New notion of index for hedgehogs of $\mathbb{R}^3$ and applications

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0. General introduction

The set $\mathcal{K}^{n+1}$ of convex bodies of $(n+1)$-Euclidean vector space $\mathbb{R}^{n+1}$ is usually equipped with Minkowski addition and multiplication by non-negative real numbers which are respectively defined by:

$$(i) \quad \forall (K,L) \in \left(\mathcal{K}^{n+1}\right)^2, \; K+L = \{u+v \,|u \in K, v \in L\}\,;$$

$$(ii) \quad \forall \lambda \in \mathbb{R}_+, \forall K \in \mathcal{K}^{n+1}, \; \lambda.K = \{\lambda u \,|u \in K\}\,.$$

Of course, $(\mathcal{K}^{n+1}, +, .)$ does not constitute a vector space since we cannot subtract convex bodies in $\mathcal{K}^{n+1}$. Now, in the same way as we construct the group of integers from the set of natural numbers, we can construct the real vector space $(\mathcal{H}^{n+1}, +, .)$ of formal differences of convex bodies of $\mathbb{R}^{n+1}$ from $(\mathcal{K}^{n+1}, +, .)$. Moreover, we can: 1. consider each formal difference of convex bodies of $\mathbb{R}^{n+1}$ as a (possibly singular and self-intersecting) hypersurface of $\mathbb{R}^{n+1}$, called a *hedgehog*; 2. extend the mixed volume $V : \left(\mathcal{K}^{n+1}\right)^{n+1} \rightarrow \mathbb{R}$ to a symmetric $(n+1)$ $-$linear form on $\mathcal{H}^{n+1}$. Thus, the development of hedgehog theory can be seen as an attempt to extend certain parts of the Brunn-Minkowski theory to $\mathcal{H}^{n+1}$. For $n \leq 2$, it goes back to a paper by H. Geppert [4] who introduced hedgehogs under the German names *stützbare Bereiche* $(n=1)$ and *stützbare Flächen* $(n=2)$.

Two principles and some applications

The relevance of hedgehog theory can be illustrated by the following *two principles*: **1.** The study of convex bodies or hypersurfaces by splitting them judiciously (that is, according to the problem under consideration) into a sum of hedgehogs in order to reveal their structure; **2.** The geometrization of analytical problems by considering real functions on the unit sphere $\mathbb{S}^n$ of $\mathbb{R}^{n+1}$ as support functions of hedgehogs or of more general hypersurfaces ('multi-hedgehogs' [6]).

The first principle permitted the author to disprove the following A.D. Alexandrov's uniqueness conjecture [11]:

Conjecture (C) [1]**.** *If S is a closed convex surface of class C^2_+ of $\mathbb{R}^3$ (that is, a C^2-surface of $\mathbb{R}^3$ with positive Gauss curvature) whose principal curvatures k_1 and k_2 satisfy the following inequality*

$$(k_1 - c)\,(k_2 - c) \leq 0,$$

with some constant $c > 0$, then S must be a sphere of radius $1/c$.

Since the problem is to compare S with a sphere Σ of radius $1/c$, the idea was to consider the hedgehog $\mathcal{H} = S - \Sigma$ and to split S into the sum $\Sigma + \mathcal{H}$. This approach gave the following reformulation of conjecture (C) in terms of hedgehogs:

Conjecture (H). *If $\mathcal{H}$ is a hedgehog of $\mathbb{R}^3$ with a C^2 support function whose curvature function (that is, whose product of the principal radii of curvature) is non-positive all over the unit sphere $\mathbb{S}^2$, then $\mathcal{H}$ is (reduced to) a single point.*

Formulations (C) and (H) are equivalent. In particular, if $\mathcal{H}$ is any counter-example to (H) and Σ any sphere with a large enough radius, then $S = \Sigma + \mathcal{H}$ is a counter-example to (C). The author gave an explicit counter-example to (H) and thus disproved conjecture (C) [11]. Later, G. Panina gave new counter-examples to conjecture (H) by constructing first 'hyperbolic polytopal hedgehogs' and then by using smoothening techniques [19].

Let us illustrate the second principle by two important problems. The first one is *the Minkowski problem for hedgehogs. The classical Minkowski problem is that of the existence, uniqueness and regularity of closed convex hypersurfaces of $\mathbb{R}^{n+1}$ whose Gauss curvature is prescribed on $\mathbb{S}^n$ as a function of the normal.* For C^2_+-hypersurfaces (that is, C^2-hypersurfaces with positive Gauss curvature), this well-known problem is equivalent to the question of solutions of certain Monge-Ampère equations of elliptic type. Using a limiting process, Minkowski proved [17] that: If K is a continuous positive function on $\mathbb{S}^n$ of $\mathbb{R}^{n+1}$ satisfying the following integral condition

$$\int_{\mathbb{S}^n} \frac{u}{K(u)}\, d\sigma(u) = 0, \tag{1}$$

where σ is the spherical Lebesgue measure on $\mathbb{S}^n$, then K is the Gauss curvature (in the sense of Gauss' definition) of a unique (up to translation) closed convex hypersurface $\mathcal{H}$ (the uniqueness is coming from the equality condition in a Minkowski's inequality [21]). The strong solution is due to Pogorelov [20] and Cheng and Yau [3] who proved independently that: if K is of class $C^m(\mathbb{S}^n;\mathbb{R})$, $(m \geq 3)$, then the support function of $\mathcal{H}$ is of class $C^{m+1,\alpha}$ for every $\alpha \in]0,1[$. This classical Minkowski Problem has a natural extension to hedgehogs, that is to Minkowski differences of closed convex hypersurfaces. But for non-convex ones, this generalized Minkowski problem is equivalent to the question of solutions of certain Monge-Ampère PDE's of non-elliptic type for which there was no global result. This geometrization permitted the author to give non-trivial examples of Monge-Ampère PDE's of mixed type with no solution [13] (resp. with non-unique solutions [12]) on $\mathbb{S}^2$. Besides, the fact that conjecture (H) is false can be formulated as follows (which disproves a conjecture of Koutroufiotis and Nirenberg [5]):

There exists a non-linear function $f : \mathbb{R}^3 \to \mathbb{R}$ whose restriction to $\mathbb{S}^2$, say h, is C^2 and satisfies the inequality

$$h^2 + h\Delta_2 h + \Delta_{22} h \leq 0,$$

where Δ_2 is the spherical Laplacian and Δ_{22} the Monge-Ampère operator, that is the sum and the product of the eigenvalues of the Hessian of $h = f_{|\mathbb{S}^2}$.

The Sturm-Hurwitz theorem states that any continuous periodic real function expandable in a Fourier series has at least as many zeros as its first nonvanishing harmonics. It has many geometrical consequences such as the 4-vertex theorem (see for instance [22]). The second problem is *the search for Sturm-Hurwitz type theorems* (in particular in higher dimensions). For C^2-functions, the author gave a geometrical interpretation and a new proof of the Sturm-Hurwitz theorem by considering plane *N-hedgehogs*, $(n \in \mathbb{N}^*)$. A plane N-hedgehog is defined as the envelope of a family of cooriented lines having exactly N cooriented support lines with a given normal vector; N is simply the number of full rotations of the coorienting normal vector so that plane 1-hedgehogs are just plane hedgehogs [14].

