Central Limit Theorem for a Class of Relativistic Diffusions

Jürgen Angst, Jacques Franchi

To cite this version:
Jürgen Angst, Jacques Franchi. Central Limit Theorem for a Class of Relativistic Diffusions. 20 pages. 2007. <hal-00131422v2>

HAL Id: hal-00131422
https://hal.archives-ouvertes.fr/hal-00131422v2
Submitted on 4 May 2007

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L’archive ouverte pluridisciplinaire HAL, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d’enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.
Central Limit Theorem for a Class of Relativistic Diffusions

JÜRGEN ANGST AND JACQUES FRANCHI

Abstract: Two similar Minkowskian diffusions have been considered, on one hand by Barbachoux, Debbasch, Malik and Rivet ([BDR1], [BDR2], [BDR3], [DMR], [DR]), and on the other hand by Dunkel and Hänggi ([DH1], [DH2]). We address here two questions, asked in [DR] and in ([DH1], [DH2]) respectively, about the asymptotic behaviour of such diffusions. More generally, we establish a central limit theorem for a class of Minkowskian diffusions, to which the two above ones belong. As a consequence, we correct a partially wrong guess in [DH1].

1 Introduction

Debbasch, Malik and Rivet introduced in [DMR] a relativistic diffusion in Minkowski space, they called Relativistic Ornstein-Uhlenbeck Process (ROUP), to describe the motion of a point particle surrounded by a heat bath, or relativistic fluid, with respect to the rest-frame of the fluid, in which the particle diffuses. This ROUP was then studied in [BDR1], [BDR2], [BDR3], [DR], and extended to the curved case in [D], Then Dunkel and Hänggi introduced and discussed in [DH1], [DH2] a similar process, in Minkowski space, they called relativistic Brownian Motion.

Note that independently, a Relativistic Diffusion on any Lorentz manifold was defined in [FLJ], as the only diffusion whose law possesses the relativistic invariance under the whole isometry group of the manifold. Accordingly, the particular case of the Schwarzschild-Kruskal-Szekeres manifold was studied. The case of Gödel’s universe was recently studied in [F].

In [DR], Debbasch and Rivet argue qualitatively that the so-called “hydrodynamical limit” of their ROUP should behave in a Brownian way. They stress that a mathematical rigorous proof remains needed, to confirm such not much intuitive statement.

In [DH1] and [DH2], Dunkel and Hänggi ask the question of the asymptotic behaviour of the variance, or “mean square displacement”, of their diffusion. Indeed, comparing to the non-relativistic case, and after numerical computations, they guess that this variance, normalised by time, should converge, to some constant for which they conjecture an empirical formula.

We answer here these two questions, asked by Debbasch and Rivet in [DR], and by Dunkel and Hänggi in [DH1], [DH2], and indeed a more general one. We establish in fact rigorously in this article, a central limit theorem for a class of Minkowskian diffusions, to which the two above mentioned ones, ROUP and Dunkel-Hänggi (DH) diffusion, belong. As a consequence of our main result, we establish, for this whole class, the convergence of the normalised variance, guessed (for their particular case) in [DH1] and [DH2]. Getting the exact expression for this limiting variance, and particularising to the (DH) diffusion, we can then invalidate and correct the wrong conjecture made in [DH1] on its expression and asymptotic behaviour (as the noise parameter goes to infinity).

To summarise the content, we begin by describing in Section 2 below the class of Minkowskian diffusions we consider, which contains both ROUP and DH diffusions as particular cases.

Then in Section 3, we present our study, leading to the following main result:
Theorem 1 Let \((x_t, p_t)_{t \geq 0} = (x_i^t, p_i^t)_{1 \leq i \leq d, t \geq 0}\) be a \(\mathbb{R}^d \times \mathbb{R}^d\)-valued diffusion solving the stochastic differential system: for \(1 \leq i \leq d\),

\[
(*) \quad \begin{cases} 
    dx_i^t = f(|p_t|) p_i^t \, dt \\
    dp_i^t = -b(|p_t|) p_i^t \, dt + \sigma(|p_t|) \left( \beta |1 + \eta(|p_t|)^2| \right)^{-1/2} \, dW_i^t + \eta(|p_t|)(p_i^t/|p_t|) \, dw_t
\end{cases}
\]

Then, under some hypotheses \((\mathcal{H})\) on the continuous functions \(f, b, \sigma, \eta\), the law of the process \((t^{-1/2} x_{at})_{a \geq 0}\) converges, as \(t \to \infty\), to the law of \((\Sigma_\beta B_a)_{a \geq 0}\), in \(C(\mathbb{R}^+, \mathbb{R}^d)\) endowed with the topology of uniform convergence on compact sets of \(\mathbb{R}^+\). Here \(W\) and \(B\) are standard \(d\)-dimensional Brownian motions, \(w\) is a standard real Brownian motion, independent of \(W\), \(\beta > 0\) is an inverse noise or heat parameter, and \(\Sigma_\beta\) is a constant, displayed in Proposition 3 below.

The following is then deduced. Recall that the symbol \(\mathbb{E}\) stands for expected or mean value, with respect to the underlying probability measure (or distribution) \(\mathbb{P}\) (governing the given Brownian motions \(W, w\)).

Corollary 2 Under the same hypotheses as in the above theorem, from any starting point, the normalised variance (mean square displacement) \(t^{-1} \mathbb{E} \left[ |x_t|^2 \right] \) goes, as \(t \to \infty\), towards \(d \times \Sigma_\beta^2\).

In Section 4, we study the behaviours of the limiting variance \(\Sigma_\beta^2\), as the inverse noise parameter \(\beta\) goes to 0 or to \(\infty\), and we also support our result by numerical simulations. Focussing on the particular case of the DH diffusion, for \(d = 1\), we get the following, which, though confirming a non-classical variance behaviour, shows up a behaviour near 0 which differs from the one implied by the wrong guess made in [DH1] about the expression of \(\Sigma_\beta^2\).

Proposition 4 Consider the DH case, for \(d = 1\), as in [DH1]. Then, we have

\[\Sigma_\beta^2 \sim 2/\beta \quad \text{as} \quad \beta \nearrow \infty; \quad \text{and, as} \quad \beta \searrow 0: \quad \Sigma_\beta^2 \sim \frac{A}{\log(1/\beta)} \quad \text{for some explicit constant} \quad A > 0.\]

Finally, we detail in Sections 5 and 6 two somewhat involved proofs. We thank Reinhard Schäfke for his kind and decisive help for the proof of Sections 5.1 and 6.2.

2 A class of Minkowskian diffusions

Let \(\mathbb{R}^{1,d}\), where \(d \geq 1\) is an integer, denote the usual Minkowski space of special relativity. In its canonical basis, denote by \(x = (x^\mu) = (x^0, x^i) = (x^0, \mathbf{x})\) the coordinates of the generic point, with greek indices running 0,..,\(d\) and latin indices running 1,..,\(d\). The Minkowskian pseudo-metric is given by: \(ds^2 = |dx^0|^2 - \sum_{i=1}^{d} |dx^i|^2\).

The world line of a particle having mass \(m\) is a timelike path in \(\mathbb{R}^{1,d}\), which we can always parametrize by its arc-length, or proper time \(s\). So the moves of such particle are described by a path \(s \mapsto (x^\mu_s)\), having momentum \(p = (p_s)\) given by:

\[p = (p^\mu) = (p^0, p^i) = (p^0, \mathbf{p}), \quad \text{where} \quad p^\mu_s := m \frac{dx^\mu_s}{ds},\]

and satisfying:

\[|p^0|^2 - \sum_{i=1}^{d} |p^i|^2 = m^2.\]
We shall consider here world lines of type \( (t, \mathbf{x}(t))_{t \geq 0} \), and take \( m = 1 \). Introducing the velocity \( \mathbf{v} = (v^1, \ldots, v^d) \) and polar coordinates \((r, \Theta)\) by setting:

\[
v^i := \frac{dx^i}{dt}, \quad r := |\mathbf{p}| = \left( \sum_{i=1}^{d} |p^i|^2 \right)^{1/2} \quad \text{and} \quad \Theta := \frac{\mathbf{p}}{r} =: (\theta^1, \ldots, \theta^d) \in \mathbb{S}^{d-1},
\]

we get at once:

\[
p^0 = \frac{dt}{ds} = \sqrt{1 + r^2} = (1 - |\mathbf{v}|^2)^{-1/2} \quad \text{and} \quad p = \sqrt{1 + r^2} (1, \mathbf{v}).
\]

Thus, a full space-time trajectory

\[
(x(t), p(t))_{t \geq 0} = (t, \mathbf{x}(t), p^0(t), \mathbf{p}(t))_{t \geq 0}
\]
is determined by the mere knowledge of its spacial component \((\mathbf{x}(t), \mathbf{p}(t))\).

We can therefore, from now on, focus on spacial trajectories \( t \mapsto (\mathbf{x}_t, \mathbf{p}_t) \in \mathbb{R}^d \times \mathbb{R}^d \).

