbb{R}^2$. The function h is clearly of class C^1 on $\mathbb{R}^2 \times B_i$ for $i = 1, 2$. To show that $M(x, z_1, z_2)$ is non degenerated, it suffices to prove that for all $x \in \mathbb{R}^2$, all $z_1 \in B_1$, $z_2 \in B_2$, the two-dimensional vectors v_1 and v_2 defined by (2.61) are well-defined and not colinear. But, for $i = 1, 2$, we get

$$v_i = \begin{pmatrix} 1 \\ x_1 - z_i \end{pmatrix}. \tag{2.63}$$

The conclusion follows: since $z_1 \neq z_2$, v_1 and v_2 are not colinear. $\square$

We conclude with a case where $(H5)$ is satisfied while $(H6)$ does not hold.

Example 2.16 *Assume that (I) holds with $d = 2$ and $n = 1$, and with $\varphi(z) = z^{-2}\mathbf{1}_{\{z\in(0,1)\}}$. For $z \in (0,1)$, denote by $[1/z]$ the integer part of $1/z$, and set $h(x,z) = \begin{pmatrix} z \\ x_1/[1/z] \end{pmatrix}$. Then h clearly satisfies $(H1)$, and it satisfies $(H5)(x_0)$ for all $x_0 \in \mathbb{R}^2$. Note that h does not satisfies $(H6)$.*

In this example, h is very degenerated, since for each fixed x, $h'_z(x,z) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ for all z such that the derivative exists. Furthermore we realize, denoting $h = (h_1, h_2)$, that for each x, the image $h_2(x, (0,1))$ is countable. However, here again, the dependance of h in x plays a fundamental role, and $X_t^x = (X_t^{x,1}, X_t^{x,2})$ has a density: roughly speaking, a first jump will allow the law of $X_t^{x,1}$ to become absolutely continuous.

A second jump will allow the law of $X_t^{x,2}$ to *catch* the density of $X_t^{x,1}$.

Proof We prove that $(H5)(x_0)$ holds with $\alpha = 2$, and with ϵ small enough, in such a way that for all $\psi \in \mathcal{D}_{x_0,2\alpha\epsilon,\epsilon}$, all $x \in B(x_0, 2\alpha\epsilon)$, $\psi'(x)$ is invertible. Set $\gamma(z) = 1/[1/z]$. We consider the open subset $O = \{z \in (0,1),\ \gamma'(z)\ exists\} = (0,1)\backslash \cup_{n\geq 2} \{1/n\}$. Then clearly,

$$\int_O \varphi(z)dz = \infty, \tag{2.64}$$

while $\gamma'(z) = 0$ for all $z \in O$. For any $\psi_1 \in \mathcal{D}_{x_0,2\alpha\epsilon,\epsilon}$, we choose $O_1(\psi_1) = O$. Next, for any $z_1 \in O_1(\psi_1)$, any $\psi_2 \in \mathcal{D}_{x_0,2\alpha\epsilon,\epsilon}$, we choose

$$O_2(\psi_1,\psi_2;z_1) = \left\{ z_2 \in O,\ \begin{pmatrix} 1 \\ \gamma(z_2) \end{pmatrix} \text{ and } \{\psi_2'[\psi_1(x_0) + h(\psi_1(x_0), z_1)]\}^{-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ are not colinear} \right\}. \tag{2.65}$$

Then clearly,

$$\int_{O_1(\psi_1)} \varphi(z_1)dz_1 = \int_{O_2(\psi_1,\psi_2;z_1)} \varphi(z_2)dz_2 = \infty. \tag{2.66}$$

Then we note that, setting to simplify $g(z_1,z_2) = g_2^{x_0,\psi_1,\psi_2}(z_1,z_2)$, g is of class C^1 on $\Delta_{\psi_1,\psi_2} = \{(z_1,z_2),\ z_1 \in O_1(\psi_1),\ z_2 \in O_2(\psi_1,\psi_2;z_1)\}$, and that the 2×2 matrix

$$g'(z_1,z_2) = \left(\psi_2'[\psi_1(x_0) + h(\psi_1(x_0), z_1)] \begin{pmatrix} 1 \\ \gamma(z_2) \end{pmatrix} \ ; \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right) \tag{2.67}$$

is invertible for each (z_1,z_2) in Δ_{ψ_1,ψ_2}. We thus may perform the substitution $(z_1,z_2) \mapsto (y_1,y_2) = g(z_1,z_2)$ to obtain (2.28). $\square$

3 The case of a non constant rate of jump

Consider now the following d-dimensional S.D.E., for some $d \in \mathbb{N}$, starting from $x \in \mathbb{R}^d$:

$$X_t^x = x + \int_0^t b(X_s^x)ds + \int_0^t \int_{\mathbb{R}^n} \int_0^\infty h(X_{s-}^x, z)\mathbf{1}_{\{u\leq\gamma(X_{s-}^x)\}} N(ds,dz,du), \tag{3.1}$$

where

Assumption (J)**:** $N(ds, dz, du)$ is a Poisson measure on $[0, \infty) \times \mathbb{R}^n \times [0, \infty)$, for some $n \in \mathbb{N}$, with intensity measure $\nu(ds, dz, du) = ds\varphi(z)dzdu$. The function $\varphi : \mathbb{R}^n \mapsto \mathbb{R}_+$ is supposed to be measurable.