Which notion of index for studying hedgehogs in $\mathbb{R}^3$?

As we have just seen, hedgehogs have already given first interesting results for both problems. But we shall see that for going further it is necessary to introduce specific tools for studying their geometry in dimensions greater than 2. In the first results mentioned above, an essential role was played by the following relationship between the winding number $i_h(x)$ of an N-hedgehog $\mathcal{H}_h \subset \mathbb{R}^2$ around a point $x \in \mathbb{R}^2 - \mathcal{H}_h$ and the number of cooriented support lines of $\mathcal{H}_h$ through x (i.e. of zeros of $h_x : [0, 2N\pi[\to \mathbb{R}$, $\theta \mapsto h(\theta) - \langle x, u(\theta)\rangle$, where $u(\theta) = (\cos\theta, \sin\theta)$):

$$i_h(x) = N - \frac{1}{2}n_h(x)\,,$$

where h denotes the support function of the N-hedgehog. Given any hedgehog $\mathcal{H}_h \subset \mathbb{R}^3$, we can still define the index $i_h(x)$ of a point $x \in \mathbb{R}^3 - \mathcal{H}_h$ with respect to $\mathcal{H}_h$ (for instance as an algebraic number of intersection). But, as we shall see, it can no longer play the same role. In the particular case where $\mathcal{H}_h \subset \mathbb{R}^3$ is projective (that is modeled on $\mathbb{P}^2(\mathbb{R}) = \mathbb{S}^2/(\text{antipodal relation})$), this i_h-index gives no information neither on $\mathcal{H}_h$ nor on zeros of $h_x(u) = h(u) - \langle x, u\rangle, (u \in \mathbb{S}^2)$. This paper introduces an j_h-index for hedgehogs of $\mathbb{R}^3$ that can in certain respects play the same role as the i_h-index does in $\mathbb{R}^2$. This new index induces a series of new notions (of interior, exterior, algebraic volume, etc) that permit us to describe the geometry of hedgehogs of $\mathbb{R}^3$ (including projective ones). Besides, it also induces a natural notion of transverse orientation (which may switch on certain curves of self-intersection) involved in the multiplicity of solutions of the Minkowski problem.

1. Generalities on hedgehogs

As we said, hedgehog theory consists of: 1. considering the Brunn-Minkowski theory in the vector space $\mathcal{H}^{n+1}$ of formal differences of convex bodies of $\mathbb{R}^{n+1}$; 2. constructing geometrically any formal difference $K - L$ of convex bodies $K, L \in \mathcal{K}^{n+1}$ as a (possibly singular and self-intersecting) hypersurface of $\mathbb{R}^{n+1}$. In the case of convex hypersurfaces (or convex bodies) of class C_+^2 (that is, of C^2-hypersurfaces with positive Gauss curvature), this can be done easily. As you can see Figure 1, we can subtract two such hypersurfaces by subtracting the points corresponding to a same outer normal to obtain a (possibly singular) hypersurface that we shall call a hedgehog.

Figure 1. Geometrical differences of two convex hypersurfaces of class C_+^2

Let us recall how such hedgehogs can be defined directly. As is well-known, every convex body $K \subset \mathbb{R}^{n+1}$ is determined by its support function

$$h_K : \mathbb{S}^n \longrightarrow \mathbb{R}, u \longmapsto \sup \{ \langle x, u \rangle \, | x \in K \} ,$$

(note that $h_K(u)$ may be interpreted as the signed distance from the origin to the support hyperplane with normal u). In particular, every closed convex hypersurface of class C^2_+ is determined by its support function h (which must be of class C^2 on $\mathbb{S}^n$ [21, p. 111]) as the envelope $\mathcal{H}_h$ of the family of hyperplanes with equation $\langle x, u \rangle = h(u)$. This envelope $\mathcal{H}_h$ is described analytically by the two following equations

$$\begin{cases} \langle x, u \rangle = h(u) \\ \langle x, . \rangle = dh_u(.) \end{cases} ,$$

of which the second is obtained from the first by performing a partial differentiation with respect to u. From the first equation , the orthogonal projection of x onto the line spanned by u is $h(u)\, u$ and from the second one its orthogonal projection onto $u^\perp$ is the gradient of h at u. Therefore, for each $u \in \mathbb{S}^n$, $x_h(u) = h(u)u + (\nabla h)(u)$ is the unique solution of this system.

Now, the envelope $\mathcal{H}_h$ is in fact well-defined for any C^2-function h on $\mathbb{S}^n$ (even if h is not the support function of a convex hypersurface). Its natural parametrization $x_h : \mathbb{S}^n \to \mathcal{H}_h, u \mapsto h(u)u + (\nabla h)(u)$ can always be interpreted as the inverse of its Gauss map, in the sense that: at each regular point $x_h(u)$ of $\mathcal{H}_h$, u is normal to $\mathcal{H}_h$. We say that $\mathcal{H}_h$ is the hedgehog with support function h. Note that x_h depends linearly on h.

It is also interesting to note that we can still define an envelope $\mathcal{H}_h$ for any function $h \in C^1(\mathbb{S}^n; \mathbb{R})$. But in general, such an envelope does not represent the difference of two convex bodies. In fact, such a hedgehog can even be fractal [10].

Gauss curvature of hedgehogs with a C^2 support function

Before defining general hedgehogs as differences of arbitrary convex bodies, let us describe briefly hedgehogs with a C^2 support function. As we saw, such hedgehogs may be singular. As the parametrization x_h can be regarded as the inverse of the Gauss map, the Gauss curvature K_h of $\mathcal{H}_h$ is given by 1 over the determinant of the tangent map of x_h:

$$\forall u \in \mathbb{S}^n, \; K_h(u) = 1/\det [T_u x_h] .$$

Therefore, the singularities of $\mathcal{H}_h$ are exactly the points where this curvature K_h becomes infinite.

An important point for our study is that the so-called 'curvature function' $R_h := 1/K_h$ is well-defined and continuous all over the unit sphere, including at the singular points, so that the Minkowski problem arises naturally for hedgehogs.

From an analytical point of view, we get exactly the same formulas as in the convex case. In particular [3], the curvature function can be given by

$$R_h(u) = \det\left[H_{ij}(u) + h(u)\,\delta_{ij}\right], \tag{2}$$

where δ_{ij} are the Kronecker symbols and $(H_{ij}(u))$ the Hessian of h at u with respect to an orthonormal frame on the unit sphere $\mathbb{S}^n$.

Example of projective hedgehogs

Concerning the spherical image of the classical models of the real projective plane in $\mathbb{R}^3$ (as the Boy surface or the Roman surface), Hilbert and Cohn-Vossen wrote in their book *Geometry and the imagination*: "Unfortunately, the way in which it is distributed over the unit sphere has not yet been studied". For 'projective hedgehogs' $\mathcal{H}_h \subset \mathbb{R}^{n+1}$, that is for hedgehogs with an antisymmetric support function h, pair of antipodal points on the unit sphere $\mathbb{S}^n$ correspond to a same point on the hypersurface $\mathcal{H}_h$. So, not too singular projective hedgehogs $\mathcal{H}_h \subset \mathbb{R}^3$ can be regarded as models of the real projective plane whose Gauss map is a bijection from the model onto the real projective plane. An interesting example is given by the following hedgehog version of the Roman surface: $\mathcal{H}_h$, where $h(x,y,z) = x(x^2 - 3y^2) + 2z^3$, $(x,y,z) \in \mathbb{S}^2 \subset \mathbb{R}^3$. This projective hedgehog is represented on Figure 2. As the Roman surface, it has a threefold axis of symmetry and three lines of self-intersection whose end points are singular points of the same topological type as Whitney umbrellas without the handle.

