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# Vlasov simulations on an adaptive phase-space grid

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The numerical resolution of the Vlasov equation is usually performed by particle methods (PIC) which consist in approximating the plasma by a finite number of particles. The trajectories of these particles are computed from the characteristic curves given by the Vlasov equation, whereas self-consistent fields are computed on a mesh of the physical space. This method allows to obtain satisfying results with a relatively small number of particles. However, it is well known that, in some cases, the numerical noise inherent to the particle method becomes too important to have an accurate description of the distribution function in phase space. To remedy to this problem, methods discretizing the Vlasov equation on a mesh of phase space have been proposed [2, 5, 6].

The major drawback of Vlasov methods using a uniform and fixed mesh is that their numerical cost is high, which makes them rather inefficient when the dimension of phase-space grows. For this reason we are investigating a method using an adaptive mesh. The adaptive method is overlaid to a classical semi-Lagrangian method which is based on the conservation of the distribution function along particle trajectories. The phase-space grid is updated using a multiresolution technique.

The model we consider throughout this paper is the nonrelativistic Vlasov equation coupled self-consistently with Poisson's equation. It reads

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \nabla_x f + \frac{q}{m} (\mathbf{E} + \mathbf{v} \times \mathbf{B}) \cdot \nabla_v f = 0, \quad (1)$$

the self electric field  $\mathbf{E}$  is computed from Poisson's equations

$$-\epsilon_0 \nabla^2 \phi = \rho(\mathbf{x}, t) = q \int f(\mathbf{x}, \mathbf{v}, t) dv, \quad \mathbf{E} = -\nabla \phi.$$

The magnetic field is external and considered to be known.

In the present work, we have chosen to introduce a phase-space mesh which can be refined or derefined adaptively in time. For this purpose, we use a technique based on multiresolution analysis which is in the same spirit as the methods developed in particular by S. Bertoluzza [1], A. Cohen *et al.* [3] and M. Griebel and F. Koster [4]. We represent the distribution function on a wavelet basis at different scales. We can then compress it by eliminating coefficients which are small and accordingly remove the associated mesh points. Another specific feature of our method is that we use an advection in physical and velocity space forward in time to predict the useful grid points for the next time step, rather than restrict ourselves to the neighboring points. This enables us to use a much larger time step, as in the semi-Lagrangian method the time step is not limited by a Courant condition. Once the new mesh is predicted, the semi-Lagrangian methodology is used to compute the new values of the distribution function at the predicted mesh points, using an interpolation based on the wavelet decomposition of the old distribution function. The mesh is then refined again by performing a wavelet transform, and eliminating the points associated to small coefficients.

This paper is organized as follows : we shall first present the context of multiresolution analysis for adaptive simulations, then describe the adaptive semi-Lagrangian method and validate our method on typical simulation in beam physics.

# 1 Multiresolution analysis

The semi-Lagrangian method consists mainly of two steps, an advection step and an interpolation step. The interpolation part is performed using for example a Lagrange interpolating polynomial on a uniform grid. Thus interpolating wavelets provide a natural way to extend this procedure to an adaptive grid in the way we shall now shortly describe.

For simplicity, we shall restrict our description to the 1D case of the whole real line. It is straightforward to extend it to periodic boundary conditions and it can also be extended to an interval with Dirichlet boundary conditions. The extension to higher dimension is performed using a tensor product of wavelets and will be addressed at the end of the section.

For any value of  $j \in \mathbb{Z}$ , we consider a uniform grid  $G^j$  of step  $2^{-j}$ . The grid points are located at  $x_k^j = k2^{-j}$ . This defines an infinite sequence of grids that we denote by  $(G_j)_{j \in \mathbb{Z}}$ , and  $j$  will be called the level of the grid.

