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Abstract

The purpose of this note is to point at an analog for continuous state branching process of the description of prolific individuals in a super-critical Galton-Watson process.

1 Introduction

Consider a supercritical Galton-Watson process Z with reproduction law π, so π is a probability measure on $\mathbb{Z}_+$ with $\sum_{n=0}^{\infty} n\pi(n) \in]1,\infty]$. We also assume that $\pi(0) > 0$ and write

$$g(s) := \sum_{n=0}^{\infty} s^n \pi(n)\,, \qquad s \in [0,1]$$

for the generating function of π. Then the following assertions are well-known and easy to check. To start with, the equation $g(s) = s$ has a unique root ρ in $]0,1[$, which coincides with the probability of extinction of Z when the process starts from a single ancestor. Further, splitting the graph of the generating function at (ρ,ρ) produces a pair of generating functions (see Figure 1 below) :

$$g_e(s) := \rho^{-1} g(\rho s) \tag{1}$$

and

$$g_p(s) := (1-\rho)^{-1} g(\rho + (1-\rho)s)\,, \qquad s \in [0,1]\,. \tag{2}$$

More precisely, on the one hand, g_e is the generating function of the subcritical reproduction law π_e of the Galton-Watson process Z_e which is obtained by conditioning Z to become extincted :

$$g_e(s) = \sum_{n=0}^{\infty} s^n \pi_e(n) \quad \text{with } \pi_e(n) := \rho^{n-1}\pi(n). \tag{3}$$

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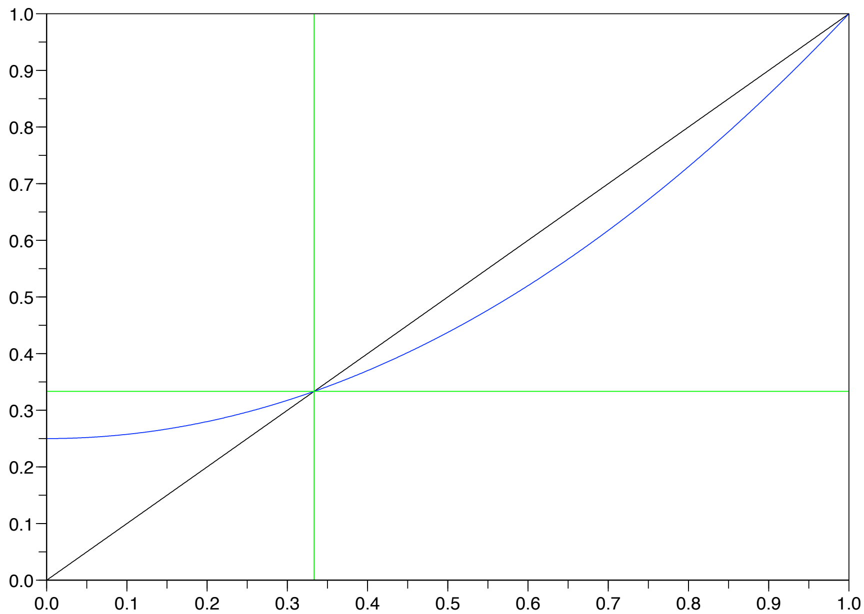

Figure 1: *Graph of the generating function* $g(s) = \frac{1}{4} + \frac{3}{4}s^2$ *splitted at* $\rho = \frac{1}{3}$*; re-scaling the lower-left part and the upper-right part yields the generating functions* g_e *and* g_p*.*

On the other hand, call *prolific* any individual with infinite descent in the Galton-Watson process. Then g_p is the generating function of the reproduction law π_p of the Galton-Watson process Z_p which is obtained by the restriction of Z to prolific individuals :

$$g_p(s) = \sum_{\ell=1}^{\infty} s^\ell \pi_p(\ell) \quad \text{with } \pi_p(\ell) := \sum_{n=\ell}^{\infty} \binom{n}{\ell} (1-\rho)^{\ell-1} \rho^{n-\ell} \pi(n) \,. \tag{4}$$

In other words, the genealogical tree induced by Z_p is distributed as that of Z after conditioning on non-extinction and removing all the finite branches.

The purpose of this note is to point at analog of these transformations in the framework of Continuous State Branching Processes (in short, CSBP). More precisely, the dynamics of a CSBP are characterized by a *branching mechanism* Ψ, which, in some loose sense, is related to the generating function g of the reproduction law for Galton-Watson processes. It is well-known that conditioning a supercritical CSBP to become (eventually) extinguished yields another CSBP whose branching mechanism Ψ_e is a simple transformation of Ψ. Our main interest here is to show that the notion of prolific individuals can also be defined for a CSBP and yields a continuous time (but discrete space) branching process whose characteristics are again expressed by simple transforms of that of the original CSBP. It will certainly not come as a surprise that a result for Galton-Watson processes possesses a counterpart in the continuous setting; however we believe that it may be interesting to spell out details. Further, in the case of stable branching mechanisms, this points at some simple path-transformations which do not seem to have been observed previously.