In this case, the generator of the Markov process X^x is given, for any $\phi \in C_b^1(\mathbb{R}^d)$ and $y \in \mathbb{R}^d$, by

$$\mathcal{L}\phi(y) = b(y).\nabla\phi(y) + \int_{\mathbb{R}^n} \gamma(y) \left[\phi(y + h(y, z)) - \phi(y)\right] \varphi(z)dz. \tag{3.2}$$

It might be possible to add a Brownian term and consider a compensated Poisson measure. However, the present situation simplifies the computations. We assume the following hypothesis.

Assumption $(A1)$**:** The function $b : \mathbb{R}^d \mapsto \mathbb{R}^d$ is of class C^1, and has at most linear growth. The function $\gamma : \mathbb{R}^d \mapsto \mathbb{R}_+$ is of class C^1. The function $h : \mathbb{R}^d \times \mathbb{R}^n \mapsto \mathbb{R}^d$ is measurable. For each $z \in \mathbb{R}^n$, $x \mapsto h(x, z)$ is of class C^1 on $\mathbb{R}^d$. There exists $\eta \in L^1(\mathbb{R}^n, \varphi(z)dz)$ and a continuous function $\zeta : \mathbb{R}^d \mapsto \mathbb{R}$ such that for all $x \in \mathbb{R}^d, z \in \mathbb{R}^n$, $\gamma(x)|h(x, z)| \leq (1 + |x|)\eta(z)$, while $|h'_x(x, z)| \leq \zeta(x)\eta(z)$.

Then it is well-known that the following result holds.

Proposition 3.1 *Assume (J) and $(A1)$. Consider the natural filtration $\{\mathcal{F}_t\}_{t\geq 0}$ associated with the Poisson measure N. Then, for any $x \in \mathbb{R}^d$, there exists a unique càdlàg $\{\mathcal{F}_t\}_{t\geq 0}$-adapted process $\{X_t^x\}_{t\geq 0}$ solution to (3.1) such that for all $x \in \mathbb{R}^d$, all $T \in [0, \infty)$,*

$$E\left[\sup_{s\in[0,T]} |X_s^x|\right] < \infty. \tag{3.3}$$

The process $\{X_t^x\}_{t\geq 0, x\in\mathbb{R}^d}$ furthermore satisfies the strong Markov property.

We refer to [5] (Section 2) for the proof of a very similar result.

We divide the section into three parts: we start with the statements and proofs, and we end with an example of application.

3.1 Statements

To obtain some absolute continuity results, we will assume the following conditions. Here x_0 is fixed in $\mathbb{R}^d$.

Assumption $(A2)$**:** There exists $c_0 > 0$ such that for all $x \in \mathbb{R}^d$, all $z \in \mathbb{R}^n$, $\det(I_d + h'_x(x,z)) \geq c_0$. For each $z \in \mathbb{R}^n$, the map $x \mapsto x + h(x,z)$ is a C^1-diffeomorphism.

Remark that if $d = 1$, the condition $1 + h'_x(x,z) \geq c_0 > 0$ ensures that $(A2)$ holds.

Assumption $(A3)(x_0)$**:** The function γ does never vanish. There exists $\epsilon > 0$ such that for all $x \in B(x_0, \epsilon)$, there exists a subset $O(x) \subset \mathbb{R}^n$ such that, (recall (1.2)),

$$\int_{O(x)} \varphi(z)dz = \infty, \text{ and for all } A \in \mathcal{A}, \quad \int_{O(x)} \mathbf{1}_{\{h(x,z) \in A\}} \varphi(z)dz = 0, \tag{3.4}$$

and such that the map $(x,z) \mapsto \mathbf{1}_{\{z \in O(x)\}}$ is measurable on $B(x_0,\epsilon) \times \mathbb{R}^n$.

The main results of this section are the following.

Theorem 3.2 *Let $x_0 \in \mathbb{R}^d$ be fixed. Assume (J), (A_1), (A_2) and $(A_3)(x_0)$. Consider the unique solution $\{X_t^{x_0}\}_{t \geq 0}$ to (2.1). Then the law of $X_t^{x_0}$ has a density with respect to the Lebesgue measure on $\mathbb{R}^d$ as soon as $t > 0$.*

As usual, an immediate consequence is the following.

Corollary 3.3 *Let $x_0 \in \mathbb{R}^d$ be fixed. Assume (J), (A_1) and that $(A_3)(x)$ holds for all $x \in \mathbb{R}^d$. Consider the unique solution $\{X_t^{x_0}\}_{t \geq 0}$ to (3.1). Then the law of $X_t^{x_0}$ has a density with respect to the Lebesgue measure on $\mathbb{R}^d$ as soon as $t > 0$.*

These results improve consequently those of [5], where many restrictive conditions were assumed, such as the monotonicity of $x \mapsto \gamma(x)$ and $x \mapsto h(x,z)$, and the positivity of $h(x,z)$. These conditions were usefull to prove the almost sure monotonicity of the (irregular) map $x \mapsto X_t^x$.

