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18.2.2002

THE HOMOTOPY OF $L_2V(1)$ FOR THE PRIME 3

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ABSTRACT. Let V(1) be the Toda-Smith complex for the prime 3. We give a complete calculation of the homotopy groups of the L_2-localization of $V(1)$ by making use of the higher real K-theory EO_2 of Hopkins and Miller and related homotopy fixed point spectra. In particular we resolve an ambiguity which was left in an earlier approach of Shimomura whose computation was almost complete but left an unspecified parameter still to be determined.

0. Introduction

The chromatic approach offers at present the most attractive perspective on the stable homotopy category of finite complexes. For any natural prime p there is a tower of localization functors L_n with natural transformations $L_n \to L_{n-1}$ where L_n is Bousfield localization with respect to a certain multiplicative homology theory $E(n)_*$. For a finite complex X the homotopy inverse limit of these localizations gives the p-localization of X. The study of the localization functors L_n is sometimes referred to as the study of the chromatic primes in stable homotopy theory. For more details the reader may consult [Ra3].

The solution of the Adams conjecture lead to a good conceptual and calculational understanding of the localization functor L_1 if p is any prime. The case of L_2 is reasonably well understood for primes $p > 3$ at least from a computational point of view [SY]. The case of L_2 at the primes $p = 3$ and $p = 2$ is harder. The standard approach to understand the L_2-localization L_2S^0 (at any prime) is to study L_2X for a “suitable” finite complex X and to work one's way back to L_2S^0 through appropriate Bockstein spectral sequences arising from the skeletal filtration of X. At odd primes the Toda-Smith complexes $V(1)$ (which are defined as cofibre of a self map A of the mod-p Moore spectrum $V(0)$ such that A induces multiplication by v_1 in Brown-Peterson theory BP_*) are suitable in this sense. For primes $p > 3$ the homotopy of $L_2V(1)$ is relatively easy to understand; the Adams-Novikov spectral sequence (ANSS for short) converging to $\pi_*L_2V(1)$ collapses at E_2 and its E_2-term is known [Ra1]. Starting from this information Shimomura and Yabe were able to compute the homotopy of L_2S^0 for all primes $p > 3$.

At the prime 3 it is natural to try the same strategy and start with studying $\pi_*L_2V(1)$. In fact, the E_2-term of the ANSS converging to $\pi_*L_2V(1)$ has been computed in [H] (see also [GSS] and [Sh1]) but this time the ANSS for $V(1)$ does not collapse. Using various information about homotopy groups of spheres and related complexes in low dimensions Shimomura studied this spectral sequence and arrived at a calculation modulo some ambiguity; there

The authors would like to thank (in chronological order) the University of Heidelberg, MPI at Bonn, Université Louis Pasteur at Strasbourg and Northwestern University for providing them with the opportunity to work together.

was an unspecified parameter $k \in \{0, 1, 2\}$ and several families of homotopy elements which lived in degrees which were only determined up to adding $24k$.

The E_2-term of the ANSS for $L_2V(1)$ can be identified by Morava's change of ring isomorphism with the continuous cohomology of a certain p-adic Lie group $\mathbb{G}_2$, (the "extended Morava stabilizer group") with coefficients in $\mathbb{F}_9[u^{\pm 1}]$. Devinatz and Hopkins [DH2] have given a homotopy theoretic interpretation of Morava's change of rings isomorphism: the localization $L_{K(2)}S^0$ of the sphere with respect to the second Morava K-theory $K(2)$ has the homotopy type of the homotopy fixed point spectrum $E_2^{h\mathbb{G}_2}$ which is defined by making use of the Hopkins-Miller rigidification of the action of $\mathbb{G}_2$ on the Lubin-Tate spectrum E_2 [Re].

In this paper we analyze the ANSS for $L_2V(1)$ by making serious use of group theoretic and cohomological properties of $\mathbb{G}_2$. In fact, $L_2V(1)$ can be identified with $L_{K(2)}V(1) \simeq L_{K(2)}S^0 \wedge V(1) \simeq E_2^{h\mathbb{G}_2} \wedge V(1)$ and we will study $L_2V(1)$ by comparing it with $E_2^{hN} \wedge V(1)$ where E_2^{hN} is the homotopy fixed point spectrum with respect to the normalizer of an element of order 3 in $\mathbb{G}_2$. The use of the spectrum E_2^{hN} was suggested by the calculation of the E_2-term of the ANSS in [H] which made heavy use of centralizers of elements of order 3 in $\mathbb{G}_2$. The other good property of E_2^{hN} on which our method relies is that it can be analyzed in terms of the Hopkins-Miller higher real K-theory spectrum EO_2 at the prime 3. Both properties together allow us to give an independant calculation of $\pi_*L_2V(1)$ which is complete and identifies Shimomura's parameter as $k = 1$.

