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TOPOLOGY, MODULI AND AUTOMORPHISMS OF REAL ALGEBRAIC SURFACES

VIATCHESLAV KHARLAMOV

These notes reproduce the content of the colloquium talks given at Göttingen University in December, 1999 and at Milan University in April, 2000; a few details have been added that have not been given in the lectures because of the lack of time. I would like to thank professors F. Catanese et B. Ruff for giving me these occasions and for their hospitality.

1. INTRODUCTION

Originally, my interest to the objects listed in the title was motivated by the first part of Hilbert's 16th problem, where Hilbert asks about the number of sheets, their shape, and their position in $\mathbb{R}p^3$ for surfaces of degree 4 and higher (due to Schläfli and Klein, the answer was already known for degree 3, see the precise statement below; for smaller degrees it is elementary).

Hilbert poses his questions for nonsingular surfaces only. He did not explain why he paid special attention to nonsingular objects, their degree and their topology. One may suspect the following speculation.

Algebraic varieties are naturally organized in families. In modern terms one can speak about Hilbert or Chow schemes, for example. In the case of projective hypersurfaces, which is our subject, it is nothing but the space of homogeneous polynomials, which splits into irreducible components numbered by the degree d of the polynomials and the number $n + 1$ of (homogeneous) variables. Each component is a projective space of dimension $N_{d,n} = C_{n+d}^d - 1$. This projective space $P^N = \{f(x) \mid \deg f = d\}$, $N = N_{d,n}$, has certain additional structures that are due to its origin. In particular, it is equipped with a special discriminant hypersurface D that consists of the polynomials

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$f \in P^N$ such that $\{f = 0\} \subset P^n$ is singular. In fact, it is a part of the stratification of P^N associated with the universal hypersurface

$$\Gamma \subset P^N \times P^n, \Gamma = \{(f, x) | f(x) = 0\}$$

and its projection $\Gamma \rightarrow P^N$. Namely, D is the critical locus of the projection, and, as a consequence, over $P^N \setminus D$ we get, in the modern terminology, a deformation family of nonsingular varieties

$$\Gamma \supset \Gamma^0 \rightarrow P^N \setminus D.$$

Everything said before works equally well over $\mathbb{C}$ and $\mathbb{R}$. However, there is a principal difference: over $\mathbb{C}$ the space $P^N \setminus D$ is connected and all hypersurfaces $\{f = 0\}$ with $f \in P^N \setminus D$ are deformation equivalent, while over $\mathbb{R}$ this is no longer the case (except $d = 1$) and it is natural to try to understand the nature of the components of $P^N(\mathbb{R}) \setminus D(\mathbb{R})$. First questions are: *how many deformation different hypersurfaces of given degree and dimension exist; and how to distinguish deformation different hypersurfaces?* (Unfortunately, not very much is known so far in general: most known results concern curves and surfaces. As to the number of deformations classes, the best, up to my knowledge, estimate is $d^{Nn}(n+1)^N(1+o(1))$; for n fixed and $d \rightarrow \infty$ it gives a sequence growing like $\exp(\gamma_n d^n \log d)$, $\gamma_n \in \mathbb{R}$. Most likely, this upper bound is not asymptotically sharp and could be essentially improved as soon as one understands better the homology of the discriminant hypersurface. For some related information, see [KO].)

To conclude these introductory speculations, let us underline that the topology of a real hypersurface (which can be understood as the topology of the real point set, as in the sixteenth Hilbert problem, or else as that of the complex conjugation involution) is its main deformation invariant. Thus, the study of the topology of a hypersurface is an important ingredient of the deformation theory.

2. TYPICAL KNOWN RESULTS ON DEFORMATIONS OF SURFACES¹

The oldest result, which is even older than Hilbert's problem, is due to Schläfli and Klein (1858–1873).