Note that we are not able to obtain a result under an assumption in the spirit of $(H5)$ (or $(H6)$) when γ is not constant, because of the irregularity of the map $x \mapsto X_t^x$ (see the introduction).

Exactly as in Subsection 2.1 (see Proposition 2.4), we have a general example of application, using the local inverse Theorem.

Proposition 3.4 *Assume (J) and (A_1). Suppose that $n = d$, that γ does never vanish. Assume that there exists $\epsilon > 0$ and an open subset $O \subset \mathbb{R}^d$ such that h is of class C^1 on $B(x_0, \epsilon) \times O$. If*

$$\forall\, x \in B(x_0, \epsilon), \quad \int_O \mathbf{1}_{\{\det h'_z(x,z) \neq 0\}} \varphi(z) dz = \infty, \tag{3.5}$$

then $(A3)(x_0)$ holds.

3.2 Proof

First of all, we proceed to a localization procedure.

Lemma 3.5 *To prove Theorem 3.2 and Corollary 3.3, we may assume the additional condition $(A4)$ below.*

Assumption $(A4)$**:** The functions b, b', γ and γ' are bounded. There exists $\tilde{\eta} \in L^1(\mathbb{R}^n, \varphi(z)dz)$ such that for all $x \in \mathbb{R}^d$, $z \in \mathbb{R}^n$, $|h(x,z)| + |h'_x(x,z)| \leq \tilde{\eta}(z)$. There exists $\gamma_0 > 0$ such that for all $x \in \mathbb{R}^d$, $\gamma(x) \geq \gamma_0$.

We omit the proof of this lemma, since it is the same as that of Lemma 2.6 (see also [5] Section 2). We will need the following Lemma.

Lemma 3.6 *There exists $\beta_0 > 0$ such that for any C^1 function $\delta : \mathbb{R}^d \mapsto \mathbb{R}^d$ satisfying $||\delta'||_\infty \leq \beta_0$, the map $x \mapsto x + \delta(x)$ is a C^1-diffeomorphism, and for all $x \in \mathbb{R}^d$, $\det(I_d + \delta'(x)) \geq 1/2$.*

Proof Set $\zeta(x) = x + \delta(x)$ First of all, it is clear, by continuity of the determinant, that if β_0 is small enough, $\det \zeta'(x) = \det[I_d + \delta'(x)] \geq 1/2$ for all $x \in \mathbb{R}^d$. Thus, it classically suffices to show that, if β_0

is small enough, ζ is injective. Consider thus x, y such that $\zeta(x) = \zeta(y)$. Then $|x - y| = |\delta(x) - \delta(y)| \leq ||\delta'||_\infty |x - y| \leq \beta_0 |x - y|$, which implies that $x = y$ if $\beta_0 < 1$. □

Next, we note that the proof of Corollary 3.3 is the same as that of Corollary 2.3, using of course Theorem 3.2 instead of that of Theorem 2.2, and using β_0 defined in Lemma 3.6 rather than δ_0. We thus omit the proof of Corollary 2.3.

The main novelty of this section consists in the following Proposition, which allows us to overcome the irregularity of the map $x \mapsto X_t^x$.

Proposition 3.7 *Assume (J), (A1), (A2) and (A4), and denote by $\{X_t^x\}_{t \geq 0, x \in \mathbb{R}^d}$ the unique solution to (3.1). Consider a probability density function f_0 on $\mathbb{R}^d$. Then for all $t \geq 0$, all $A \in \mathcal{A}$,*

$$\int_{\mathbb{R}^d} f_0(x) P[X_t^x \in A] dx = 0. \tag{3.6}$$

In other words, if X_0 is a random variable (independent of N) with law $f_0(x)dx$, then $X_t^{X_0}$ has a density for each $t \geq 0$. To prove this, we first consider the case where f_0 satisfies some additional conditions.

Lemma 3.8 *Assume (J), (A1), (A2) and (A4), and denote by $\{X_t^x\}_{t \geq 0, x \in \mathbb{R}^d}$ the unique solution to (3.1). Consider a d-dimensional random variable X_0, independent of N, satisfying $E[|X_0|] < \infty$. Assume that the law of X_0 is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}^d$, and that its density f_0 satisfies*

$$\int_{\mathbb{R}^d} f_0^2(x) dx < \infty. \tag{3.7}$$

Then for all $t \geq 0$, the law of $X_t^{X_0}$ has a density $f(t, x)$, and furthermore, for any $T \in [0, \infty)$,

$$\sup_{[0,T]} \int_{\mathbb{R}^d} f^2(t, x) dx < \infty. \tag{3.8}$$

Proof We split the proof into several steps. We first introduce an approximating process X_t^l in Step 1. We next show some non-uniform L^∞ estimates for the density of X_t^l in Step 2, which allow us to prove rigorously some uniform (in l) L^2 estimates in Step 3. We go to the limit in Step 4.