In order to map a function defined on the grid from one level to the next or the previous, we define a projection operator and a prediction operator. Consider two grid levels  $G_j$  and  $G_{j+1}$  and discrete values (of a function) denoted by  $(c_k^j)_{k \in \mathbb{Z}}$  and  $(c_k^{j+1})_{k \in \mathbb{Z}}$ . Even though we use the same index  $k$  for the grid points in the two cases, there are of course twice as many points in any given interval on  $G_{j+1}$  as on  $G_j$ . Using the terminology in [3], we then define the projection operator

$$\begin{aligned} P_{j+1}^j : G_{j+1} &\rightarrow G_j, \\ c_{2k}^{j+1} &\mapsto c_k^j, \end{aligned}$$

which is merely a restriction operator, as well as the prediction operator

$$\begin{aligned} P_j^{j+1} : G_j &\rightarrow G_{j+1}, \\ \text{such that } c_{2k}^{j+1} &= c_k^j, \\ c_{2k+1}^{j+1} &= P_{2N-1}(x_{2k+1}^{j+1}), \end{aligned}$$

where  $P_{2N-1}$  stands for the Lagrange interpolation polynomial of odd degree  $2N - 1$  centered at the point  $(x_{2k+1}^{j+1})$ .

Using the just defined prediction operator, we can construct on  $G_j$  a subspace of  $L^2(\mathbb{R})$  that we shall denote by  $V_j$ , a basis of which being given by  $(\varphi_k^j)_{k \in \mathbb{Z}}$  such that  $\varphi_k^j(x_{k'}^j) = \delta_{kk'}$  where  $\delta_{kk'}$  is the Kronecker symbol. The value of  $\varphi_k^j$  at any point of the real line is then obtained by applying, possibly an infinite number of times, the prediction operator.

In the wavelets terminology the  $\varphi_k^j$  are called scaling functions. Let us denote by  $\varphi = \varphi_0^0$ . Notice that  $[-N + 1, N - 1]$ ,

$$\varphi_k^j(x) = \varphi(2^j x - k).$$

The scaling functions have the following noteworthy properties: 1) they have compact support,  $\varphi$  vanishes outside, 2) they are interpolating as by construction  $\varphi(0) = 1$  and  $\varphi(k) = 0$  if  $k \neq 0$ , 3) All polynomials of degree less than  $N - 1$  can be obtained exactly as linear combinations of the  $\varphi_k^0$ , 4) Change of scale:  $\varphi$  can be expressed using the half scale:

$$\varphi(x) = \sum_{l=-N+1}^{N-1} h_l \varphi(2x - l),$$

the  $h_l$  are linked to the projection operator  $P_j^{j+1}$ , as  $c_k^{j+1} = \sum h_{l-2k} c_l^j$ .

As  $V_j \subset V_{j+1}$ , there exists a supplementary of  $V_j$  in  $V_{j+1}$  that we shall call the detail space and denote by  $W_j$  :

$$V_{j+1} = V_j \oplus W_j.$$

The construction of  $W_j$  can be made in the following way: an element of  $V_{j+1}$  is characterized by the sequence  $(c_k^{j+1})_{k \in \mathbb{Z}}$  and by construction we have

$$c_k^j = c_{2k}^{j+1}.$$

Thus, if we define

$$d_k^j = c_{2k+1}^{j+1} - P_{2N+1}(x_{2k+1}^{j+1}),$$

where  $P_{2N+1}$  is the Lagrange interpolation polynomial by which the value of an element of  $V_j$  at the point  $(x_{2k+1}^{j+1})$  can be computed,  $d_k^j$  represents exactly the difference between the value in  $V_{j+1}$  and the value predicted in  $V_j$ . Finally, any element of  $V_{j+1}$  can be characterized by the two sequences  $(c_k^j)_k$  of values in  $V_j$  and  $(d_k^j)_k$  of details in  $W_j$ . Moreover this strategy for constructing  $W_j$  is particularly interesting for adaptive refinement as  $d_k^j$  will be small at places where the prediction from  $V_j$  is good and large elsewhere, which gives us a natural refinement criterion. Besides, there exists a function  $\psi$ , called wavelet such that  $\{\psi_k^j = 2^{j/2} \psi(2^j x - k)\}_{k \in \mathbb{Z}}$  is a basis of  $W_j$ .