The Minkowskian diffusions we consider here are associated as above, to Euclidian diffusions \( t \mapsto (x^i_t, p^i_t)_{1 \leq i \leq d} \), which are the solution to a stochastic differential system of the following type:

\[
(\star) \begin{cases}
    dx^i_t = f(r_t) p^i_t \, dt \\
    dp^i_t = -b(r_t) p^i_t dt + \sigma(r_t) \left( \beta [1 + \eta(r_t)^2] \right)^{-1/2} \, [dW^i_t + \eta(r_t) \theta^i_t \, dw] 
\end{cases}, \quad \text{for} \ 1 \leq i \leq d,
\]

where \( W := (W^1, \ldots, W^d) \) denotes a standard \( d \)-dimensional Euclidian Brownian motion, \( w \) denotes a standard real Brownian motion, independent of \( W \), \( \beta > 0 \) is an inverse noise or heat parameter, and the real functions \( f, b, \sigma, \eta \) are continuous on \( \mathbb{R}_+ \) and satisfy the following hypotheses, for some fixed \( \varepsilon > 0 \):

\[
(\mathcal{H}) \quad \sigma \geq \varepsilon \ \text{on} \ \mathbb{R}_+; \quad g(r) := \frac{2r b(r)}{\sigma^2(r)} \geq \varepsilon \ \text{for} \ \text{large} \ r; \quad \lim_{r \to \infty} e^{-\varepsilon' r} f(r) = 0 \ \text{for some} \ \varepsilon' < \frac{4\varepsilon}{3}.
\]

Of course, in the particular case of constant functions \( f, b, \sigma, \) and \( \eta = 0 \), the process \((\mathbf{x}_t)\) is an integrated Ornstein-Uhlenbeck process. The process considered by Debbasch, Malik and Rivet ([BDR1], [BDR2], [BDR3], [DMR], [DR]), they call Relativistic Ornstein-Uhlenbeck Process (ROUP), corresponds to:

\[
(\text{ROUP}) \quad f(r) = b(r) = (1 + r^2)^{-1/2}, \quad \sigma(r) = \sqrt{2}, \quad \eta = 0, \quad g(r) = r (1 + r^2)^{-1/2},
\]

and the relativistic process considered by Dunkel and Hänggi ([DH1], [DH2]) corresponds to:

\[
(\text{DH}) \quad f(r) = (1 + r^2)^{-1/2}, \quad b(r) = 1, \quad \sigma(r) = \sqrt{2\sqrt{1 + r^2}}, \quad \eta(r) = r, \quad g(r) = r (1 + r^2)^{-1/2}.
\]

These processes are intended to describe the motion of a point particle surrounded by a heat bath, or relativistic fluid, with respect to the rest-frame of the fluid, in which the particle diffuses. The Minkowskian diffusion \((\mathbf{x}_t, \mathbf{p}_t)\) solving the stochastic differential system \((\star)\) is isotropic precisely when \( \eta \equiv 0 \) (for \( d \geq 2 \); when \( d = 1 \), \( \eta \) does not matter). If \( \eta \neq 0 \), the momentum \((\mathbf{p}_t)\) undergoes a radial drift.
In ([DR], Section 4), Debbasch and Rivet argue heuristically that the so-called “hydrodynamical limit” of their Roup should behave in a Brownian way, and ask the question of a mathematical proof confirming such not much intuitive statement.

In ([DH1], [DH2]), Dunkel and Hänggi ask the question of the convergence, as $t$ goes to infinity, of the normalised variance (or mean square displacement):

$$\Sigma^2(t) := t^{-1} \mathbb{E} \left( \sum_{i=1}^{d} |x_i|^2 \right) = \mathbb{E} \left[ |X_t|^2 / t \right].$$

We shall answer these two questions, by means of the more general one we address, which is the asymptotic behaviour, as $t \to \infty$, of the process:

$$\left( x^t_a \right)_{a \geq 0} := \left( t^{-1/2} x_{at} \right)_{a \geq 0} = t^{-1/2} \left( x^1_{at}, \ldots, x^d_{at} \right)_{a \geq 0},$$

where the diffusion $(x_t, \rho_t)_{t \geq 0}$ solves $(\ast)$, under the hypotheses $(\mathcal{H})$.

## 3 Asymptotic behaviour of the process $(x^t_a)_{a \geq 0}$

### 3.1 An auxiliary function $F$

Let us look for a function $F = (F^1, \ldots, F^d) \in C^2(\mathbb{R}^d, \mathbb{R}^d)$ such that for $1 \leq i \leq d$:

$$dF^i(p_t) + dx^i_t - dM^i_t = 0,$$

for some martingale $M_t = (M^1_t, \ldots, M^d_t)$, so that $t^{-1/2} x^i_{at} = t^{-1/2} M^i_{at} - t^{-1/2} F^i(p_{at})$.

Now, Itô’s Formula gives:

$$dF^i(p_t) = \left[ -\sum_{j=1}^{d} \partial_j F^i(p_t) p^j_t b(r_t) + \frac{\sigma^2(r_t)}{2 \beta \left( 1 + \eta(r_t)^2 \right)} \sum_{1 \leq j, k \leq d} \left( \delta_{jk} + \eta(r_t)^2 \theta^j_r \theta^k_r \right) \partial^2_{jk} F^i(p_t) \right] dt + dM^i_t,$$

with

$$dM^i_t = \left( \beta \left( 1 + \eta(r_t)^2 \right) \right)^{-1/2} \sigma(r_t) \sum_{j=1}^{d} \partial_j F^i(p_t) [dW^j_t + \eta(r_t) \theta^j_t dw_t].$$

Note that, in other words, this means that the so-called infinitesimal generator of the momentum diffusion $(p_t)$ is

$$\frac{\sigma^2(r)}{2 \beta \left( 1 + \eta(r)^2 \right)} \left( \Delta + \eta(r)^2 \sum_{1 \leq j, k \leq d} \theta^j_r \theta^k_r \frac{\partial^2}{\partial p^j \partial p^k} \right) - b(r) \sum_{j=1}^{d} p^j \frac{\partial}{\partial p^j},$$

$\Delta$ denoting the usual Euclidian Laplacian of $\mathbb{R}^d$. Hence a function $F$ satisfying $(1)$ must solve:

$$\frac{\sigma^2(r)}{2 \beta \left( 1 + \eta(r)^2 \right)} \sum_{1 \leq j, k \leq d} \left( \delta_{jk} + \eta(r)^2 \theta^j_r \theta^k_r \right) \partial^2_{jk} F^i(p) - b(r) \sum_{j=1}^{d} p^j \partial_j F^i(p) = - p^i \times f(r).$$

Let us take $F^i$ of the form

$$F^i(p) = \theta^i \times \psi_{\beta}(r) = p^i \times \psi_{\beta}(r) / r,$$

4
and set for \( r \in \mathbb{R}_+ \):
\[
g(r) := \frac{2r \, b(r)}{\sigma^2(r)}, \quad h(r) := \frac{2r \, f(r)}{\sigma^2(r)}, \quad \text{and} \quad G(r) := \int_0^r g(\rho) \, d\rho. \tag{3}
\]

Then a direct computation shows that
\[
dM_t^i = \frac{\sigma(r_t)}{\sqrt{1 + \eta(r_t)^2}} \left[ \psi'_\beta(r_t) dW_t^i + \left[ \frac{\psi_\beta(r_t)}{r_t} - \psi'_\beta(r_t) \right] \sum_{j=1}^d \delta_{ij} - \theta_i^j \theta_j^i \right] dW_t^j + \eta(r_t) \psi'_\beta(r_t) \theta_i^i dw_t^i, \tag{4}
\]
and that Equation \(^3\) is equivalent to:
\[
\psi'_\beta(r) - \left( \beta g(r) - \frac{d-1}{r \left[ 1 + \eta(r)^2 \right]} \right) \psi'_\beta(r) - \frac{d-1}{r^2 \left[ 1 + \eta(r)^2 \right]} \psi_\beta(r) + \beta h(r) = 0. \tag{5}
\]

Note that, if \( b \equiv f \), or equivalently if \( g \equiv h \), then Equation \(^3\) admits the trivial solution \( \psi_\beta(r) = r \). If \( d = 1 \), Equation \(^3\) is easily solved too. But it is not easily solved in the general case we are considering, and not even in the case of the diffusion (DH) considered in [DH1] (and isotropically extended to higher dimensions) or in [DH2].

However, we have the following, whose delicate proof is postponed to Section \(^3\).

**Proposition 1** Under hypotheses \((H)\), Equation \(^3\) admits a solution \( \psi_\beta \in C^2(\mathbb{R}_+, \mathbb{R}) \) such that \( \psi_\beta(0) = 0 \), and \( |\psi_\beta(r)| = O(e^{\varepsilon r}) \), \( |\psi'_\beta(r)| = O(e^{\varepsilon r}) \) near infinity, for some \( \varepsilon' < \frac{d}{2} \). Moreover, if \( 0 \leq f \leq b \), then we have \( 0 \leq \psi_\beta \leq 1 \).

### 3.2 Polar decomposition of the process \((p_t)\) and equilibrium distribution \(\nu\)

Since the diffusion \((x_t, p_t)\) solves \((\ast)\), the radial process \( r_t = |p_t| \) solves:
\[
dr_t = \left( \frac{(d-1) \sigma^2(r_t)}{2 \beta \left[ 1 + \eta(r_t)^2 \right]} - r_t b(r_t) \right) dt + \sigma(r_t) \beta^{-1/2} dB_t,
\]
with
\[
dB_t := (1 + \eta(r_t)^2)^{-1/2} \left[ \sum_{i=1}^d \theta_i^i dW_t^i + \eta(r_t) \theta_i^i dw_t^i \right]. \tag{6}
\]
As \( \langle B, B \rangle_t = t \), \( B \) is a standard real Brownian motion. Consider then the angular process \( \Theta_s := (\tilde{\theta}_1^1, \ldots, \tilde{\theta}_d^d) \in \mathbb{S}^{d-1} \) defined by the time change \( \tilde{\Theta}_s := \Theta_{C^{-1}(s)} \), i.e. by \( p_t = r_t \times \tilde{\Theta}_{C_t} \), by means of the clock \( C_t = C(t) := \int_0^t \frac{\sigma^2(r_s)}{\beta r_s^2} \, ds \). The process \((\tilde{\Theta}_s) \in \mathbb{S}^{d-1} \) is a spherical Brownian motion, since it solves:
\[
d\tilde{\theta}_s = \left( \frac{1-d}{2} \right) \tilde{\theta}_s ds + \sum_{j=1}^d \left( \delta_{ij} - \tilde{\theta}_s^i \tilde{\theta}_s^j \right) dW_s^j,
\]
for some standard Brownian motion \( \tilde{W} = (\tilde{W}^1, \ldots, \tilde{W}^d) \in \mathbb{R}^d \). Hence the infinitesimal generator of the diffusion \((r_t, \Theta_t) = (r_t, \Theta_{C_t}) \) is
\[
A := L_r + \frac{\sigma^2(r)}{2 \beta r^2} \Delta_{\mathbb{S}^{d-1}}, \quad \text{with} \quad L_r := \frac{\sigma^2(r)}{2 \beta} \left( \partial_r^2 + \frac{d-1}{1 + \eta(r)^2} \right) - \beta g(r) \partial_r. \tag{7}
\]
Under this form, it appears that the anisotropy function \( \eta \) results in a radial drift.

Set for \( r \in \mathbb{R}_+ \):

\[
\mu(r) := \exp \left[ \int_1^r \frac{ds}{s \left[ 1 + \eta(s)^2 \right]} \right] \in \mathbb{R}_+. \tag{8}
\]

Note that \( \min \{ r, 1 \} \leq \mu(r) \leq \max \{ r, 1 \} \), and that \( 0 \leq r \leq s \Rightarrow 1 \leq \frac{\mu(s)}{\mu(r)} \leq \frac{s}{r} \).