Step 1: We consider a sequence $\{f_l^0\}_{l\geq 1}$ of bounded and continuous density functions, converging to f_0 in $L^2(\mathbb{R}^d)$. We build a sequence $\{X_0^l\}_{l\geq 1}$ of random variables (independent of N), such that for each l, the law of X_0^l is given by $f_l^0(x)dx$. Since $E[|X_0|] < \infty$, we may handle this construction in such a way that $\lim_l E[|X_0 - X_0^l|] = 0$. We also consider an increasing sequence K_l of subsets of $\mathbb{R}^n$ such that $\cup_l K_l = \text{supp } \tilde{\eta}$ (recall $(A4)$), and such that for each l, $\Lambda_l = \int_{K_l} \varphi(z)dz < \infty$ (choose for example $K_l = \{z \in \mathbb{R}^n, \tilde{\eta}(z) \geq 1/l\}$). We finally denote, for each $l \in \mathbb{N}$, by $\{X_t^l\}_{t\geq 0}$ a $\mathbb{R}^d$-valued Markov process starting from X_0^l and with generator $\mathcal{L}^l$, defined for any bounded measurable function $\phi : \mathbb{R}^d \mapsto \mathbb{R}$ and any $x \in \mathbb{R}^d$, by

$$\mathcal{L}^l\phi(x) = l\gamma(x)\left[\phi(x + b(x)/l\gamma(x)) - \phi(x)\right] + \gamma(x)\int_{K_l}\varphi(z)dz\left[\phi(x + h[x,z]) - \phi(x)\right]. \tag{3.9}$$

We now show that for each $t \geq 0$, X_t^l converges to X_t in law as l tends to infinity. To this aim, we build $\{X_t^l\}_{t\geq 0}$ with the help of N, and of another independent Poisson measure $M^l(ds, du)$ on $[0,\infty) \times [0,\infty)$ with intensity measure $ldsdu$:

$$X_t^l = X_0^l + \int_0^t\int_0^\infty \frac{b(X_{s-}^l)}{l\gamma(X_{s-}^l)}\mathbf{1}_{\{u\leq\gamma(X_{s-}^l)\}}M^l(ds,du) + \int_0^t\int_{K_l}\int_0^\infty h(X_{s-}^x, z)\mathbf{1}_{\{u\leq\gamma(X_{s-})\}}N(ds,dz,du), \tag{3.10}$$

Noting that

$$Y_t^l = \int_0^t\int_0^\infty \frac{b(X_{s-}^l)}{l\gamma(X_{s-}^l)}\mathbf{1}_{\{u\leq\gamma(X_{s-}^l)\}}M^l(ds,du) - \int_0^t b(X_s^l)ds \tag{3.11}$$

is a martingale with bracket

$$\left\langle Y^l\right\rangle_t = \frac{1}{l}\int_0^t \frac{b^2(X_s^l)}{\gamma(X_s^l)}ds \leq ||b/\gamma||_\infty^2\frac{t}{l} \to 0, \tag{3.12}$$

and using $(A1)$ and $(A4)$ repeatedly, one may then show that for any $T \geq 0$,

$$\lim_{l\to\infty} E[\sup_{[0,T]}|X_t^l - X_t^{X_0}|] = 0. \tag{3.13}$$

Step 2: Consider now $l_0 > ||(b/\gamma)'||_\infty/\beta_0$, where β_0 was defined in Lemma 3.6. This is possible due to $(A4)$. We aim to prove that for any $l \geq l_0$, any $t \geq 0$, X_t^l has a bounded density $f_l(t,x)$, and that for any $T > 0$,

$$\sup_{[0,T]}\sup_{x\in\mathbb{R}^d} f_l(t,x) < \infty. \tag{3.14}$$

We thus consider $l \geq l_0$ to be fixed. We also denote, for any $a \in (0,\infty)$, by $\mathcal{C}_a = \{A \in \mathcal{B}(\mathbb{R}^d);\ \int_A dx \leq a\}$. A direct computation, using (3.9), the fact that γ is bounded, and neglecting all the non positive terms, yields that there exists a constant C (depending on l) such that for any $A \in \mathcal{B}(\mathbb{R}^d)$,

$$\begin{aligned} P[X^l_t \in A] &= P[X^l_0 \in A] + l\int_0^t E[\gamma(X^l_s)\left\{\mathbf{1}_{\{X^l_s + b(X^l_s)/l\gamma(X^l_s)\in A\}} - \mathbf{1}_{\{X^l_s\in A\}}\right\}]ds \\ &\quad + \int_0^t \int_{K_l} E[\gamma(X^l_s)\left\{\mathbf{1}_{\{X^l_s + h(X^l_s,z)\in A\}} - \mathbf{1}_{\{X^l_s\in A\}}\right\}]\varphi(z)dzds \qquad (3.15) \\ &\leq P[X^l_0 \in A] + C\int_0^t P[X^l_s + b(X^l_s)/l\gamma(X^l_s) \in A]ds + C\int_0^t \sup_{z\in K_l} P[X^l_s + h(X^l_s,z)\in A]ds. \end{aligned}$$