We refer to [5, 7] and references therein for background on CSBP and start by recalling the material that will be needed here.

2 Preliminaries

Consider a conservative CSBP $X = (X(t,a) : t \geq 0 \text{ and } a \geq 0)$, where t is the time-parameter and a the size of the initial population. This means that for each fixed $a \geq 0$, the process $X(\cdot, a)$ is a time-homogeneous Markov process with values in $\mathbb{R}_+$ started from $X(0,a) = a$. Further the fundamental branching property holds, namely for every $a, b \geq 0$, $X(\cdot, a+b) - X(\cdot, a)$ has same the law as $X(\cdot, b)$ and is independent of the family of processes $(X(\cdot, c), 0 \leq c \leq a)$.

The dynamics of X are characterized by its branching mechanism $\Psi : [0, \infty[\to \mathbb{R}$, which is a convex function of the type

$$\Psi(q) \;=\; \alpha q + \beta q^2 + \int_{]0,\infty[} \left(\mathrm{e}^{-qx} - 1 + qx\mathbf{1}_{\{x\leq 1\}}\right) \Pi(\mathrm{d}x),$$

where $\alpha \in \mathbb{R}$, $\beta \geq 0$, and Π is a measure on $]0, \infty[$ such that $\int(1 \wedge x^2)\Pi(dx) < \infty$. Specifically, the semigroup of $X(\cdot, a)$ can be characterized via its Laplace transform as follows. For every $q > 0$, we have

$$\mathbb{E}\left(\exp\{-qX(t,a)\}\right) \;=\; \exp\left\{-au_t(q)\right\}, \tag{5}$$

where the function $u_t(q)$ solves

$$\frac{\partial u_t(q)}{\partial t} \;=\; -\Psi(u_t(q)) \quad , \quad u_0(q) \;=\; q\,. \tag{6}$$

We will assume throughout this work that X is supercritical, i.e. that

$$\Psi'(0+) = \alpha - \int_{]1,\infty[} x\Pi(dx) \in [-\infty, 0[$$

and not immortal, in the sense that $\Psi(q) > 0$ when q is sufficiently large. As the branching mechanism is a convex function with $\Psi(0) = 0$, this implies that there exists a unique $q_0 > 0$ that solves the equation

$$\Psi(q_0) = 0\,.$$

We also recall that the hypothesis that X is conservative (i.e. the process $X(\cdot, a)$ does not explode in finite time a.s.) is then equivalent to $\int_{0+} |\Psi(q)|^{-1}\mathrm{d}q = \infty$ (see Grey [4]).

The importance of the role of the positive root q_0 of the branching mechanism should be already clear from the following easy consequence of (5) and (6) : For each $a \geq 0$, the process $\exp\{-q_0 X(\cdot, a)\}$ is a martingale with values in $[0, 1]$; it thus converges a.s. and it is easily seen that its limit can only take the values 0 or 1 a.s. More precisely, writing $X(\infty, a) = \lim_{t\to\infty} X(t, a)$, we have

$$\mathbb{P}(X(\infty,a) = 0) = 1 - \mathbb{P}(X(\infty,a) = \infty) = \mathrm{e}^{-q_0 a}\,, \qquad \forall a \geq 0.$$

In the sequel, we shall say that the CSBP with initial population of size a becomes *eventually extinguished* when $X(\infty, a) = 0$, and is *prolific* when $X(\infty, a) = \infty$. We mention that the process may become eventually extinguished without being ever entirely extinguished, i.e. the event that $X(\infty, a) = 0$ and $X(t, a) > 0$ for all $t \geq 0$ may have a positive probability.

In order to define rigorously prolific individuals, we turn our attention to specifying the genealogy in a CSBP, which requires the connexion with subordinators and Bochner subordination. Specifically, the branching property entails that for each fixed $t \geq 0$, the process $X(t, \cdot)$ has independent and homogeneous increments with values in $\mathbb{R}_+$. We shall always deal with its right-continuous modification which is then a subordinator. We see from (5) that its Laplace exponent is the function $q \to u_t(q)$, and the semigroup identity $u_{t+s}(q) = u_t(u_s(q))$ points at the following representation (see Proposition 1 in [2] for details).

Lemma 1 *On some probability space, there exists a process* $(S^{(s,t)}(a), 0 \leq s \leq t$ *and* $a \geq 0)$ *such that:*
(i) *For every* $0 \leq s \leq t$, $S^{(s,t)} = \left(S^{(s,t)}(a), a \geq 0\right)$ *is a subordinator with Laplace exponent* $u_{t-s}(\cdot)$.
(ii) *For every integer* $p \geq 2$ *and* $0 \leq t_1 \leq \cdots \leq t_p$, *the subordinators* $S^{(t_1,t_2)}, \ldots, S^{(t_{p-1},t_p)}$ *are independent and*

$$S^{(t_1,t_p)}(a) \;=\; S^{(t_{p-1},t_p)} \circ \cdots \circ S^{(t_1,t_2)}(a)\,, \quad \forall a \geq 0 \quad a.s.$$