The radial process \((r_t)\) admits the invariant measure \( \nu(r)dr \), having density on \( \mathbb{R}_+ \):

\[
\nu(r) := \sigma^{-2}(r) \mu(r)^{d-1} e^{-\beta G(r)}. \tag{9}
\]

Note that this equilibrium distribution equals the so-called Jüttner one, in the ROUP case (we have indeed \( G(r) = \sqrt{1 + r^2} - 1 \) in the ROUP and DH cases).

The hypotheses \((H)\) ensure that \( \nu \) is finite, and then that the radial process \((r_t)\) is ergodic. Denoting by \( d\Theta \) the uniform probability measure on the sphere \( S^{d-1} \), and setting:

\[
\pi(dr,d\Theta) := \left( \int_0^\infty \nu \right)^{-1} \times \nu(r) \, dr \, d\Theta, \tag{10}
\]

it is easily seen that \( \pi \) is an invariant probability measure (or equivalently: equilibrium distribution, meaning that the operator \( A \) is symmetrical with respect to \( \pi : \int \Phi_1 \mathcal{A}_F \Phi_2 \, d\pi = \int \Phi_2 \mathcal{A}_F \Phi_1 \, d\pi \) for any test-functions \( \Phi_1, \Phi_2 \) on \( \mathbb{R}_+ \times S^{d-1} \) for the process \((r_t,\tilde{\Theta}_t) = (r_t,\Theta_t)\), which is then a symmetrical ergodic diffusion on \( \mathbb{R}_+ \times S^{d-1} \).

**Lemma 1** For any starting point \( p_0 = r_0 \tilde{\Theta}_0 \), uniformly with respect to \( a \geq 0 \), we have:

\[
t^{-1} \mathbb{E}_{p_0} \left[ |F^i(p_{at})|^2 \right] \longrightarrow 0, \quad \text{as} \quad t \to \infty, \quad \text{for} \quad 1 \leq i \leq d.
\]

**Proof** Since \( |F^i(p)|^2 = |\theta^i \times \psibeta(r)|^2 \leq \psi^2beta(r) = \mathcal{O}(e^{2\varepsilon' r}) \) by Proposition \([1]\), we have

\[
\mathbb{E}_{p_0} \left[ |F^i(p_{at})|^2 \right] \leq C \mathbb{E}_{r_0} [e^{2\varepsilon' r_{at}}], \quad \text{for some constant} \quad C.
\]

Let \((Q_t)\) denote the semi-group of the radial diffusion \((r_t)\), solution to \( \partial_t Q_t = \mathcal{L}_r Q_t \). It is known (see for example ([V], chapter 31)) that \( Q_t(r_0,r) \) is a continuous function of \((t,r)\), and that (see for example ([V], chapter 32)) \( Q_1(r_0,r) = q_1(r) \nu(r) \), for some bounded function \( q_1 \).

Hence on one hand we have:

\[
\mathbb{E}_{r_0} [e^{2\varepsilon' r_s}] \leq \sup_{0 \leq s \leq 1} Q_s(e^{2\varepsilon'})(r_0) < \infty, \quad \text{for} \quad 0 \leq s \leq 1,
\]

and on the other hand, by the Markov property, for \( s \geq 1 \) we have:

\[
\mathbb{E}_{r_0} [e^{2\varepsilon' r_s}] = Q_s Q_{s-1}(e^{2\varepsilon'})(r_0) = \int_0^\infty \mathbb{E}_0 [e^{2\varepsilon' r_{s-1}}] q_1(\rho) \nu(\rho) \, d\rho \leq ||q_1||_1 \int_0^\infty e^{2\varepsilon' \rho} \nu(\rho) \, d\rho < +\infty,
\]

by \((H)\) and since \( 2\varepsilon < \beta \varepsilon \). This shows that \( s \mapsto \mathbb{E}_{r_0} [e^{2\varepsilon' r_s}] \) is bounded, whence the result. \( \diamond \)

### 3.3 Asymptotic study of the martingale \( M \)

By Formula \([\text{1}]\) and Lemma \([\text{1}]\), we are now left with the study of the martingale part \((M_t)\). Recall that the coordinates \( M^i \) of the martingale \( M \) are given by Equation \([\text{1}]\).
3.3.1 Asymptotic independence of the martingales $M^i$

**Lemma 2**  For $1 \leq i, l \leq d$, as $t \to \infty$ we have almost surely:

$$\lim_{t \to \infty} \frac{\langle M^i, M^l \rangle_t}{t} = \delta_{il} \Sigma_1^2, \quad \text{with} \quad \Sigma_1^2 := \frac{1}{\beta d} \left[ \int |\psi_{r}^2|^{2} \sigma^{2} d\pi + (d - 1) \int \psi_{r}^2 (1 + \eta^2)^{-1} I d^{-2} \sigma^{2} d\pi \right].$$

**Proof**  The computation of brackets gives easily:

$$\beta \langle M^i, M^l \rangle_t = \delta_{il} S^i_t - (1 - \delta_{il}) T^i,l_t,$$

with

$$S^i_t := \int_{0}^{t} \sigma^{2}(r_{s}) \psi_{r}^2(r_{s}) |\psi_{s}^2|^{2} ds + \int_{0}^{t} [1 + \eta(r_{s})^2]^{-1} r_{s}^{-2} \sigma^{2}(r_{s}) \psi_{r}^2(r_{s}) (1 - |\theta_{s}^1|^2) ds,$$

$$T^i,l_t := \int_{0}^{t} \sigma^{2}(r_{s}) \left[ \psi_{r}^2(r_{s}) [1 + \eta(r_{s})^2]^{-1} r_{s}^{-2} - \psi_{r}^2(r_{s}) \right] \theta_{s}^{1} \theta_{s}^{l} ds .$$

Setting

$$k^i(r, \Theta) := \sigma^{2}(r) \psi_{r}^2(r)^2 |\theta_{r}^1|^2 + [1 + \eta(r)^2]^{-1} r^{-2} \sigma^{2}(r) \psi_{r}^2(r) (1 - |\theta_{r}^1|^2)$$

and

$$\ell^{i,l}(r, \Theta) := \sigma^{2}(r) \left[ \psi_{r}^2(r) [1 + \eta(r)^2]^{-1} r^{-2} - \psi_{r}^2(r) \right] \theta_{r}^1 \theta_{r}^l,$$

and noticing that these functions are $\pi$-integrable by Proposition 1, we can apply the ergodic theorem, to get the following almost sure convergences:

$$\lim_{t \to \infty} S^i_t/t = \int_{R_{+} \times S^{d-1}} k^i d\pi, \quad \lim_{t \to \infty} T^i,l_t/t = \int_{R_{+} \times S^{d-1}} \ell^{i,l} d\pi .$$

Now the spherical symmetry with respect to $\Theta$ implies that for $1 \leq i \neq l \leq d$:

$$\int k^i d\pi = d^{-1} \int |\psi_{r}^2|^{2} \sigma^{2} d\pi + (1 - d^{-1}) \int \psi_{r}^2 (1 + \eta^2)^{-1} I d^{-2} \sigma^{2} d\pi =: \beta \Sigma_1^2, \quad \text{and} \quad \int \ell^{i,l} d\pi = 0 .$$

Hence we have got:

$$t^{-1} \langle M^i, M^l \rangle_t \overset{p.s.}{\longrightarrow} \delta_{il} \Sigma_1^2 = \delta_{il} (\beta d)^{-1} \left[ \pi \left( \sigma^{2} |\psi_{r}^2|^{2} \right) + (d - 1) \pi \left( \sigma^{2} \psi_{r}^2 / (1 + \eta^2) I d^2 \right) \right] . \quad \diamond$$

Consider now the martingale $M^i$ defined by:

$$M^i_a := (M^{i,t}_{a}, \ldots, M^{d,t}_{a}) := t^{-1/2} M^{i,t}_{a},$$

and the Dambis-Dubins-Schwarz Brownian motions $B^{i,t}$, such that

$$M^{i,t}_{a} = B^{i,t}((M^{i,t}, M^{l,t})_a) = B^{i,t}(t^{-1} \langle M^i, M^l \rangle_a).$$

Applying the asymptotic Knight theorem (see for example ([RY], Theorem 2.3 and Corollary (2.4) p. 524-525)), we deduce now from Lemma 2 the asymptotic independence of the martingales $M^i$ and $M^l$, for $1 \leq i \neq l \leq d$, in the following sense.

**Corollary 1**  The process $(B^{1,t}, \ldots, B^{d,t})$ converges in law, as $t$ goes to infinity, towards a standard $d$-dimensional Brownian motion $B$.
3.3.2 Convergence of the finite-dimensional marginal laws

**Proposition 2** The finite-dimensional marginal laws of the martingale $M^t$ converge, as $t$ goes to infinity, to those of the Brownian motion $\Sigma_\beta \times \mathcal{B}$, where $\Sigma_\beta$ is the (positive) constant given by:

$$\Sigma_\beta = \left[ d \int_0^\infty e^{-\beta G(r)} \mu(r)^{d-1} \sigma(r)^{-2} \, dr \right]^{-1} \times \int_0^\infty \psi_\beta(r) e^{-\beta G(r)} \mu(r)^{d-1} h(r) \, dr . \quad (11)$$

Recall that $\psi_\beta$ comes from Proposition 5 and that we set in Formulas (3) and (8):

$$G(r) = \int_0^r g(\rho) \, d\rho , \quad h(r) = \frac{2 \, r \, f(r)}{\sigma^2(r)} , \quad \mu(r) = \exp \left[ \int_1^r \frac{ds}{s \left[ 1 + \eta(s)^2 \right]} \right] . \quad (12)$$

**Proof** Fix any integer $N \geq 1$, positive numbers $0 < a_1 < \ldots < a_N$, and consider the vector random processes:

$$X^t := \left( \langle M^{i,t}, M^{i,t} \rangle_{\rho_k}, B^i_r \right)_{1 \leq i \leq d, 1 \leq k \leq N, s \geq 0} , \quad X^\infty := \left( \langle \Sigma_\beta^2 a_k, B^i_k \rangle \right)_{1 \leq i \leq d, 1 \leq k \leq N, s \geq 0} .$$