To state our main result we need some notation. First of all, from now on all spaces or spectra will be localized at 3.

The homotopy of $L_2V(1)$ is annihilated by 3 and a module over the homotopy of $L_{K(2)}S^0$. Therefore it can be regarded as a module over the algebra $\mathbb{F}_3[\beta] \otimes \Lambda(\zeta)$ where β is the image of the generator $\beta_1 \in \pi_{10}S^0$ and ζ is in $\pi_{-1}L_{K(2)}S^0$. The homotopy groups turn out to be periodic of period 144 and on the E_2-level this periodicity corresponds to multiplication by v_2^9 where v_2 is the polynomial generator in $\pi_{16}BP$. We do not prove that this periodicity arises geometrically but it is convenient to describe $\pi_*L_2V(1)$ nevertheless as a module over $P := \mathbb{F}_3[v_2^{\pm 9}, \beta] \otimes \Lambda(\zeta)$. We will use notation like $P/(\beta^5)\{v_2^l\}_{l=0,1,5}$ to denote the direct sum of P-modules each of which is killed by β^5 (more precisely its annihilator ideal is the ideal generated by β^5) and which have generators named $1 = v_2^0$, v_2 and v_2^5.

Finally we note that the E_2-term has a product structure and it contains elements which deserve to be named v_2 (which is closely related to the generator $v_2 \in \pi_{16}(BP)$) , α (which detects the image of the generator $\alpha_1 \in \pi_3(S^0)$), $v_2^{1/2}\beta$, $v_2^{1/2}\alpha$, βa_{35}, αa_{35}, $v_2^{1/2}\beta a_{35}$ and $v_2^{1/2}\beta\alpha a_{35}$ and which live in total degree 16, 3, 18, 11, 45, 38, 53 and 56 respectively. The reason for these names will become clear once we have discussed the spectrum E_2^{hN} and in particular $E_2^{hN} \wedge V(1)$.

Theorem. *As a module over $P = \mathbb{F}_3[v_2^{\pm 9}, \beta] \otimes \Lambda(\zeta)$ there is an isomorphism*

$$\begin{aligned}
\pi_*L_2V(1) \cong\ & P/(\beta^5)\{v_2^l\}_{l=0,1,5} \oplus P/(\beta^3)\{v_2^l\alpha\}_{l=0,1,2,5,6,7} \\
& \oplus\ P/(\beta^4)\{v_2^{l+1/2}\beta\}_{l=0,4,5} \oplus\ P/(\beta^2)\{v_2^{l+1/2}\alpha\}_{l=0,1,2,4,5,6} \\
& \oplus\ P/(\beta^4)\{v_2^l\beta a_{35}\}_{l=0,1,5} \oplus\ P/(\beta^3)\{v_2^l\alpha a_{35}\}_{l=0,1,2,5,6,7} \\
& \oplus\ P/(\beta^5)\{v_2^{l+1/2}\beta a_{35}\}_{l=0,4,5} \oplus\ P/(\beta^2)\{v_2^{l+1/2}\beta\alpha a_{35}\}_{l=0,1,2,4,5,6}\ .
\end{aligned}$$

We note that this result has already been used to carry out the programm to calculate $\pi_*(L_2S^0)$ that we mentioned above [Sh2], [SW].

The paper is organized as follows. In the first section we discuss the spectrum EO_2 and $EO_2 \wedge V(1)$. In particular we give a detailed discussion of the ANSS converging towards $\pi_*(EO_2)$. In the second section we introduce the homotopy fixed point spectrum E_2^{hN} and we relate E_2^{hN} to EO_2, and consequently $E_2^{hN} \wedge V(1)$ to $EO_2 \wedge V(1)$. In the final section we compare $L_2V(1)$ with $E_2^{hN} \wedge V(1)$ and prove the main theorem.

1. The homotopy of EO_2 and $EO_2 \wedge V(1)$

1.1. The spectrum EO_2.

We begin by recalling the construction of the spectrum EO_2 due to Hopkins and Miller. We refer to [Re] for more details.