(iii) *The processes* $(S^{(0,t)}(a), t \geq 0$ *and* $a \geq 0)$ *and* $(X(t,a), t \geq 0$ *and* $a \geq 0)$ *have the same finite-dimensional marginals.*

For the sake of simplicity, we shall further assume from now on that

$$\beta > 0 \quad \text{or} \quad \int_{]0,\infty[} (1 \wedge x)\Pi(\mathrm{d}x) = \infty,$$

in order to ensure that the subordinators $S^{(s,t)}$ are pure jump processes (i.e. they have no drift); see Silverstein [9]. Analyzing their jumps in the framework of the representation above yields a natural notion of genealogy of CSBP (we refer to [2] for details) : For every $b, c \geq 0$ and $0 \leq s < t$, we say that the individual c in the population at time t has ancestor (or is a descendant of) the individual b in the population at time s if b is a jump time of $S^{(s,t)}$ and

$$S^{(s,t)}(b-) \;<\; c \;<\; S^{(s,t)}(b).$$

Note that when the subordinator $S^{(s,t)}$ has a jump at the location b, then the size of this jump $\Delta S^{(s,t)}(b) = S^{(s,t)}(b) - S^{(s,t)}(b-)$ describes the size of the sub-population at time t which descends from the individual b in the population at time s. Considering the limit as $t \to \infty$, this enables us to define prolific individuals.

Definition. *For every* $b \geq 0$ *and* $s \geq 0$, *we say that the individual* b *in the population at time* s *is* prolific *if*

$$\lim_{t\to\infty} \Delta S^{(s,t)}(b) \;=\; \infty.$$

For every $a \geq 0$ and $s \geq 0$, we then introduce the number of prolific individuals in the population at time s which descend from the initial population $[0, a]$ of size a:

$$P(s, a) := \mathrm{Card}\{b \in [0, X(s, a)] : b \text{ is a prolific in the population at time } s\}\,.$$

We point that there are prolific individuals in the initial population if and only if the CSBP is prolific. This is certainly not surprising, but it deserves however a rigorous argument which has some importance in this study.

Lemma 2 *For every initial population $a \geq 0$, the events*

$$\{X(\cdot, a) \text{ becomes eventually extinguished}\}$$

and

$$\{P(0, a) = 0\}$$

coincide a.s. Furthermore, $P(0, a)$ has the Poisson distribution with parameter aq_0.

Proof: The inclusion

$$\{X(\cdot, a) \text{ becomes eventually extinguished}\} \subseteq \{P(0, a) = 0\}$$

is obvious, so we just need to check that the probability of the two events coincide.

Fix an arbitrary time $t > 0$ and focus on the evolution of the initial population $[0, a]$. The fact that the subordinator $S^{(0,t)}$ is pure jump means that almost all the population at time t descends from at most countably many individuals in the initial population. More precisely, denote by $(a_i)_{i \in I}$ the set of jump location of $S^{(0,t)}(\cdot)$ on $[0, a]$, so $\Delta S^{(0,t)}(a_i)$ is the size of the sub-population at time t having a_i as ancestor, and

$$\sum_{i \in I} \Delta S^{(0,t)}(a_i) = X(t, a)\,.$$

Since for every $t' > t$, the pure jump subordinator $S^{(t,t')}$ is independent of $S^{(0,t)}$ and $S^{(0,t')} = S^{(t,t')} \circ S^{(0,t)}$, we see that the ancestors in the population at time t of the almost entire population at time t' descend from the individuals $(a_i)_{i \in I}$. As a consequence, any prolific individual in the initial population belongs to the set of ancestors $(a_i)_{i \in I}$. By applying the branching property at time t, we get that the conditional probability given the evolution of the process up-to time t that the individual a_i is prolific equals $1 - \exp\{-q_0 \Delta S^{(0,t)}(a_i)\}$, and for different indices i, these events are (conditionally) independent. Thus

$$\begin{aligned}
\mathbb{P}\left(P(0, a) = 0\right) &= \mathbb{E}\left(\prod_{i \in I} \exp\{-q_0 \Delta S^{(0,t)}(a_i)\}\right) \\
&= \mathbb{E}\left(\exp\left\{-q_0 \sum_{i \in I} \Delta S^{(0,t)}(a_i)\right\}\right) \\
&= \mathbb{E}\left(\exp\{-q_0 X(t, a)\}\right) \\
&= \exp\{-q_0 a\} \\
&= \mathbb{P}\left(X(\cdot, a) \text{ becomes eventually extinguished}\right).
\end{aligned}$$

This shows the first assertion. Finally the branching property entails that the process $P(0,\cdot)$ is Poisson, and since $\mathbb{P}\left(P(0,a)=0\right)=\mathrm{e}^{-q_0 a}$, its intensity is q_0. $\square$

Remark. An application of the Markov property shows that conditionally on $X(t,a)=b$, the number of prolific individuals at time t has the Poisson law with parameter $q_0 b$. By the law of large number for the Poisson laws, we deduce that conditionally on the event that $X(\cdot,a)$ is prolific, we have $P(t,a)\sim q_0 X(t,a)$ as $t\to\infty$.

