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Local Euler-Maclaurin expansion of Barvinok valuations and Ehrhart coefficients of a rational polytope

Velleda Baldoni, Nicole Berline and Michèle Vergne

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1 Introduction

Let $p$ be a rational polytope in $V = \mathbb{R}^d$ and $h(x)$ a polynomial function on $V$. A classical problem in Integer Programming is to compute the sum of values of $h(x)$ over the set of integral points of $p$,

$$S(p, h) = \sum_{x \in p \cap \mathbb{Z}^d} h(x).$$

When $p$ is dilated by an integer $n \in \mathbb{N}$, we obtain a function of $n$ which is quasi-polynomial, the so-called Ehrhart quasi-polynomial of the pair $(p, h)$

$$S(np, h) = \sum_{m=0}^{d+N} E_m(p, h, n)n^m$$

of degree $d + N$ where $N = \deg h$. The coefficients $E_m(p, h, n)$ are periodic functions of $n \in \mathbb{N}$, with period the smallest integer $q$ such that $qp$ is a lattice polytope.

Replacing $h(x)$ by an exponential, we are led to study the analytic function on $V^*$ defined by

$$S(p)(\xi) = \sum_{x \in p \cap \mathbb{Z}^d} e^{\langle \xi, x \rangle}.$$

If $p$ is any rational polyhedron, this sum still makes sense as a meromorphic function defined near 0 and the map $p \mapsto S(p)(\xi)$ is a valuation.
In [4], we proved that the meromorphic function \( S(p)(\xi) \) has a local Euler-Maclaurin expansion
\[
S(p)(\xi) = \sum f(\xi) \int e^\langle \xi, x \rangle dx.
\]

The sum is taken over the set of faces \( f \) of the polyhedron \( p \). For each face \( f \), the function \( \mu(t(p, f))(\xi) \) is holomorphic near 0, and it depends only on the transverse cone \( t(p, f) \) of \( p \) along \( f \). More precisely, once a rational scalar product is chosen on \( V \), we define canonically a map \( a \mapsto \mu(a) \) from the set of rational affine cones \( a \) in quotient spaces of \( V \), with values in the space of functions on \( V \) which are analytic near 0, then we prove that these functions satisfy the above formula. The map \( a \mapsto \mu(a) \) is invariant under lattice translations, equivariant with respect to lattice preserving isometries, and it is a valuation on the set of affine cones with a fixed vertex ([4], Theorems 17 and 18).

It is easy to see that the Ehrhart quasi-polynomial can be computed in terms of the Taylor coefficients of the functions \( \mu(t(p, f))(\xi) \). For example, if \( p \) is a lattice polytope and \( h(x) = 1 \), we have ([4], Corollary 28)
\[
\text{Card}(np \cap \mathbb{Z}^d) = \sum f(\xi) \int \xi \int_a \text{vol}(f)n^{\dim f}.
\]

Using the valuation property of \( \mu(a) \) and Barvinok’s decomposition of a cone into unimodular cones, we thus obtained in [4] an algorithm for computing the Ehrhart quasi-polynomial. It has polynomial length with respect to the input \((p, h)\), when the dimension \( d \) and the degree \( N \) are fixed.

The valuation \( S(p, h) \) has a natural generalization used by Barvinok in [2], the mixed valuation \( S^L(p, h) \), where \( L \subseteq V \) is a rational vector subspace. Denote the projected lattice on \( V/L \) by \( \Lambda_L \). For a polytope \( p \subset V \) and a polynomial \( h(x) \)
\[
S^L(p, h) = \sum_\gamma \int_{p \cap (\gamma + L)} h(x) dx.
\]

In other words, the polytope \( p \) is sliced along lattice affine subspaces parallel to \( L \) and the integrals of \( h \) over the slices are added up. For \( L = V \), there is only one term and \( S^V(p, h) \) is just the integral of \( h(x) \) over \( p \), while, for \( L = \{0\} \), we recover \( S(p, h) \), the sum of values of \( h(x) \) over the set of integral points of \( p \).
In the case \( h(x) = 1 \), we write \( S(p) \) and \( S^L(p) \) in place of \( S(p, 1) \) and \( S^L(p, 1) \).

Using these mixed valuations, Barvinok gave an algorithm which computes the \( r + 1 \) highest degree Ehrhart coefficients of \( S(np) = \text{Card}(np \cap \mathbb{Z}^d) \), when \( p \) is a simplex in \( \mathbb{R}^d \). Barvinok’s algorithm has polynomial length when \( d \) is an input, provided \( r \) is fixed. The method consists in reducing the problem to summations over lattice points in dimension \( \leq r \).

Barvinok considers particular linear combinations

\[
\sum_{L \in \mathcal{L}} \rho(L)S^L(p),
\]

where \( \mathcal{L} \) is a finite set of rational vector subspaces of \( V \) which is closed under sum, and the coefficients \( \rho(L) \) are integers which satisfy the following relation between characteristic functions:

\[
\chi(\bigcup_{L \in \mathcal{L}} L^\perp) = \sum_{L \in \mathcal{L}} \rho(L)\chi(L^\perp),
\]

where \( L^\perp \subseteq V^* \) is the orthogonal of \( L \). We call a function \( \mathcal{L} \to \mathbb{Z} \) with this property a patchwork function on \( \mathcal{L} \).

When \( p \) is dilated by an integer \( n \), \( S^L(np) \) is again given by a quasi-polynomial in \( n \), as is a linear combination

\[
\sum_{L \in \mathcal{L}} \rho(L)S^L(np) = \sum_{m=0}^{d} \nu_m(p, n)n^m.
\]

The main theoretical result of [2], Theorem (1.3), is the following: if \( \mathcal{L} \) is a family of subspaces which is closed under sum and contains the vector subspace \( \text{lin}(f) \) parallel to \( f \), for every face \( f \) of codimension \( \leq r \) of \( p \), and if \( \rho \) is a patchwork function on \( \mathcal{L} \), then the \( r + 1 \) highest degree coefficients \( \nu_m(p, n) \), for \( m = d, \ldots, d - r \), are equal to the corresponding Ehrhart coefficients of \( S(np) = \text{Card}(np \cap \mathbb{Z}^d) \).

In the present article, we introduce the meromorphic functions which extend \( S^L(p) \). For any polyhedron \( p \),

\[
S^L(p)(\xi) = \sum_{y \in \Lambda \cap L} \int_{p \cap (y + L)} e^{\langle \xi, x \rangle} dx
\]
is defined as a meromorphic function near $\xi = 0$. We show that $S^L(p)(\xi)$ also enjoys a local Euler-Maclaurin expansion (Theorem 8),

$$S^L(p)(\xi) = \sum_{j} \mu^L(t(p,f))(\xi) \int_{f} e^{\langle \xi,x \rangle} dx.$$  

Furthermore, for a linear combination of Barvinok type, if $f$ is a face of $p$ such that $\text{lin}(f) \in \mathcal{L}$, we prove that the $f$-term in the Euler-Maclaurin expansions of

$$S^{L,f}(p)(\xi) = \sum_{L \in \mathcal{L}} \rho(L)S^L(p)(\xi)$$

and of the usual valuation $S(p)(\xi)$ are equal (Theorem 17):

$$\sum_{L \in \mathcal{L}} \rho(L)\mu^L(t(p,f))(\xi) = \mu(t(p,f))(\xi).$$

This is the main result of the present article. From the relation between Ehrhart quasi-polynomials and Euler-Maclaurin expansions, it implies Barvinok’s Theorem (1.3).