The point of departure is the Lubin-Tate deformation theory of formal group laws (cf. [LT]), in particular the universal deformation of the formal group law Γ of height 2 over the field $\mathbb{F}_9$ with $[p]$-series $[p](x) = x^9$. The universal deformation is a lift of Γ to a formal group law $\widetilde{\Gamma}$ over $\mathbb{W}_{\mathbb{F}_9}[[u_1]]$ (where $\mathbb{W}_{\mathbb{F}_9}$ are the Witt vectors of $\mathbb{F}_9$ and u_1 is a formal power series variable). Over the graded ring $\mathbb{W}_{\mathbb{F}_9}[[u_1]][u^{\pm 1}]$ (where u is of degree -2 and u_1 of degree 0) this formal group law is isomorphic to the one induced from the universal p - typical formal group law over $BP_* \cong \mathbb{Z}_{(p)}[v_1, v_2, \ldots]$ via the map of algebras which sends v_1 to u_1u^{1-p}, v_2 to u^{1-p^2} and v_n to 0 for $i > 2$. The Landweber exact functor theorem implies that there is a homology theory E_2^* represented by a ring spectrum E_2 with coefficients $\pi_*(E_2) = \mathbb{W}_{\mathbb{F}_9}[[u_1]][u^{\pm 1}]$ such that the cohomology theory ${E_2}^*$ is complex oriented with orientation u and such that the associated formal group law is isomorphic to $\widetilde{\Gamma}$. To simplify notation we will abbreviate $\mathbb{W}_{\mathbb{F}_9}$ by $\mathbb{W}$ and E_2 by E throughout.

The group $\mathbb{S}_2$ of automorphisms of Γ (also known as the Morava stabilizer group) acts on the ring spectrum E, up to homotopy, and this action extends in a canonical way to an action of the extended stabilizer group, given as the semidirect product $\mathbb{G}_2 := \mathbb{S}_2 \rtimes C_2$ where the cyclic group C_2 of order 2 acts on $\mathbb{S}_2$ via Galois automorphisms of Γ. (Note that Γ is defined ove $\mathbb{F}_3$ and thus we get an action of the Galois group of the extension $\mathbb{F}_3 \subset \mathbb{F}_9$ on $\mathbb{S}_2$.)

Hopkins and Miller have shown how to rigidify this action to a genuine action via A_∞ - maps [Re] and subsequently Devinatz and Hopkins [DH2] have shown how to construct homotopy fixed point spectra E^{hH} with respect to closed subgroups H of $\mathbb{G}_2$. Their construction has the following properties: it agrees in the case of finite subgroups with the naive construction of the homotopy fixed point spectrum, if $H = \mathbb{G}_2$ then $E^{hH} \simeq L_{K(2)}S^0$, and for any closed subgroup H and any finite spectrum X there is a spectral sequence

$$E_2^{s,t}(X) = H^s_{cts}(H, E_t(X)) \Longrightarrow \pi_{t-s}(E^{hH} \wedge X)$$

where H^*_{cts} denotes continous cohomology of the p-adic group H.

The group $\mathbb{S}_2$ can also be identified with the group of units of the maximal order $\mathcal{O}_2$ of the division algebra $\mathbb{D}_2$ over the 3-adic rationals $\mathbb{Q}_3$. The maximal order is a free $\mathbb{W}$-module of rank 2 with basis 1 and S; as a ring it is determined by the relations $S^2 = 3$ and $Sa = \phi(a)S$ if ϕ notes the lift of Frobenius from $\mathbb{F}_9$ to $\mathbb{W}$ [Ra2, Appendix 2]. From this point of view the extended group $\mathbb{G}_2$ is the split extension $\mathbb{S}_2 \rtimes C_2$ where the action of C_2 is given by conjugation with S in $\mathbb{D}_2$.

Let ω be a fixed 8th root of unity in $\mathbb{W}$. The element $s := -\frac{1}{2}(1+\omega S)$ is easily checked to be of order 3. Furthermore $\omega^2 s\omega^{-2} = s^2$ so that s and $t := \omega^2$ generate a subgroup G_{12} of $\mathbb{S}_2$ which is isomorphic to $C_3 \rtimes C_4$ with C_4 acting non-trivially on C_3. The spectrum EO_2 is defined as the homotopy fixed point spectrum $E^{hG_{12}}$. We add that G_{12} is a maximal finite subgroup of $\mathbb{S}_2$ and every other maximal finite subgroup is conjugate to G_{12}.

1.2. The E_2-term of the ANSS converging to $\pi_*(EO_2)$.

We do not claim any originality for the results in this and the following subsection. The ANSS for EO_2 was first investigated by Hopkins and Miller but unfortunately their work remains unpublished. There is a rather brief discussion of this spectral sequence in the still unpublished paper [N]. A discussion of its E_2-term from a different point of view can be found in [GS]. Neither of these sources suits well our needs and therefore we have decided to give a self-contained treatment here.

In order to describe the E_2-term $E_2^{s,t} \cong H^s(G_{12}, E_t)$ we start by analyzing E_* as a G_{12}-algebra. The first step is to locate an appropriate subrepresentation in E_{-2}.