3 Main results

Fix $a>0$ and introduce the probability measure

$$\mathbb{P}_e=\mathrm{e}^{q_0 a}\mathbf{1}_{\{X(a,\infty)=0\}}\mathbb{P}$$

which is obtained by conditioning the CSBP with initial population of size a to become eventually extinguished. Observe that on the sigma-field $\mathcal{F}_t=\sigma(X(r,a):0\le r\le t)$, $\mathbb{P}_e$ is absolutely continuous with respect to the initial probability measure $\mathbb{P}$ with density given by the martingale $\mathrm{e}^{q_0 a}\exp\{-q_0 X(t,a)\}$.

We now have all the material needed to state and prove the main results of this note. First, let us present the continuous time analogue of the interpretation of the component g_e for Galton-Watson processes, which belongs to the folklore of CSBP.

Proposition 3 *Under $\mathbb{P}_e$, $X(\cdot,a)$ is a CSBP with initial population of size a. Its branching mechanism is given by*

$$\Psi_e(q)\;=\;\Psi(q_0+q)\,,\qquad q\ge 0$$

and can be expressed in the form

$$\Psi_e(q)\;=\alpha_e q+\beta q^2+\int_{]0,\infty[}\left(\mathrm{e}^{-qx}-1+qx\mathbf{1}_{\{x\le 1\}}\right)\Pi_e(\mathrm{d}x),$$

where

$$\Pi_e(\mathrm{d}x)=\mathrm{e}^{-q_0 x}\Pi(\mathrm{d}x)$$

and

$$\alpha_e=\alpha+2\beta q_0+q_0\int_{]0,\infty[}\left(1-\mathrm{e}^{-q_0 x}\right)x\mathbf{1}_{\{x\le 1\}}\Pi(\mathrm{d}x).$$

More generally, we point out how a simple modification of the law $\mathbb{P}_e$ of the branching process $X(\cdot,a)$ conditioned to become eventually extinguished, enables us to describe the conditional distribution of $X(t,a)$ on the number of prolific individuals $P(t,a)$ at a fixed time $t>0$.

Proposition 4 *For every $a,t\ge 0$ and $n\in\mathbb{Z}_+$, the conditional law of $X(t,a)$ given $P(t,a)=n$ is*

$$\mathbb{P}(X(t,a)\in\mathrm{d}x|P(t,a)=n)=x^n\frac{\mathbb{P}_e(X(t,a)\in\mathrm{d}x)}{\mathbb{E}_e(X(t,a)^n)}.$$

Proof: To start with, Remark following Lemma 2 yields the identity

$$\begin{aligned}\mathbb{E}(\exp\{-qX(t,a)\}s^{P(t,a)}) &= \mathbb{E}(\exp\{-qX(t,a)\}\exp\{-q_0(1-s)X(t,a)\}) \\ &= \mathbb{E}(\exp\{-(q+q_0)X(t,a)\}\exp\{q_0 sX(t,a)\}) \\ &= \sum_{n=0}^{\infty}\frac{(sq_0)^n}{n!}\mathbb{E}(\exp\{-(q+q_0)X(t,a)\}X(t,a)^n).\end{aligned}$$

Next, we define

$$f(t,q,a,n) := \mathbb{E}(\exp\{-qX(t,a)\}|P(t,a)=n)\,.$$

Using again the remark after Lemma 2, but conditioning first on $P(t,a)$ and then on $X(t,a)$ we get that

$$\mathbb{E}(\exp\{-qX(t,a)\}s^{P(t,a)}) = \sum_{n=0}^{\infty}\frac{(sq_0)^n}{n!}f(t,q,a,n)\mathbb{E}(\exp\{-q_0X(t,a)\}X(t,a)^n).$$

We deduce

$$\begin{aligned}f(t,q,a,n) &= \frac{\mathbb{E}(\exp\{-(q+q_0)X(t,a)\}X(t,a)^n)}{\mathbb{E}(\exp\{-q_0X(t,a)\}X(t,a)^n)} \\ &= \frac{\mathbb{E}_e(\exp\{-qX(t,a)\}X(t,a)^n)}{\mathbb{E}_e(X(t,a)^n)}\,,\end{aligned}$$

where in the second identity, we made use of the fact that on the sigma-field $\mathcal{F}_t = \sigma(X(r,a) : 0 \le r \le t)$, the probability measure $\mathbb{P}_e$ for the CSBP conditioned to become eventually extinguished is absolutely continuous with respect to the initial probability measure $\mathbb{P}$ with density $e^{q_0 a}\exp\{-q_0X(t,a)\}$. Inverting the Laplace transform (in the variable q) yields the formula of the statement. $\square$

Recall that with any probability law m on $\mathbb{R}_+$ with finite non-zero mean, one can associate the law $\overline{\mathrm{m}}$ of its size-biased picking, defined by

$$\overline{\mathrm{m}}(dy) = \frac{y}{c}\,\mathrm{m}(dy)$$

with $c = \int_0^\infty y\mathrm{m}(dy)$. We may then note the following recursive identity : for every $n \in \mathbb{Z}_+$, the law $\mathcal{L}(X(t,a)|P(t,a)=n+1))$ is obtained from $\mathcal{L}(X(t,a)|P(t,a)=n))$ by size biased picking.