By Section 3.3.1, $X^t$ converges in law, as $t$ goes to infinity, to $X^\infty$. By the Skorokhod coupling theorem (see for example ([K], Theorem (4.30) p. 78)), there exist vector random processes $\tilde{X}^t$ and $\tilde{X}^\infty$ satisfying the identities in law:

$$(\tilde{X}^t) \overset{d}{=} (X^t) , \quad \tilde{X}^\infty \overset{d}{=} X^\infty ,$$

and such that $\tilde{X}^t$ converges almost surely to $\tilde{X}^\infty$. As a consequence, we get the following convergence in distribution:

$$(B^{i,t}(\langle M^{i,t}, M^{i,t} \rangle_{\rho_k})_{1 \leq i \leq d, 1 \leq k \leq N} \overset{d}{\to} (B^i(\Sigma_\beta^2 a_k))_{1 \leq i \leq d, 1 \leq k \leq N} ,$$

or equivalently:

$$(M^{i,t}_{\rho_k})_{1 \leq i \leq d, 1 \leq k \leq N} \overset{d}{\to} \Sigma_\beta \times (B^i_{\rho_k})_{1 \leq i \leq d, 1 \leq k \leq N} .$$

Note that from Formulas (5), (11), and Lemma 2, we get directly the following expression for $\Sigma_\beta$:

$$\Sigma_\beta^2 = \int_0^\infty \psi_\beta(r)^2 e^{-\beta G(r)} \mu(r)^{d-1} \, dr + (d - 1) \int_0^\infty \psi_\beta(r)^2 e^{-\beta G(r)} \mu(r)^{d-1} \left[ 1 + \eta(r)^2 \right]^{-1} r^{-2} \, dr \overset{\beta d \int_0^\infty e^{-\beta G(r)} \mu(r)^{d-1} \sigma(r)^{-2} \, dr}{} .$$

It remains to derive from this expression the expression (11) of the statement for $\Sigma_\beta$. This is achieved as follows, integrating by parts and using Proposition 1, which implies that

$$\lim_{r \to \infty} \left[ \psi_\beta(r) \psi_\beta'(r) e^{-\beta G(r)} \mu(r)^{d-1} \right] = 0 ,$$

together with Equation (3):

$$\int_0^\infty \psi_\beta'(r)^2 e^{-\beta G(r)} \mu(r)^{d-1} \, dr = \left[ \psi_\beta(r) \psi_\beta'(r) e^{-\beta G(r)} \mu(r)^{d-1} \right]_0^\infty - \int_0^\infty \psi_\beta(r) \frac{d}{dr} \left[ \psi_\beta'(r) e^{-\beta G(r)} \mu(r)^{d-1} \right] \, dr$$

$$= \int_0^\infty \psi_\beta(r) \left[ \frac{(d-1)r}{1 + \eta(r)} - \beta g(r) \right] \frac{(d-1) \psi_\beta'(r)}{1 + \eta(r)^2} \, dr + \beta h(r) + \beta(1 + \eta(r)^2)^{-1} \left[ \psi_\beta'(r) e^{-\beta G(r)} \mu(r)^{d-1} \right] \, dr$$

$$= \beta \int_0^\infty \psi_\beta(r) e^{-\beta G(r)} \mu(r)^{d-1} h(r) \, dr - (d - 1) \int_0^\infty \psi_\beta(r)^2 e^{-\beta G(r)} \mu(r)^{d-1} \left[ 1 + \eta(r)^2 \right]^{-1} r^{-2} \, dr . \quad \diamond$$
3.3.3 Tightness

**Proposition 3**  The family of martingales $M^t$ is tight, in $C(\mathbb{R}_+, \mathbb{R}^d)$, endowed with the topology of uniform convergence on compact sets of $\mathbb{R}_+$.

**Proof**  Fix any $T > 0$, and use the Arzelà-Ascoli theorem (see for example ([K], Theorem (16.5) p. 311)) : the family $(t^{-1/2} M_{at}, a \in [0, T])$ is tight, in $C([0, T], \mathbb{R}^d)$, if and only if

$$\lim_{h \to 0} \limsup_{t \to \infty} \mathbb{E} \left[ t^{-1/2} \sup_{0 \leq a \leq T, b-a \leq h} |M_{at} - M_{bt}| \right] = 0.$$ 

Fix $i \in \{1, \ldots, d\}$, $h > 0$, and denote by $n$ the integral part of $T/h$. There exists a standard Brownian motion $\widetilde{W}^i$ such that $(S^i)$ being as in Section 3.3) : the family of martingales $\widetilde{W}^i$ such that ($S^i$)

$$\left| \sup_{0 \leq s \leq u} \left| \widetilde{W}^i_s - \widetilde{W}^i_u \right| \right| = \frac{1}{\sqrt{hT}} \sup_{0 \leq a \leq T, b-a \leq h} \left| M^i_{at} - M^i_{bt} \right| = \frac{1}{\sqrt{hT}} \sup_{0 \leq a \leq T, b-a \leq h} \left| \widetilde{W}^i (S^i_{at} - S^i_{bt}) \right| .$$

Setting $\widetilde{W}^i(u) = \sup_{0 \leq s \leq u} |\widetilde{W}^i_s|$, we have also :

$$\frac{1}{\sqrt{hT}} \sup_{0 \leq a \leq T, b-a \leq h} \left| \widetilde{W}^i (S^i_{bt} - S^i_{at}) \right| = \frac{1}{\sqrt{hT}} \left( \sup_{0 \leq a \leq T-h} \frac{1}{h} (S^i_{at+h} - S^i_{at}) \right) \leq \frac{1}{h} (A^i_{at}) ,$$

where

$$A^i_{at} = \sup_{0 \leq j \leq n} A^i_{at} , \quad A^i_{at} = \frac{1}{h} (S^i_{j+2} - S^i_{j} ) .$$

By the ergodic theorem we have (as in the proof of Lemma 3), the following convergence, as $t \to \infty$ , valid almost surely and in $L^1$-norm as well : $A^i_{ht} \to 2 \Sigma^2_\beta$, which implies the uniform integrability of $\{A^i_{ht}, \ h \geq 1\}$.

Otherwise, by Doob’s inequality (applied to the martingale $\int_0^s 1_{(A^i_{ht})} d\widetilde{W}^i_u$), we have :

$$\mathbb{E} \left[ \left| \widetilde{W}^i (A^i_{ht}) \right| \right] \leq \left\| \left| \widetilde{W}^i (A^i_{ht}) \right| \right\|_2 \leq 2 \sqrt{\mathbb{E} [A^i_{ht}]},$$

whence

$$\mathbb{E} \left[ t^{-1/2} \sup_{0 \leq a \leq T, b-a \leq h} |M^i_{at} - M^i_{bt}| \right] \leq 2 \beta^{-1/2} \sqrt{h \times \mathbb{E} [A^i_{ht}]}. $$

Now, as for fixed $h$ and for any $\lambda > 2 \Sigma^2_\beta$ we have :

$$\mathbb{E} [A^i_{ht}] = \int_0^\infty \mathbb{P}(A^i_{ht} \geq s)ds \leq \lambda + \int_\lambda^\infty \mathbb{P}(A^i_{ht} \geq s)ds \leq \lambda + \sum_{j=0}^{\infty} \int_\lambda^{\infty} \mathbb{P}(A^i_{ht} \geq s)ds ,$$

we deduce that

$$\mathbb{E} [A^i_{ht}] \leq \lambda + \sum_{j=0}^{\infty} \mathbb{E} \left[ A^i_{ht} \times 1_{(A^i_{ht} \geq \lambda)} \right] \to \lambda , \quad as \ t \to \infty .$$

Hence, $\limsup_{t \to \infty} \mathbb{E} [A^i_{ht}] \leq 2 \Sigma^2_\beta$, and then

$$\lim_{h \to 0} \limsup_{t \to \infty} t^{-1/2} \mathbb{E} \left[ \sup_{0 \leq a \leq T, b-a \leq h} |M^i_{at} - M^i_{bt}| \right] = 0 . \ \Box$$
3.3.4 Main result

Gathering Formula (1), Lemma 3 and Propositions 4 and 6 we get at once the following main result of this article.

**Theorem 1** Let \((x_t, p_t) = (x^i_t, p^i_t)_{1 \leq i \leq d}\) be a \(\mathbb{R}^d \times \mathbb{R}^d\)-valued diffusion solving the stochastic differential system

\[
\begin{align*}
   dx^i_t &= f(r_t) p^i_t \, dt \\
   dp^i_t &= -b(r_t) p^i_t \, dt + \sigma(r_t) \left( \beta [1 + \eta(r_t)] \right)^{-1/2} \left[ dW^i_t + \eta(r_t) \theta^i_t \, dw_t \right],
\end{align*}
\]

where \(W := (W^1, \ldots, W^d)\) denotes a standard \(d\)-dimensional Euclidian Brownian motion, \(w\) denotes a standard real Brownian motion, independent of \(W\), \(\beta > 0\) is an inverse noise or heat parameter, and the real functions \(f, b, \sigma, \eta\) are continuous on \(\mathbb{R}^+\) and satisfy the following hypotheses, for some fixed \(\varepsilon > 0\):

\[
(H) \quad \sigma \geq \varepsilon \text{ on } \mathbb{R}^+; \quad (g)(r) := \frac{2r b(r)}{\sigma^2(r)} \geq \varepsilon \text{ for large } r; \quad \lim_{t \to \infty} e^{-\varepsilon r} f(r) = 0 \text{ for some } \varepsilon' < \frac{\beta \varepsilon}{2}.
\]

Then the law of the process \((t^{-1/2} x^i_{at})_{a \geq 0}\) converges, as \(t \to \infty\), to the law of \((\Sigma_{\beta} B^i_a)_{a \geq 0}\), in \(C(\mathbb{R}^+, \mathbb{R}^d)\), endowed with the topology of uniform convergence on compact sets of \(\mathbb{R}^+\). Here \(B^i\) is a standard \(d\)-dimensional Brownian motion, and the constant \(\Sigma_{\beta}\) is given by Formula (7). This result holds from any starting point \((x_0, p_0)\) (\(p_0\) can also obey the equilibrium law \(\pi\)).

We deduce now the result conjectured in [DH1], [DH2], and an expression of the limit.