Let χ be the representation of G_{12} on $\mathbb{W}$ which is trivial on s and on which t acts by multiplication by ω^2. Define a G_{12} - module ρ by the short exact sequence

$$0 \to \chi \to \mathbb{W}[G_{12}] \otimes_{\mathbb{W}[C_4]} \chi \to \rho \to 0$$

in which in the middle term χ is considered as a representation of the subgroup C_4 generated by t and where the first map takes a generator e of χ to

$$(1+s+s^2)e := (1+s+s^2) \otimes e \in \mathbb{W}[G_{12}] \otimes_{\mathbb{W}[C_4]} \chi \ .$$

Lemma 1. *There is a morphism of G_{12}-modules*

$$\rho \longrightarrow E_{-2}$$

so that the induced map

$$\rho \otimes_{\mathbb{W}} \mathbb{F}_9 \to E_{-2} \otimes_{E_0} E_0/(3, u_1^2)$$

is an isomorphism.

Proof. We need to know something about the action of $\mathbb{G}_2$ on E_*. Let $\mathfrak{m} = (3, u_1) \subseteq E_0$ be the maximal ideal. Then Proposition 3.3 and Lemma 4.9 of [DH1] together imply that, modulo $\mathfrak{m}^2 E_{-2}$

$$\begin{aligned} s_*(u) &\equiv -\frac{1}{2}(u+\omega^3 uu_1) \\ s_*(uu_1) &\equiv -\frac{1}{2}(3\omega u + uu_1) \\ t_*(u) &\equiv \omega^2 u \\ t_*(uu_1) &\equiv -\omega^2 uu_1 \ . \end{aligned}$$

In particular, we see that $E_{-2}\otimes_{E_0} E_0/(3,u_1^2)$ is isomorphic to $\rho\otimes_{\mathbb{W}}\mathbb{F}_9$ as a G_{12} - module and that the residue class of u is a G_{12}-module generator. Thus we would like to find a class $x\in E_{-2}$ with the same reduction as u so that $t_*(x)=\omega^2x$ and $x+s_*(x)+s_*^2(x)=0$. In fact, because the action of t on $E_{-2}\otimes_{E_0} E_0/(3,u_1^2)$ is diagonalizable with distinct eigenvectors it suffices to find x such that $x\equiv u \bmod (3,u_1)$, up to a scalar in $\mathbb{F}_9^\times$.

Such an x can be obtained as follows: we start with the element $u^{-2}u_1$ which is the image of $v_1\in BP_*$ with respect to the map $BP_*\to E_*$ which classifies $\widetilde{\Gamma}$ (we will denoted this element simply by v_1 in the sequel); v_1 is invariant modulo 3 with respect to the action of all of $\mathbb{S}_2$. More precisely, the structure formulae in BP_*BP [Ra2, Appendix 2] yield

$$g_*(v_1)=v_1+(3-3^3)t_1(g)\equiv v_1+3t_1(g) \bmod (3^2)$$

for every $g\in\mathbb{S}_2$. Here we use the identification of E_*E with the continous functions from the profinite group $\mathbb{S}_2$ with values in E_* equipped with the profinite topology (see [St, Thm. 12] for a convenient reference) and $t_1\in E_4E$ is the image of the element with the same name in BP_*BP.

By definition of t_1 we have $t_1(-\frac{1}{2}(1+\omega S))\equiv \omega u^{-2} \bmod (3,u_1)$. Hence, if $z=\frac{1}{3}(v_1-s_*(v_1))$ then

$$z\equiv \omega u^{-2} \mod (3,u_1)\ ,$$

in particular z is non-zero. Clearly we have $z+s_*(z)+s_*^2(z)=0$ but z does not yet have the right degree.

Therefore we consider the class

$$y=us_*(u)s_*^2(u)z\ .$$

Then

$$y\equiv \omega u \mod (3,u_1)$$

and we still have

$$y+s_*(y)+s_*^2(y)=0\ .$$

However, y might not yet have the correct invariance property with respect to the element t of order 4. Therefore we average and set

$$x=\frac{1}{4}(y+\omega^{-2}t_*(y)+\omega^{-4}t_*^2(y)+\omega^{-6}t_*^3(y))\ .$$

Then we get $x\equiv\omega u \bmod(3,u_1^2)$ and we are done. □

The morphism of G_{12}-modules constructed in the lemma defines a morphism of $W[G_{12}]$-algebras

$$S(\rho)\longrightarrow E_*$$

where $S(\rho)$ denotes the symmetric algebra on ρ. We note that as an algebra $S(\rho)$ is polynomial over $\mathbb{W}$ on two generators e and $s_*(e)$, and we can choose e such that its image is the element x of the proof of the lemma above and is therefore invertible in E_*. Let

$$N=\prod_{g\in G_{12}} g_*e\in S(\rho)\ ;$$

then we have a morphism of $\mathbb{W}[G_{12}]$-algebras

$$S(\rho)[N^{-1}]\longrightarrow E_*\ .$$

Note that inverting N inverts e as well, but in an invariant manner. Let $I\subset S(\rho)[N^{-1}]$ be the preimage of the maximal ideal $\mathfrak{m}=(3,u_1)\subset E_*$ (now considered as a homogeneous graded ideal).