Actually, we derive from Theorem 17 another computation of the $r + 1$ highest coefficients of the Ehrhart quasi-polynomial for a pair $(p,h)$, based on Brion’s decomposition of a polytope into cones, in the line of [1] and [6].

For each vertex $s$ of $p$, let $c_s$ be the cone of feasible directions of $p$ at $s$. Instead of the full family $\mathcal{L}$ generated by taking sums of the subspaces $\text{lin}(f)$, when $f$ runs over the set of faces of codimension $\leq r$ of the polytope $p$, we consider, for each vertex $s$ of $p$, the family $\mathcal{L}_s$ generated by faces of $c_s$ of codimension $\leq r$. The point in taking a family which depends on $s$ lies in the case where $p$ is simplicial. Then $\mathcal{L}_s$ consists only of the spaces $\text{lin}(f)$ where $f$ is a face of $c_s$ of codimension $\leq r$, as this set is already closed under sum. Moreover the coefficients $\rho(L)$ are just signed binomial coefficients (Lemma 15), and the computation of $S^L(c_s)$ is easier when $L$ is parallel to a face of $c_s$ (Example 3).

Let us describe our method in the simpler case of a lattice polytope $p$ and polynomial $h(x) = 1$. By Brion’s theorem, we have

$$S(p)(\xi) = \sum_{s} e^{\langle \xi,s \rangle} S(c_s)(\xi).$$

For each vertex $s$, let $\rho_s : \mathcal{L}_s \rightarrow \mathbb{Z}$ be a patchwork function. We define

$$B(p)(\xi) = \sum_{s} e^{\langle \xi,s \rangle} S^{c_s,\rho_s}(c_s)(\xi).$$
For the dilated polytope $np$, we have

$$S(np)(\xi) = \sum_s e^{n\langle \xi, s \rangle} S(c_s)(\xi) = \sum_{m \geq 0} \frac{n^m}{m!} \sum_s \langle \xi, s \rangle^m S(c_s)(\xi).$$

Hence, the meromorphic function $\frac{1}{m!} \sum_s \langle \xi, s \rangle^m S(c_s)(\xi)$ is actually regular at $\xi = 0$ and its value at $\xi = 0$ is the $m$th Ehrhart coefficient of $p$.

We have similarly

$$B(np)(\xi) = \sum_{m \geq 0} \frac{n^m}{m!} \sum_s \langle \xi, s \rangle^m S_{\rho_s}(c_s)(\xi).$$

The meromorphic functions $S^L(c_s)(\xi)$ and $S_{\rho_s}(c_s)(\xi)$ have a special form: they can be written as the quotient of an analytic function by a product of $d' \leq d$ linear forms. Such a function $\phi$ has an expansion into rational functions $\phi = \sum_{j \geq -d} \phi[j]$ where $\phi[j]$ is homogeneous of total degree $j$.

Now it follows from our main theorem that, for $m \geq d - r$, we have

$$S(c_s)[-m](\xi) = S_{\rho_s}(c_s)[-m](\xi),$$

hence the zero degree terms of $\sum_s \langle \xi, s \rangle^m S(c_s)(\xi)$ and $\sum_s \langle \xi, s \rangle^m S_{\rho_s}(c_s)(\xi)$ are equal. Therefore the latter is also analytic at $\xi = 0$ and its value at $\xi = 0$ is the $m$th Ehrhart coefficient of $p$. This is the content of Theorem 20.

Thus, besides taking care of any polynomial $h(x)$, not only $h(x) = 1$, this method to compute the $r + 1$ highest degree Ehrhart coefficients for the pair $(p, h)$ leads to a simpler algorithm than the one proposed in [2]. When $p$ is a rational simplex, the contributions of the terms of the form $S^L(c_s)(\xi)$ when $L \in L_s$ are immediately reduced to the computation of a function $S(a)$ with $a$ a simplicial cone of dimension smaller or equal to $r$.

There is also another possible implementation of an algorithm to compute the $r + 1$ Ehrhart highest degree coefficients for the pair $(p, h)$ based on the results of [4]. As seen from Equation (3), this involves the computation of the analytic function $\mu(t(p, f))$, also associated to simplicial cones in dimension smaller or equal to $r$. We plan to compare the implementation of both methods in the near future.
2 Local Euler-Maclaurin expansion of a mixed valuation $S^L$

We consider a rational vector space $V$ of dimension $d$, that is to say a finite dimensional real vector space with a lattice denoted by $\Lambda_V$ or simply $\Lambda$. We will need to consider subspaces and quotient spaces of $V$, this is why we cannot just let $V = \mathbb{R}^d$ and $\Lambda = \mathbb{Z}^d$. By lattice, we mean a discrete additive subgroup of $V$ which generates $V$ as a vector space. Hence, a lattice is generated by a basis of the vector space $V$. A basis of $V$ which is a $\mathbb{Z}$-basis of $\Lambda_V$ is called an integral basis. The elements of $\Lambda$ are called integral. An element $x \in V$ is called rational if $qx \in \Lambda$ for some integer $q \neq 0$. The space of rational points in $V$ is denoted by $V^\mathbb{Q}$. A subspace $L$ of $V$ is called rational if $L \cap \Lambda$ is a lattice in $L$. If $L$ is a rational subspace, the image of $\Lambda$ in $V/L$ is a lattice in $V/L$, so that $V/L$ is a rational vector space. We will call the image of $\Lambda$ in $V/L$ the projected lattice.

Example 1 Let $V = \mathbb{R}^2$ with standard lattice $\mathbb{Z}^2$. Let $v_1, v_2$ be two primitive integral independent vectors. Using an integral basis with first basis vector $v_1$, a straightforward computation shows that the projected lattice on $\mathbb{R}^2/\mathbb{R}v_1$ is $\mathbb{Z}\overline{v_2}/\det(v_1,v_2)$, where $\overline{v_2}$ is the projection of $v_2$ on $\mathbb{R}^2/\mathbb{R}v_1$.

A rational space $V$, with lattice $\Lambda$, has a canonical Lebesgue measure, for which $V/\Lambda$ has measure 1. An affine subspace $L$ of $V$ is called rational if it is a translate of a rational subspace by a rational element. It is similarly provided with a canonical Lebesgue measure. We will denote this measure by $dm_L$.

We will denote elements of $V$ by latin letters $x, y, v, \ldots$ and elements of $V^*$ by greek letters $\xi, \alpha, \ldots$. We denote the duality bracket by $\langle \xi, x \rangle$.

If $S$ is a subset of $V$, we denote by $< S >$ the affine subspace generated by $S$. If $S$ consists of rational points, then $< S >$ is rational. Remark that $< S >$ may contain no integral point. We denote by $\text{lin}(S)$ the vector subspace of $V$ parallel to $< S >$.

If $S$ is a subset of $V$, we denote by $S^\perp$ the subspace of $V^*$ orthogonal to $S$:

$$S^\perp = \{ \xi \in V^* ; \langle \xi, x \rangle = 0 \text{ for all } x \in S \}.$$ 

If $L$ is a subspace of $V$, the dual space $(V/L)^*$ is canonically identified with the subspace $L^\perp \subset V^*$. 


A convex rational polyhedron $p$ in $V$ (we will simply say polyhedron) is, by definition, the intersection of a finite number of half spaces bounded by rational affine hyperplanes. We say that $p$ is solid (in $V$) if $<p> = V$. A polytope $p$ is a compact polyhedron.

The set of non negative real numbers is denoted by $\mathbb{R}_+$. A convex rational cone $c$ in $V$ is a closed convex cone $\sum_{i=1}^{k} \mathbb{R}_+ v_i$ which is generated by a finite number of elements $v_i$ of $V_{\mathbb{Q}}$. In this article, we simply say cone instead of convex rational cone.