**Corollary 2** Under the same hypotheses as in the above theorem, for any starting point, the normalised variance (mean square displacement) \(t^{-1} \mathbb{E} \left[ |x_t|^2 \right] \) goes, as \(t \to \infty\), towards \(d \times \Sigma_{\beta}^2\).

**Proof.** By Theorem 1, we have convergence in law of the random variable \(t^{-1} |x_t|^2\), towards \(\Sigma_{\beta}^2 |B^i_1|^2\). By Formula (4) and Lemma 6, we have only to make sure that for \(1 \leq i \leq d\), the following holds:

\[t^{-1} \mathbb{E} \left[ |M^i_t|^2 \right] = (\beta t)^{-1} \mathbb{E} \left[ S^i_t \right] \to \Sigma_{\beta}^2.\]

Now, on one hand we already noticed (recall the proof of Lemma 3) that, by ergodicity, we have

\[t^{-1} S^i_t \overset{P.S.}{\to} \int k^i \, d\pi = \beta \Sigma_{\beta}^2.\]

And on the other hand, exactly the same reasoning as in the proof of Lemma 3 (to show that \(s \mapsto \mathbb{E}_r [e^{2\varepsilon r}]\) is bounded), merely using the semi-group \((P_t)\) of the diffusion \((r_t, \Theta_{C^l})\), solution to \(\partial_t P_t = AP_t\) instead of the radial semi-group \((Q_t)\), shows that \(s \mapsto \mathbb{E}_{p_0} [k^i(p_0)]\) is bounded. Moreover, in the same spirit, by the Markov property and by the proof of Lemma 6, we have:

\[\mathbb{E}_{p_0} \left[ \frac{S^i_t}{t} \right] = \frac{1}{t} \int_0^t P_s(k^i)(p_0) \, ds + \int \left( \frac{1}{t} \int_0^{t-1} P_s(k^i)(p) \, ds \right) \tilde{q}_i(p) \, \pi(dp),\]

\(\tilde{q}_i\) being the bounded density of \(P_1(p_0, dp)\) with respect to \(\pi(dp)\). It is clear that the first term of the right hand side goes to 0. Finally, by the Chacon-Ornstein theorem and by dominated convergence, the second term goes indeed to \(\beta \Sigma_{\beta}^2\). \(

10

4 Behaviours of $\Sigma^2_{\beta}$, as $\beta \searrow 0$ and as $\beta \nearrow \infty$

Theorem 1 and Corollary 2 show up the interest of the limiting constant $d \times \Sigma^2_{\beta}$. Recall then from Sections 2 and 3.1 that the processes considered by ([BD R1], [BDR2], [BDR3], [DMR], [DR]) and by ([DH1], [DH2]), correspond respectively to :

\( \text{(ROUP) } h(r) = g(r) = r \left(1 + r^2\right)^{-1/2}, \quad G(r) = \sqrt{1 + r^2} - 1, \quad \eta = 0, \quad \sigma(r)^2 = 2, \quad \psi_{\beta}(r) = r, \)

\( \text{(DH) } h(r) = \frac{r}{1 + r^2}, \quad g(r) = \mu(r) = \frac{r}{\sqrt{1 + r^2}}, \quad G(r) = \sqrt{1 + r^2} - 1, \quad \sigma(r)^2 = 2 \sqrt{1 + r^2}, \quad \eta(r) = r, \)

for some positive (noise or heat) inverse parameter $\beta$. It is natural to wonder, as in [DH1], how behaves the limiting variance $\Sigma^2_{\beta}$, as $\beta \searrow 0$ and as $\beta \nearrow \infty$.

In the ROUP case, we have simply $d \times \Sigma^2_{\beta} = \frac{2d}{\beta}$ and $G(r) = \sqrt{1 + r^2} - 1$. The variance behaviour is Euclidian.

In the DH case of [DH1], [DH2], we have by Formula (11) :

\[ d \times \Sigma^2_{\beta} = 2 \int_0^\infty \psi_{\beta}(r) e^{-\beta \sqrt{1 + r^2}} \left(1 + r^2\right)^{-(d+1)/2} r^d dr \]

\[ + \int_0^\infty e^{-\beta \sqrt{1 + r^2}} \left(1 + r^2\right)^{d/2} r^{d-1} dr. \]

Note that the precise value of $\psi_{\beta}$ is given in Section 5.2 :

\[ \psi_{\beta}(r) = \zeta_1(r) \int_0^r \zeta_2(\rho) w_{\beta}(\rho)^{-1} h(\rho) d\rho + \zeta_2(r) \int_r^\infty \zeta_1(\rho) w_{\beta}(\rho)^{-1} h(\rho) d\rho, \]

with functions $\zeta_1, \zeta_2, w_{\beta}$ given in Section 5.1.

In [DH1], for $d = 1$, after numerical simulations, Dunkel and Hänggi conjecture that $\Sigma^2_{\beta}$ could be equal to $\frac{2}{\beta}$. The expression we got above for $\Sigma^2_{\beta}$ invalidates this conjecture, and, even the asymptotic behaviour near 0 it implies. However, it is true that a non-classical variance behaviour occurs. We have indeed the following, whose technical proof is postponed to Section 6.

**Proposition 4** Consider the DH case, for $d = 1$, as in [DH1]. Then, we have

$\Sigma^2_{\beta} \sim \frac{2}{\beta}$ as $\beta \nearrow \infty$; and, as $\beta \searrow 0$ : $\Sigma^2_{\beta} \sim \frac{A}{\log(1/\beta)}$, for some explicit constant $A > 0$.

4.1 Numerical Simulations

To confirm the validity of our estimates in Proposition 4, invalidating the conjecture of [DH1], we performed numerical simulations relating to the DH diffusion, in the case $d = 1$. We used the Monte-Carlo method, with $N = 1000$ simulations. For different values of $\beta$ (from $10^{-5}$ to $10^6$), we computed $x_j(t)$ for $t = 1, \ldots, N$ for $0 \leq t \leq T = 1000$, and then the quantity :

\[ \overline{x^2}(T) = \frac{1}{N} \sum_{j=1}^N x_j^2(T). \]

The following diagram represents our results in logarithmic coordinates. Thus, the horizontal axis represents $\log(1/\beta)$, the points $\ast$ represent the simulated values $\log(\overline{x^2}(T)/T)$ in function of $\log(1/\beta)$.
The straight line corresponds to the Euclidian behaviour, the continuous curve to the function $\beta \rightarrow 2/((\beta + 2))$, and the dashed curve corresponds to a decrease in $\log(1/\beta)^{-1}$ for small $\beta$.

These simulations confirm the Euclidian behaviour of the DH diffusion as $\beta >> 1$. For small $\beta$, the expression conjectured in [DH1] is a good approximation as long as $\beta > 1/10$; however, for smaller $\beta$, a divergence appears clearly. On the contrary, the $\log(1/\beta)^{-1}$-like asymptotic behaviour of the limit, which we established above, appears as confirmed.

4.1.1 The program used for the simulations

```matlab
function res=asymp(N,h,D,T) % (written in “matlab”)
N is the iteration number in the Monte Carlo method. The process \(x(t)\) is simulated on \([0, T]\), with mesh \(h\). Different values for \(\beta\) have been tested.
Initialisation. Arrays \(p\) and \(x\) contain the values of \(p(t)\) and \(x(t)\) for \(0 \leq t \leq T\)
\(t=0 \ : \ h \ : \ T\); \(n=\text{length}(t)\); \(r=[]\);
for \(k=1 \ : 1 \ : N\)
\(p=\text{zeros}(1,n)\); \(x=\text{zeros}(1,n)\);
\(Simulation\ of\ Brownian\ motion\)
\(u=\text{randn}(1,n)\); \(W=\sqrt{2*D*h}*u\);
\(Simulation\ of\ processes\ \(p(t)\), and \(x(t)\) by integration\)
for \(j=1 \ : 1 \ : n-1\)
\(g=\text{sqrt}(1+p(1,j).*p(1,j));\)
\(p(1,j+1)=p(1,j)-(p(1,j))*h + \sqrt{gam}\times W(1,j);\)
\(x(1,j+1)=x(1,j)+(p(1,j)/g);\)
end
The \(N\) simulations of \(x(t)\) are placed in the array \(r\)
\(r=[r ; x]\); end
\(Computation\ of\ the\ mean\ of\ \(x^2(T)\), normalised by \(T\)
\(\text{car}=r.*r; \ \text{limite}=\text{mean(car)}; \ \text{res}=\text{limite}(n)/T;\)
end of program.
```
5 Proof of Proposition \([1]\)

We are indebted to Reinhard Schäfe for this proof, who kindly indicated to us how to proceed for Sections 5.1 and 5.2 below. We thank him warmly. Consider first the homogeneous equation associated to \([1]\):

\[
\zeta''(r) + \left( \frac{d-1}{r[1 + \eta(r)^2]} - \beta g(r) \right) \zeta'(r) - \frac{d-1}{r^2[1 + \eta(r)^2]} \zeta(r) = 0. \tag{13}
\]

It has a pole of order 2 at 0 (except for \(d = 1\)), and a pole at infinity. Using the fixed point method, we construct two solutions \(\zeta_1\) and \(\zeta_2\) of Equation \([13]\), bounded respectively near infinity and near 0. Using these two solutions of the homogeneous equation, a solution \(\psi_\beta\) to \([1]\) is then deduced, which vanishes at 0. Finally, we establish the wanted control on \(\psi_\beta, \psi'_\beta\).

Recall from Formula \([8]\) that we set: \(\mu(r) = \exp \left[ \int_1^r \frac{ds}{s[1 + \eta(s)^2]} \right]\), so that \(\mu\) increases and \(\min\{r, 1\} \leq \mu(r) \leq \max\{r, 1\}\), and \(0 \leq r \leq s \Rightarrow 1 \leq \frac{\mu(s)}{\mu(r)} \leq \frac{1}{r}\).