Proposition 2. *The induced map of complete algebras*

$$S(\rho)[N^{-1}]^{\wedge}_{I} \longrightarrow E_*$$

is an isomorphism.

Proof. It is enough to show that the induced maps

$$\frac{I^k}{I^{k+1}} \to \frac{\mathfrak{m}^k}{\mathfrak{m}^{k+1}}$$

are isomorphisms for each k.

If we identify $S(\rho)$ with $\mathbb{W}[e, s_*(e)]$ then we get $N = -\big(es_*(e)(e+s_*(e)\big)^4$. Furthermore it is straightforward to check that the homogeneous graded ideal I is generated by 3 and $e - s_*(e)$ and that the maps in question are isomorphisms. $\square$

Our next step is to study $H^*(G_{12}, S(\rho))$. We will see in Theorem 6 below how the calculation of the E_2-term can be reduced to that of $H^*(G_{12}, S(\rho))$.

If $e \in \rho$ is the generator, let d be the multiplicative norm of e with respect to the subgroup generated by s, i.e. $d = es_*(e)s^2{}_*(e)$. We note that d is of degree -6, it is invariant with respect to s and furthermore $d^4 = -N$.

For a finite group G and any G module M, let

$$\mathrm{tr}_G = \mathrm{tr} : M \longrightarrow M^G = H^0(G, M)$$

be the transfer: $\mathrm{tr}(x) = \sum_{g\in G} gx$. The following calculates $H^*(G_{12}, S(\rho))$ completely if $* > 0$ and gives partial information if $* = 0$; an element listed as being in bidegree (s,t) is in $H^s(G, S_t(\rho))$.

Lemma 3. *Let $C_3 \subseteq G_{12}$ be the normal subgroup generated by s. Then there is an exact sequence*

$$S(\rho) \xrightarrow{\mathrm{tr}} H^*(C_3, S(\rho)) \to \mathbb{F}_9[b,d] \otimes \Lambda(a) \to 0$$

where a has bidegree $(1,-2)$, b has bidegree $(2,0)$ and d has bidegree $(0,-6)$. Furthermore the action of t is described by

$$t_*(a) = -\omega^2 a \qquad t_*(b) = -b \qquad t_*(d) = \omega^6 d\ .$$

(By abuse of notation we have denoted the image of the invariant class d in the quotient $H^0(C_3, S_{-6}(\rho))/Im\ tr$ still by d.)

Proof. Let F be the G_{12}-module $\mathbb{W}[G_{12}] \otimes_{\mathbb{W}[C_4]} \chi$. We can choose a $\mathbb{W}$-basis x_1, x_2, x_3 of F such that $x_2 := s_*(x_1)$ and $x_3 := s_*(x_2)$, and then we get an identification

$$S(F) = \mathbb{W}[x_1, x_2, x_3]$$

with all x_i in degree -2. The kernel of the canonical C_3-linear algebra map which sends F to ρ is the principal ideal generated by $\sigma_1 = x_1 + x_2 + x_3$, i.e. we have a short exact sequence of graded C_3-modules

$$(*) \qquad 0 \to S(F)\otimes\chi \to S(F) \to S(\rho) \to 0\ .$$

(In the tensor product χ has to be treated as a representation in degree -2 in order to make the maps degree preserving.)

As a C_3-module $S(F)$ splits into a direct sum of free modules and trivial modules where the trivial modules are generated by the powers of the monomial $\sigma_3 := x_1x_2x_3$. Therefore we obtain a short exact sequence

$$S(F) \xrightarrow{\mathrm{tr}} H^*(C_3, S(F)) \to \mathbb{F}_9[b, \sigma_3] \to 0$$

where b has bidegree $(0, 2)$ and σ_3 has bidegree $(0, -6)$. Here b is a generator of $H^2(C_3, \mathbb{W}) \cong \mathbb{W}/3$ and $\mathbb{W} \subseteq S(F)$ is the submodule generated by the unit of the algebra $S(F)$. The action of t is given by the following formula

$$t_*(\sigma_3) = \omega^6\sigma_3 = -\omega^2\sigma_3 \qquad \text{and} \qquad t_*(b) = -b.$$