An affine (rational) cone $a$ is, by definition, the translate of a cone in $V$ by a rational element $s \in V_{\mathbb{Q}}$. This cone is uniquely defined by $a$.

A cone $c$ is called simplicial if it is generated by independent elements of $V_{\mathbb{Q}}$. A simplicial cone $c$ is called unimodular if it is generated by independent integral vectors $v_1, \ldots, v_k$ such that $\{v_1, \ldots, v_k\}$ can be completed in an integral basis of $V$. An affine cone $a$ is called simplicial (resp. simplicial unimodular) if it is the translate of a simplicial (resp. simplicial unimodular) cone.

An affine cone $a$ is called pointed if it does not contain any straight line.

The set of faces of an affine cone $a$ is denoted by $F(a)$. If $a$ is pointed, then the vertex of $a$ is the unique face of dimension 0, while $a$ is the unique face of maximal dimension $\dim a$.

Let us recall the definition of the transverse cone $t(p, f)$ of a polyhedron $p$ along one of its faces $f$. Let $x$ be a point in the relative interior of $f$. The cone of feasible directions of $p$ at $x$ is the set $c(p, f) := \{v \in V : x + \epsilon v \in p \text{ for } \epsilon > 0 \text{ small enough}\}$. It does not depend on the choice of $x$. We denote the projection $V \to V/\text{lin}(f)$ by $\pi_f$. Then $t(p, f)$ is the image $\pi_f(f + c(p, f))$ of the affine cone $f + c(p, f)$ in $V/\text{lin}(f)$. It is a solid pointed affine cone in the quotient space $V/\text{lin}(f)$ with vertex $\pi_f(<f>)$. In particular, if $v$ is a vertex of $p$, the transverse cone $t(p, v)$ coincides with the supporting cone $v + c(p, v) \subseteq V$.

If $a$ is an affine cone and $f$ is a face of $a$, then $c(a, f) = a + \text{lin}(f)$ and the transverse cone $t(a, f)$ of $a$ along $f$ is just the projection $\pi_f(a)$ of $a$ on $V/\text{lin}(f)$.

**Definition 2** Denote by $\mathcal{H}(V^*)$ the ring of analytic functions around $0 \in V^*$. Denote by $\mathcal{M}(V^*)$ the ring of meromorphic functions defined around $0 \in V^*$ and by $\mathcal{M}_e(V^*) \subseteq \mathcal{M}(V^*)$ the subring consisting of those meromorphic functions $\phi(\xi)$ such that there exists a product of linear forms $D(\xi)$ with $D(\xi)\phi(\xi) \in \mathcal{H}(V^*)$. 
A function $\phi(\xi) \in \mathcal{M}_\ell(V^*)$ has a unique expansion into homogeneous rational functions

$$
\phi(\xi) = \sum_{m \gg -\infty} \phi_{[m]}(\xi)
$$

where $m$ is the total degree.

If $P$ is a homogeneous polynomial on $V^*$ of degree $p$, and $D$ a product of $r$ linear forms, then $P D$ is an element in $\mathcal{M}_\ell(V^*)$ homogeneous of degree $m = p - r$.

Let us recall the definition of the function $I(p) \in \mathcal{M}_\ell(V^*)$ associated to a polyhedron $p$, (see for instance the survey [3]).

**Proposition 3** There exists a map $I$ which to every polyhedron $p \subset V$ associates a meromorphic function with rational coefficients $I(p) \in \mathcal{M}_\ell(V^*)$, so that the following properties hold:

(a) If $p$ contains a straight line, then $I(p) = 0$.

(b) If $\xi \in V^*$ is such that $|e^{\langle \xi, x \rangle}|$ is integrable over $p$, then

$$
I(p)(\xi) = \int_p e^{\langle \xi, x \rangle} dm_{p}(x).
$$

(c) For every point $s \in V_\mathbb{Q}$, we have

$$
I(s + p)(\xi) = e^{\langle \xi, s \rangle} I(p)(\xi).
$$

(d) The map $I$ is a simple valuation: if the characteristic functions $\chi(p_i)$ of a family of polyhedra $p_i$ satisfy a linear relation $\sum_i r_i \chi(p_i) = 0$, then the functions $I(p_i)$ satisfy the relation

$$
\sum_{\{i, <p_i> = V\}} r_i I(p_i) = 0.
$$

In the following proposition, we define the mixed valuation $p \mapsto S^L(p)$ associated to a rational vector subspace $L \subseteq V$. To any polyhedron $p$, we associate a meromorphic function $S^L(p)(\xi) \in \mathcal{M}(V^*)$. If $p$ is compact, this function is actually regular at 0, and its value for $\xi = 0$ is the valuation $E_{L,\perp}(p)$ considered by Barvinok [3].

We denote by $\Lambda_{V/L}$ the projection on $V/L$ of the lattice $\Lambda$. 

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Proposition 4 Let $L \subseteq V$ be a rational subspace. There exists a map $S^L$ which to every rational polyhedron $p \subset V$ associates a meromorphic function with rational coefficients $S^L(p) \in \mathcal{M}(V^*)$ so that the following properties hold:

(a) If $p$ contains a line, then $S^L(p) = 0$.

(b) $S^L(p)(\xi) = \sum_{y \in \Lambda_{V/L}} \int_{p \cap (y + L)} e^{\langle \xi, x \rangle} dm_L(x)$,

for every $\xi \in V^*$ such that the above sum converges.

(c) For every point $s \in \Lambda$, we have

$$S^L(s + p)(\xi) = e^{\langle \xi, s \rangle} S^L(p)(\xi).$$

(d) The map $S^L$ is a valuation: if the characteristic functions $\chi(p_i)$ of a family of polyhedra $p_i$ satisfy a linear relation $\sum r_i \chi(p_i) = 0$, then the functions $S^L(p_i)$ satisfy the same relation

$$\sum r_i S^L(p_i) = 0.$$

For $L = \{0\}$, we recover the valuation $S$ given by

$$S(p)(\xi) = \sum_{x \in p \cap \Lambda} e^{\langle \xi, x \rangle},$$

provided this sum is convergent.

For $L = V$, we have $S^V(p) = I(p)$, if $p$ is solid, and $S^V(p) = 0$ otherwise. The proof is entirely analogous to the case $L = \{0\}$, see Theorem 3.1 in [3], and we omit it.

Remark 5 The function $S^L(p)$ is actually an element of $\mathcal{M}_1(V^*)$, but we do not prove it at this point. Let $a$ be an affine cone and $\{v_i\}$ the generators of its edges. It will follow from the Euler-Maclaurin expansion of $S^L(a)$ (Theorem 8) that $\prod_{i} \langle \xi, v_i \rangle S^L(a)(\xi)$ is analytic near zero for any $L$. It would be interesting to prove it a priori. By Brion’s theorem and the valuation property, it follows in particular that $S^L(p) \in \mathcal{M}_1(V^*)$. 