In order to state the main result of this note, we first recall some further well-known material (see, e.g. Chapter III in [1], or [6]). A continuous time branching process $Z = (Z(t,k) : t \ge 0, k \in \mathbb{Z}_+)$, where t is the time parameter and k the number of ancestors, can be viewed as a Galton-Watson process in which individuals have independent exponentially distributed lifetimes. The rate of reproduction is governed by a finite measure μ on $\mathbb{Z}_+$ with $\mu(1) = 0$. Specifically, each individual lives for an exponential time with parameter $\mu(\mathbb{Z}_+)$ and begets at its death a random number of children which is distributed according to the normalized probability measure $\mu(\cdot)/\mu(\mathbb{Z}_+)$ (that coincides with the reproduction law of the underlying Galton-Watson process). Thus for each $k \in \mathbb{Z}_+$,

$Z(\cdot, k)$ is a Markov chain in continuous time, whose dynamics are entirely characterized by the reproduction measure μ. In turn, the latter is determined by the function

$$\Phi(s) := \sum_{n=0}^{\infty}(s^n - s)\mu(n)\,, \qquad s \in [0,1]\,. \tag{7}$$

More precisely the branching property entails that the generating function of $Z(t,k)$ has the form

$$\mathbb{E}(s^{Z(t,k)}) = \gamma_t(s)^k\,, \qquad s \in [0,1], k \in \mathbb{Z}_+\,, \tag{8}$$

and solves

$$\frac{\partial \gamma_t(s)}{\partial t} = \Phi(\gamma_t(s))\,. \tag{9}$$

In the case when $\mu(0) = 0$, we say that Z is immortal as each individual has at least two children a.s.

Theorem 5 *For every $a \geq 0$, the process $P(\cdot, a)$ is an immortal branching process in continuous time, with initial distribution given by the Poisson law with parameter $q_0 a$. Its reproduction measure μ_p is characterized in terms of the branching mechanism of X by*

$$\Phi_p(s) = \sum_{n=0}^{\infty}(s^n - s)\mu_p(n) = \frac{1}{q_0}\Psi(q_0(1-s))\,, \qquad s \in [0,1]\,, \tag{10}$$

and is given explicitly by

$$\mu_p(n) = q_0^{n-1}\int_{]0,\infty[}\frac{x^n}{n!}\mathrm{e}^{-q_0 x}\Pi(\mathrm{d}x) \quad \textit{for } n \geq 3\,, \tag{11}$$

and

$$\mu_p(2) = \beta q_0 + q_0\int_{]0,\infty[}\frac{x^2}{2}\mathrm{e}^{-q_0 x}\Pi(\mathrm{d}x)\,.$$

Figure 2 below depicts the transformation $\Psi \to (\Psi_e, \Phi_p)$ and should be compared to Figure 1 for generating functions.

Proof: The proof of assertion that the process $P(\cdot, a)$ of the number of prolific individuals is a branching process in continuous time follows the same route as Galton-Watson processes by using the argument in the proof of Lemma 2. That this branching process is immortal is obvious. Lemma 2 also states that its initial distribution is the Poisson law with parameter aq_0.

Let us now compute the generating function of its semigroup. Recall that the generating function of the Poisson distribution with parameter $c \geq 0$ is $s \to \exp\{-(1-s)c\}$. On the one hand, combining Lemma 2 with the Markov property at time t yields

$$\begin{aligned}\mathbb{E}(s^{P(t,a)}) &= \mathbb{E}\left(\exp\{-(1-s)q_0 X(t,a)\}\right)\\ &= \exp\{-a u_t((1-s)q_0)\}\end{aligned}$$

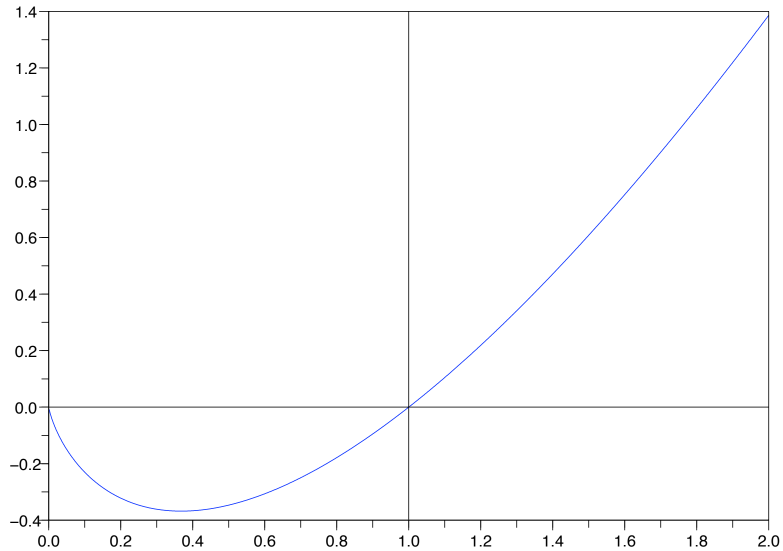

Figure 2: *Graph of the branching mechanism* $\Psi(q) = q \ln q$ *splitted at* $q_0 = 1$. *The right-part gives the graph of* Ψ_e *and the symmetric of the left-part that of* Φ_p.