2.1. Theorem. *Two nonsingular real cubic surfaces are real deformation equivalent if and only if their real point sets are homeomorphic.*

Since the real part is 2-dimensional, there is no difference between 'homeomorphic' and 'diffeomorphic'; the deformation equivalence can be understood

¹The talks being devoted to surfaces, it is worth noting however that for curves the deformation problem is solved: real curves are deformation equivalent, if and only if their real structures are topologically equivalent, see [Nat].

here as a continuous path in $P^N \setminus D$, cf. above. (Note also that the analogous result for surfaces of degree ≤ 2 is trivial; in degree ≤ 2 the topology of the real point set determines the surface even up to projective transformation, and the transformation can be chosen preserving the orientation of P^3 , i.e., in the component of unity of the group $PGL(4; \mathbb{R})$. A simple proof of Theorem 2.1 going back to Klein is found in [DK2].)

As a consequence, one obtains the following.

2.2 Corollary. *The real structures of two real nonsingular cubic surfaces are diffeomorphic if (and only if) the real point sets of the surfaces are homeomorphic.*

In fact, there are 5 classes of nonsingular cubic surfaces: $\mathbb{R}p^2 \sqcup S^2$, $\mathbb{R}p^2$, $\#_3\mathbb{R}p^2$, $\#_5\mathbb{R}p^2$, and $\#_7\mathbb{R}p^2$. Here, $\#$ stands for the connected sum and $\sqcup$, for the disjoint sum.

Cubic surfaces are rational. In fact, a similar quasi-simplicity result holds for all rational surfaces. In what follows we call *quasi-simple* a family of surfaces (or an individual surface) whose real deformation class² is determined by the diffeomorphism type of the real structure.

2.3 Theorem. *Rational surfaces are quasi-simple.*

For minimal surfaces this result is essentially due to Comessatti, Manin, and Iskovskikh (see, e.g., the survey [MT]). To prove the theorem for ruled surfaces, as well as for conic bundles, one can use Kodaira deformations, which provide the deformation equivalence between ruled surfaces with rulings of the same parity. For Del Pezzo surfaces the proof is found in [DIK], where one can also find a list of deformation classes of minimal rational and Del Pezzo surfaces. A complete proof of theorem 2.3 is given in [DK4].

In fact, as is proved in [DK4], the deformation type of a real rational surface is determined by certain homological information.

2.4 Proposition. *Two real rational surfaces X' , X'' are deformation equivalent if and only if there is a homeomorphism $X'(\mathbb{R}) \rightarrow X''(\mathbb{R})$ preserving the additional structures described below and one has $E(X') = E(X'')$ where $E(X)$ is the part of the eigenlattice $L(X) = \text{Ker}[(1 - \text{conj}_*): H_2(X) \rightarrow H_2(X)]$ obtained by removing from $L(X)$ the summands isomorphic to $\langle -2 \rangle$ (the condition $E(X') = E(X'')$ holds automatically unless $b_0(X'_\mathbb{R}) = b_0(X''_\mathbb{R}) = 4$ or 5).*

The additional structures to be preserved are:

- (1) *a distinguished ‘essentially nonorientable’ component of $X_\mathbb{R}$, defined in the case $E(X) = E_8$: this is the only component whose class in*

²Here, one can use the general definition of real deformation equivalence given below.

- $H_2(X; \mathbb{Z}/2)$ does not belong to the image of $L(X)$;
- (2) a cyclic order on the set of the other components, defined whenever $b_0(X) \geq 4$ as follows: two components F_1, F_2 are neighbors if and only if there is an element $e \in H_2(X/\text{conj})$ such that $e^2 = -1$ and $e \circ [F_1] = e \circ [F_2] = 1 \pmod{2}$.

Next (and practically last for the moment) group of surfaces whose real deformation theory is well understood is formed by the minimal surfaces of Kodaira dimension 0. These are Abelian, hyperelliptic, $K3$ -, and Enriques surfaces. Since they will be our principal subject, let me recall their definitions and give some examples.

At this point we have to choose the category of surfaces to be considered, i.e., either algebraic surfaces only or, more generally, compact complex surfaces. We make the latter choice, as it leads to simpler and more transparent statements (for rational surfaces this makes no difference, since they form a class closed under the deformation equivalence as it is defined below).

Thus, in what follows we mean by an *Abelian surface* the quotient of C^2 by a rank 4 lattice $L \subset C^2$, by a *$K3$ -surface* a compact complex surface with $\pi_1 = 1$ and $c_1 = 0$, and by an *Enriques surface* a surface with $\pi_1 = \mathbb{Z}/2$ whose universal (double) covering is a $K3$ -surface. By hyperelliptic surfaces we mean the (nonsingular) quotients of Abelian surfaces by free finite group actions.