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Example 6 Let \( a \) be a simplicial affine cone in the space \( V \), and assume that \( L = \text{lin}(f_1) \) for some face \( f_1 \) of \( a \). In this case, \( S^L(a)(\xi) \) decomposes as product of an integral and a discrete sum. For simplicity, assume that \( a \) is solid. Let \( f_2 \) be the face of \( a \) such \( V = \text{lin}(f_1) \oplus \text{lin}(f_2) \). We write \( x = x_1 + x_2 \) and \( \xi = \xi_1 + \xi_2 \) for the corresponding decompositions of \( x \in V \) and \( \xi \in V^* \). Then \( a = a_1 + a_2 \) where \( a_i \) is a simplicial affine cone in \( \text{lin}(f_i) \). Let us denote by \( \Lambda_2 \) the projected lattice in \( V / \text{lin}(f_1) \sim \text{lin}(f_2) \). From (2), we obtain immediately

\[
S^L(a)(\xi_1 + \xi_2) = I(a_1)(\xi_1) \sum_{x_2 \in a_2 \cap \Lambda_2} e^{\langle \xi_2, x_2 \rangle}.
\]

Notice that the lattice \( \Lambda_2 \) is usually bigger than \( \Lambda \cap \text{lin}(f_2) \).

Example 7 Let \( V = \mathbb{R}^2 \) with the standard lattice. We compute \( S^L(a) \) when \( a \) is a cone and \( L \) is a line. Let \( a = \mathbb{R}_+ v_1 + \mathbb{R}_+ v_2 \), where \( v_1, v_2 \) are two linearly independent primitive integral vectors.

(a) Assume that \( L \) is the line supporting an edge of \( a \), say \( L = \mathbb{R}v_1 \). We identify \( V / L \) to \( \mathbb{R}v_2 \). The projected lattice is \( \Lambda_2 = \mathbb{Z}v_2 / \det(v_1, v_2) \), (Example 6), hence, by (3) in Example 6, we have

\[
S^L(a)(\xi) = -\frac{1}{\langle \xi, v_1 \rangle} \frac{1}{1 - e^{\langle \xi, v_2 \rangle / \det(v_1, v_2)}}.
\]

(b) Assume now that \( L \) is transverse to both edges of \( a \). Assume that \( \det(v_1, v_2) > 0 \). Let \( L = \mathbb{R}u \) where \( u \) is a primitive integral vector chosen so that \( \det(u, v_2) > 0 \). Let \( a_i = \mathbb{R}_+ u + \mathbb{R}_+ v_i \) for \( i = 1, 2 \). We decompose the characteristic function of the cone \( a \) as \( \chi(a) = \chi(a_2) + \chi(a_1) - \chi(\mathbb{R}_+ u) \) or \( \chi(a) = \chi(a_2) - \chi(a_1) + \chi(\mathbb{R}_+ v_1) \), depending on whether \( u \) belongs to \( a \) or not. Using the valuation property, case (a) and the relation

\[
\frac{1}{1 - e^x} + \frac{1}{1 - e^{-x}} = 1,
\]

we obtain in both cases

\[
S^L(a)(\xi) = -\frac{1}{\langle \xi, u \rangle} \left( \frac{1}{1 - e^{\langle \xi, v_2 \rangle / \det(u, v_2)}} - \frac{1}{1 - e^{\langle \xi, v_1 \rangle / \det(u, v_1)}} \right).
\]

In this example, we see that \( \langle \xi, v_1 \rangle \langle \xi, v_2 \rangle S^L(a)(\xi) \) is indeed analytic near \( \xi = 0 \).
In the following theorem and its applications, we will consider the functions $S^L(p)$ when the space $V$ is replaced with a quotient space $W$. We denote by $C(W)$ the set of affine cones in $W$. Thus if $a \in C(W)$, and $L$ a rational subspace of $W$, the function $S^L(a)$ is a meromorphic function on $W^*$. We are going to show that the function $S^L(a)$ has a local Euler-Maclaurin expansion, which generalizes the case $L = \{0\}$.

**Theorem 8** Let $V$ be a rational space and $Q$ a rational scalar product on $V^*$. There exists a unique family of maps $\mu^L_W$, indexed by pairs $(W, L)$ where $W$ is a rational quotient space of $V$ and $L$ is a rational vector subspace of $W$ such that the family enjoys the following properties:

(a) $\mu^L_W$ maps $C(W)$ to $\mathcal{H}(W^*)$, the space of analytic functions on $W^*$.

(b) If $W = \{0\}$, then $\mu_W^{(0)}(\{0\}) = 1$.

(c) For $\dim W > 0$ and $L = W$, then $\mu_W^W(a) = 0$.

(d) If the affine cone $a \in C(W)$ contains a straight line, then $\mu^L_W(a) = 0$.

(e) For any affine cone $a$ in $W$, the following formula holds

$$S^L(a) = \sum_{f \in F(a)} \mu^{L+\dim(f)/\dim(f)}_{W/\dim(f)}(t(a, f)) I(f)$$

where the sum is over all faces of the cone $a$.

In this last formula, the function $\mu^{L+\dim(f)/\dim(f)}_{W/\dim(f)}(t(a, f))$ is considered as a function on $W^*$ itself by means of the orthogonal projection $W^* \to (W/\dim(f))^* = (\dim(f))^\perp$ with respect to the scalar product on $W^* \subset V^*$.

**Proof.** The proof is entirely similar to the case $L = \{0\}$ studied in [4]. Note that $\mu_W^{(0)}$ coincides with the map denoted by $\mu_W$ in [4]. The only new item is (c). It follows immediately from the relation $S^W_W(a) = I(a)$.

**Remark 9** Let $a$ be a solid cone in $W$, and let $f$ be a face of $a$ such that $\dim f < \dim W$. If $L$ is transverse to the face $f$, that is, if $L + \dim(f) = W$, then $\mu^{L+\dim(f)/\dim(f)}_{W/\dim(f)}(t(a, f)) = 0$. This follows from (c).

From now on we omit the subscript $W$, thus we write $\mu^L$ in place of $\mu^L_W$. The next theorem and its proof are also entirely similar to the case $L = \{0\}$ in [4].
Theorem 10 The analytic functions defined in Theorem 8 have the following properties:

(a) For any $x \in \Lambda$, one has $\mu^L(x + a) = \mu^L(a)$.

(b) The map $(a, L) \mapsto \mu^L(a)$ is equivariant with respect to lattice-preserving isometries. In other words, let $g$ be an isometry of $W$ which preserves the lattice $\Lambda_W$. Then $\mu^L(g(a)) = g^{-1}\mu^L(a)$.

(c) If $W$ is an orthogonal sum $W = W_1 \oplus W_2$ of rational spaces, $L_i \subseteq W_i$ and $a_i$ is an affine cone in $W_i$ for $i = 1, 2$, then

$$\mu^{L_1 \oplus L_2}(a_1 + a_2) = \mu^{L_1}(a_1)\mu^{L_2}(a_2).$$

(d) For a fixed $s \in W$, the map $c \mapsto \mu^L(s + c)$ is a valuation on the set of cones in $W$.

(e) Let $p \subset W$ be a rational polyhedron, then

$$S^L(p) = \sum_{f \in F(p)} \mu^{L + \text{lin}(f)}/\text{lin}(f)(t(p, f))I(f).$$

Example 11 Let us compute the function $\mu^L$ for the various transverse cones of Example 7. We define a function $B(u)$ on $\mathbb{C}$, holomorphic near 0, by

$$B(u) = \frac{1}{1 - e^u} + \frac{1}{u}.$$

We have

$$I(a)(\xi) = \frac{|\det(v_1, v_2)|}{\langle \xi, v_1 \rangle \langle \xi, v_2 \rangle}.$$

Consider case (b) where $L$ is transverse to both edges $f_i = \mathbb{R}_+v_i$ of $a$.