Figure 2. A projective hedgehog version of the Roman surface

Parallel hypersurfaces

Hedgehogs with a C^2 support function can also be seen as parallel hypersurfaces to closed convex hypersurfaces of class C^2_+. Let $\mathcal{H}_h \subset \mathbb{R}^{n+1}$ be such a convex hypersurface and let x_h be its parametrization for which u is the outer unit normal at $x_h(u)$. By definition, the parallel hypersurface at distance t from $\mathcal{H}_h$ is obtained by associating to each $x_h(u)$, the point $x_h(u) + tu$. In fact, this parallel hypersurface is just the hedgehog with support function $h + t$. Of course, outer parallel hypersurfaces remain convex but 'inner' ones may be singular. So, any hedgehog with a C^2 support function can be seen as an inner or outer 'wave front' at distance t from a convex hypersurface.

Generic singularities

Hedgehogs with a C^∞-smooth support function are actually wave fronts in the sense of contact geometry. Therefore, they have only Legendre singularities. The generic singularities of hedgehogs are cusp points in 2-space, cuspidal edges and swallowtails in 3-space. Swallowtails are the cusp points of cuspidal edges. Note that we can distinguish two types of swallowtails (negative or positive) according to the sign of the Gauss curvature on the tail. More precisely, for an open dense subset Ω of $C^\infty(\mathbb{S}^2; \mathbb{R})$ for the C^4 topology, we have: for every $h \in \Omega$, all the singularities of $\mathcal{H}_h \subset \mathbb{R}^3$ are equivalent to one of the three models represented on Figure 3 and $\mathcal{H}_h$ satisfies the following counting formula on $\mathbb{S}^2$ [6]:

$$r^+ - r^- = \frac{q^+ - q^-}{2} + 1$$

where q^- is the number of negative swallowtails, q^+ the number of positive ones, r^- the number of hyperbolic regions and r^+ the number of elliptic ones.

Let us mention the following problem raised by R. Langevin, G. Levitt and H. Rosenberg in [6]:

Is there exists a projective hedgehog $\mathcal{H}_h \subset \mathbb{R}^3$ whose singular locus is reduced to one (or several) immersed cuspidal edge(s) (without any swallowtail) ?

Figure 3. Generic singularities of hedgehogs in $\mathbb{R}^3$

General hedgehogs as differences of arbitrary convex bodies

Now, let us see briefly the way of defining geometrically general hedgehogs of $\mathbb{R}^{n+1}$ as differences of arbitrary convex bodies. As addition of convex bodies corresponds to addition of their support sets with same normal (which are lower-dimensional convex bodies), subtraction of convex bodies must correspond to subtraction of these support sets. So, the idea is simply to proceed by induction on the dimension replacing support sets by support hedgehogs.

More precisely, the method is the following: First. In dimension one, convex bodies are segments that can be subtracted as oriented segments; Second. In dimension $n+1$, support sets of a convex body can be regarded as convex bodies of an Euclidean space of dimension n. Thus, if differences of convex bodies have already been defined in dimension n, then they can again be defined in dimension $n+1$ as collections of differences of support sets (that is of support hedgehogs). Fore precise definitions and more details, we refer the reader to [16].

Figure 4 represents a polygonal hedgehog obtained by subtracting two squares.

Figure 4. A polygonal hedgehog, difference of two squares

2.1. The Minkowski problem.

As already recalled, the classical Minkowski problem is to prescribe the Gauss curvature of convex hypersurfaces in $(n+1)$-Euclidean vector space $\mathbb{R}^{n+1}$. Main results on the classical Minkowski problem have been summarized in the introduction. Now, let us consider the Minkowski problem extended to hedgehogs.

Extension of the Minkowski problem to hedgehogs

In this section, 'hedgehog' will mean 'hedgehog with a C^2 support function'. As already noticed in Section 1, the curvature function $R_h := 1/K_h$ of a hedgehog $\mathcal{H}_h \subset \mathbb{R}^{n+1}$ is well-defined and continuous all over the unit sphere $\mathbb{S}^n$, including at the singular points, so that the Minkowski problem arises naturally for hedgehogs.

What can we expect for hedgehogs ? For $n = 1$, the curvature function depends linearly on the support function so that the problem is simple even for general hedgehogs [16]. But in higher dimensions the problem is very difficult and we shall only consider the case $n = 2$. From (2), the curvature function $R_h := 1/K_h$ of a hedgehog $\mathcal{H}_h \subset \mathbb{R}^3$ is then given by

$$R_h = (\lambda_1 + h)(\lambda_2 + h) = h^2 + h\Delta_2 h + \Delta_{22} h,$$

where Δ_2 denotes the spherical Laplacian and Δ_{22} the Monge-Ampère operator (in other words, $\Delta_2 h$ and $\Delta_{22} h$ are respectively the sum and the product of the eigenvalues λ_1, λ_2 of the Hessian of h).

So, the equation we are dealing with is the following

$$h^2 + h\Delta_2 h + \Delta_{22} h = 1/K.$$

Its type is given by the sign of $1/K$. Thus, the classical Minkowski problem boils down to the study of Monge-Ampère equations of elliptic type since closed convex hypersurfaces of class C^2_+ have a positive Gauss curvature. But for non-convex hedgehogs (which must have hyperbolic regions), we have to deal with Monge-Ampère equations of mixed type on $\mathbb{S}^2$ (a class of equations for which there is no global result but only local ones by C.S. Lin [7] and C. Zuily [23]).

What (necessary and sufficient) conditions must a continuous function on $\mathbb{S}^2$ satisfy to be the curvature function of a hedgehog ? Of course, integral condition (1) is still necessary. It just expresses that any hedgehog of $\mathbb{R}^3$ is a closed surface. But it is no longer sufficient: for instance -1 satisfies this condition and cannot be the curvature function of a hedgehog $\mathcal{H}_h \subset \mathbb{R}^3$ since there is no compact surface with negative Gauss curvature in $\mathbb{R}^3$. Can the curvature function of a hedgehog $\mathcal{H}_h \subset \mathbb{R}^3$ be non-positive all over the unit sphere ? As recalled in the introduction, this problem is in fact equivalent to the study of A.D. Alexandrov's uniqueness conjecture (C) and the answer is positive.

However the answer is negative in the case where the support function is analytic on $\mathbb{S}^2$. Indeed, A.D. Alexandrov (1966) [2] and H.F. Münzner (1967) [18] proved conjecture (C) in the case of analytic surfaces. But in the general case, no progress was made for almost 30 years.