Any nonsingular degree 4 surface in P^3 and the double covering of a nonsingular quadric in P^3 branched in a transversal section by a quartic are examples of $K3$ -surfaces. Consider, for example, the quadric $x^2 + y^2 = z^2 + 1$ and pick a section which is invariant with respect to the involution $x \mapsto x, y \mapsto -y, z \mapsto -z$ and which contains none of the (four) fixed points of the involution; then we can lift the involution to a free involution on the covering $K3$ -surface and thus get an Enriques surface as its orbit space.

When dealing with a compact complex manifold X , we define a *real structure* on X as an antiholomorphic involution $c : X \rightarrow X$. Respectively, by a *real deformation equivalence* we mean the equivalence generated by local real deformations: given a proper holomorphic equivariant submersion

$$f : (M, c) \rightarrow (|z| < 1, \text{conj}),$$

where c is a real structure on M and conj is the usual complex conjugation, the real varieties $(X_0, c), (X_t, c)$ with $t \in \mathbb{R}, |t| < 1, X_t = f^{-1}(t)$ are said to be *real deformation equivalent*.

In the above examples the $K3$ and Enriques surfaces inherit a natural real structure from the standard complex conjugation in P^3 as soon as the corresponding equations have real coefficients. Small variations of the coefficients

provide local deformations. (Note that contrary to Enriques surfaces, not all $K3$ surfaces are projective in general. The reader should be also warned that the above definition of real deformation equivalence differs slightly from that discussed in Introduction: every real deformation equivalence in the sense of Introduction is a real deformation equivalence in the sense of the above definition, while the converse is not true; for example, a small deformation of a surface of degree 4 in P^3 can provide a surface non embedable in P^3 , and the same happens for non small deformations of surfaces of degree > 4 .)

2.4 Theorem. *Abelian, $K3$ -, and Enriques surfaces are quasi-simple.*

For Abelian surfaces this result is essentially due to Comessatti [C] and (as long as it concerns not necessarily algebraic surfaces, that is, without taking into account a polarization) it follows from fairly elementary analysis of involutions on lattices. Namely, one considers the universal covering, which is an affine complex plane, and the covering affine action of the group L of deck transformations extended by $\mathbb{Z}/2$ by means of the lifts of the real structure from the quotient to the covering plane,

$$0 \rightarrow L \rightarrow G \rightarrow \mathbb{Z}/2 \rightarrow 0.$$

(Note that the real structure lifts to a real structure of the covering plane if and only if the Abelian surface has real points; the existence of such a lift is equivalent to the existence of a splitting of the above short exact sequence.) The deck transformations form a lattice in $\mathbb{C}^2$ and the action of the extended group of transformations G on $\mathbb{C}^2$ is determined, up to deformation, by the action of $\mathbb{Z}/2$ on L associated with the extension and the element $a \in L/(1+c)L$ attached to it. This implies the quasi-simplicity. (For the similar result in the case of principally polarized Abelian surfaces, see, e.g., [Shi] and [Si]).³

For $K3$ - and Enriques surfaces the result follows from the global Torelli theorem, see [N] and [DK], respectively, or [DIK], where the proof is given for both the cases. In fact, in all the three cases (Abelian, $K3$, and Enriques) the quasi-simplicity can be refined and made more explicit. Namely, as it follows from the above consideration, the deformation type of a real Abelian surface is determined by the topology of the real part with the exception of one case, when the real part is empty and there are 2 deformation classes which are distinguished by the action of the real structure in H^1 of the surface. The deformation type of a real $K3$ -surface is determined by the action in H^2 . For real Enriques surfaces, the real structure lifts to two real structures on the

³**Added in proof.** The classification of real hyperelliptic surfaces up to deformation has recently been accomplished by F. Catanese and P. Frediani [CF]. As follows from their classification, the real hyperelliptic surfaces are also quasi-simple.

covering $K3$ -surface, and the deformation type is determined by the induced $\mathbb{Z}/2 \times \mathbb{Z}/2$ -action in H^2 of the covering $K3$ -surface. (Contrary to the case of Abelian surfaces, a real structure of an Enriques surface lifts always to a real structure (and, thus, to two real structures) of the universal covering (which is the covering $K3$ -surface), even if the real part of the surface is empty. Indeed, as is known, the obstruction is the same as for lifting the real structure into the line bundle associated with the covering, and in the case of Enriques surfaces this bundle is the canonical line bundle, to which the real structure lifts tautologically.)