Using the equation $\det(v_1, v_2)u = \det(u, v_2)v_1 - \det(u, v_1)v_2$, we have

$$I(a)(\xi) = \frac{1}{\langle \xi, u \rangle} \left( \frac{\det(u, v_2)}{\langle \xi, v_2 \rangle} - \frac{\det(u, v_1)}{\langle \xi, v_1 \rangle} \right).$$

Thus we can rewrite (3) as

$$S^L(a) = \mu^L(a) + \sum_{i=1,2} \mu^{L + \text{lin}(f_i)/\text{lin}(f_i)}((t(a, f_i))I(f_i) + I(a),$$

with

$$\mu^L(a)(\xi) = \frac{1}{\langle \xi, u \rangle} \left[ B \left( \frac{\langle \xi, v_1 \rangle}{\det(u, v_1)} \right) - B \left( \frac{\langle \xi, v_2 \rangle}{\det(u, v_2)} \right) \right],$$

$$\mu^{L + \text{lin}(f_i)/\text{lin}(f_i)}((t(a, f_i)) = 0 \text{ for } i = 1, 2.$$
Observe that (7) is indeed regular at $\xi = 0$.

In case (a) where $L = \mathbb{R}v_1$, we have $L + \text{lin}(f_1)/\text{lin}(f_1) = \{0\}$. Let us assume that $\det(v_1, v_2) > 0$. Then we have, by (3),

$$
\mu^{(0)}(t(a, f_1)) = B \left( \frac{-C_1 \langle \xi, v_1 \rangle + \langle \xi, v_2 \rangle}{\det(v_1, v_2)} \right)
$$

with $C_1 = \frac{Q(v_1, v_2)}{Q(v_1, v_1)}$.

As $I(f_1)(\xi) = \frac{1}{\langle \xi, v_1 \rangle}$, we can rewrite (3) as

$$
S^L(a) = \mu^L(a) + \mu^{(0)}(t(a, f_1))I(f_1) + \mu^{L+\text{lin}(f_2)/\text{lin}(f_2)}((t(a, f_2))I(f_2) + I(a),
$$

with

$$
\mu^L(a)(\xi) = \frac{1}{\langle \xi, v_1 \rangle} \left[ B \left( \frac{-C_1 \langle \xi, v_1 \rangle + \langle \xi, v_2 \rangle}{\det(v_1, v_2)} \right) - B \left( \frac{\langle \xi, v_2 \rangle}{\det(v_1, v_2)} \right) \right],
$$

which is indeed regular at $\xi = 0$, and, again,

$$
\mu^{L+\text{lin}(f_2)/\text{lin}(f_2)}((t(a, f_2)) = 0.
$$

3 Barvinok valuations

Let $\mathcal{L}$ be a finite family of rational vector subspaces of $V$, and let $\rho(L), L \in \mathcal{L}$, be a set of rational coefficients. The linear combination $\sum_{L \in \mathcal{L}} \rho(L)S^L(p)$ is again a valuation on the set of polyhedra $p \subset V$, with values in $\mathcal{M}_\ell(V^*)$. By taking linear combinations, we obtain a local Euler-Maclaurin expansion for the function $\sum_{L \in \mathcal{L}} \rho(L)S^L(p)(\xi)$.

**Definition 12** Let $p \subset V$ be a polyhedron.

(a) We denote

$$
S^{(\mathcal{L}, \rho)}(p) = \sum_{L \in \mathcal{L}} \rho(L)S^L(p).
$$

(b) We define the $f$-term in the local Euler-Maclaurin expansion of $S^{(\mathcal{L}, \rho)}(p)$ to be

$$
\mu^{(\mathcal{L}, \rho)}(t(p, f)) = \sum_{L \in \mathcal{L}} \rho_L(L)\mu^{L+\text{lin}(f)/\text{lin}(f)}((t(p, f)) larvae
Thus we have
\[ S^{(L, \rho)}(p) = \sum_{f \in F(p)} \mu^{(L, \rho)}(t(p, f)) I(f). \]

We are going to compute the \( f \)-term in the case of the following particular linear combinations introduced by Barvinok [2]:

**Definition 13** The valuation \( S^{(L, \rho)} \) is called a Barvinok valuation if
(a) the family of subspaces \( L \) is stable under sum,
(b) \( \rho \) is an integer valued function on the set \( L \) such that the characteristic function of the union of the subspaces \( L^\perp \subseteq V^* \) can be written as the linear combination
\[ \chi(\bigcup_{L \in L} L^\perp) = \sum_{L \in L} \rho(L) \chi(L^\perp). \]

**Definition 14** We call a function \( L \to \mathbb{Z} \) which satisfies (8) a patchwork function on \( L \).

As the set of orthogonal subspaces \( L^\perp \subseteq V^* \) is stable under intersection, a particular function \( \rho \) with this property can be computed in terms of the Moebius function of the partially ordered set \( L \) ([7], vol I, section 3.7), as explained in [2].

Let us compute a patchwork function \( \rho \) in the following case. \( V = \mathbb{R}^d \) with standard basis \( e_i, i = 1, \ldots, d \), and \( L_{d,q} \) is the set of subspaces \( L_I = \bigoplus_{i \in I} \mathbb{R}e_i \) with cardinal \( |I| \geq q \). The function \( \rho_{d,q} \) defined below is actually the one associated to the Moebius function, but we will not need this fact.

We denote the binomial coefficient \( \frac{m!}{k!(m-k)!} \) by \( C^k_m \).

**Lemma 15** The function \( \rho_{d,q} \) on \( L_{d,q} \) defined by
\[ \rho_{d,q}(L_I) = (-1)^{n-q} C^{q-1}_{n-1} \quad \text{if} \quad |I| = n, \]
satisfies Equation (8).

**Proof.** If \( e^i \) is the dual basis, the orthogonal space \( L_I^\perp \) is equal to \( \sum_{i \notin I} \mathbb{R}e^i \). Let \( \xi = \sum_{i=1}^d \xi_i e^i \in \bigcup_{L \in L_{n}} L^\perp \). Let \( I_0 \) be the set of indices \( i \in [1, \ldots, d] \) such that \( \xi_i = 0 \). Then \( |I_0| \geq q \), and \( \xi \in L^\perp \) if and only \( I \subseteq I_0 \). Let \( |I_0| = N \). The value at \( \xi \) of the right-hand side of (8) is equal to
\[ E(N, q) = \sum_{I \subseteq I_0} \rho_{d,q}(L_I) = \sum_{n=q}^{N} (-1)^{n-q} C^q_n C^{q-1}_{n-1}. \]

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We want to prove that \( E(N, q) = 1 \). Writing \( n = q + i \), we have

\[
E(N, q) = \sum_{i=0}^{N-q} (-1)^i \frac{N!}{(q+i)(N-q-i)!i!(q-1)!}.
\]

Let us compute \((q - 1)! \left( E(N + 1, q) - E(N, q) \right) \). This is equal to

\[
(-1)^{N+1-q} \frac{(N + 1)!}{(N + 1)(N + 1 - q)!} \sum_{i=0}^{N-q} (-1)^i \frac{1}{(q+i)!} \left( \frac{(N + 1)!}{(N + 1 - q - i)!} - \frac{N!}{(N - q - i)!} \right)
\]

\[
= N! \sum_{i=0}^{N+1-q} (-1)^i \frac{N!}{i!(N + 1 - q - i)!} = 0.
\]

We obtain \( E(N, q) = E(q, q) = 1 \). □

In the case of a Barvinok valuation, it turns out that the \( f \)-term in the Euler-Maclaurin expansion of \( S(L, \rho)(p) \) coincides with that of \( S(p) \), if the vector subspace \(\text{lin}(f)\) belongs to the set \( L \). This is the crucial result of the present article. It is an easy consequence of the following combinatorial lemma.