The crucial fact is the existence of a (non compact) cross-cap hedgehog whose curvature function is defined and non-positive on $\mathbb{S}^2$ minus a semi-great-circle. By fitting 4 cross-caps together with a central part, the author constructed a closed surface to which he gave an appropriate saddle form to obtain a non-trivial hedgehog whose curvature function is non-positive all over $\mathbb{S}^2$ [11]. Such a hedgehog is a counter-example to conjecture (H) since it is not reduced to a point. By adding a large enough sphere to it, we get a counter-example to conjecture (C).

It is important to note that the notion of index $i_h(x)$ of a point x with respect to a hedgehog $\mathcal{H}_h \subset \mathbb{R}^2$, that is of winding number of $\mathcal{H}_h$ around x, played an important role in the way the author studied conjecture (H) through orthogonal projection techniques adapted to hedgehogs [11, Theorem 1].

Discrete version

The Minkowski Problem for hedgehogs has of course a discrete version concerning polytopal hedgehogs. In [12], the author presented a discretization of the previous counter-example to conjecture (H) [which is composed of a central part (Figure 5.a) and 4 discrete cross-caps (Figure 5.b)] whose spherical representation on $\mathbb{S}^2$ is shown on Figure 5.c. It is an example of hyperbolic polytopal hedgehog. What does that mean ? As any plane hedgehog $\mathcal{H}_h \subset \mathbb{R}^2$ (that is, any difference of plane convex bodies, smooth or not), a polygonal one can be regarded as an oriented rectifiable curve [16] whose algebraic area is defined as the integral

$$a(h) = \int_{\mathbb{R}^2 - \mathcal{H}_h} i_h(x)\, d\lambda(x),$$

where $i_h(x)$ is the index of x with respect to $\mathcal{H}_h$ and λ the Lebesgue measure on $\mathbb{R}^2$. A polytopal hedgehog of $\mathbb{R}^3$ (that is a difference of two convex polytopes of $\mathbb{R}^3$) is said to be hyperbolic if all its faces are plane hedgehogs with no positive area (that is, whose i_h-index is everywhere non-positive). Of course, such faces may be convex curves: it depends on the orientation.

Figure 5. Discretization of the counter-example to (H)

Monge-Ampère equations of mixed type

Here are examples of Monge-Ampère equations of mixed type with no solution that concern functions changing sign cleanly (as in the only known local results on such PDE's by C.S. Lin [7] and C. Zuily [23]) : For every fixed $v \in \mathbb{S}^2$, the following smooth function $R(u) = 1 - 2\langle u, v\rangle^2$ satisfies integral condition (1) but is not a curvature function on $\mathbb{S}^2$ [13]. The proof makes use of orthogonal projection techniques adapted to hedgehogs.

Now here is a non-trivial example of an equation with non-unique solutions: these two non-isometric hedgehogs of $\mathbb{R}^3$ have a smooth (but not analytic) support function and the same curvature function $R \in C(\mathbb{S}^2;\mathbb{R})$: $\mathcal{H}_f$ and $\mathcal{H}_g$, where $f(x,y,z) = \exp(-1/z^2)$ and $g(x,y,z) = sgn(z)\exp(-1/z^2)$, $((x,y,z) \in \mathbb{S}^2 \subset \mathbb{R}^3)$. Note that the second one is projective. Of course, if a function $f \in C^2(\mathbb{S}^2;\mathbb{R})$ is a solution of the Monge-Ampère equation

$$h^2 + h\Delta_2 h + \Delta_{22} h = R,$$

where $R \in C(\mathbb{S}^2;\mathbb{R})$ satisfies the integral condition

$$\int_{\mathbb{S}^2} uR(u)\, d\sigma(u) = 0,$$

then it is also the case of $g = -f$. But then these two solutions correspond to isometric hedgehogs. In the case where these hedgehogs bound a convex body, one of these hedgehogs will be transversally oriented towards the interior and the other one towards the exterior.

In the convex case, the uniqueness comes from the equality condition in a well-known Minkowski's inequality [21]. But this inequality cannot be extended to hedgehogs [6] and we lose the uniqueness.

2.2. The Sturm-Hurwitz theorem

Another important problem is that of the existence of Sturm-Hurwitz type theorems in dimensions > 2. The classical Sturm-Hurwitz theorem states that any continuous real function of the form

$$h(\theta) = \sum_{n=N}^{+\infty} (a_n \cos n\theta + b_n \sin n\theta)$$

for some sequences of real numbers (a_n) and (b_n), has at least as many zeros as its first nonvanishing harmonics:

$$\#\{\theta \in [0, 2\pi[\ | h(\theta) = 0\} \geq 2N.$$

For C^2-functions, we can give a geometrical interpretation and a geometrical proof by considering the $2N\pi$-periodic function $h(\theta/N)$ as the support function of an 'N-hedgehog' $\mathcal{H}_h \subset \mathbb{R}^2$ that is, of the envelope of a family of cooriented lines having exactly N cooriented support lines with a given normal $u \in \mathbb{S}^1$ [14]. The integer N is just the number of full rotations of the coorienting normal vector. Figure 6.a shows a plane projective hedgehog and Figure 6.b a plane 3-hedgehog.

Figure 6. A projective hedgehog and a 3-hedgehog

Here is a geometrical interpretation of the Sturm-Hurwitz theorem in terms of hedgehogs in the case of C^2-functions [14].

Hedgehog version of the Sturm-Hurwitz theorem. *If $\mathcal{H}_h \subset \mathbb{R}^2$ is an N-hedgehog whose support function satisfies*

$$h(N\theta) = \sum_{n=N}^{+\infty} \left(a_n \cos n\theta + b_n \sin n\theta\right),$$

for some sequences of real numbers (a_n) and (b_n), then $\mathcal{H}_h$ has no 'positive area' (that is, $i_h(x) \leq 0$ for all $x \in \mathbb{R}^2 - \mathcal{H}_h$).

3. Usefulness and limitations of the usual index

The above hedgehog version of the Sturm-Hurwitz theorem is based on the following relationship between the index $i_h(x)$ of x with respect to $\mathcal{H}_h$ and the number of zeros of $h_x(\theta) = h(\theta) - \langle x, u(\theta)\rangle$, where $u(\theta) = (\cos\theta, \sin\theta)$ ([9], [14]).

Theorem [14]. *Let $\mathcal{H}_h \subset \mathbb{R}^2$ be an N-hedgehog whose support function h is of class C^2 on $\mathbb{S}^1$. This N-hedgehog $\mathcal{H}_h$ satisfies:*

$$\forall x \in \mathbb{R}^2 - \mathcal{H}_h,\ i_h(x) = N - \frac{1}{2} n_h(x), \tag{3}$$

where $n_h(x)$ is the number of cooriented support lines through x (i.e. the number of zeros of $h_x : [0, 2N\pi[\rightarrow \mathbb{R}, \theta \longmapsto h(\theta) - \langle x, u(\theta)\rangle$, where $u(\theta) = (\cos\theta, \sin\theta)$). Note that relationship (3) permits us to define $i_h(x) \in \mathbb{Z} \cup \{-\infty\}$ for any $x \in \mathbb{R}^2$.

The geometrical proof of the Sturm-Hurwitz theorem given in [14] consists in proving the above hedgehog version using the two following key points: 1. The evolute of $\mathcal{H}_h \subset \mathbb{R}^2$ is the N-hedgehog with support function $(\partial h)(\theta) = h'\left(\theta - \frac{\pi}{2}\right)$; 2. For every $x \in \mathbb{R}^2$, we have: $i_{\partial h}(x) \leq i_h(x)$.