The above invariants can be simplified further. For example, in the case of $K3$ -surfaces, the deformation type is determined by the topology of the real part plus one more bit of information: the vanishing (or not) of the real part in the second $\mathbb{Z}/2$ -homology group of the surface (note that, in fact, this is the only possible relation between the classes realized by separate real components). The additional invariants are more subtle in the case of Enriques surfaces. They involve some natural decompositions of the real part in halves and quarters, quadratic forms on the $\mathbb{Z}/2$ -homology of the real part, etc. The details, the lists of the values of the invariants, and some related information are found in [DIK].

As follows from the proofs, the number of deformation classes is 5 for Abelian surfaces, 75 for $K3$ -surfaces, and several hundreds for Enriques surfaces. (It is worth noting that over $\mathbb{C}$ all Abelian, as well as $K3$ - or Enriques surfaces, are deformation equivalent.)

As an example of application, let us mention that the following consequence of the classification of real Enriques surfaces completes the Bieberbach-Hitchin classification of compact Ricci flat Einstein 4-manifolds.

2.5 Proposition. *There is only one connected family of compact Ricci flat Einstein 4-manifolds with $\pi_1 = \mathbb{Z}/2 \times \mathbb{Z}/2$.*

Indeed, as proved by Hitchin [Hi], any compact Ricci flat Einstein 4-manifold with $\pi_1 = \mathbb{Z}/2 \times \mathbb{Z}/2$ (or, equivalently,⁴ compact Einstein 4-manifolds with $|\sigma| = \frac{2}{3}\chi$ and $\pi_1 = \mathbb{Z}/2 \times \mathbb{Z}/2$) is the orbit space of a real structure without real points on an Enriques surface. According to our classification, the real Enriques surfaces without real points form a connected family (first, direct proof is found in [DK3]).

The deformation classification of real Enriques surfaces happens to be most advanced among the other deformation classifications of real objects known for the moment. In some sense, it contains a significant number of the others, like that of cubic surfaces, plane quartics, cubic sections of a quadratic cone,

⁴As proved by Hitchin [Hi], $|\sigma| \leq \frac{2}{3}\chi$ for any compact Einstein 4-manifold, and this bound turns into equality if and only if the Einstein manifold is Ricci flat.

intersection of two quadrics in P^4 , generalized (nodal) Del Pezzo surfaces, curves of bidegree $(2, 4)$ on $P^1 \times P^1$, etc. Since one of the methods employed for the study of deformations of Enriques surfaces reduces the deformation problem to some arithmetic of lattices, new proofs for the deformation results listed above can be deduced from the corresponding lattice calculations.

It is natural to expect that for surfaces of general type there is no quasi-simplicity: there should exist examples of $\mathbb{R}$ -deformation distinct surfaces with differentiably equivalent real structures. The problem is to find convenient deformation invariants which are not topological.

This does not exclude the quasi-simplicity for some classes of surfaces of general type. One such example is given by Bogomolov-Miyaoka-Yau surfaces, that is, surfaces covered by a ball in $\mathbb{C}^2$.

3. SOME FINITENESS RESULTS

All the deformation classifications mentioned above are finite. Apparently, this should reflect a general statement: given a differential type M of a compact 4-manifold, the number of deformation classes of real surfaces (X, c) with X diffeomorphic to M is finite. It would be natural to subdivide the proof in two parts and try, first, to prove that the number of non isomorphic real forms of a fixed complex surface is finite, and, second, that the map from the moduli space of real surfaces to the real part of the moduli space of complex surfaces is nice in some appropriate sense (e.g., semi-algebraic). Note that the number of complex deformation classes within M is known to be finite.⁵

Let me mention a phenomenon that exposes one of the difficulties of the real deformation theory and explains the strategy suggested above. Consider the moduli space M of complex structures on a given compact manifold X (topological or smooth). When it is well defined, it is a complex space and it has a natural real structure, which consists in replacing a complex structure on X by the conjugate one. Certainly, real varieties (X, c) represent fixed points of this action. But the converse is wrong, in general. First counter examples appear already in the case of curves of genus ≥ 3 , see [Shi]. As a consequence, the moduli space of real objects is extremely rarely the real part of the moduli space of complex ones.