In practise, for adaptive refinement we set the coarsest level  $j_0$  and the finest level  $j_1$ ,  $j_0 < j_1$ , and we decompose the space corresponding to the finest level on all the levels in between:

$$V_{j_1} = V_{j_0} \oplus W_{j_0} \oplus W_{j_0+1} \oplus \dots \oplus W_{j_1-1}.$$

A function  $f \in V_{j_1}$  can then be decomposed as follows

$$f(x) = \sum_{l=-\infty}^{+\infty} c_l^{j_0} \varphi_l^{j_0}(x) + \sum_{j=j_0}^{j_1-1} \sum_{l=-\infty}^{+\infty} d_l^j \psi_l^j(x),$$

where the  $(c_l^{j_0})_l$  are the coefficients on the coarse mesh and the  $(d_l^j)_l$  the details at the different level in between.

**2D generalisation:** In two dimensions, the prediction operator which is the base of the multiresolution analysis can be defined as a tensor product of 1D operators. In practice, one needs to consider three distinct cases. (see figure 1 for notations):

1. Refinement in  $x$  (corresponds to the points  $c_{2k_1+1, 2k_2}^{j+1}$  and  $c_{2k_1+1, 2k_2+2}^{j+1}$ ): we use the 1D prediction operator in  $x$  for fixed  $k_2$ .
2. Refinement in  $v$  (corresponds to the points  $c_{2k_1, 2k_2+1}^{j+1}$  and  $c_{2k_1+2, 2k_2+1}^{j+1}$ ): we use the 1D prediction operator in  $v$  for fixed  $k_1$ .
3. Refinement in  $x$  and  $v$  (corresponds to the point  $c_{2k_1+1, 2k_2+1}^{j+1}$ ): we first use the 1D prediction operator in  $v$  for fixed  $k_1$  to construct the necessary points to apply the 1D prediction operator in  $x$  for fixed  $k_2$  that is the applied.

The corresponding wavelet bases are respectively of type  $\psi(x)\varphi(v)$ ,  $\varphi(x)\psi(v)$  et  $\psi(x)\psi(v)$  where  $\varphi$  and  $\psi$  are respectively the 1D scaling function and wavelet. Indeed if we have a decomposition like

$$V_{j+1} \times V_{j+1} = V_j \times V_j + V_j \times W_j + W_j \times V_j + W_j \times W_j.$$

we obtain a 2D wavelet decomposition of the following type

$$f(x, v) = \sum_{k_1, k_2} \left( c_{k_1, k_2}^{j_0} \varphi_{k_1}^{j_0}(x) \varphi_{k_2}^{j_0}(v) + \sum_{j_0}^{j_1-1} \left( d_{k_1, k_2}^{row, j} \psi_{k_1}^j(x) \varphi_{k_2}^j(v) + d_{k_1, k_2}^{col, j} \varphi_{k_1}^j(x) \psi_{k_2}^j(v) + d_{k_1, k_2}^{mid, j} \psi_{k_1}^j(x) \psi_{k_2}^j(v) \right) \right). \quad (2)$$

**nD generalization :** This will be useful for three, four, five or six dimensional phase space as is necessary for Vlasov simulation. A nD multiresolution analysis can also be constructed from the 1D multiresolution analysis by tensor products

$$\mathcal{V}_j = V_j \times V_j \cdots \times V_j.$$

Defining  $\mathcal{I}bin = \{I = (i_1, i_2, \dots, i_n) \in \mathbb{Z}^n, I \neq 0, i_k \in \{0, 1\} \forall k\}$ , we obtain the formula

$$\mathcal{V}_j = \mathcal{V}_{j-1} + \sum_{I \in \mathcal{I}bin} \mathcal{W}_{j-1}^I,$$

where  $\mathcal{W}_{j-1}^I = U_{j-1}^{i_1} \times \cdots \times U_{j-1}^{i_n}$  with  $U_{j-1}^i = \begin{cases} V_{j-1} & \text{if } i = 0, \\ W_{j-1} & \text{if } i = 1. \end{cases}$