**Lemma 16** Let \( \mathcal{L} \) be a finite family of vector subspaces of \( V \), stable under sum and let \( \rho \) be a patchwork function on \( \mathcal{L} \).

1. Let \( L_0 \in \mathcal{L} \). Then
   \[
   \sum_{\{L \in \mathcal{L}, L \subseteq L_0\}} \rho(L) = 1.
   \]

2. Let \( L_0 \subsetneq L_1 \) be two subspaces in the family \( \mathcal{L} \). Then
   \[
   \sum_{\{L \in \mathcal{L}, L + L_0 = L_1\}} \rho(L) = 0.
   \]

**Proof.** There exists a \( \xi_0 \in L_0^\perp \) such that, for \( L \in \mathcal{L} \), \( \xi_0 \in L^\perp \) if and only if \( L \subseteq L_0 \). We obtain (a) by evaluating both sides of (8) at this particular element \( \xi_0 \). Next, we deduce (b) from (a), by induction on \( \dim L_1 - \dim L_0 \).

For two subspaces \( M \subseteq M' \) in the family \( \mathcal{L} \), let us denote

\[
f(M, M') = \sum_{\{L \in \mathcal{L}, L + M = M'\}} \rho(L).
\]
If \( M = M' \), we have \( f(M, M) = \sum_{\{L \in \mathcal{L}, L \subseteq M\}} \rho(L) = 1 \) by (a).

We apply this with \( M = L_1 \). Thus

\[
\sum_{\{L \in \mathcal{L}, L \subseteq L_1\}} \rho(L) = 1.
\]

In this sum, we group the \( L \in \mathcal{L} \) such that \( L + L_0 \) is equal to a given \( M_1 \in \mathcal{L} \) together.

First we consider the case when \( \dim L_1 - \dim L_0 = 1 \). Then \( M_1 \) is either \( L_0 \) or \( L_1 \), hence we obtain

\[
f(L_0, L_0) + f(L_0, L_1) = 1.
\]

Since \( f(L_0, L_0) = 1 \) by (a), we obtain \( f(L_0, L_1) = 0 \) as required.

Next we consider the case when \( \dim L_1 - \dim L_0 > 1 \). We obtain

\[
\sum_{\{M_1 \in \mathcal{L}, L_0 \subseteq M_1 \subseteq L_1\}} f(L_0, M_1) = 1.
\]

For \( M_1 = L_0 \), we have \( f(L_0, L_0) = 1 \) by (a). For \( M_1 \not\subseteq L_1 \), we have \( f(L_0, M_1) = 0 \) by induction hypothesis. Hence there remains only the term \( f(L_0, L_1) \) which must be equal to 0. □

We study now the Euler-Maclaurin expansion of a Barvinok valuation.

**Theorem 17** Let \( p \subseteq V \) be a rational polyhedron and let \( \mathcal{L} \) be a face of \( p \). Let \( \mathcal{L} \) be a finite family of rational vector subspaces of \( V \), stable under sum. Let \( \rho \) be a patchwork function on \( \mathcal{L} \), and let \( S^{(L, \rho)} = \sum_{L \in \mathcal{L}} \rho(L)S^L \).

Assume that \( \operatorname{lin}(\mathcal{L}) \) belongs to \( \mathcal{L} \). Then

\[
\mu^{(L, \rho)}(t(p, f)) = \mu^{(0)}(t(p, f)).
\]

In other words, the \( \mathcal{L} \)-term in the local Euler-Maclaurin expansion of \( S^{(L, \rho)}(p) \) coincides with that of \( S(p) \).

**Proof.** In the sum \( \sum_{L \in \mathcal{L}} \rho(L)\mu^{L+\operatorname{lin}(\mathcal{L})/\operatorname{lin}(\mathcal{L})}(t(p, f)) \), we group the terms for which \( L + \operatorname{lin}(\mathcal{L}) \) is equal to a given \( L_1 \) together. By Lemma \( \square \) we obtain \( \mu^{(0)}(t(p, f)) \) for \( L_1 = \operatorname{lin}(\mathcal{L}) \) and 0 otherwise. □
Corollary 18 Let $p \subset V$ be a rational polyhedron. Let $0 \leq k \leq d$. Let $\mathcal{L}$ be a finite family of rational vector subspaces of $V$, stable under sum, such that $\text{lin}(f) \in \mathcal{L}$ for every $k$-dimensional face $f$ of $p$ and let $\rho$ be a patchwork function on $\mathcal{L}$.

- Let $0 < k \leq d$. Then the meromorphic function

$$S(p)(\xi) - S^{(\mathcal{L},\rho)}(p)(\xi)$$

has lowest degree $\geq -k + 1$.

- Let $k = 0$. Then

$$S(p)(\xi) = S^{(\mathcal{L},\rho)}(p)(\xi).$$

**Proof.** By Theorem 17, the local Euler-Maclaurin expansion of the difference involves only faces of dimension $< k$.

$$S(p)(\xi) - S^{(\mathcal{L},\rho)}(p)(\xi) = \sum_{\{f \in \mathcal{F}(p), \dim f < k\}} (\mu^{(1)}(t(p,f))(\xi) - \mu^{(\mathcal{L},\rho)}(t(p,f))(\xi)) I(f)(\xi).$$

For a face of dimension $j$, the function $I(f)(\xi)$ is homogeneous of degree $-j$. Multiplied by the **holomorphic** function $\mu^{(1)}(t(p,f))(\xi) - \mu^{(\mathcal{L},\rho)}(t(p,f))(\xi)$, its lowest degree can only increase. □

Remark that the statement of Corollary 18 above does not involve any scalar product. In the next section, we will show that this corollary implies our main Theorem (Theorem 20).

4 Ehrhart quasi-polynomial

Let $p$ be a rational polytope and let $h(x)$ be a polynomial function on $V$. Let

$$S(p, h) = \sum_{x \in p \cap \mathbb{Z}^d} h(x).$$

When $p$ is dilated by a non negative integer $n$, we obtain the Ehrhart quasi-polynomial of the pair $(p, h)$

$$S(np, h) = \sum_{m=0}^{d+N} E_m(p, h, n)n^m,$$
where $N = \deg h$. The coefficients $E_m(p, h, n)$ are periodic functions of $n \in \mathbb{N}$, with period the smallest integer $q$ such that $q \mathfrak{p}$ is a lattice polytope.

If an integer $r \leq d$ is fixed, and $h = 1$, Barvinok \cite{Barvinok1996} proved that the $r + 1$ highest Ehrhart coefficients $E_d(p, 1, n), \ldots, E_{d-r}(p, 1, n)$ of $S(np, 1)$ can be computed in polynomial time with respect to $d$, when $p$ is a rational simplex.

Let $L \subseteq V$ be a rational vector subspace. Denote the projected lattice on $V/L$ by $\Lambda_L$. Consider the mixed valuation $S^L(p, h) = \sum_{y \in \Lambda_{V/L}} \int_{p \cap (y + L)} h(x) dx$.

As shown by Barvinok, and as we will see here, we can use linear combination of these mixed valuations to approximate $S(np, h)$ when $n$ is big.

For any polyhedron $a$, we define the meromorphic function $S^L(a, h)(\xi) \in \mathcal{M}_\ell(V^*)$ similarly to $S^L(a, h)$. For $\xi \in V^*$ such that the sum converges, we have

$$S^L(a, h)(\xi) = \sum_{y \in \Lambda_{V/L}} \int_{a \cap (y + L)} h(x) e^{\langle \xi, x \rangle} dm_L(x).$$

**Remark 19** It is clear that $S^L(a, h)(\xi) = h(\partial \xi) \cdot S^L(a)(\xi)$.

If $p$ is a polytope, then $S^L(p, h)(\xi)$ is regular at $\xi = 0$ and $S^L(p, h)(0) = S^L(p, h)$.