Remark. The i_h-index can be defined for any plane hedgehog $\mathcal{H}_h \subset \mathbb{R}^2$ (i.e. for any difference $K - L$ of two plane convex bodies). See [16] for applications.

Conclusion. This notion of index of a point x with respect to an N-hedgehog $\mathcal{H}_h \subset \mathbb{R}^2$ and its relationship with the number of zeros of the support function h_x played an essential role in the way the author: 1. geometrized the Sturm-Hurwitz theorem and gave a proof of it [14]; 2. studied conjecture (H) through orthogonal projection techniques [11].

What about the index in higher dimensions ?

Of course, this notion of index $i_h(x)$ can be extended in higher dimensions. Given a hedgehog $\mathcal{H}_h \subset \mathbb{R}^{n+1}$, $(n \geq 1)$, the i_h-index of $x \in \mathbb{R}^{n+1} - \mathcal{H}_h$ with respect to $\mathcal{H}_h$ can be defined as the degree of the map

$$\mathcal{U}_{(h,x)} : \mathbb{S}^n \longrightarrow \mathbb{S}^n,\ u \longmapsto \frac{x_h(u) - x}{\|x_h(u) - x\|}\ ,$$

and interpreted as the algebraic intersection number of an oriented half-line with origin x with the hypersurface $\mathcal{H}_h$ equipped with its transverse orientation (number independent of the oriented half-line for an open dense set of directions). It can be given by

$$i_h(x) = \frac{1}{\omega_n} \int_{\mathbb{S}^n} \frac{(h(u) - \langle x, u \rangle)\, R_h(u)}{\|x_h(u) - x\|^{n+1}} d\sigma(u)\,,$$

where ω_n is the surface area of the unit ball of $\mathbb{R}^{n+1}$, σ the spherical Lebesgue measure on $\mathbb{S}^n$ and R_h the curvature function of $\mathcal{H}_h$.

Remark. Many notions from the theory of convex bodies carry over to hedgehogs, and quite a number of classical results find their counterparts. Of course, certain adaptations are necessary. In particular, areas and volumes have to be replaced by their algebraic versions, which can also attain negative values. For example, the (algebraic $(n+1)$-dimensional) volume of a hedgehog $\mathcal{H}_h \subset \mathbb{R}^{n+1}$ can be defined by

$$V(h) := \int_{\mathbb{R}^{n+1}-\mathcal{H}_h} i_h(x)\, d\lambda(x),$$

where λ denotes the Lebesgue measure on $\mathbb{R}^{n+1}$, and satisfies

$$V(h) = \frac{1}{n+1} \int_{\mathbb{S}^n} h(u) R_h(u) d\sigma(u),$$

where R_h is the curvature function of $\mathcal{H}_h$ and σ the spherical Lebesgue measure on $\mathbb{S}^n$. In [8], the author studied the extension to hedgehogs of classical geometrical inequalities for convex bodies, such as Brunn-Minkowski, Minkowski and Alexandrov-Fenchel type inequalities.

The i_h-index remains natural in higher dimensions but it is no longer relevant for studying our two problems even in dimension 3. It appears that we need new specific tools for studying hedgehogs in higher dimensions and that the parity of the dimension might be an important datum. To understand it, consider the case of projective hedgehogs of $\mathbb{R}^3$, that is the case of hedgehogs $\mathcal{H}_h \subset \mathbb{R}^3$ whose support function is antisymmetric on $\mathbb{S}^2$. Recall that $i_h(x)$ can be regarded as the algebraic intersection number of almost every oriented half-line with origin x with $\mathcal{H}_h$ equipped with its transverse orientation. Therefore, if $\mathcal{H}_h$ is projective, then the map $x \mapsto i_h(x)$ is identically equal to 0 on $\mathbb{R}^3 - \mathcal{H}_h$ so that it gives no information neither on $\mathcal{H}_h$ nor on zeros of the function $h_x(u) = h(u) - \langle x, u \rangle$, $(u \in \mathbb{S}^2)$.

Index of $\mathcal{H}_h \subset \mathbb{R}^3$ at a point x and sign of $h_x(u) = h(u) - \langle x, u \rangle$

Let $\mathcal{H}_h \subset \mathbb{R}^3$ be a hedgehog whose support function h is of class C^2 on $\mathbb{S}^2$. For every $x \in \mathbb{R}^3$, define $h_x \in C^2(\mathbb{S}^2; \mathbb{R})$ by $h_x(u) := h(u) - \langle x, u \rangle$, $(u \in \mathbb{S}^2)$: $h_x(u)$ may be interpreted as the signed distance from x to the support hyperplane cooriented by u. This function h_x is such that: $\forall u \in \mathbb{S}^2$, $x_{h_x}(u) = h_x(u)\, u + (\nabla h_x)(u) = x_h(u) - x$. Thus, for every $x \in \mathbb{R}^3 - \mathcal{H}_h$, $(\nabla h_x)(u) \neq 0$ whenever $h_x(u) = 0$. Consequently, we can state the following.

Remark. *For every $x \in \mathbb{R}^3 - \mathcal{H}_h$, the set $h_x^{-1}(\{0\})$ consists of a finite number of disjoint simple smooth closed curves of $\mathbb{S}^2$ on which h_x changes sign cleanly.*

Theorem 1. *Let $\mathcal{H}_h \subset \mathbb{R}^3$ be a hedgehog with support function $h \in C^2(\mathbb{S}^2; \mathbb{R})$. For every $x \in \mathbb{R}^3$, define $h_x \in C^2(\mathbb{S}^2; \mathbb{R})$ by $h_x(u) := h(u) - \langle x, u \rangle$, $(u \in \mathbb{S}^2)$: $h_x(u)$ may be interpreted as the signed distance from x to the support hyperplane cooriented by u. We have: $\forall x \in \mathbb{R}^3 - \mathcal{H}_h$,*

$$i_h(x) = r_h^+(x) - r_h^-(x),$$

where $r_h^-(x)$ *(resp.* $r_h^+(x)$*) denotes the number of connected components of* $\mathbb{S}^2 - h_x^{-1}(\{0\})$ *on which* h_x *is negative (resp. positive).*

Sketch of the proof. Let $x \in \mathbb{R}^3 - \mathcal{H}_h$. Let $c_h(x)$ denote the number of connected components of $h_x^{-1}(\{0\}) \subset \mathbb{S}^2$, that is the number of spherical curves corresponding to points of $\mathcal{H}_h$ at which the support plane passes through x. Note that:

$$c_h(x) = r_h^-(x) + r_h^+(x) - 1.$$

The proof is based on the two following lemmas.

Lemma 1. *The map* $x \mapsto i_h(x) - \left(r_h^+(x) - r_h^-(x)\right)$ *is constant on* $\mathbb{R}^3 - \mathcal{H}_h$.