On the other hand, using the fact that $P(0,a)$ has the Poisson distribution with parameter aq_0 yields that the generating function γ_t of the continuous time branching process fulfills

$$\begin{aligned} \mathbb{E}(s^{P(t,a)}) &= \mathbb{E}\left(\gamma_t(s)^{P(0,a)}\right) \\ &= \exp\{-aq_0(1-\gamma_t(s))\}. \end{aligned}$$

We deduce from these two observations that

$$1 - \gamma_t(s) = \frac{1}{q_0} u_t((1-s)q_0)\,.$$

Taking the derivative in the variable t yields by (9) and (6)

$$\Phi_p(\gamma_t(s)) = \frac{1}{q_0}\Psi(u_t((1-s)q_0)) = \frac{1}{q_0}\Psi(q_0(1-\gamma_t(s)))$$

We conclude that

$$\Phi_p(s) = \frac{1}{q_0}\Psi(q_0(1-s))\,.$$

Finally one recovers the measure μ_p by inverting of the transform Φ_p. This can be performed by combining Proposition 3 and the observation that $\Psi(q_0(1-s)) = \Psi_e(-q_0 s)$. □

In the setting of super-critical Galton-Watson processes, we can assign a type to each individual depending on whether it has finite descent or is prolific, and this yields two-type Galton-Watson processes. A similar observation can be made in the continuous setting; in this direction, recall that for every fixed $t \geq 0$, there are only countably many prolific individuals at time t, which thus do not contribute to the size of the population at time t (but of course the descent of prolific individuals at time t may have a crucial role in the size of the population at time $t' > t$). Then one can check that the pair $((X(t,a), P(t,a)) : a, t \geq 0)$ also enjoys the branching property. More precisely, for every $a \geq 0$ and $n \in \mathbb{Z}_+$, let us write $(X(\cdot, a, n), P(\cdot, a, n))$ for a version of the pair of processes $(X(t,a), P(t,a))_{t \geq 0}$ conditioned on $P(0,a) = n$. Then for every $a, a' \geq 0$ and $n, n' \in \mathbb{Z}_+$, there is the identity in distribution

$$\begin{aligned} & (X(\cdot, a+a', n+n'), P(\cdot, a+a', n+n')) \\ \stackrel{\mathcal{L}}{=} \; & (X(\cdot, a, n), P(\cdot, a, n)) + (X'(\cdot, a', n'), P'(\cdot, a', n')) \,, \end{aligned}$$

where, in the right-hand side, $(X'(\cdot, a', n'), P'(\cdot, a', n'))$ is independent of $(X(\cdot, a, n), P(\cdot, a, n))$ and has the same law as $(X(\cdot, a', n'), P(\cdot, a', n'))$.

For $n = 0$, $X(\cdot, a, 0)$ is just a version of the initial CSBP with an initial population of size a and conditioned to become eventually extinguished (i.e. with branching mechanism $\Psi_e(q) = \Psi(q + q_0)$), and obviously $P(\cdot, a, 0) \equiv 0$. This yields

$$\mathbb{E}(\exp\{-qX(t,a,0)\} s^{P(t,a,0)}) = \exp\left\{-a(u_t(q+q_0) - q_0)\right\} \,, \tag{12}$$

for all $q \geq 0$ and $s \in]0,1]$, where $u_t(\cdot)$ is defined as in (6). Next, from recall from the remark following Lemma 2 that there is the identity

$$\begin{aligned} \mathbb{E}(\exp\{-qX(t,a)\} s^{P(t,a)}) &= \mathbb{E}(\exp\{-(q + q_0(1-s))X(t,a)\}) \\ &= \exp\{-a u_t(q + q_0(1-s))\}. \end{aligned} \tag{13}$$

On the other hand, since $P(0,a)$ has the Poisson distribution with parameter aq_0, the branching property enables us to express the preceding quantity as

$$\begin{aligned} & \mathbb{E}(\exp\{-qX(t,a)\} s^{P(t,a)}) \\ = \; & \sum_{n=0}^{\infty} \mathrm{e}^{-aq_0} \frac{(aq_0)^n}{n!} \mathbb{E}(\exp\{-qX(t,a,0)\} s^{P(t,a,0)}) \left(\mathbb{E}(\exp\{-qX(t,0,1)\} s^{P(t,0,1)})\right)^n \end{aligned}$$