The short exact sequence (*) and the fact that $H^1(C_3, S(F)) = 0$ now imply that there is an exact sequence

$$S(\rho) \xrightarrow{\mathrm{tr}} H^*(C_3, S(\rho)) \to \mathbb{F}_9[a, b, d]/(a^2) \to 0.$$

where d is the image of σ_3 and a is the preimage of $b \in H^2(C_3, \mathbb{W}) = H^2(C_3, S_0(F) \otimes \chi)$ with respect to the isomorphism

$$H^1(C_3, \rho) = H^1(C_3, S_1(\rho)) \to H^2(C_3, S_0(F) \otimes \chi)$$

given by the obvious connecting homomorphism. Thus a has bidegree $(1, -2)$ and the action of t is twisted by χ:

$$t_*(a) = -\omega^2 a = \omega^6 a \ . \quad \square$$

The next step is to compute the invariants $S(\rho)^{C_3}$ together with the action of t. For this we start with the invariants of $S(F)$ and then we use the exact sequence (*). The action of the cyclic group C_3 on $S(F) = \mathbb{W}[x_1, x_2, x_3]$ extends in an obvious way to an action of the symmetric group Σ_3 on three letters; thus we have an inclusion of algebras

$$\mathbb{W}[\sigma_1, \sigma_2, \sigma_3] = \mathbb{W}[x_1, x_2, x_3]^{\Sigma_3} \subseteq S(F)^{C_3}.$$

It is clear that the following element

$$\epsilon = x_1^2x_2 + x_2^2x_3 + x_3^2x_1 - x_2^2x_1 - x_1^2x_3 - x_3^2x_2$$

(the "anti-symmetrization" of $x_1^2x_2$) is also invariant under the action of C_3. We use the same notation for the images of these elements in $S(\rho)$ and we note that σ_1 becomes 0 in $S(\rho)$.

Lemma 4.

a) There is an isomorphism of $\mathbb{W}$-algebras

$$\mathbb{W}[\sigma_1, \sigma_2, \sigma_3, \epsilon]/(\epsilon^2 - f) \cong S(F)^{C_3}$$

where f is the following polynomial in σ_1, σ_2, σ_3

$$f = -27\sigma_3^2 - 4\sigma_2^3 - 4\sigma_3\sigma_1^3 + 18\sigma_1\sigma_2\sigma_3 + \sigma_1^2\sigma_2^2 \ .$$

Furthermore, the action of $t \in G_{12}$ is given by

$$t_*(\sigma_1) = \omega^2\sigma_1 \qquad t_*(\sigma_2) = -\sigma_2 \qquad t_*(\sigma_3) = \omega^6\sigma^3 \qquad t_*(\epsilon) = \omega^2\epsilon\ .$$

b) There is an isomorphism of $\mathbb{W}$-algebras

$$\mathbb{W}[\sigma_2, \sigma_3, \epsilon]/(\epsilon^2 - g) \cong S(\rho)^{C_3}$$

where g is the following polynomial in σ_2, σ_3

$$g = -27\sigma_3^2 - 4\sigma_2^3$$

Furthermore, the action of $t \in G_{12}$ is given by

$$t_*(\sigma_2) = -\sigma_2 \qquad t_*(\sigma_3) = \omega^6\sigma^3 \qquad t_*(\epsilon) = \omega^2\epsilon\ .$$

Proof. a) It is clear that ϵ^2 is Σ_3 invariant and therefore it can be expressed as a polynomial in σ_1, σ_2 and σ_3. To find the precise relation is an elementary exercise. We also leave it to the reader to verify that the action of t is as claimed. Thus it remains to determine the algebra structure of $S(F)^{C_3}$.

As a graded C_3-module $S(F)$ decomposes into a direct sum of free modules of rank 3 and of trivial modules of rank 1, and each of these summands contributes a summand $\mathbb{W}$ to $S(F)^{C_3}$. From this it is easy to calculate the Poincare series of the invariants and we find

$$\chi_{S(F)^{C_3}}(t) = \frac{1+t^6}{(1-t^2)(1-t^4)(1-t^6)}\ .$$

(In this calculation we regrade $S(F)$ such that F is homogeneous of degree $+2$.)

On the other hand there is still an action of C_2 on $S(F)^{C_3}$, and $S(F)$ splits as direct sum of eigenspaces

$$S(F) \cong S(F)^+ \oplus S(F)^-\ .$$

Furthermore $S(F)^+ \cong S(F)^{\Sigma_3}$ and $S(F)^-$ is a module over $S(F)^-$. By the Poincaré series calculation it is therefore enough to verify that $S(F)^-$ is free as $S(F)^+$-module with generator ϵ.

In fact, it is clear that ϵ is in $S(F)^-$ and because $\mathbb{W}[x_1, x_2, x_3]$ is without zero divisors it is also clear that ϵ generates a free $S(F)^+$ module with the correct Poincaré series. Now suppose $p \in S(F)^-$. We can choose a constant $c \in \mathbb{W}$ of mininimal valuation, say r, such that $cp = \epsilon q$ for a unique polynomial $q \in \mathbb{W}[\sigma_1, \sigma_2, \sigma_3]$. Then

$$\epsilon(cp) = \epsilon^2 q = fq\ .$$

If c is divisible by 3 then the formula for f shows that q must be divisible by 3 and then r was not minimal. Hence $r = 0$, $c \in \mathbb{W}^\times$ and p is in the submodule generated by ϵ.