Contrary to the quasi-simplicity, for the finiteness of the number of real structures on a given variety the case of general type is an easy one. Since the composition of two real structures is a biholomorphic automorphism and

⁵The situation is changing in higher dimensions. There are known examples of compact smooth 6-manifolds for which the number of deformation classes of complex structures is infinite. Even in the symplectic category, where it is the question of deforming the symplectic structures on a given manifold, there are similar examples.

since the group of automorphisms of a variety of general type is finite, there is only finitely many real structures on any variety of general type. On the other hand, in the case of surfaces the Hilbert scheme of surfaces of general type with given characteristic numbers is quasi-projective, which implies directly (without using the finiteness of the number of real structures on a given surface) the finiteness of the number of deformation classes of real structures. Otherwise, to my knowledge, the problem of finiteness of the number of conjugacy classes of real structures remains essentially open.

Before discussing the case of surfaces, let me compare the above finiteness problem with the analogous problems in other categories.

In the category of compact smooth manifolds, the number of conjugacy classes of involutions is finite for any manifold of dimension ≤ 2 , while the finiteness fails in higher dimensions. In particular, in dimension 4 there are two famous series of examples of infinitely many pairwise non equivalent smooth involutions on S^4 : one with various homology 3-dimensional spheres as the fixed point set (see [P], [DeR]), and the other with the fixed point set diffeomorphic to the standard 2-sphere (see [Gi], [Go], [V], etc). It is worth noting that the latter series of examples provide an infinite number of pairwise non diffeomorphic imbeddings of R^2 into R^4 with R^2 standard at infinity.

In the category of compact Riemann manifolds, the number of conjugacy classes of involutions is finite for any manifold of any dimension.

It would be interesting to examine such an intermediate case as that of compact symplectic manifolds. Here, again it is obvious that in dimension ≤ 2 the finiteness takes place. Since, as it has already been mentioned, in the symplectic category in dimension ≥ 6 there is no finiteness of deformation classes of symplectic structures, it is doubtful that some kind of finiteness for the number of anti-involutions can hold without any boundness hypothesis. It is worth noting that using Gromov's technique of J -holomorphic curves one can prove that any anti-involution of the standard symplectic R^4 is standard as soon as the involution is standard at infinity. Thus, one may expect, however, some reasonable finiteness results in 4-dimensional real symplectic geometry.

As to algebraic surfaces, in this category we have the following result proved in [DIK].

3.1 Theorem. *For any algebraic surface of Kodaira dimension ≥ 1 and any minimal algebraic surface, the number of isomorphism classes of real structures is finite.*

In what concerns this finiteness problem, the case of non minimal rational surfaces looks to be the most complicated and enigmatic. However, one may add to the surfaces enumerated in the theorem the Del Pezzo surfaces and the projective planes blown up at ≤ 10 points. The case of Kodaira dimension 2

is trivial, because of the finiteness of the group of automorphisms. The proof in the case of Kodaira dimension 1 is based on the Borel-Serre finiteness for real structures on (connected) algebraic linear groups and the following, well known, proposition.

3.2 Proposition. *For any algebraic surface X , the kernel of the natural representation of $\text{Aut}'(X) = \text{Aut} / \text{Aut}_0$ in $L = K_X^\perp \subset \text{Num } X$ is finite.*

(Here, Num stands for the Neron-Severi lattice which is the image of Pic in $H^2(X)/\text{Tors}$, Aut is the group of automorphisms, and Aut_0 its component of the identity.)

After that, to finish with the case of Kodaira dimension 1 it remains to use the interpretation of the set of conjugacy classes of real structures as the Galois cohomology set $H^1(\mathbb{Z}/2, \text{Aut})$, where Aut is regarded as a $\mathbb{Z}/2$ -space with the action $g \mapsto cgc^{-1}$, c being a fixed real structure, and to study some standard exact cohomology sequences. (To apply the Borel-Serre theorem one should use, in addition, the Matsusaka theorem, which states that Aut_0 is an algebraic group, and the Chevalley theorem, according to which such a group is an extension of an abelian variety by a linear algebraic group.)