For $A \in \mathcal{B}(\mathbb{R}^d)$, set $\tau(A) = \{x \in \mathbb{R}^d,\ x + b(x)/l\gamma(x) \in A\}$, and $\tau_z(A) = \{x \in \mathbb{R}^d,\ x + h(x,z) \in A\}$. Then, using $(A2)$, we deduce that for any $z \in \mathbb{R}^n$, any $A \in \mathcal{B}(\mathbb{R}^d)$,

$$\int_{\tau_z(A)} dx = \int_{\mathbb{R}^d} \mathbf{1}_{\{x+h(x,z)\in A\}}dx = \int_{\mathbb{R}^d} \mathbf{1}_{\{y\in A\}} \frac{dy}{|\det(I_d + h'_x(x,z))|} \leq \frac{1}{c_0}\int_A dx. \qquad (3.16)$$

By the same way, using that $l \geq l_0$ and Lemma 3.6, we get

$$\int_{\tau(A)} dx \leq 2\int_A dx. \qquad (3.17)$$

Gathering (3.15), (3.16) and (3.17), we obtain, setting $n = [2 \vee 1/c_0] + 1$, that for some constant C, for any $a \in (0,\infty)$,

$$\begin{aligned} \sup_{A\in\mathcal{C}_a} P[X^l_t \in A] &\leq \sup_{A\in\mathcal{C}_a} P[X^l_0 \in A] + C\int_0^t \sup_{A\in\mathcal{C}_{na}} P[X^l_s \in A]ds \\ &\leq a||f^0_l||_\infty + nC \sup_{A\in\mathcal{C}_a} P[X^l_s \in A]ds. \qquad (3.18) \end{aligned}$$

To obtain the last term, we have used that any $A \in \mathcal{C}_{na}$ may be written as a union of n elements of $\mathcal{C}_a$. We finally obtain, using the Gronwall Lemma, that for any T, there exists C_T such that for all $a \in (0,\infty)$,

$$\sup_{[0,T]} \sup_{A\in\mathcal{C}_a} P[X^l_t \in A] \leq C_T \times a. \qquad (3.19)$$

This ensures (3.14).

Step 3: We now show, and it is the heart of the proof, that for any $T \geq 0$, there exists a constant C_T, not depending on $l \geq l_0$, such that

$$\sup_{[0,T]} \int_{\mathbb{R}^d} f_l^2(t,x)dx \leq C_T. \qquad (3.20)$$

We will rather work with the weight function $\gamma(x)$, which seems artificial: we are however not able to conclude working directly with $\int f_l^2 dx$. Setting for simplicity $\gamma f_l(t,x) = \gamma(x) f_l(t,x)$, we get

$$\begin{aligned}
\partial_t \int_{\mathbb{R}^d} \gamma(x) f_l^2(t,x) dx &= 2\int_{\mathbb{R}^d} [\partial_t f_l(t,x)][\gamma f_l(t,x)] dx = 2\int_{\mathbb{R}^d} f_l(t,x) \mathcal{L}^l \{\gamma f_l(t,x)\} dx \\
&= 2l \int_{\mathbb{R}^d} f_l(t,x)\gamma(x)[\gamma f_l(t, x + b(x)/l\gamma(x)) - \gamma f_l(t,x)] dx \\
&\quad +2\int_{\mathbb{R}^d} f_l(t,x)\gamma(x) \int_{K_l} \varphi(z) \left[\gamma f_l(t, x + h(x,z)) - \gamma f_l(t,x)\right] dz dx \\
&= 2A_l(t) + 2B_l(t), \qquad (3.21)
\end{aligned}$$

the last equality standing for a definition. First, using the Cauchy-Schwartz inequality, we obtain, setting $||g||_2^2 = \int_{\mathbb{R}^d} g^2(x) dx$,

$$A_l(t) \leq l \left[||\gamma f_l(t,.)||_2 ||\gamma f_l(t, . + b(.)/l\gamma(.))||_2 - ||\gamma f_l(t,.)||_2^2 \right]. \qquad (3.22)$$

But the substitution $x \mapsto y = x + b(x)/l\gamma(x)$, which is valid for $l \geq l_0$ due to Lemma 3.6, leads to the conclusion that

$$\begin{aligned}
||\gamma f_l(t, . + b(.)/l\gamma(.))||_2^2 &= \int_{\mathbb{R}^d} \gamma f_l^2(t,y) \frac{1}{\det(I_d + (b/\gamma)'(x)/l)} dy \\
\leq \frac{||\gamma f_l(t,.)||_2^2}{\inf_{x\in\mathbb{R}^d} \det(I_d + (b/\gamma)'(x)/l)} &\leq ||\gamma f_l(t,.)||_2^2 \times \left(2 \wedge \frac{1}{1 - C/l}\right). \qquad (3.23)
\end{aligned}$$

The last inequality is due to the fact that $(b/\gamma)'$ is bounded due to $(A4)$. We finally obtain, the value of C changing from line to line, that for any $l \geq l_0$,

$$A_l(t) \leq ||\gamma f_l(t,.)||_2^2 \times l \left[\sqrt{2} \wedge (1 - C/l)^{-1/2} - 1\right] \leq C ||\gamma f_l(t,.)||_2^2. \qquad (3.24)$$