For a family $\mathcal{L}$ and a function $L \mapsto \rho(L), L \in \mathcal{L}$, we define

$$S^{(\mathcal{L}, \rho)}(a, h)(\xi) = \sum_{L \in \mathcal{L}} \rho(L) S^L(a, h)(\xi).$$

If $p$ is a polytope, and we dilate by $n \in \mathbb{N}$, we have again a quasi-polynomial

$$S^{(\mathcal{L}, \rho)}(np, h)(0) = \sum_{m=0}^{d+N} E_m(\mathcal{L}, \rho, p, h, n)n^m.$$

We can replace the quasi-polynomial $S^{(\mathcal{L}, \rho)}(np, h)(0)$ by $q$ legal polynomials in the variable $u$, by splitting $N$ into classes modulo $q$. Writing $n = qu + k$, for $k = 0, \ldots, q - 1$, we obtain the polynomial function of $u$:

$$S^{(\mathcal{L}, \rho)}((qu + k)p, h)(0) = \sum_{m=0}^{d+N} E_m^{(k)}(\mathcal{L}, \rho, p, h)u^m.$$
We briefly recall how the usual Ehrhart quasi-polynomial of a polytope can be computed using Brion’s theorem. We will then use a similar method in order to compute efficiently the \( r+1 \) highest coefficients only, using Barvinok valuations.

Let \( \mathcal{V}(p) \) be the set of vertices of \( p \). For each vertex \( s \), let \( \mathcal{C}_s \) be the cone of feasible directions of \( p \) at \( s \), so that the supporting cone at \( s \) is \( s + \mathcal{C}_s \). By Brion’s theorem [5], we have

\[
S(p,h)(\xi) = \sum_{s \in \mathcal{V}(p)} S(s + \mathcal{C}_s, h)(\xi).
\]

Let \( n \in \mathbb{N} \) and consider the dilated polytope \( np \). The supporting cone at the vertex \( ns \) is \( ns + \mathcal{C}_s \). Let \( q \in \mathbb{N} \) such that \( qp \) is a lattice polytope and fix \( k \in \mathbb{N}, 0 \leq k \leq q - 1 \). Let \( n = qu + k \). As \( qus \) is an integral point, we have

\[
S((qu + k)s + \mathcal{C}_s, h)(\xi) = e^{nu(\xi,s)} S^{L_s, \rho_s}(ks + \mathcal{C}_s, h)(\xi).
\]

Expanding in powers of \( u \), we obtain

\[
S((qu + k)p, h)(\xi) = \sum_{m \geq 0} u^m q^m \sum_{s \in \mathcal{V}(p)} S(ks + \mathcal{C}_s, h)(\xi).
\]

It follows that for each \( m \), the sum of meromorphic functions

\[
\frac{q^m}{m!} \sum_{s \in \mathcal{V}(p)} (\xi, s)^m S(ks + \mathcal{C}_s, h)(\xi)
\]

is actually analytic. Its value at \( \xi = 0 \) is obtained by taking the zero degree term. We obtain

\[
S((qu + k)p, h)(0) = \sum_{m \geq 0} E_m^{(k)}(p, h) u^m,
\]

with

\[
E_m^{(k)}(p, h) = \frac{q^m}{m!} \sum_{s \in \mathcal{V}(p)} (\xi, s)^m S(ks + \mathcal{C}_s, h)_{[-m]}(\xi).
\]

The right-hand side of this relation, a priori a meromorphic function of \( \xi \), is actually constant. Moreover, we have \( S(ks + \mathcal{C}_s, h)_{[-m]}(\xi) = 0 \) if \( m > d + N \), hence the Ehrhart quasi-polynomial has degree \( \leq d + N \).
We apply now Brion’s theorem to $S^{L,\rho}(p, h)(\xi)$. We obtain

$$S^{L,\rho}(p, h)(\xi) = \sum_{s \in V(p)} S^{L_s,\rho_s}(s + c_s, h)(\xi).$$

For reasons to be explained later on, instead of one family $L$, we take a family of subspaces $L_s$ for each vertex $s$. Let $\rho_s : L_s \to \mathbb{Z}$ be a function on $L_s$. We denote now by $(\mathcal{L}, \rho)$ the map $s \mapsto (L_s, \rho_s)$.

We define:

(9) $\mathcal{B}^{L,\rho}(p, h)(\xi) = \sum_{s \in V(p)} S^{L_s,\rho_s}(s + c_s, h)(\xi).$

If the family does not depend on $s$, $(L_s, \rho_s) = (L_0, \rho_0)$ for every vertex $s$, then, by Brion’s theorem, we have

$$\mathcal{B}^{L,\rho}(p, h)(\xi) = S^{L_0,\rho_0}(p, h)(\xi).$$

We dilate (9). Let $n = qu + k$. We obtain

$$\mathcal{B}^{L,\rho}((qu + k)p, h)(\xi) = \sum_{s \in V(p)} e^{qu\langle \xi, s \rangle} S^{L_s,\rho_s}(ks + c_s, h)(\xi).$$

Expanding in powers of $u$, we obtain

(10) $\mathcal{B}^{L,\rho}((qu + k)p, h)(\xi) = \sum_{m \geq 0} u^m E^{(k)}_m(\mathcal{L}, \rho, p, h)(\xi)$

with

(11) $E^{(k)}_m(\mathcal{L}, \rho, p, h)(\xi) = \frac{q^m}{m!} \sum_{s \in V(p)} \langle \xi, s \rangle^m S^{L_s,\rho_s}(ks + c_s, h)(\xi).$

If the family does not depend on $s$, $(L_s, \rho_s) = (L_0, \rho_0)$ for all vertices, then $\mathcal{B}^{L,\rho}(p, h)(\xi) = S^{L_0,\rho_0}(p, h)(\xi)$ is analytic near $\xi = 0$, and so are the coefficients (11).

On the contrary, if we take a different family $L_s$ for each vertex $s$, the coefficient $E^{(k)}_m(\mathcal{L}, \rho, p, h)(\xi)$ of $u^m$ in (10) is no longer analytic near $\xi = 0$, in general. However, the meromorphic function $\xi \mapsto E^{(k)}_m(\mathcal{L}, \rho, p, h)(\xi)$ belongs to $\mathcal{M}_\ell(V^*)$, thus it has a term of degree 0 with respect to $\xi$, given by

(12) $E^{(k)}_m(\mathcal{L}, \rho, p, h)|_0(\xi) = \frac{q^m}{m!} \sum_{s \in V(p)} \langle \xi, s \rangle^m S^{L_s,\rho_s}(ks + c_s, h)[-m](\xi).$
For a family $$(\mathcal{L}, \rho)$$ as described in the next theorem, it turns out that, for large $$m$$, this zero-degree part $$E_m^k(\mathcal{L}, \rho, p, h)_{[0]}(\xi)$$ is actually analytic, hence constant, and its value is equal to the $$m$$-th Ehrhart coefficient $$E_m^k(p, h)$$ of $$S((k + qu)p, h)(0)$$.