The decomposition of a function  $f$  at the scale  $j_1$  can be expressed as

$$f(X) = \sum_K C_K^{j_0} \Phi_K^{0, j_0}(X) + \sum_{j=j_0}^{j_1-1} \sum_{I \in \mathcal{I}bin} \sum_K D_K^{I, j} \Phi_K^{I, j}(X),$$

where  $\Phi_K^{I, j}(X) = \phi_{k_1}^{i_1, j}(x_1) \cdot \phi_{k_2}^{i_2, j}(x_2) \cdots \phi_{k_n}^{i_n, j}(x_n)$  with  $\phi_k^{i, j} = \begin{cases} \varphi_k^j & \text{if } i = 0, \\ \psi_k^j & \text{if } i = 1. \end{cases}$

## 2 The adaptive algorithm

In the initialization phase, we first compute the wavelet decomposition of the initial condition  $f_0$  which is known analytically, and then proceed by compressing it, i.e. eliminating the details which are smaller than a threshold that we set. We then construct an adaptive mesh which, from all the possible points at all the levels between our coarsest and finest, contains only those of the coarsest and those corresponding to details which are above the threshold. We denote by  $\tilde{G}$  this mesh.

- **Prediction in  $x$ :** We predict the positions of points where the details should be important at the next time split step by advancing in  $x$  the characteristics originating from the points of the mesh  $\tilde{G}$ . Then we retain the grid points, at one level finer as the starting point, the support of which contains the end point the characteristic.

- **Construction of mesh  $\hat{G}$ :** From the predicted mesh  $\tilde{G}$ , we construct the mesh  $\hat{G}$  where the values of the distribution at the next time step shall be computed. This mesh  $\hat{G}$  contains exactly the points necessary for computing the wavelet transform of  $f^*$  at the points of  $\tilde{G}$ .
- **Advection in  $x$ :** We compute the origin of the characteristics for each point of  $\hat{G}$  and interpolate its value at the level where it arrives.
- **Wavelet transform of  $f^*$ :** We compute the  $c_k$  and  $d_k$  coefficients at the points of  $\tilde{G}$  from the values of  $f^*$  at the points of  $\hat{G}$ .
- **Compression:** We eliminate the points of  $\tilde{G}$  where the details  $d_k$  are lower than the fixed threshold.
- **Computation of the electric field:** We compute  $\rho$  on the finest grid and solve Poisson.
- Same steps for the  $v$  part of the splitting method.

### 3 Numerical results

In order to assess the benefits of the adaptive solver we computed the transverse evolution of a semi-Gaussian beam in a uniform and periodic focusing channel. For such a beam the initial distribution function reads

$$f(r, v) = \frac{I}{\pi a^2 \sqrt{2\pi} b} e^{-\frac{1}{2}(v^2/b^2)} \text{ if } r < a,$$

and  $f(r, v) = 0$  else.

#### 3.1 Semi-Gaussian beam in uniform focusing channel

We first consider the evolution of a semi-Gaussian beam in a uniform focusing channel. The tune depression  $\sigma/\sigma_0$  is 0,25.

Moreover, here, in dimensionless units  $\omega = 2$ ,  $\omega_0 = 1$ .  $a = 4/\sqrt{15}$ ,  $b = 1/(2 * \sqrt{15})$  et  $n_0 = 2\pi$ . The tests have been performed with a time step of  $1/32^{\text{th}}$  of period, that is  $\Delta t = 2\pi/(32\omega_0) = \pi/16$  for a coarse mesh of  $8 \times 8$  points with 5 refinement levels, that is  $256 \times 256$  points on the finest grid, and a refinement threshold of  $10^{-3}$ . This means that details such that  $|d_k^j| < 10^{-3}$  are considered to be 0 and the corresponding grid point is removed. Figure 2 represents snapshots of the evolution along with the grid points kept by the adaptive algorithm for the computation. We notice that the adaptive grid follows very well the evolution of the fine structures.