5.1 Constructing solutions to the homogeneous equation \([13]\)

5.1.1 Constructing a solution \(\zeta_1\) to \([13]\), bounded near \(\infty\)

Using hypotheses \((\mathcal{H})\), fix \(\varepsilon > 0\) and \(r_0 \geq 1\) such that \(g \geq \varepsilon\) on \([r_0, \infty]\). For \(r \geq r_0\), set

\[
\lambda(r) := \int_r^\infty \mu(\rho)^{1-d} e^{\beta G(\rho)} \left[ \int_\rho^\infty e^{-\beta G(s)} \mu(s)^{d-1} [1 + \eta(s)^2]^{-1} s^{-2} ds \right] d\rho.
\]

We have

\[
\lambda(r) \leq \int_r^\infty \left[ \int_\rho^\infty e^{-\beta \varepsilon(s-\rho)} \left( \frac{s}{\rho} \right)^{d-1} s^{-2} ds \right] d\rho = \int_r^\infty \left[ \int_0^1 e^{-\beta \varepsilon s} (1 + s/\rho)^{d-3} ds \right] \rho^{-2} d\rho
\]

\[
\leq \frac{1}{r} \int_0^\infty e^{-\beta \varepsilon s} \max\{1, (1+s/r_0)^{d-3}\} ds = O(1/r).
\]

As \(r \to \infty\), \(\lambda(r)\) decreases to 0, so that (up to increase \(r_0\)) we can suppose that \(\lambda(r_0) \leq 1/(2d)\). On \([r_0, \infty]\), let us define by induction on \(n \in \mathbb{N}\) the functions: \(\varphi_0 \equiv 1\), and

\[
\varphi_{n+1}(r) := 1 + (d-1) \int_r^\infty \mu(\rho)^{1-d} e^{\beta G(\rho)} \left[ \int_\rho^\infty e^{-\beta G(s)} \mu(s)^{d-1} \varphi_n(s) [1 + \eta(s)^2]^{-1} s^{-2} ds \right] d\rho.
\]

We have for \(r \geq r_0\):

\[
1 \leq \varphi_{n+1}(r) \leq 1 + (d-1) \|\varphi_n\|_{L^\infty_{[r_0, \infty]}} \times \lambda(r) \leq 1 + \frac{1}{2} \|\varphi_n\|_{L^\infty_{[r_0, \infty]}},
\]

whence \(1 \leq \varphi_n \leq \|\varphi_n\|_{L^\infty_{[r_0, \infty]}},\) for any \(n \in \mathbb{N}\). Then similarly:

\[
\|\varphi_{n+1} - \varphi_n\|_{L^\infty_{[r_0, \infty]}} \leq \|\varphi_n - \varphi_n - 1\|_{L^\infty_{[r_0, \infty]}},
\]

which allows to apply the fixed point method, to get \(\zeta_1 := \lim_{n \to \infty} \varphi_n\), which satisfies

\[
1 \leq \zeta_1(r) = 1 + (d-1) \int_r^\infty \mu(\rho)^{1-d} e^{\beta G(\rho)} \left[ \int_\rho^\infty e^{-\beta G(s)} \mu(s)^{d-1} \zeta_1(s) [1 + \eta(s)^2]^{-1} s^{-2} ds \right] d\rho. \tag{14}
\]
In particular, as \( r \to \infty \) we have:
\[
\zeta_1(r) \leq 1 + (d - 1) \| \zeta_1 \|_{L^\infty[r_0, \infty]} \lambda(r) \to 1,
\]
hence \( \lim_{r \to \infty} \zeta_1(r) = 1 \), and
\[
\zeta_1'(r) = (1 - d) \mu(r)^{1-d} e^{\beta G(r)} \int_r^\infty e^{-\beta G(s)} \mu(s)^{d-1} \zeta_1(s) [1 + \eta(s)^2]^{-1} s^{-2} ds < 0,
\]
then
\[
\zeta_1''(r) + \left( \frac{d - 1}{r [1 + \eta(r)^2]} - \beta \frac{g(r)}{r^2 [1 + \eta(r)^2]} \right) \zeta_1'(r) = \frac{d - 1}{r} \zeta_1(r) = 0.
\]
This solution can be continued over the whole \( \mathbb{R}_+^* \), yielding \( \zeta_1 \) still satisfying (13) and (14) on \( \mathbb{R}_+^* \). We have also \( \lim_{r \to 0} \zeta_1 = +\infty \).

5.1.2 Constructing a solution \( \zeta_2 \) to (13), bounded near 0

For \( r \in [0, 1] \), set:
\[
\Lambda(r) := \beta \int_0^r \mu(\rho)^{-d-1} \rho^{-2} e^{\beta G(\rho)} \left[ \int_0^\rho e^{-\beta G(s)} \mu(s)^{d+1} |g(s)| s ds \right] d\rho.
\]
We have \( 0 \leq \Lambda'(r) \leq \beta e^{2\beta \int_0^1 |g(r)| s ds} \to 0 \) as \( r \to 0 \), by hypotheses (H). We can then fix \( r_1 \in [0, 1] \) such that \( \Lambda(r_1) \leq 1/2 \). On \( [0, r_1] \), let us define by induction on \( n \in \mathbb{N} \) the functions:
\[
\phi_0 \equiv 1,
\]
\[
\phi_{n+1}(r) := 1 + \beta \int_0^r \mu(\rho)^{-d-1} \rho^{-2} e^{\beta G(\rho)} \left[ \int_0^\rho e^{-\beta G(s)} \mu(s)^{d+1} g(s) \phi_n(s) s ds \right] d\rho.
\]
We have \( \phi_n \in C^2([0, r_1]) \), \( \| \phi_{n+1} \|_{L^\infty[0, r_1]} \leq 1 + \Lambda(r_1) \| \phi_n \|_{L^\infty[0, r_1]} \) so that \( \| \phi_n \|_{L^\infty[0, r_1]} < 2 \), and
\[
\| \phi_{n+1} - \phi_n \|_{L^\infty[0, r_1]} \leq \| \phi_n - \phi_{n-1} \|_{L^\infty[0, r_1]} \times \Lambda(r_1),
\]
which allows to apply the fixed point method, to get \( \bar{\phi} := \lim_{n \to \infty} \phi_n \), which satisfies:
\[
\bar{\phi}(r) = 1 + \beta \int_0^r \mu(\rho)^{-d-1} \rho^{-2} e^{\beta G(\rho)} \left[ \int_0^\rho e^{-\beta G(s)} \mu(s)^{d+1} g(s) \phi(s) s ds \right] d\rho = 1 + \mathcal{O}[\Lambda(r)] \quad (15)
\]
for any \( r \in [0, r_1] \). Hence,
\[
\bar{\phi}'(r) = \beta \mu(r)^{-d-1} r^{-2} e^{\beta G(r)} \int_0^r e^{-\beta G(s)} \mu(s)^{d+1} g(s) \phi(s) s ds = \mathcal{O}[\Lambda'(r)] \to 0 \quad \text{as } r \to 0.
\]

Therefore, \( \bar{\phi}(0) = 1 \), \( \bar{\phi}'(0) = 0 \), and for any \( r \in [0, r_1] \):
\[
\bar{\phi}''(r) + \left( \frac{d + 1}{r [1 + \eta(r)^2]} + \frac{2}{r} - \beta \frac{g(r)}{r} \right) \bar{\phi}'(r) = \frac{d + 1}{r} \bar{\phi}(r) = 0. \quad (16)
\]
This function \( \bar{\phi} \) can be continued on the whole \( \mathbb{R}_+ \), into a function \( \bar{\phi} \) satisfying still Equations (13) and (14). Set now \( \zeta_2(r) := r \bar{\phi}(r) \). It is immediate that \( \zeta_2 \) solves (13) on \( \mathbb{R}_+ \), and satisfies:
\[
\zeta_2(0) = 0, \quad \zeta_2'(0) = 1.
\]
5.1.3 The Wronskian $w_\beta$ of $\zeta_1, \zeta_2$

Consider the Wronskian:

$$w_\beta := \zeta_1' \zeta_2 - \zeta_1 \zeta_2' \quad \text{on } \mathbb{R}_+^*.$$

We have

$$w_\beta' = \zeta_1 \zeta_2'' - \zeta_1'' \zeta_2 = \left(\beta g - \frac{d - 1}{[1 + \eta^2]I_d}\right) \times w_\beta,$$

so that

$$w_\beta(r) = a_\beta \mu(r)^{1-d} e^{\beta G(r)} \quad \text{for any } r > 0$$

and for some constant $a_\beta$. As $\zeta_1 \geq 1$, $\zeta_2' > 0$, $\zeta_1' < 0$, $\zeta_2 > 0$ near 0, we must have $a_\beta > 0$.

5.2 Constructing a solution $\psi_\beta$ to Equation (3) on $\mathbb{R}_+$

For any continuous function $k$ on $\mathbb{R}_+$, such that $\lim_{r \to \infty} e^{-\varepsilon r} k(r) = 0$ for some $\varepsilon < \frac{\beta}{2}$, and for any $0 < r < \infty$, set :

$$\Psi(k)(r) := \zeta_1(r) \int_0^r \zeta_2(\rho) w_\beta(\rho)^{-1} k(\rho) d\rho + \zeta_2(r) \int_r^\infty \zeta_1(\rho) w_\beta(\rho)^{-1} k(\rho) d\rho.$$

Note that $\Psi(k)$ is well defined; we have indeed, using that $g \geq \varepsilon$ on $[r_0, \infty[$ :

$$\int_{r_0}^\infty \zeta_1(\rho) w_\beta(\rho)^{-1} |k(\rho)| d\rho = \mathcal{O}(1) \int_{r_0}^\infty \mu(\rho)^{1-d} e^{-\beta G(\rho)} e^{\beta \varepsilon / 2} d\rho < \infty.$$

Note also that by $(\mathcal{H})$, we can take in particular $k = \beta h$ (recall Formula (3) defining $h$).

Moreover, again for $0 < r < \infty$, we have :

$$\Psi(k)'(r) = \zeta_1'(r) \int_0^r \zeta_2(\rho) w_\beta(\rho)^{-1} k(\rho) d\rho + \zeta_2'(r) \int_r^\infty \zeta_1(\rho) w_\beta(\rho)^{-1} k(\rho) d\rho,$$

and

$$\Psi(k)''(r) = \zeta_1''(r) \int_0^r \psi_2(\rho) w_\beta(\rho)^{-1} k(\rho) d\rho + \zeta_2''(r) \int_r^\infty \zeta_1(\rho) w_\beta(\rho)^{-1} k(\rho) d\rho - k(r),$$

so that $\Psi(k)$ solves on $\mathbb{R}_+^*$ Equation (3), with $k$ instead of $\beta h$. Near 0, we have $\zeta_2(r) \sim r$.