Using (12) and considering the asymptotic when $a \to 0$ easily yield

$$\mathbb{E}(\exp\{-qX(t,0,1)\} s^{P(t,0,1)}) = \frac{1}{q_0}(u_t(q+q_0) - u_t(q + q_0(1-s))) \,. \tag{14}$$

Putting the pieces together, we conclude that the joint law of $(X(t,a,n), P(t,a,n))$ is characterized by

$$\begin{aligned} & \mathbb{E}(\exp\{-qX(t,a,n)\} s^{P(t,a,n)}) \\ = \; & \exp\left\{-a(u_t(q+q_0) - q_0)\right\} \left(\frac{1}{q_0}(u_t(q+q_0) - u_t(q+q_0(1-s)))\right)^n \,. \end{aligned} \tag{15}$$

4 Some examples

We shall now present some examples in which explicit computations are possible. The third one will point at a path-transformation relating strictly stable CSPB to some supercritical CSBP which may be new.

Example 1 (Quadratic branching). The simplest example is when $\Psi(q) = q^2 - q$, so $\Pi = 0$, $\beta = 1$ and $\alpha = -1$. Then we get $q_0 = 1$ and $\Phi_p(s) = s^2 - s$, which yields $\mu_p = \delta_2$. We conclude that as time passes, the number of prolific individuals evolves as a standard Yule process.

Example 2 (Neveu's branching). Next, we consider Neveu's branching process [8, 2] which has branching mechanism $\Psi(q) = q \ln(q)$. Then $q_0 = 1$ and

$$\Phi_p(s) = (1-s)\ln(1-s) = \sum_{n=2}^{\infty} \frac{s^n - s}{n(n-1)}\,.$$

We thus obtain $\mu_p(n) = \frac{1}{n(n-1)}$ for every $n \geq 2$. As a check, recall that Neveu's branching process has no Gaussian component and that its Lévy measure is $\Pi(\mathrm{d}x) = x^{-2}\mathrm{d}x$, and thus we recover from Equation (11) that for $n \geq 2$

$$\mu_p(n) = \int_0^\infty \frac{x^n}{n!} \mathrm{e}^{-x} x^{-2} \mathrm{d}x = \frac{(n-2)!}{n!} = \frac{1}{n(n-1)}\,.$$

We also point out that $u_t(q) = q^{\mathrm{e}^{-t}}$, and thus

$$\gamma_t(s) = 1 - (1-s)^{\mathrm{e}^{-t}}\,, \qquad s \in [0,1]\,.$$

Example 3 (Stable branching). Next consider the supercritical stable branching mechanism $\Psi(q) = \Gamma(-\vartheta)(q^\vartheta - q)$, so that $q_0 = 1$ and $\Pi(\mathrm{d}x) = x^{-\vartheta-1}\mathrm{d}x$, and

$$\mu_p(n) = \int_0^\infty \frac{x^n}{n!} \mathrm{e}^{-x} x^{-\vartheta-1} \mathrm{d}x = \frac{\Gamma(n-\vartheta)}{n!} \qquad n \geq 2.$$

It is easily checked that the total mass of μ_p is

$$\mu_p(\mathbb{Z}_+) = \Gamma(2-\vartheta)/\vartheta = (\vartheta - 1)\Gamma(-\vartheta),$$

so the normalized probability measure $\mu_p(\cdot)/\mu_p(\mathbb{Z}_+)$ is given by

$$\frac{\mu_p(n)}{\mu_p(\mathbb{Z}_+)} = \frac{\vartheta(2-\vartheta)\cdots(n-1-\vartheta)}{n!} := \nu_\vartheta(n)\,, \qquad n \geq 2.$$

The reproduction law $(\nu_\vartheta(n))_{n\geq 2}$ in the third example already appeared at the bottom of page 74 in Duquesne and Le Gall [3] (see also Section 7 in [6]), which points at a rather surprising connexion with *strictly* stable trees reduced at some *finite* level. More precisely, Duquesne and Le Gall (see Sections 2.6 and 2.7 in [3]) were interested in the limit of certain reduced critical Galton-Watson trees observed up-to some large generation. Following Theorem 2.7.1 in [3], we consider a time-inhomogeneous Markov process $(Z^1_t)_{0\le t<1}$ with values in $\mathbb{N}$, which models the evolution of a population with the following dynamics. The death-time of an individual which is alive at time $t \in [0,1[$ has the uniform distribution on $[t,1]$, and at its death, this individual begets a random number of children distributed according to the reproduction law ν_ϑ, independently of the death-time. Further, different individuals evolve independently one of the others. Heuristically, the quantity Z^1_t can be interpreted as the number of individuals at time t which have a non-zero descent at time 1 in a strictly stable(ϑ)-CSBP, i.e. with branching mechanism $\Psi_\vartheta(q) := cq^\vartheta$ where $c > 0$ is arbitrary.