b) This is an immediate consequence of (a) and the vanishing of $H^1(C_3, S(F))$. $\square$

The next step is to invert the element N. This element is the image of σ_3^4; thus, we are effectively inverting the element $\sigma_3 \in S(\rho)^{C_3}$. We begin with the observation that if we divide

$$\epsilon^2 = -27\sigma_3^2 - 4\sigma_2^3$$

by $4\sigma_3^6$ we obtain

$$(\frac{\epsilon}{2\sigma^3})^2 + (\frac{\sigma_2}{\sigma_3^2})^3 = -\frac{27}{4\sigma_3^4}.$$

Thus if we set

$$c_4 = -\frac{\sigma_2}{\sigma_3^2}, \quad c_6 = \frac{\epsilon}{2\sigma_3^3}, \quad \Delta = -\frac{1}{4\sigma_3^4}$$

then we get the following relation

$$c_6^2 - c_4^3 = 27\Delta$$

which corresponds to the famous relation from the theory of modular forms [D] (except that in our case 2 is invertible and hence the usual factor 1728 can be simplified to 27). The reader is referred to [GS] for an explanation of this coincidence.

Furthermore, c_4, c_6, and Δ are all invariant under the action of the entire group G_{12}. The elements

$$\alpha := ad^{-1} \in H^1(C_3, (S(\rho)[N^{-1}])_4)$$

and

$$\beta := bd^{-2} \in H^2(C_3, (S(\rho)[N^{-1}])_{12})$$

are clearly fixed by the action of t and by degree reasons they are acted on trivially by c_4 and c_6. The following result is now straightforward to verify.

Proposition 5.

a) The inclusion

$$\mathbb{W}[c_4, c_6, \Delta^{\pm 1}]/(c_6^2 - c_4^3 = 27\Delta) \to S(\rho)[N^{-1}]^{G_{12}}$$

is an isomorphism.

b) There is an exact sequence

$$S(\rho)[N^{-1}] \xrightarrow{\mathrm{tr}} H^*(G_{12}, S(\rho)[N^{-1}]) \to \mathbb{F}_9[\alpha, \beta, \Delta^{\pm 1}]/(\alpha^2) \to 0$$

and c_4 and c_6 act trivially on α and β. □

The final step is now to investigate what happens under completion. We continue to use c_4, c_6 etc. for the images of these elements in $H^*(G_{12}, E_*)$ with respect to the map $S(\rho)[N^{-1}] \to E_*$ studied in Proposition 2.

Theorem 6.

a) There is an isomorphism

$$(E_*)^{G_{12}} \cong \mathbb{W}[[c_4^3\Delta^{-1}]][c_4, c_6, \Delta^{\pm 1}]/(c_6^2 - c_4^3 = 27\Delta)$$

b) There is an exact sequence

$$E_* \xrightarrow{\mathrm{tr}} H^*(G_{12}, E_*) \to \mathbb{F}_9[\alpha, \beta, \Delta^{\pm 1}] \to 0$$

and c_4 and c_6 act trivially on α and β.

Proof. a) We use Proposition 2 and we use that completion commutes with taking invariants. We abbreviate $S(\rho)[N^{-1}]$ by A and we recall that the ideal $I \subset A$ is generated by 3 and

$e - s_*(e)$. With this it is straightforward to check that both c_4 and c_6 belong to I. The relation $c_6^2 - c_4^3 = 27\Delta$ implies then that $I \cap A_0^{C_3}$ is generated by 3 and $c_4^3\Delta^{-1}$. This implies (a).

b) It is also straightforward to verify that $\sigma_2 \equiv -(e - s_*(e))^2 \bmod (3)$ and this implies that the ideals I and $(3, c_4^3\Delta^{-1})$ define the same completion. Abbreviate $c_4^3\Delta^{-1}$ by z. We have an isomorphism

$$E_* \cong \lim_k A/(z^k) \ .$$

Now we consider the short exact sequence

$$0 \to A \xrightarrow{z^k} A \to A/(z^k) \to 0 \ .$$

Because z acts trivially on $H^q(G_{12}, A)$ for $q > 0$ (by Proposition 5b) we obtain for each $q > 0$ a tower (indexed by k) of short exact sequences

$$0 \to H^q(G_{12}, A) \to H^q(G_{12}, A/(z^k)) \to H^{q+1}(G_{12}, A) \to 0.$$

The maps on the right hand side of this tower are also induced by multiplication with z, hence they are trivial and therefore we obtain an isomorphism

$$H^q(G_{12}, A) \cong \lim_k H^q(G_{12}, A/(z^k)) \ .$$

On the other hand the graded quotients $A/(z^k)$ are of finite type for each k and this implies that the usual short exact sequences

$$0 \to \lim_k^1 H^{q-1}(G_{12}, A/(z^k)) \to H^q(G_{12}, \lim_k A/(z^k)) \to \lim_k H^q(G_{12}, A/I^kA) \to 0$$