#### 3.2 Semi-Gaussian beam in periodic focusing channel

We now consider the transverse evolution of a semi-Gaussian beam in a periodic focusing field of the form  $\alpha(1 + \cos 2\pi z/S)$  for a tune depression  $\sigma/\sigma_0$  of 0,17. Figure 3 represents snapshots of the evolution along with the grid points kept by the adaptive algorithm for the computation. As in the previous case, the adaptive grid follows very well the evolution of the fine structures.

## 4 Conclusion

We have introduced in this paper the concept of an adaptive semi-Lagrangian Vlasov solver and proved the feasibility of the method for 1D model problems. The method we proposed can be generalized to higher dimensions. Adaptivity is defined through a multiresolution analysis based on interpolating wavelets which is coupled very naturally to the semi-Lagrangian method as fine grid points are added where interpolation on the coarser grid does not do a good enough job.

Adaptive methods enable semi-Lagrangian methods not to waste computing time in regions where nothing is happening and thus make them a lot more efficient. On the other hand, an important overhead is introduced, and programming such a method in an optimal way is a lot more challenging. We are still working on it.

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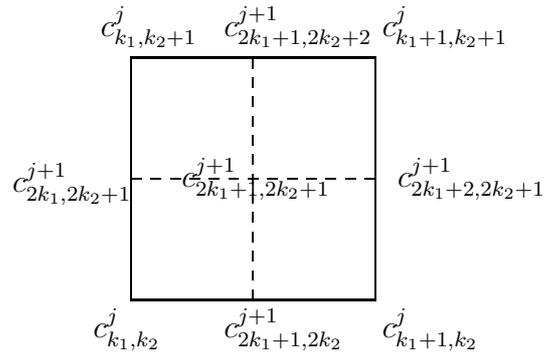


Figure 1: Raffinement du maillage en 2D.