Next, using the Fubini Theorem and then the Cauchy-Shcwarz inequality, we get

$$\begin{aligned}
B_l(t) &= \int_{K_l} \varphi(z) dz \int_{\mathbb{R}^d} \left[\gamma f_l(t,x)\gamma f_l(t, x + h(x,z)) - (\gamma f_l)^2(t,x)\right] dx \\
&\leq \int_{K_l} \varphi(z) dz \left[||\gamma f_l(t,.)||_2 ||\gamma f_l(t, . + h(.,z))||_2 - ||\gamma f_l(t,.)||_2^2 \right]
\end{aligned} \qquad (3.25)$$

But the substitition $x \mapsto y = x + h(x,z)$, valid due to $(A2)$, shows that

$$||\gamma f_l(t, . + h(.,z))||_2^2 = \int_{\mathbb{R}^d} (\gamma f_l)^2(t,y) \frac{1}{\det(I_d + h'_x(x,z))} dy \leq \alpha(z) ||\gamma f_l(t,.)||_2^2, \qquad (3.26)$$

where $\alpha(z) = \sup_{x\in\mathbb{R}^d}[1/\det(I_d + h'_x(x,z))]$ is well-defined due to $(A2)$. We thus obtain that

$$B_l(t) \le ||\gamma f_l(t,.)||_2^2 \int_{K_l} \varphi(z)dz|\sqrt{\alpha(z)}-1| \le ||\gamma f_l(t,.)||_2^2 \int_{\mathbb{R}^n} \varphi(z)dz|\sqrt{\alpha(z)}-1| = C||\gamma f(t,.)||_2^2. \quad (3.27)$$

The constant C is finite here due to $(A2)$ and $(A4)$: one may check that for some constants c_1, c_2, c_3, $|\sqrt{\alpha(z)}-1| \le \frac{1}{\sqrt{c_0}} \wedge c_1\tilde{\eta}(z)\mathbf{1}_{\{\tilde{\eta}(z)<c_2\}} \le c_3\tilde{\eta}(z) \in L^1(\mathbb{R}^n, \varphi(z)dz)$.

Gathering together the previous estimates, integrating against time, and using that γ is bounded, we obtain, for some constant C not depending on $l \ge l_0$,

$$\begin{aligned}\int_{\mathbb{R}^d} \gamma(x) f_l^2(t,x)dx &\le \int_{\mathbb{R}^d} \gamma(x)(f_l^0)^2(x)dx + C\int_0^t ||\gamma f_l(s,.)||_2^2 ds \\ &\le \int_{\mathbb{R}^d} \gamma(x)(f_l^0)^2(x)dx + C\int_0^t ds \int_{\mathbb{R}^d} \gamma(x) f_l^2(s,x)dx. \quad (3.28)\end{aligned}$$

Since γ is bounded, we deduce that $\sup_l \int_{\mathbb{R}^d} \gamma(x)(f_l^0)^2(x)dx < \infty$. Furthermore, we deduce from (3.14) that for all $T \ge 0$, for each $l \ge l_0$, $\int_{\mathbb{R}^d} \gamma(x) f_l^2(t,x)dx$ is bounded on $[0,T]$. We thus may conclude, using the Gronwall Lemma and the fact that γ is bounded below, that for any T,

$$\sup_{l\ge l_0} \sup_{[0,T]} \int_{\mathbb{R}^d} f_l^2(t,x)dx \le C \sup_{l\ge l_0} \sup_{[0,T]} \int_{\mathbb{R}^d} \gamma(x) f_l^2(t,x)dx < \infty. \quad (3.29)$$

Step 4: We now fix $t \ge 0$. The space $L^2(\mathbb{R}^d)$ being weakly compact, using (3.29) allows us to find a subsequence $f^{k_l}(t,.)$, going weakly to a function $f(t,.) \in L^2(\mathbb{R}^d)$. On the other hand, we know that X_t^l converges in law to $X_t^{X_0}$. Hence the law of $X_t^{X_0}$ is given by $f(t,x)dx$, and (3.29) allows us to conclude that (3.8) holds. □

Proposition 3.7 follows easily from Lemma 3.8.

Proof of Proposition 3.7 For each $n \in \mathbb{N}$, consider the probability density function f_0^n on $\mathbb{R}^d$ defined by $f_0^n(x) = c_n[f_0(x) \wedge n]\mathbf{1}_{\{|x|\le n\}}$. Here c_n is a normalization constant. Consider a random variable X_0^n, independent of N, with law $f_0^n(x)dx$ independent of N. Then X_0^n satisfies the assumptions of Lemma 3.8, for each $n \in \mathbb{N}$. Thus $X_t^{X_0^n}$ has a density for each $t \ge 0$, which implies that for all $n \in \mathbb{N}$, all $t \ge 0$, all $A \in \mathcal{A}$,

$$\int_{\mathbb{R}^d} [f_0(x) \wedge n]\mathbf{1}_{\{|x|\le n\}} P[X_t^x \in A]dx = c_n^{-1} \int_{\mathbb{R}^d} f_0^n(x) P[X_t^x \in A]dx = c_n^{-1} P[X_t^{X_0^n} \in A] = 0. \quad (3.30)$$

The Lebesgue Theorem allows us to conclude that (3.6) holds, since $[f_0(x) \wedge n]\mathbf{1}_{\{|x|\leq n\}}$ increases pointwise to $f_0(x)$ as n tends to infinity. $\square$

We are finally able to conclude.