Sketch of the proof of Lemma 1. The first step consists in proving that the map $x \mapsto i_h(x) - \left(r_h^+(x) - r_h^-(x)\right)$ is constant on each connected component Ω of $\mathbb{R}^3 - \mathcal{H}_h$ by noticing that $x \mapsto r_h^-(x)$, $x \mapsto r_h^+(x)$ and thus $x \mapsto c_h(x)$ are constant on Ω. The second one consists in proving that the map $x \mapsto i_h(x) - \left(r_h^+(x) - r_h^-(x)\right)$ remains constant as x crosses $\mathcal{H}_h$ transversally at a regular point m (by distinguishing the case where m is elliptic and the one where m is hyperbolic). $\square$

Lemma 2. *If the Euclidean norm of* x *is sufficiently large, then* $c_h(x) = 1$.

Sketch of the proof of Lemma 2. Lemma 2 essentially follows from the fact that the natural parametrization $x_h : \mathbb{S}^2 \to \mathcal{H}_h, u \mapsto h(u)u + (\nabla h)(u)$ can be interpreted as the inverse of the Gauss map of $\mathcal{H}_h$. $\square$

Lemma 2 implies that $r_h^-(x) = r_h^+(x) = 1$ when the Euclidean norm of x is sufficiently large. Thus Theorem 1 follows from Lemma 1. $\square$

4. New notion of index in $\mathbb{R}^3$ and applications

Now, here is a more appropriate notion of index for studying hedgehogs of $\mathbb{R}^3$.

Definition. *Let* $\mathcal{H}_h \subset \mathbb{R}^3$ *be a hedgehog with support function* $h \in C^2(\mathbb{S}^2;\mathbb{R})$. *For every* $x \in \mathbb{R}^3$, *define* $h_x \in C^2(\mathbb{S}^2;\mathbb{R})$ *by* $h_x(u) := h(u) - \langle x, u\rangle, (u \in \mathbb{S}^2)$: $h_x(u)$ *may be interpreted as the signed distance from* x *to the support hyperplane cooriented by* u. *For every* $x \in \mathbb{R}^3 - \mathcal{H}_h$, *define the* j_h*-index* of x *with respect to* $\mathcal{H}_h$ *by:*

$$j_h(x) := 1 - c_h(x),$$

where $c_h(x)$ denotes the number of connected components of $h_x^{-1}(\{0\}) \subset \mathbb{S}^2$, that is the number of spherical curves corresponding to points of $\mathcal{H}_h$ at which the support plane passes through x.

In certain respects, this j_h-index can play in $\mathbb{R}^3$ the same role as the i_h-index does in $\mathbb{R}^3$ (compare the definition of $j_h(x)$ with the relationship between the i_h-index of x with respect to $\mathcal{H}_h \subset \mathbb{R}^2$ and the number of zeros of the function $h_x(u) = h(u) - \langle x, u\rangle, (u \in \mathbb{S}^1)$).

Let $\mathcal{H}_h \subset \mathbb{R}^3$ be a hedgehog with a C^2-support function. When the Euclidean norm of $x \in \mathbb{R}^3$ is sufficiently large, $c_h(x)$ must be equal to 1 (see Lemma 2) and thus $j_h(x)$ to 0. In other words, the map $x \mapsto j_h(x)$ is identically equal to 0 on the unbounded connected component of $\mathbb{R}^3 - \mathcal{H}_h$. Note that we may have $j_h(x) = 0$ on a bounded connected component of $\mathbb{R}^3 - \mathcal{H}_h$.

Remark. The value of $j_h(x)$ must obviously decrease as x crosses $\mathcal{H}_h$ transversally at an elliptic point from locally convex to locally concave side.

Additional definitions. Here are some additional definitions to describe the geometry of hedgehogs of $\mathbb{R}^3$. The interior (resp. the exterior) of $\mathcal{H}_h \subset \mathbb{R}^3$ relative to its j_h-index, or *j_h-interior* (resp. *j_h-exterior*) *of* $\mathcal{H}_h$, will be defined by:

$$\mathcal{J}_h = \{x \in \mathbb{R}^3 - \mathcal{H}_h \,|\, j_h(x) \neq 0\}$$

$$(\text{resp. } \mathcal{F}_h = \{x \in \mathbb{R}^3 - \mathcal{H}_h \,|\, j_h(x) = 0\}).$$

Recall that the interior (resp. exterior) of $\mathcal{H}_h$ relative to the i_h-index is usually defined by

$$\mathcal{I}_h = \{x \in \mathbb{R}^3 - \mathcal{H}_h \,|\, i_h(x) \neq 0\}$$

$$(\text{resp. } \mathcal{E}_h = \{x \in \mathbb{R}^3 - \mathcal{H}_h \,|\, i_h(x) = 0\}).$$

For all $x \in \mathbb{R}^3 - \mathcal{H}_h$, $j_h(x) = 1 - c_h(x) = 0$ implies $i_h(x) = r_h^+(x) - r_h^-(x) = 0$. Therefore $\mathcal{I}_h \subset \mathcal{J}_h$. This inclusion may be strict as shown by the example of non-trivial projective hedgehogs of $\mathbb{R}^3$ (see geometrical applications below): indeed, for such a hedgehog $\mathcal{H}_h \subset \mathbb{R}^3$, we have $\mathcal{I}_h = \emptyset$ and $\mathcal{J}_h \neq \emptyset$.

Recall that we defined the convex interior of a plane hedgehog $\mathcal{H}_h \subset \mathbb{R}^2$ as the following convex subset of $\mathbb{R}^2$ [9]:

$$\mathcal{C}_h = \left\{x \in \mathbb{R}^2 - \mathcal{H}_h \,|\, i_h(x) = 1\right\}.$$

Similarly, we define the *convex interior* of a hedgehog $\mathcal{H}_h \subset \mathbb{R}^3$ as the following convex subset of $\mathbb{R}^3$:

$$\mathcal{C}_h = \left\{ x \in \mathbb{R}^3 - \mathcal{H}_h \,|\, j_h(x) = 1 \right\}$$

(The method to prove the convexity of $\mathcal{C}_h$ is absolutely the same in both cases).

This new notion of index also implies a new notion of (algebraic) volume. The volume of $\mathcal{H}_h$ relative to its j_h-index, or *j_h-volume of $\mathcal{H}_h$*, will be defined as the real number:

$$V_{\mathcal{J}}\left(h\right) := \int_{\mathbb{R}^3 - \mathcal{H}_h} j_h\left(x\right) d\lambda\left(x\right),$$

where λ denotes the Lebesgue measure on $\mathbb{R}^3$.

Remark. We may define a mixed volume $V_{\mathcal{J}} : C^2\left(\mathbb{S}^2; \mathbb{R}\right)^3 \longrightarrow \mathbb{R}$ as the symmetric map

$$\begin{aligned} & V_{\mathcal{J}} : C^2\left(\mathbb{S}^2; \mathbb{R}\right)^3 \longrightarrow \mathbb{R}, \\ & \quad (h_1, h_2, h_3) \longmapsto V_{\mathcal{J}}\left(h_1, h_2, h_3\right) = \tfrac{1}{3!} \sum_{k=1}^{3} (-1)^{3+k} \sum_{i_1 < \ldots < i_k} V_{\mathcal{J}}\left(h_{i_1} + \ldots + h_{i_k}\right). \end{aligned}$$

But this mixed volume $V_{\mathcal{J}} : C^2\left(\mathbb{S}^2; \mathbb{R}\right)^3 \longrightarrow \mathbb{R}$ does not satisfy

$$V_{\mathcal{J}}\left(\lambda_1 h_1 + \ \ldots \ + \lambda_m h_m\right) = \sum_{i_1, i_2, i_3 = 1}^{m} \lambda_{i_1} \lambda_{i_2} \lambda_{i_3} V_{\mathcal{J}}(h_{i_1}, h_{i_2}, h_{i_3}),$$

for all $h_1, \ \ldots, \ h_m \in C^2\left(\mathbb{S}^2; \mathbb{R}\right)$ and all $\lambda_1, \ \ldots, \ \lambda_m \in \mathbb{R}$.