Now, the only remaining case is that of minimal surfaces of Kodaira dimension 0, i.e., that of hyperelliptic, abelian, $K3$, and Enriques surfaces. The real structures on abelian surfaces (and abelian varieties of any dimension) were treated by A. Comessatti [C] in the mid 20th. In fact, the finiteness part is not complicated at all and easily extends to hyperelliptic surfaces.

For $K3$ -surfaces (as well as for Abelian surfaces) one can prove a stronger statement, which clearly implies the similar statement for Enriques surfaces (resp., hyperelliptic surfaces).

3.3 Theorem. *Up to isomorphism, a $K3$ -surface admits at most finitely many finite group Klein actions.*

Here by a group Klein action we mean a group acting by both holomorphic and antiholomorphic maps. The group in the theorem is not fixed, so that the theorem states, in particular, that the number of finite groups acting on a given $K3$ -surface is finite. The surface in the theorem is not assumed to be algebraic.

The proof looks different in the algebraic and non algebraic cases. Details are given in [DIK]. Let me underline that in the non algebraic case the main finiteness ingredient is coming from the Dirichlet theory of units in algebraic number fields and gives, as a by-product, the following result (which, in what concerns non algebraic $K3$ -surfaces, we did not find in the literature).

3.4 Proposition. *The group of automorphisms of any $K3$ -surface is finitely generated.*

In the algebraic case the proof of Theorem 3.3 follows more traditional lines and is based, aside from the global Torelli theorem, on the finiteness of the kernel of the natural representation of Aut in Aut Pic (Piatetskii-Shapiro, Shafarevich) and the finiteness of the Siegel fundamental polyhedron (Siegel, Garland, Raghunathan, Weelenberg).

4. “TO HAVE AND HAVE NOT” (ADDED IN PROOF).

Additional natural question is: *Can any complex nonsingular surface be deformed to a real surface?* As is shown in [KK], the answer is in the negative.

4.1 Theorem. *There exist rigid surfaces without any real structure.*

Here by rigid surface we mean a surface having no nontrivial local deformations. In fact, the surfaces in our examples are strongly rigid (in the sense of Mostow): X is strongly rigid if any homotopy equivalent surface Y is isomorphic to either X or $\bar{X}$. In our principal example the surface is a $\mathbb{Z}/5 \times \mathbb{Z}/5$ Galois covering of P^2 ramified in a configuration of 9 lines dual to the inflection points of a cubic. In this, slight, modification of Hirzebruch constructions (we select additional ramification weights on the lines) we control the automorphism group of the covering surface. We prove that the group is exhausted by the deck transformations and, moreover, the surface has no any anti-holomorphic automorphism. The strong rigidity follows, as usually, from the relation $c_1^2 = 3c_2$ (which implies that the universal covering is the complex ball; certainly, the surfaces in question are different from $\mathbb{C}P^2$).

As a consequence, we get new counter examples to the (complex) $\text{Dif}=\text{Def}$ conjecture. More precisely, we get the following result.

4.2 Proposition. *There exist smooth oriented 4-manifolds for which the moduli space M_{dif} of complex structures with given underlying smooth structure (and given orientation) consists of 2 points interchanged by complex conjugation.*

After [KK], F. Catanese [Ca] found different, non-rigid, examples of families of surfaces where the complex conjugation also interchanges the components of the moduli space. His surfaces are covered by the bi-disc $D \times D \subset \mathbb{C}^2$.

Other counter examples given in [KK] are fake projective planes (that is, surfaces of general type with $q = p_g = 0$ and $K^2 = 9$). From the surfaces considered in [KK] one can easily obtain some examples of varieties X of any dimension ≥ 3 having the same property, i.e., varieties such that X and $\bar{X}$ belong to distinct components of the moduli space. It is sufficient to consider products of any of our surfaces (rigid examples) or products of fake projective planes with a torus (non-rigid examples). The statement on the components

of the moduli space will then follow from the well known properties of the Albanese map and Siu's rigidity theorem.

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