Otherwise, noticing that $(\zeta_1 / \zeta_2)' = -w_\beta / \zeta_2^2$, we have for $r > 0$ :

$$\int_r^1 \frac{w_\beta}{\zeta_2^2} = \frac{\zeta_1}{\zeta_2}(r) - \frac{\zeta_1}{\zeta_2}(1), \quad \text{i.e.,} \quad \zeta_1(r) = \frac{\zeta_1}{\zeta_2}(1) \zeta_2(r) + \zeta_2(r) \int_r^1 \frac{w_\beta}{\zeta_2^2}.$$

Hence, near 0 we have : $\zeta_1(r) \sim r a_\beta \int_r^1 \mu(s)^{1-d} s^{-2} ds \leq a_\beta \mu(r)^{1-d}$, and then $\Psi(k)(r) = \mathcal{O}(r)$.

In particular, we have $\Psi(k)(0) = 0$. Using $\zeta_1(s) = \mathcal{O}\left(\mu(s)^{1-d}\right)$ in the expression of $\zeta_1'$ (recall Section 5.1.1), we get at once $|\zeta_1'(r)| = \mathcal{O}\left(\mu(r)^{1-d}/r\right)$ near 0. We have therefore near 0 :

$$|\zeta_1'(r)| \int_0^r \zeta_2(\rho) w_\beta(\rho)^{-1} k(\rho) d\rho = \mathcal{O}\left(\mu(r)^{1-d}/r\right) \int_0^r \mu(\rho)^{d-1} \rho d\rho = \mathcal{O}(r),$$
Near infinity, we have on one hand:

$$\zeta_2(r) \int_r^\infty \zeta_1(\rho) w_\beta(\rho)^{-1} k(\rho) \, d\rho = O(1),$$

whence  $$\Psi(k)'(0) = \int_0^\infty \zeta_1(\rho) w_\beta(\rho)^{-1} k(\rho) \, d\rho \in \mathbb{R}.$$  Using (5), we get also $$\psi'_0(0) \in \mathbb{R}.$$  

Setting $$\psi_\beta := \beta \Psi(h),$$ we have thus $$\psi_\beta \in C^2(\mathbb{R}_+), \psi_\beta(0) = 0,$$ and $$\psi_\beta$$ solves (5) on $$\mathbb{R}_+.$$  

### 5.3 Estimates for $$\psi_\beta$$ and $$\psi'_\beta$$ near $$\infty$$

Recall from Section 5.2 that

$$\psi_\beta(r) = \beta \zeta_1(r) \int_0^r \zeta_2(\rho) w_\beta(\rho)^{-1} h(\rho) \, d\rho + \beta \zeta_2(r) \int_r^\infty \zeta_1(\rho) w_\beta(\rho)^{-1} h(\rho) \, d\rho$$

and

$$\psi'_\beta(r) = \beta \zeta'_1(r) \int_0^r \zeta_2(\rho) w_\beta(\rho)^{-1} h(\rho) \, d\rho + \beta \zeta'_2(r) \int_r^\infty \zeta_1(\rho) w_\beta(\rho)^{-1} h(\rho) \, d\rho.$$  

Near infinity, we have on one hand: $$\zeta_1 \sim 1$$ and (by Section 5.1.1)

$$|\zeta'_1(r)| = O\left( \mu(r)^{1-d} \right) \int_r^\infty e^{-\beta \varepsilon (s-r)} \mu(s)^{d-1} s^{-2} \, ds = O(r^{-2}) \int_0^\infty e^{-\beta \varepsilon s} (1 + s/r)^{d-1} \, ds = O(r^{-2}).$$

On the other hand, by (H) we have also near infinity:

$$w_\beta(r) - \frac{2}{\beta \varepsilon} w'_\beta(r) = \left( 1 - \frac{2}{\varepsilon} g(r) + \frac{2 (d-1)}{\beta \varepsilon [1 + \eta(r)^2]^2} \right) w_\beta(r) < 0,$$

so that

$$\int_0^r w_\beta \leq \frac{2}{\beta \varepsilon} (r w_\beta(r) - w_\beta(r_0)) = O(w_\beta(r)).$$

Noticing that  $$\zeta_2(\zeta'_1)^2 = w_\beta/\zeta'_1^2,$$  we have for any  $$r > 0:$$  $$\zeta_2(r) = \frac{\zeta_2}{\zeta_1}(1) \zeta'_1(r) + \zeta_2(r) \int_1^r \frac{w_\beta}{\zeta_1}.$$  

This implies that  $$\zeta_2(r) = O(w_\beta(r))$$ near infinity. Therefore, by (H) there exists an  $$\varepsilon' < \varepsilon \beta/2$$  such that:

$$\int_0^r \zeta_2(\rho) w_\beta(\rho)^{-1} h(\rho) \, d\rho = \int_0^r O(e^{\varepsilon' \beta}) \, d\rho = O(e^{\varepsilon' r}).$$

To control the second integral in the expressions for  $$\psi_\beta$$  and  $$\psi'_\beta,$$  observe similarly that:

$$w_\beta(r) \int_r^\infty \zeta_1(\rho) w_\beta(\rho)^{-1} h(\rho) \, d\rho = \mu(r)^{1-d} \int_r^\infty e^{-\beta[G(\rho)-G(r)]} \mu(\rho)^{d-1} O(e^{\varepsilon' \rho}) \, d\rho$$

$$= O(e^{\varepsilon' r}) \mu(r)^{1-d} \int_r^\infty e^{-\varepsilon \beta \rho/2} \mu(\rho)^{d-1} \, d\rho = O(e^{\varepsilon' r}) \int_0^\infty e^{-\varepsilon \beta \rho/2} (1 + \rho/r)^{d-1} \, d\rho = O(e^{\varepsilon' r}).$$

Then by definition of  $$w_\beta,$$  we have  $$\zeta_2^2 = \frac{w_\beta}{\zeta_1} + \frac{\zeta_2}{\zeta_1} \zeta'_1,$$  whence  $$\zeta_2^2 \sim w_\beta$$ near infinity.  

As a conclusion, gathering the above we have indeed, for some  $$\varepsilon' < \varepsilon \beta/2:$$  $$|\psi_\beta(r)| = O(e^{\varepsilon' r})$$  and  $$|\psi'_\beta(r)| = O(e^{\varepsilon' r}),$$  for large  $$r.$$
5.4 We have \( \beta \Psi(g) = Id \) on \( \mathbb{R}_+ \), and \( 0 \leq \psi_\beta \leq Id \) if \( 0 \leq f \leq b \)

Set \( \tilde{\psi}_\beta := \beta \Psi(g) \), and note that the identical function \( Id \) solves (K) if \( g = h \). Hence, the function \( r \to \tilde{\psi}_\beta(r) - r \) solves the homogeneous equation (K), so that

\[
\tilde{\psi}_\beta(r) - r = c \zeta_2(r) + c' \zeta_1(r),
\]

for some real constants \( c, c' \) and for any \( r > 0 \).

As \( \tilde{\psi}_\beta(0) = 0 \), we must have \( c' = 0 \). Then, as \( \zeta_2(r) \sim \int_1^r w_\beta \gg e^{\beta G(r)/2} \gg r \) for large \( r \), we must have \( c \geq 0 \). Otherwise, integrating by parts, near infinity we have:

\[
\beta \int_r^\infty \mu(\rho)^{d-1} e^{-\beta G(\rho)} g(\rho) d\rho = \mu(r)^{d-1} e^{-\beta G(r)} + \mathcal{O}(r^{-1}) \int_r^\infty e^{-\beta G(\rho)} \mu(\rho)^{d-1} d\rho \sim \mu(r)^{d-1} e^{-\beta G(r)},
\]

so that

\[
\beta \int_r^\infty \zeta_1(\rho) \mu(\rho)^{d-1} e^{-\beta G(\rho)} g(\rho) d\rho \sim \beta \int_r^\infty \mu(\rho)^{d-1} e^{-\beta G(\rho)} g(\rho) d\rho \sim \mu(r)^{d-1} e^{-\beta G(r)}.
\]

Hence, by definition of \( \tilde{\psi}_\beta \) and \( \Psi \) and by the above, we have near infinity:

\[
\tilde{\psi}_\beta(r) = \mathcal{O}(1) \int_0^r g(\rho) d\rho + \mathcal{O}(w_\beta(r)) \mu(r)^{d-1} e^{-\beta G(r)} = \mathcal{O}(G(r) + 1) = o(e^{\beta G(r)/2}),
\]

whence

\[
c e^{\beta G(r)/2} = o(c \zeta_2(r)) = o(\tilde{\psi}_\beta(r) - r) = o(e^{\beta G(r)/2}), \quad \text{which forces} \quad c = 0.
\]

Therefore \( \tilde{\psi}_\beta(r) = r \) on \( \mathbb{R}_+ \), as wanted. Finally, if \( 0 \leq f \leq b \), then \( 0 \leq h \leq g \), so that \( \Psi(h) \geq 0 \) and \( \Psi(g - h) \geq 0 \) by Section 5.2 above, and then by linearity of \( \Psi : Id = \beta \Psi(g) \geq \beta \Psi(h) = \psi_\beta \).

\[\Diamond\]

6 Proof of Proposition 4

For \( d = 1 \) (and any \( \eta \)), it is immediate from (K) that \( \psi_\beta(r) = \beta \int_0^r \int_0^\infty e^{-\beta G(\rho)} h \rho^{d-1} e^{\beta G(\rho)} d\rho \), so that in the DH case, the limit expresses as:

\[
\Sigma_\beta^2 = 2 \beta \left( \int_{\mathbb{R}_+} \frac{e^{\beta(1-\sqrt{1+x^2})}}{\sqrt{1+x^2}} dx \right) \times \int_{\mathbb{R}_+} \left[ \int_x^\infty y e^{\beta(1-\sqrt{1+y^2})} \frac{dy}{1+y^2} \right] e^{\beta(\sqrt{1+x^2}-1)} dx = J_\beta^{-1} K_\beta,
\]

where

\[
J_\beta := \int_{\mathbb{R}_+} e^{\beta(1-\sqrt{1+x^2})} \frac{dx}{\sqrt{1+x^2}}, \quad K_\beta := 2 \beta \int_{\mathbb{R}_+} I_\beta(x)^2 e^{\beta(\sqrt{1+x^2}-1)} dx, \quad I_\beta(x) := \int_x^\infty y e^{\beta(1-\sqrt{1+y^2})} \frac{dy}{1+y^2}.
\]
6.1 Behaviour as $\beta \to \infty$