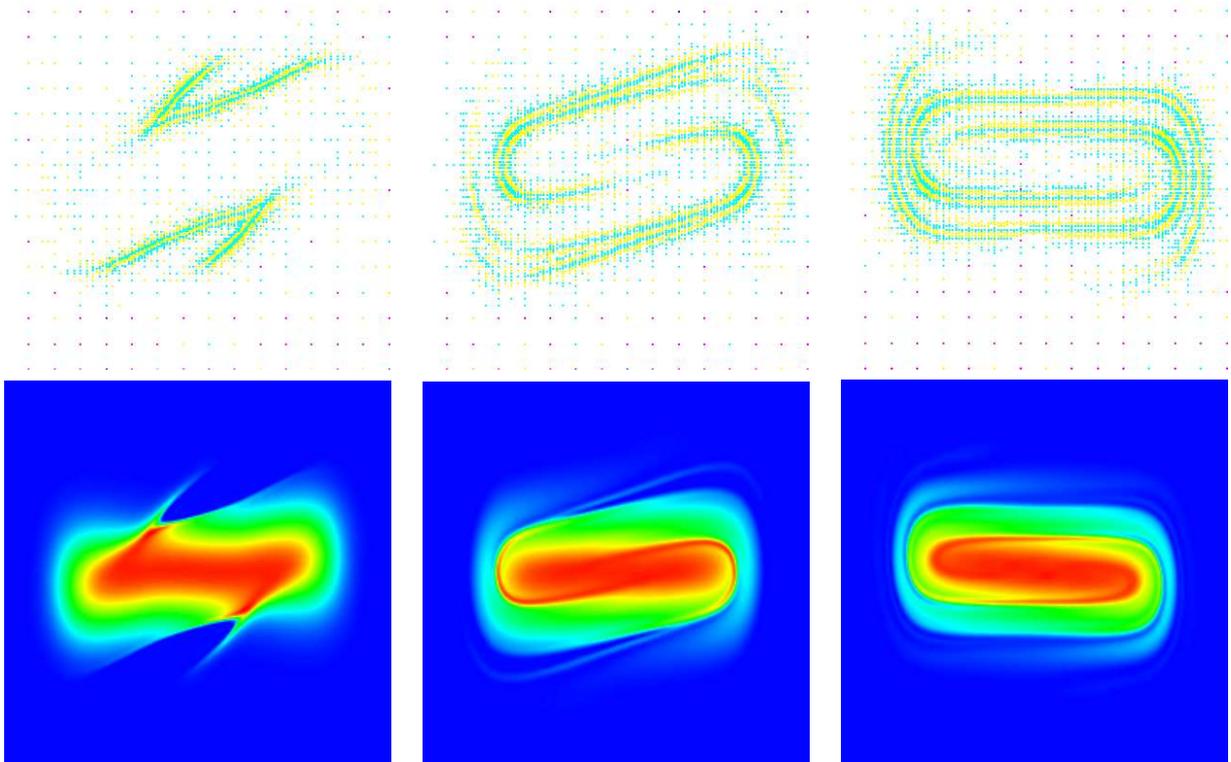


Figure 2: Snapshots of the evolution of a semi-Gaussian beam and the associated adaptive grid in a uniform focusing channel

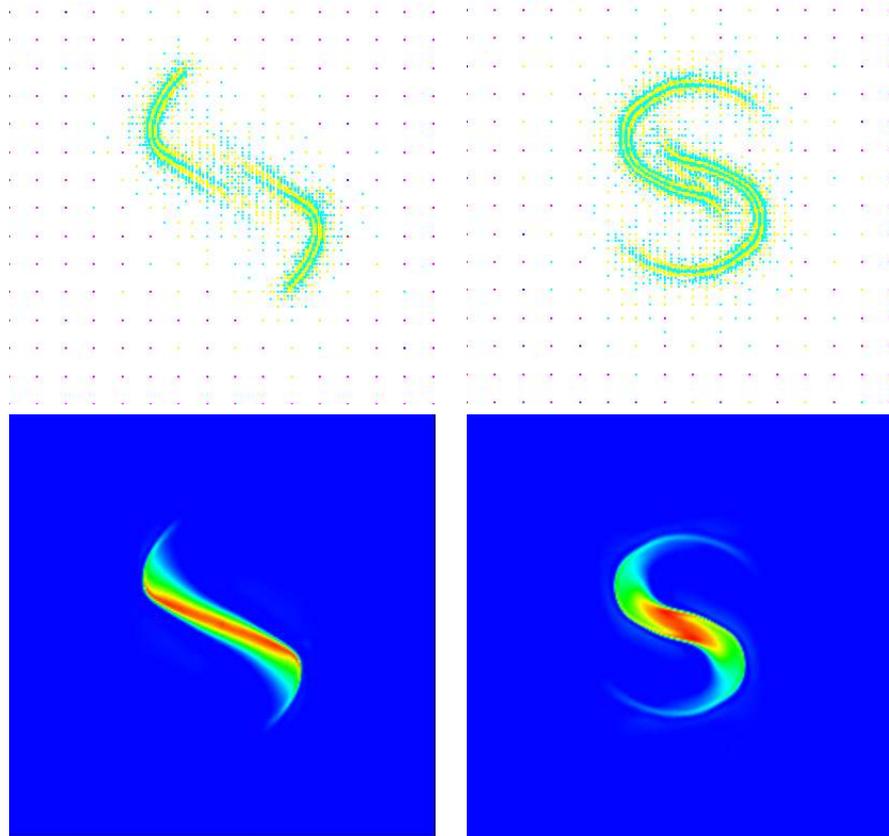


Figure 3: Snapshots of the evolution of a semi-Gaussian beam and the associated adaptive grid in a periodic focusing channel