Case of polytopal hedgehogs

Of course, we can extend the definition of the j_h-index to hedgehogs of $\mathbb{R}^3$ whose support function is not of class C^2 on $\mathbb{S}^2$. In particular, we can define it for any polytopal hedgehog $\mathcal{H}_h \subset \mathbb{R}^3$ (that is for any difference $P - Q$ of two convex polytopes of $\mathbb{R}^3$) and the conclusion of Theorem 1 still holds for such a hedgehog.

Examples of geometrical applications

As an example of application, let us see some geometrical consequences for projective hedgehogs of $\mathbb{R}^3$. By convention, we shall say that $x = x_h\left(u\right)$ is a simple point of a projective hedgehog $\mathcal{H}_h \subset \mathbb{R}^3$ if $-u$ and u are the only two elements of $\mathbb{S}^2$ that are mapped to x by the parametrization $x_h : \mathbb{S}^2 \to \mathcal{H}_h$.

Theorem 2. *Let $\mathcal{H}_h \subset \mathbb{R}^3$ be a projective hedgehog whose (antisymmetric) support function is of class C^2 on $\mathbb{S}^2$. The following properties are satisfied:*

(i) For every $x \in \mathbb{R}^3 - \mathcal{H}_h$, we have $j_h(x) = 1 - c_h(x) \leq 0$. In particular, the j_h-volume of $\mathcal{H}_h$ is non-positive: $V_{\mathcal{J}}(h) \leq 0$;

(ii) Let $x_h(u)$ be a simple elliptic point of $\mathcal{H}_h$ adherent to the j_h-exterior. Then $\mathcal{H}_h$ turns its convexity towards its j_h-interior at $x_h(u)$ (in other words, there exists a neighbor of $x_h(u)$ in $\mathbb{R}^3$ in which the support plane with equation $\langle x, u \rangle = h(u)$ does not intersect the j_h-exterior of $\mathcal{H}_h$);

(iii) $\mathcal{H}_h$ is included in the convex hull of its singularities;

(iv) The j_h-volume of $\mathcal{H}_h$ is negative if $\mathcal{H}_h$ is not reduced to a single point.

Proof of Theorem 2. Property (i). Since h_x is antisymmetric (and non identically equal to zero) on $\mathbb{S}^2$, it must change sign on $\mathbb{S}^2$, so that $c_h(x) \geq 1$.

Property (ii). From (i), as x crosses $\mathcal{H}_h$ transversally at $x_h(u)$ in the direction of its j_h-interior, $j_h(x)$ must decrease from 0 to -2 (knowing that the j_h-index of a projective hedgehog $\mathcal{H}_h \subset \mathbb{R}^3$ takes its values in $2\mathbb{Z}$ since the parametrization x_h describes the surface twice). In other words, x is then crossing $\mathcal{H}_h$ transversally at $x_h(u)$ from locally convex to locally concave side.

Property (iii) is an immediate consequence of property (ii).

Property (iv). A nontrivial projective hedgehog $\mathcal{H}_h \subset \mathbb{R}^3$ must have elliptic points (see [11]) so that its j_h-index cannot be identically equal to 0 on $\mathbb{R}^3 - \mathcal{H}_h$.□

Remarks. 1. Property (iii) already appeared in [9].

2. Properties $(i) - (iv)$ have to be compared with the corresponding properties of plane projective hedgehogs (for which, of course, i_h is replacing j_h) [9].

3. It is not difficult to check that properties $(i) - (iv)$ still hold for any hedgehog $\mathcal{H}_h \subset \mathbb{R}^3$ whose support function h satisfies

$$\int_{\mathbb{S}^2} h(u)\; d\sigma(u) = 0,$$

where σ denotes the spherical Lebesgue measure on $\mathbb{S}^2$ (that is, whose integral of mean curvature is equal to 0 [8]). As a consequence, if $\mathcal{H}_h \subset \mathbb{R}^3$ is such a hedgehog (with a sufficiently smooth support function) having only generic singularities, then no negative swallowtail of $\mathcal{H}_h$ can be visible from the j_h-exterior.

This example of projective hedgehogs shows that this j_h-index is more appropriate than the i_h-index for studying the geometry of hedgehogs in $\mathbb{R}^3$.

Transverse orientation relative to the j_h-index

This new notion of index induces a natural notion of transverse orientation (which may switch on certain curves of self-intersection) for any hedgehog $\mathcal{H}_h$ of $\mathbb{R}^3$ with a C^2-support function: at each simple regular point $x_h(u)$ of $\mathcal{H}_h$, orient the normal line in the direction of decreasing values of $j_h(x)$ and then define $\varepsilon_h(u) \in \{-1, 1\}$ in order that

$$\nu_h(u) = \varepsilon_h(u)\, sign[1/K_h(u)]u$$

be the corresponding unit normal, where $K_h(u)$ is the Gauss curvature of $\mathcal{H}_h$ at $x_h(u)$. For other $u \in \mathbb{S}^2$, let $\varepsilon_h(u) = 0$.

Unless otherwise stated, from now on 'transverse orientation' will mean 'transverse orientation relative to the j_h-index'.

Case of convex (resp. projective) hedgehogs

Remark. For any hedgehog $\mathcal{H}_h \subset \mathbb{R}^3$ with a C^2-support function, consider the hedgehog $\mathcal{H}_{\widetilde{h}} \subset \mathbb{R}^3$ with support function $\widetilde{h}(-u) = -h(u)$, $(u \in \mathbb{S}^2)$. These two hedgehogs $\mathcal{H}_h$ and $\mathcal{H}_{\widetilde{h}}$ have the same geometrical realization: $\forall u \in \mathbb{S}^2$, $x_{\widetilde{h}}(-u) = x_h(u)$. For every $u \in \mathbb{S}^2$, the support hyperplane of $\mathcal{H}_h$ cooriented by u is the support hyperplane of $\mathcal{H}_{\widetilde{h}}$ cooriented by $-u$, so that: $\forall u \in \mathbb{S}^2$, $\varepsilon_{\widetilde{h}}(-u) = -\varepsilon_h(u)$. In the case of a nonsingular hedgehog $\mathcal{H}_h \subset \mathbb{R}^3$, which must bound a convex body K, $\mathcal{H}_h$ is transversally oriented towards the exterior of K. Moreover, in this case, for any interior point x of K, we have: $\varepsilon_h = sign(h_x)$, where $h_x(u) := h(u) - \langle x, u\rangle$, $(u \in \mathbb{S}^2)$. In the case of a projective hedgehog $\mathcal{H}_h \subset \mathbb{R}^3$, we have $\widetilde{h} = h$ and thus: $\forall u \in \mathbb{S}^2$, $\varepsilon_h(-u) = -\varepsilon_h(u)$.