Proof of Theorem 3.2 Due to Lemma 3.5, we assume the additional condition $(A4)$, and we in particular denote by $\gamma_0 > 0$ a lowerbound of γ. We consider $x_0 \in \mathbb{R}^d$ and $t > 0$ to be fixed. The proof follows closely the line of that of Theorem 2.2, so that we will only sketch it.

Step 1: Due to $(A3)(x_0)$, we may build, for each $x \in B(x_0, \epsilon)$, an increasing sequence $\{O_p(x)\}_{p\geq 1}$ of subsets of $\mathbb{R}^n$ satisfying (2.8), in such a way that for each $p \geq 1$, the map $(x, z) \mapsto \mathbf{1}_{\{z\in O_p(x)\}}$ is measurable on $B(x_0, \epsilon) \times \mathbb{R}^n$. We also consider the a.s. positive stopping time $\tau > 0$ defined by (2.9). We finally consider the stopping time, for $p \geq 1$,

$$S_p = \inf\left\{ s \geq 0;\ \int_0^s \int_{\mathbb{R}^d} \int_0^\infty \mathbf{1}_{\left\{z\in O_p\left(X^{x_0}_{(r\wedge\tau)-}\right)\right\}} \mathbf{1}_{\{u\leq\gamma_0\}} N(dr, dz, du) \geq 1 \right\}, \tag{3.31}$$

and the associated *mark* $Z_p \in \mathbb{R}^n$, uniquely defined by $N(\{S_p\} \times \{Z_p\} \times [0, \infty)) = 1$.

Due to (2.8), and to the fact that $X^{x_0}_{(u\wedge\tau)-}$ always belongs to $B(x_0, \epsilon)$, one may prove that (see the proof of Theorem 2.2 Step 1 for details)

(i) $p \mapsto S_p$ is a.s. nonincreasing,

(ii) $\lim_{p\to\infty} S_p = 0$ a.s.,

(iii) conditionally to $\mathcal{F}_{S_p-}$, the law of Z_p is given by $\frac{1}{p}\varphi(z)\mathbf{1}_{\left\{z\in O_p\left(X^{x_0}_{(S_p\wedge\tau)-}\right)\right\}} dz$.

Step 2: We now claim that conditionally to $\sigma(S_p)$, the law of $X^{x_0}_{S_p}$ has a density with respect to the Lebesgue measure on $\mathbb{R}^d$, on the set $\Omega^0_p = \{\tau \geq S_p\}$. It indeed suffices to follow line by line Step 2 of the proof of Theorem 2.2.

Step 3: We may now deduce that for any $p \geq 1$, the law of $X_t^{x_0}$ has a density on the set $\Omega_p^1 = \{S_p \leq \tau \wedge t\}$. We deduce from Step 2 that on $\Omega_p^1 \subset \Omega_p^0$ the law of $(S_p, X_{S_p}^{x_0})$ is of the shape $\nu_p(ds) f_p(s,x)dx$. Hence, for any $A \in \mathcal{A}$, using the strong Markov property, we obtain, conditionning with respect to $\mathcal{F}_{S_p}$,

$$P[\Omega_p^1, X_t^{x_0} \in A] \quad = \quad E\left[\mathbf{1}_{\Omega_p^1} E\left\{\int_0^t \nu_p(ds) \int_{\mathbb{R}^d} f_p(s,x)dx \mathbf{1}_{\{X_{t-s}^x \in A\}}\right\}\right] = 0. \tag{3.32}$$

The last inequality follows from Proposition 3.7, applied with $f_0(x) = f_p(s,x)$ for each s fixed.

Step 4: The conclusion readily follows, copying line by line Step 4 of the proof of Theorem 2.2. □

3.3 Application to some fragmentation equations

We would like to end this paper with an example of application of Theorem 3.3. We will show a regularization property for a class of fragmentation equations. We refer to [6] for details concerning this type of equations. We call *fragmentation kernel* any nonnegative symmetric function $F(x,y) = F(y,x)$ on $(0,\infty) \times (0,\infty)$. A function $c(t,x) : [0,\infty) \times (0,\infty) \mapsto [0,\infty)$, representing the *concentration* of particles with size x at time t, is said to solve the fragmentation equation if for all $t \geq 0$, all $x \in (0,\infty)$,

$$\partial_t c(t,x) = \int_x^\infty F(x, y-x)c(t,y)dy - \frac{1}{2}c(t,x)\int_0^x F(y, x-y)dy. \tag{3.33}$$

We will assume in the sequel the following assumptions on the fragmentation kernel (see Remark 3.3 of [6]).

Assumption (K)**:** $F(x,y) = \alpha(x+y)\beta(x/(x+y))$ for some C^1 functions $\alpha : (0,\infty) \mapsto [0,\infty)$ and $\beta : (0,1) \mapsto [0,\infty)$, β being symmetric at 1/2.