On the other hand, recall that a random variable $\mathbf{e}$ which has the exponential distribution with parameter $c > 0$ enjoys the property of absence of memory, and further $1-\exp\{-c\mathbf{e}\}$ is then uniformly distributed on $[0,1]$. Putting these observations together, we now realize that if Z^1 starts with a number of ancestors distributed according to the Poisson law with parameter a, then the time-changed process

$$t \to Z^1_{1-\exp\{-(\vartheta-1)\Gamma(-\vartheta)t\}}$$

is a version of the process $(P(t,a))_{t\ge0}$ of the number of prolific individuals for a CSBP with branching mechanism $\Psi(q) = \Gamma(-\vartheta)(q^\vartheta - q)$ and started from an initial population of size a.

We now conclude this work by providing a direct explanation for the preceding relation, which is based on the following simple transformation of strictly stable CSBP.

Proposition 6 *Let $(Y(t,a) : t \ge 0$ and $a \ge 0)$ be a strictly stable CSBP with branching mechanism $\Psi_\vartheta(q) = cq^\vartheta$, where $c > 0$ and $\vartheta \in]1,2]$, and fix $b > 0$. Then the process*

$$\widetilde{Y}(t,a) := \mathrm{e}^{bt} Y(1-\mathrm{e}^{-b(1-\vartheta)t}, a), \qquad t \ge 0 \text{ and } a \ge 0$$

is a CSBP with branching mechanism

$$\widetilde{\Psi}_\vartheta(q) = bc(\vartheta-1)q^\vartheta - bq\,.$$

This provides a pathwise proof the identity in distribution which was observed above. Indeed, we choose $b = \Gamma(-\vartheta)$ and $c = 1/(\vartheta-1)$ so that $\widetilde{\Psi}_\vartheta(q) = \Gamma(-\vartheta)(q^\vartheta - q)$. Then it suffices to observe that $Z^1_{1-\exp\{-(\vartheta-1)\Gamma(-\vartheta)t\}}$, the number of individuals at time $1-\mathrm{e}^{-b(1-\vartheta)t}$ which have a non-zero descent at time 1 in the strictly stable CSBP Y, coincides with the number of prolific individuals at time t in the supercritical CSBP $\widetilde{Y}$. For this, one has to use the feature that, since $\int^\infty \mathrm{d}q/\widetilde{\Psi}_\vartheta(q) < \infty$, the following equivalence holds with probability one :

$$\widetilde{Y}(t,a) = 0 \text{ when } t \text{ is sufficiently large} \iff \lim_{t\to\infty} \widetilde{Y}(t,a) = 0.$$

In other words, when the CSBP $\widetilde{Y}$ becomes eventually extinguished, it must become entirely extinguished at some finite time. See for instance the exercise in [7] on its page 28.

Proof: Let us write

$$v_t(q) = -\ln \mathbb{E}(\exp\{-qY(t,1)\})$$

for the solution to the equation (6) for the branching mechanism $\Psi_\vartheta(q) = cq^\vartheta$. This equation can be solved explicitly and one finds

$$v_t(q) = ((\vartheta - 1)ct + q^{1-\vartheta})^{1/(1-\vartheta)}, \qquad q > 0.$$

It is immediate to check that the transformed process $\widetilde{Y}$ is a (possibly time-inhomogeneous) Markov process that enjoys the branching property. The identity (5) yields

$$\mathbb{E}(\exp\{-q\widetilde{Y}(t,a)\}) = \mathbb{E}(\exp\{-q\mathrm{e}^{bt}Y(1-\mathrm{e}^{-b(1-\vartheta)t}, a)\}) = \exp\{-au_t(q)\},$$

with

$$\begin{aligned} u_t(q) &= v_{1-\mathrm{e}^{-b(\vartheta-1)t}}(q\mathrm{e}^{bt}) \\ &= \left((\vartheta-1)c(1-\mathrm{e}^{-b(\vartheta-1)t}) + q^{1-\vartheta}\mathrm{e}^{-b(\vartheta-1)t})\right)^{1/(1-\vartheta)}. \end{aligned}$$

Taking the derivative with respect to t, we obtain

$$\begin{aligned} \frac{\partial u_t(q)}{\partial t} &= \frac{1}{1-\vartheta}\left(b(\vartheta-1)^2c\mathrm{e}^{-b(\vartheta-1)t} - b(\vartheta-1)q^{1-\vartheta}\mathrm{e}^{-b(\vartheta-1)t})\right) u_t(q)^\vartheta \\ &= -b\left((\vartheta-1)c - u_t(q)\right)u_t(q)^\vartheta. \end{aligned}$$

We thus see that $u_t(q)$ solves

$$\frac{\partial u_t(q)}{\partial t} = -\widetilde{\Psi}_\vartheta(u_t(q)), \; u_0(q) = q,$$

and as in this PDE, the function $\widetilde{\Psi}_\vartheta$ does not depend on t, this ensures that $\widetilde{Y}$ has in fact the time-homogeneous branching property. More precisely, $\widetilde{Y}$ is a CSBP with branching mechanism $\widetilde{\Psi}_\vartheta$. □

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