**Theorem 20** Let $$p$$ be a rational polytope in a rational vector space of dimension $$d$$. For each vertex $$s$$ of the polytope $$p$$, let $$c_s$$ be the cone of feasible directions of $$p$$ at $$s$$, so that the supporting cone at $$s$$ is $$s + c_s$$. For each vertex $$s$$, let $$\mathcal{L}_s$$ be a finite family of rational vector subspaces of $$V$$, stable under sum, such that $$\text{lin}(f)$$ belongs to $$\mathcal{L}_s$$ for every face $$f$$ of codimension $$r$$ of the cone $$c_s$$, and let $$\rho_s$$ be a patchwork function on $$\mathcal{L}_s$$. Let $$q \in \mathbb{N}$$ such that $$q p$$ is a lattice polytope and fix $$k \in \mathbb{N}$$, $$0 \leq k \leq q - 1$$. Let $$h(x)$$ be a homogeneous polynomial of total degree $$N$$.

Then, for $$m \geq d + N - r$$, the zero-degree term $$E_m^k(\mathcal{L}, \rho, p, h)_{[0]}(\xi)$$ defined by (12) is regular near $$\xi = 0$$, hence constant. Its value is the coefficient $$E_m^k(p, h)$$ of $$u^m$$ in the Ehrhart quasi-polynomial

$$S((k + qu)p, h)(0) = \sum_{x \in ((k + qu)p) \cap \Lambda} h(x) = \sum_{m=0}^{d+N} a^m E_m^k(p, h).$$

**Proof.** We first consider the case $$h(x) = 1$$. We have, for every $$m \geq 0$$,

$$E_m^k(p, 1) = \frac{q^m}{m!} \sum_{s \in V(p)} \langle \xi, s \rangle^m S(k s + c_s)_{[-m]}(\xi)$$

where the right-hand side is actually a constant function of $$\xi$$.

For $$m > d - r - 1$$, we have, by Corollary 18,

$$S(k s + c_s)(\xi)_{[-m]} = S^{\mathcal{L}, \rho}(k s + c_s)(\xi).$$

This proves the theorem when $$h(x) = 1$$. The case of a non constant polynomial $$h(x)$$ is quite similar. If $$h(x) = x_1^{N_1} \ldots x_d^{N_d}$$, we just have to replace the meromorphic functions $$S(k s + c_s)(\xi)$$ and $$S^{\mathcal{L}, \rho}(k s + c_s)(\xi)$$ by their derivatives under $$\partial^{N_1}_{\xi_1} \ldots \partial^{N_d}_{\xi_d}$$. □

If for each vertex $$s$$, we take $$\mathcal{L}_s = \mathcal{L}$$, the full collection generated by all $$r$$ codimensional faces of $$p$$, we obtain Corollary 21 below, that is Barvinok’s theorem [2], with an extension to the sum of values of any polynomial $$h(x)$$ over the set of integral points of a rational polytope (Barvinok considers only the case $$h(x) = 1$$).
Corollary 21 Let $p \subset V$ be a rational polytope and let $h(x)$ be a polynomial function on $V$. Let $\mathcal{L}$ be a finite family of rational vector subspaces of $V$, stable under sum. Assume that $\text{lin}(f)$ belongs to $\mathcal{L}$ for every face $f$ of codimension $r$ of $p$. Let $p$ be a patchwork function on $\mathcal{L}$ and let $S^{\mathcal{L},p} = \sum_{L \in \mathcal{L}} \rho(L)S^L$. Then the $r+1$ highest Ehrhart coefficients of $S(tp,h)(0)$ and $S^{\mathcal{L},p}(tp,h)(0)$ are equal.

The point in taking a family $\mathcal{L}_s$ which depends on the vertex $s$ lies in the case where $p$ is simplicial. In this case, we can take $\mathcal{L}_s$ to be just the set of subspaces $\text{lin}(f)$, for all faces $f$ of codimension $\leq r$ of the supporting cone $c_s$ at vertex $s$. This family is stable under sum. Moreover the patchwork function on $\mathcal{L}_s$ is simple, (Lemma 15) and the computation of the function $S^L(ks + c_s)(\xi)$, when $L \in \mathcal{L}_s$, is immediately reduced (Example 6) to the computation of a function $S(a)(\xi)$ for a simplicial cone $a$ in a rational vector space of dimension smaller or equal than $r$. When $p$ is a simplex, we obtain in this way a method for computing the $r+1$ highest Ehrhart coefficients for the pair $(p,h)$.

5 Local Euler-Maclaurin formula for mixed sums

Finally in this last section, we discuss an application of the existence of the coefficients $\mu^L$ (Theorem 8) in the line of [4].

Let $p$ be a rational polytope in a rational vector space $V$ of dimension $d$ and let $h(x)$ be a polynomial function on $V$. Let $L$ be a rational subspace of $V$. Consider the mixed sum

$$S^L(p,h) = \sum_{y \in \Lambda_{V/L}} \int_{p \cap (y + L)} h(x) dm_L(x).$$

As in [4], we associate to the analytic function $\mu^L(t(p,f))$ a constant coefficients differential operator (of infinite order) on $V$.

Definition 22 Let $f$ be a face of $p$. We denote by $D^L(p,f)$ the differential operator on $V$ associated to analytic function $\mu^L(t(p,f))$:

$$D^L(p,f)(\partial_\xi) \cdot e^{\langle \xi, x \rangle} = \mu^L(t(p,f))(\xi) e^{\langle \xi, x \rangle}.$$
The operators $D^L(p, f)$ are **local**, that is they depend only of the transverse cone $t(p, f)$ of $p$ along $f$, and they involve only derivatives in directions orthogonal to the face $f$. We can state the following theorem with the same proof as in [4].

**Theorem 23 (Local Euler-Maclaurin formula)** Let $p$ be a polytope in $V$. For any polynomial function $h(x)$ on $V$, we have

\[ S^L(p, h) = \sum_{f \in \mathcal{F}(p)} \int_{\mathcal{F}(p)} D^L(p, f) \cdot h \]

where the integral on the face $f$ is taken with respect to the Lebesgue measure on $<f>$ defined by the lattice $\Lambda \cap \text{lin}(f)$.

In particular, for $h = 1$, we obtain

\[ S^L(p, 1) = \sum_{f \in \mathcal{F}(p)} \mu^L(t(p, f))(0) \text{vol}(f). \]

Let us dilate the polytope $p$ by a non negative integer $n$. If $f$ is a face of $p$, let $q_f$ be the smallest positive integer such that $q_f < f >$ contains integral points. Define $D(p, f, n) = D(np, q_f)$, if $n > 0$, and $D(p, f, 0) = D(q_f p, q_f f)$. The function $n \mapsto D(p, f, n)$ is periodic of period $q_f$.

**Proposition 24** Let $p$ be a rational polytope and $h$ a polynomial function of degree $N$ on $V$. Then, for any integer $n \geq 0$, we have

\[ S^L(np, h) = \sum_{f \in \mathcal{F}(p)} \int_{n|f} D^L(p, f, n) \cdot h. \]

Furthermore, if $f \in \mathcal{F}(p)$, we have

\[ \int_{n|f} D^L(p, f, n) \cdot h = \sum_{i=\dim f}^{\dim f + N} E_i(p, h, f, n) n^i \]

where the coefficients $E_i(p, h, f, n)$ are periodic with period $q_f$.

Hence the Ehrhart coefficients are given by

\[ E^L_m(p, h, n) = \sum_{f, \dim f \leq m} E_m(p, h, f, n). \]
When we apply the last proposition to the function \( h(x) = 1 \), we obtain
\[
S^L(np, 1) = \sum_{f \in \mathcal{F}(p)} \mu^L(nt(p, f))(0) \text{vol}(f)n^{\dim f}.
\]

As \( \mu^L(nt(p, f)) \) is invariant by integral translations, the function \( \mu^L(nt(p, f))(0) \) is of period \( q_f \).

References


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