The conservation of mass $\int_0^\infty xc(t,x)dx = \int_0^\infty xc(0,x)dx = 1$ being expected to hold, we may rewrite (3.33) in terms of the probability measures $Q_t(dx) = xc(t,x)dx$ (see Definition 2.1 in [6]). It is shown in [6] (see Remark 2.4, Theorem 3.2, Remark 3.3, Proposition 3.8, and Remark 3.10) that the following result holds.

Proposition 3.9 *Assume (K). Consider a probability measure Q_0 on $(0,\infty)$, satisfying $\langle Q_0, x^p\rangle < \infty$ for some $p \geq 1$. Assume that $\int_0^1 z(1-z)\beta(z)dz < \infty$, that $\lim_{x\to 0} x^2\alpha(x) = 0$, while $x^2\alpha(x) \leq C(1+x^p)$ for some constant C. Then there exists a $\mathbb{R}$-valued Markov process $\{X_t\}_{t\geq 0}$ enjoying the following properties:*

(i) X is a.s. càdlàg, nonnincreasing, and takes its values in $[0,\infty)$;

(ii) the law of X_0 is given by Q_0, while its generator is given, for any $\phi \in C_b^1([0,\infty))$, any $y \in (0,\infty)$, by

$$L^F(y) = y\alpha(y)\int_0^1 [\phi(y-zy) - \phi(y)]\,(1-z)\beta(z)dz; \tag{3.34}$$

(iii) if $x^2\alpha(x) \leq C(x+x^p)$ for some constant C, then X does a.s. never reach 0, that is $P[X_t = 0] = 0$ for all $t \geq 0$;

(iv) if $x^2\alpha(x) \geq \epsilon x^\delta$ for some $\delta \in (0,1)$, some $\epsilon > 0$, then $P[X_t = 0] > 0$ for each $t > 0$;

(iv) setting $Q_t = \mathcal{L}(X_t)$ for each $t > 0$, the family $\{x^{-1}Q_t(dx)\}_{t\geq 0}$ solves (3.33) in a weak sense.

We will prove here the following regularization result, which improves consequently [6] Proposition 3.12.

Proposition 3.10 *Additionally to the hypotheses of Proposition 3.9, suppose that for all $x > 0$, $\alpha(x) > 0$, and that $\int_0^1 \beta(z)dz = \infty$.*

1. Then the law of X_t has a density with respect to $dx + \delta_0(dx)$ as soon as $t > 0$. Here dx stands for the Lebesgue measure on $\mathbb{R}$.

2. In the case where $x^2\alpha(x) \leq C(x+x^p)$ for some constant C, this implies that the law of X_t has a density with respect to dx as soon as $t > 0$. Hence the measure *weak solution $\{x^{-1}Q_t(dx)\}_{t\geq 0}$ to (3.33) becomes a* function *weak solution (starting from a* measure *initial condition).*

Proof First note that point 2 follows immediately from point 1 and Proposition 3.9-(iii). On the other hand, it clearly suffices to prove 1 when $Q_0 = \delta_{x_0}$, for some arbitrary $x_0 > 0$, by linearity.

The Markov process X taking its values in $[0,\infty)$, we just have to check that for each $\epsilon > 0$, each Lebesgue-null subset $A \subset (\epsilon,\infty)$, each $t > 0$, $P[X_t \in A] = 0$. Let thus such a couple ϵ, A be fixed.

We unfortunately can not apply Corollary 3.3 directly, since the map $\gamma(x) = x\alpha(x)$ may explode or vanish when x tends to 0, while $h(x,z) = -xz$ is degenerated when $x = 0$. We thus consider a C_b^1 strictly positive function $\gamma_\epsilon : \mathbb{R} \mapsto (0, \infty)$, and such that $\gamma_\epsilon(y) = \gamma(y)$ for all $y \in [\epsilon, x_0]$ (this is possible since γ is strictly positive and of class C^1 on $(0, \infty)$). Consider also a C_b^1 function $f_\epsilon : \mathbb{R} \mapsto (\epsilon/2, \infty)$, such that $f_\epsilon(y) = y$ for all $y \in [\epsilon, x_0]$. Finally, set $h_\epsilon(y, z) = -f_\epsilon(y)z$. Then there exists a unique Markov process X^ϵ starting from x, nonincreasing, with generator

$$L^F(y) = \gamma_\epsilon(y) \int_0^1 [\phi(y + h_\epsilon(y, z)) - \phi(y)] (1 - z)\beta(z)dz. \tag{3.35}$$

Noting that $\int_0^1 (1 - z)\beta(z)dz = \infty$ (because β is symmetric at $1/2$ and since $\int_0^1 \beta(z)dz = \infty$ by assumption), one may easily check that $(A1)$ and $(A3)(y)$ (for any $y \in \mathbb{R}$) holds for X^ϵ. Corollary 3.3 thus ensures that $P[X_t^\epsilon \in A] = 0$ for any $t > 0$.

Finally, X and X^ϵ being almost surely nonincreasing, starting both from x_0, and having the same generator for $y \in [\epsilon, x_0]$, they clearly coincide while one of them is greater than ϵ (in distribution). Since $A \subset (\epsilon, \infty)$, we deduce that $P[X_t \in A] = P[X_t^\epsilon \in A]$ for any $t > 0$. This concludes the proof. □

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