Integrating by parts yields:

$$I_\beta(x) = \frac{e^{-\beta(\sqrt{1+x^2}-1)}}{\beta\sqrt{1+x^2}} - \frac{1}{\beta} \int_x^\infty \frac{y e^{-\beta(\sqrt{1+y^2}-1)}}{(1+y^2)^{3/2}} \, dy.$$

As

$$\left(\frac{e^{\beta(1-\sqrt{1+x^2})}}{1+x^2}\right)^{-1} \int_x^\infty y e^{\beta(1-\sqrt{1+y^2})} \, dy = \int_x^\infty e^{-\beta(\sqrt{1+y^2}-1)} \frac{y (1+x^2)}{(1+y^2)^{3/2}} \, dy$$

$$\leq \int_x^\infty e^{-\beta(\sqrt{1+y^2}-\sqrt{1+x^2})} \frac{y}{\sqrt{1+y^2}} \, dy = e^{\beta\sqrt{1+x^2}} \int_x^\infty e^{-\beta\sqrt{1+y^2}} \frac{y}{\sqrt{1+y^2}} \, dy = \frac{1}{\beta},$$

we get

$$I_\beta(x)^2 e^{\beta(\sqrt{1+x^2}-1)} = \frac{e^{\beta(1-\sqrt{1+x^2})}}{\beta^2 (1+x^2)} \times [1 + \mathcal{O}(1/\beta)].$$

Hence,

$$K_\beta = 2\beta \int_{\mathbb{R}_+} I_\beta(x)^2 e^{\beta(\sqrt{1+x^2}-1)} \, dx = \frac{2}{\beta} \int_{\mathbb{R}_+} \frac{e^{\beta(1-\sqrt{1+x^2})}}{1+x^2} \, dx \times [1 + \mathcal{O}(1/\beta)].$$

Setting $u = \beta \left(\sqrt{1+x^2} - 1\right)$, we get:

$$\frac{1}{\beta} \int_{\mathbb{R}_+} \frac{e^{\beta(1-\sqrt{1+x^2})}}{1+x^2} \, dx = \int_{\mathbb{R}_+} \frac{e^{-u} \, du}{(u+\beta)\sqrt{u(u+2\beta)}} \sim \sqrt{\frac{\pi}{2}} \beta^{-3/2}$$

by dominated convergence, as $\beta \to \infty$. We have similarly:

$$J_\beta = \int_{\mathbb{R}_+} \frac{e^{\beta(1-\sqrt{1+z^2})}}{\sqrt{1+z^2}} \, dz = \int_{\mathbb{R}_+} \frac{e^{-u} \, du}{\sqrt{u(u+2\beta)}} \sim \sqrt{\frac{\pi}{2}} \beta^{-1/2},$$

whence

$$\Sigma_\beta^2 = J^{-1}_\beta K_\beta \sim 2/\beta.$$

6.2 Behaviour as $\beta \to 0$

We have

$$J_\beta = \int_{\mathbb{R}_+} \frac{e^{-u} \, du}{\sqrt{u(u+2\beta)}} = 2 \int_{\mathbb{R}_+} \frac{e^{-u^2} \, du}{\sqrt{u^2+2}} = 2 \left[ \int_0^1 \frac{e^{-u^2} \, du}{\sqrt{u^2+2}} + \int_0^{1/\sqrt{3}} \frac{e^{-t^2} \, dt}{t\sqrt{1+2\beta^2}} \right],$$

where we performed the change of variable $\beta u^2 = 1/t^2$. Hence, as $\beta \to 0$, we have:

$$J_\beta = 2 \left[ \int_0^1 \frac{e^{-u^2} \, du}{\sqrt{u^2+2}} + \int_0^{1/\sqrt{3}} \frac{e^{-1/t^2} \, dt}{t\sqrt{1+2\beta^2}} + \int_1^{1/\sqrt{3}} \frac{dt}{t\sqrt{1+2\beta^2}} + \int_1^{1/\sqrt{3}} \frac{(e^{-1/t^2} - 1) \, dt}{t\sqrt{1+2\beta^2}} \right]$$

$$= 2 \int_1^{1/\sqrt{3}} \frac{dt}{t\sqrt{1+2\beta^2}} + C_1 + o(1),$$
for a positive constant $C_1$. Integrating by parts yields then:

$$
\int_1^{1/\sqrt{\beta}} \frac{dt}{t \sqrt{1 + 2\beta t^2}} = \left[ \frac{\log t}{\sqrt{1 + 2\beta t^2}} \right]_1^{1/\sqrt{\beta}} + 2\beta \int_1^{1/\sqrt{\beta}} \frac{t \log t \, dt}{(1 + 2\beta t^2)^{3/2}}.
$$

Setting $u = \sqrt{\beta} t$, we get:

$$
\int_1^{1/\sqrt{\beta}} \frac{dt}{t \sqrt{1 + 2\beta t^2}} = \frac{1}{\sqrt{3}} \log \left( \frac{1}{\sqrt{3}} \right) + 2 \int_{\sqrt{3}}^{1} \frac{u \log(u/\sqrt{\beta}) \, du}{(1 + 2u^2)^{3/2}} = \frac{1}{\sqrt{3}} \log \left( \frac{1}{\sqrt{3}} \right) + 2 \log \left( \frac{1}{\sqrt{3}} \right) \int_{\sqrt{3}}^{1} \frac{u \, du}{(1 + 2u^2)^{3/2}} + 2 \int_{\sqrt{3}}^{1} \frac{u \log u \, du}{(1 + 2u^2)^{3/2}}.
$$

Now, as $\beta \to 0$ we have

$$
\int_{\sqrt{3}}^{1} \frac{u \, du}{(1 + 2u^2)^{3/2}} = \log \left( \frac{1}{\sqrt{3}} \right) \times [1 + o(1)] + C_2,
$$

and

$$
J_\beta \sim 2 \log \left( \frac{1}{\beta} \right).
$$

Otherwise,

$$
K_\beta = 2\beta \int_{\mathbb{R}^+} I_\beta(x)^2 e^{\beta(x^2 - 1)} \, dx = 2\beta \int_{\mathbb{R}^+} \left( \int_{\sqrt{x+2}}^{\infty} e^{\beta(y-1)} \, dy \right) \, dx = 2\beta e^{3} \int_{1}^{\infty} \frac{y e^{\beta y} \, dy}{\sqrt{y^2 - 1}}.
$$

Observing that

$$
\left| \left[ \int_{y}^{\infty} e^{-\beta u} \frac{du}{u} \right] - \left[ \int_{y}^{y/\beta} e^{-\beta u} \frac{du}{u} \right] \right|^2 = 2 \left[ \int_{y}^{y/\beta} e^{-\beta u} \frac{du}{u} \right] \left[ \int_{y/\beta}^{\infty} e^{-\beta u} \frac{du}{u} \right] \left[ \int_{y/\beta}^{\infty} e^{-\beta u} \frac{du}{u} \right] \leq 2 \log \left( \frac{1}{\beta} \right) \cdot \frac{e^{-y}}{y} + \frac{e^{-2y}}{y^2},
$$

and then that

$$
\left( \int_{1}^{\infty} \int_{y}^{\infty} e^{-\beta u} \, du \right)^{-1} \frac{y e^{\beta y} \, dy}{\sqrt{y^2 - 1}} = \left( \int_{1}^{\infty} \int_{y}^{\infty} e^{-\beta u} \, du \right)^{-1} \frac{y e^{\beta y} \, dy}{\sqrt{y^2 - 1}} + O(1),
$$

changing $\beta y$ into $y$ we get:

$$
\left( \int_{1}^{\infty} \int_{y}^{\infty} e^{-\beta u} \, du \right)^{-1} K_\beta = 2 e^{3} \left( \int_{1}^{\infty} \int_{y}^{\infty} e^{-\beta u} \, du \right)^{-1} A_\beta + O(\beta), \quad \text{with} \quad A_\beta := \int_{\beta}^{\infty} \left( \int_{y}^{y/\beta} e^{-u} \, du \right) \frac{y e^{\beta y} \, dy}{\sqrt{y^2 - \beta^2}}.
$$
Finally, setting \( x = \sqrt{y^2 - \beta^2} \), we have by dominated convergence:

\[
A_\beta = \int_0^\infty \left( \int \frac{e^{-u \sqrt{x^2 + \beta^2}}}{u} \, dx \right)^2 e^{\sqrt{x^2 + \beta^2}} \, du \rightarrow A := \int_0^\infty \left( \int e^{-u \sqrt{x^2 + \beta^2}} \, dx \right)^2 e^x \, dx \in \mathbb{R}_+^*,
\]

whence

\[
\Sigma_\beta^2 = J_\beta^{-1} K_\beta \sim A / \log(1/\beta).
\]

REFERENCES

[BDR1] Barbachoux C., Debbasch F., Rivet J.P.  
Hydrodynamic behavior of Brownian particles in a position-dependent constant force-field.  

[BDR2] Barbachoux C., Debbasch F., Rivet J.P.  

[BDR3] Barbachoux C., Debbasch F., Rivet J.P.  
The spatially one-dimensional Relativistic Ornstein-Uhlenbeck process in an arbitrary inertial frame.  

[D] Debbasch F.  
A diffusion process in curved space-time.  

[DH1] Dunkel J., Hänggi P.  
Theory of relativistic Brownian Motion : The (1 + 1)-dimensional case.  

[DH2] Dunkel J., Hänggi P.  
Theory of relativistic Brownian Motion : The (1 + 3)-dimensional case.  
ArXiv, 0505532, 2005.

[DMR] Debbasch F., Mallick K., Rivet J.P.  
Relativistic Ornstein-Uhlenbeck Process.  

[DR] Debbasch F., Rivet J.P.  
A diffusion equation from the Relativistic Ornstein-Uhlenbeck Process.  

[F] Franchi J.  
Relativistic Diffusion in Gödel’s Universe.  

[FLJ] Franchi J., Le Jan Y.  
Relativistic Diffusions and Schwarzschild Geometry.  

[K] Kallenberg O.  
Foundation of Modern Probability.  

[RY] Revuz D., Yor M.  
Continuous martingales and Brownian motion.  

[V] Varadhan S.R.S.  
Diffusion problems and partial differential equations.  