Switches of transverse orientation on a hedgehog of $\mathbb{R}^3$

It follows that non-trivial projective hedgehogs of $\mathbb{R}^3$ necessary present switches of transverse orientation on certain curves of self-intersection. For instance, in the example $\mathcal{H}_h \subset \mathbb{R}^3$ of a projective hedgehog version of the Roman surface represented in Figure 2, the transverse orientation switches on the three curves of self-intersection.

Integral condition

The j_h-volume of a hedgehog $\mathcal{H}_h \subset \mathbb{R}^3$ can be given by:

$$\int_{\mathbb{S}^2} \varepsilon_h(u)\, h(u)\, \frac{u}{K_h(u)}\, d\sigma(u) = 0,$$

where σ denotes the spherical Lebesgue measure on $\mathbb{S}^2$ and K_h the Gauss curvature of $\mathcal{H}_h$. From the translation invariance of this algebraic volume, we get immediately the following integral condition (which has to be compared with integral condition (1)):

$$\int_{\mathbb{S}^2} \varepsilon_h(u) \frac{u}{K_h(u)} d\sigma(u) = 0.$$

On ε_h functions and the non-uniqueness in the Minkowski problem

In the following example of non-uniqueness in the Minkowski problem, it is interesting to note that these two non-isometric hedgehogs which have the same curvature function on $\mathbb{S}^2$ correspond to different ε_h functions: $\mathcal{H}_f$ and $\mathcal{H}_g$, where $f(x,y,z) = \exp(-1/z^2)$ and $g(x,y,z) = sgn(z)\exp(-1/z^2)$, $((x,y,z) \in \mathbb{S}^2 \subset \mathbb{R}^3)$. In the same order of ideas, if two hedgehogs $\mathcal{H}_f$ and $\mathcal{H}_g$ are two hypersurfaces bounding the same centrally symmetric convex body $K \subset \mathbb{R}^3$ but equipped with opposite (usual) transverse orientations, then they have the same curvature function but opposite ε_h functions. This suggests that the study of the multiplicity of solutions in the Minkowski problem for hedgehogs should take into account these ε_h functions.

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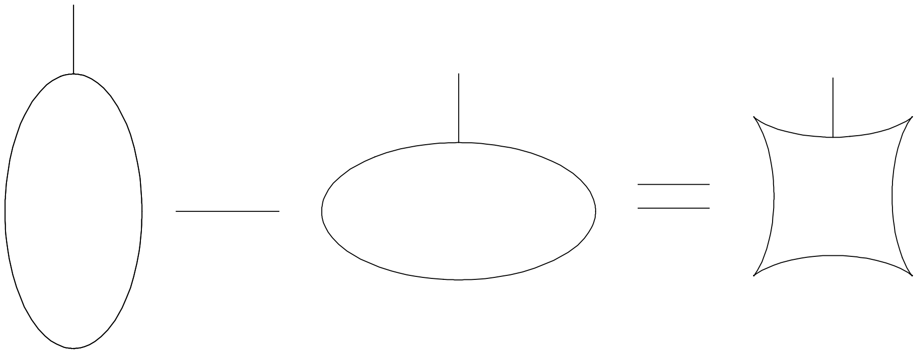

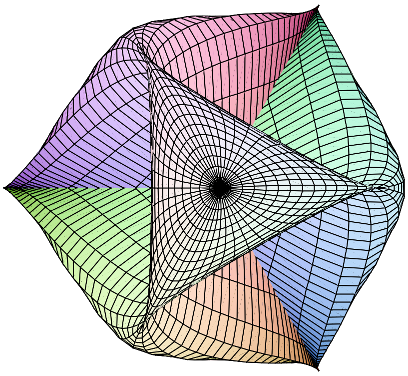

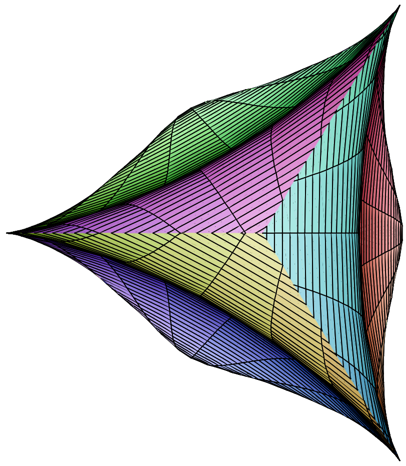

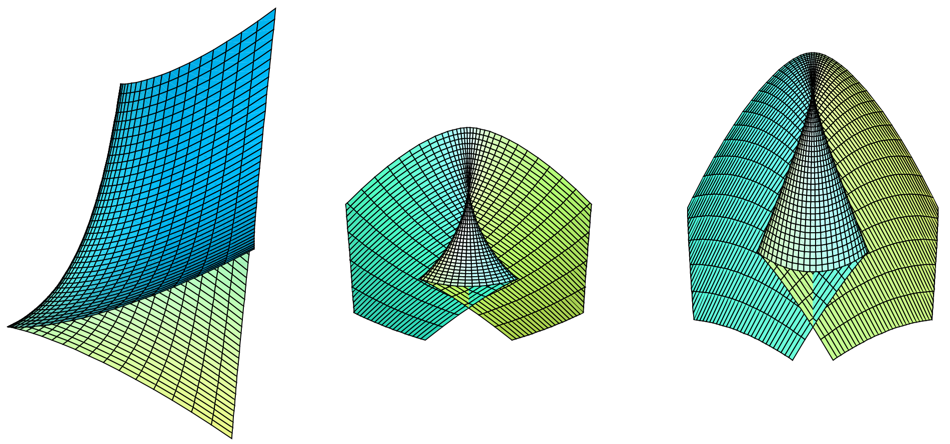

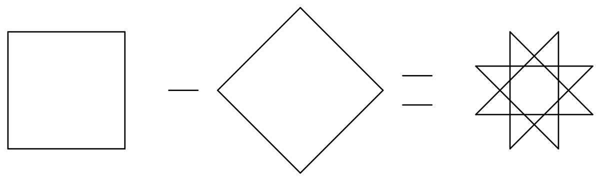

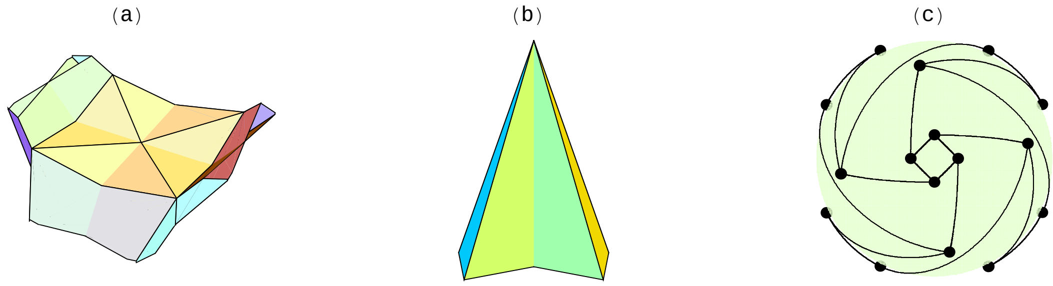
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