degenerate into isomorphisms

$$H^q(G_{12}, \lim_k A/(z^k)) \cong \lim_k H^q(G_{12}, A/I^kA)$$

and the proof is complete. $\square$

<u>Remark</u> With the same reasoning we can also compute the E_2-terms for the homotopy fixed point spectra $E_2^{hC_3}$ and $E_2^{hC_6}$ where C_3 is as before the subgroup generated by s and C_6 that generated by s and t^2. In fact, the G_{12}-invariant $\Delta = -1/4\sigma_3^4$ has a C_3-invariant 4-th root $\Delta^{1/4}$ in $S(\rho)[N^{-1}]$ and we get

$$(E_*)^{C_3} \cong \mathbb{W}[[c_4^3\Delta^{-1}]][c_4, c_6, \Delta^{\pm 1/4}]/(c_6^2 - c_4^3 = 27\Delta) \ .$$

Furthermore there is an exact sequence

$$E_* \xrightarrow{\mathrm{tr}} H^*(C_3, E_*) \to \mathbb{F}_9[\alpha, \beta, \Delta^{\pm 1/4}] \to 0$$

and c_4 and c_6 act trivially on α and β.

Similarly,

$$(E_*)^{C_6} \cong \mathbb{W}[[c_4^3\Delta^{-4}]][c_4, c_6, \Delta^{\pm 1/2}]/(c_6^2 - c_4^3 = 27\Delta) \ ,$$

there is an exact sequence

$$E_* \xrightarrow{\mathrm{tr}} H^*(C_3, E_*) \to \mathbb{F}_9[\alpha, \beta, \Delta^{\pm 1/2}] \to 0$$

and c_4 and c_6 act trivially on α and β.

1.3. The homotopy of EO_2.

Before we turn to the discussion of the differentials of the spectral sequence we relate the elements Δ, c_4, c_6, α and β to well known quantities in homotopy theory.

We start by recalling that the elements $v_1^k \in Ext^{0,4}_{BP_*BP}(BP_*, BP_*/(3))$ define permanent cycles in the classical ANSS of the mod-3 Moore space $V(0)$. Similarly, the element $v_2 \in Ext^{0,16}_{BP_*BP}(BP_*, BP_*/(3, v_1))$ defines a permanent cycle in the classical ANSS for the cofibre $V(1)$ of the Adams self map $\Sigma^4 V(0) \to V(0)$. Furthermore v_1 and v_2 give rise via the Greek letter construction to generators $\alpha_1 \in \pi_3(S^0) \cong \mathbb{Z}/3$ resp. $\beta_1 \in \pi_{10}(S^0) \cong \mathbb{Z}/3$ which are detected in the classical ANSS by elements with the same name in $Ext^{1,4}_{BP_*BP}(BP_*, BP_*)$ resp. $Ext^{2,12}_{BP_*BP}(BP_*, BP_*)$.

Finally we note that the localization map from a finite spectrum X to $L_{K(2)}X$ together with the Morava change of rings isomorphism and the obvious restriction homomorphism in group cohomology induce a natural homomorphism

$$\lambda_X : Ext^{s,t}_{BP_*BP}(BP_*, BP_*X) \to H^s(\mathbb{G}_2; E_tX) \to H^s(G_{12}; E_tX)\ .$$

We will denote the images of the elements v_1, v_2 with respect to $\lambda_{V(0)}$ resp. $\lambda_{V(1)}$ still by v_1 resp. v_2.

Proposition 7.

a) Reduction modulo $(3, u_1)$ sends Δ^2 to the image of v_2^3 in $H^0(G_{12}, E_{32}/(3, u_1))$.

b) The mod-3 reduction map

$$H^0(G_{12}, E_t) \to H^0(G_{12}, E_t/(3))$$

sends c_4 resp. c_6 to the image of v_1^2 resp. v_1^3, up to multiplication by a unit in $H^0(G_{12}, E_0/(3)) \cong \mathbb{F}_9[[c_4^3\Delta^{-1}]]$. Furthermore there is an element $\widetilde{\alpha} \in H^1(G_{12}, E_{12}/(3))$ and an isomorphism (of modules over $\mathbb{F}_9[[v_1^6\Delta^{-1}]][v_1, \Delta^{\pm 1}, \beta] \otimes \Lambda(\alpha)$)

$$H^*(G_{12}, E_*/3) \cong \mathbb{F}_9[[v_1^6\Delta^{-1}]][v_1, \Delta^{\pm 1}, \beta] \otimes \Lambda(\alpha)\{1, \widetilde{a}\}/(v_1\alpha, v_1\widetilde{a}, \alpha\widetilde{a} + v_1\beta)$$

c) The map λ_{S^0} sends α_1 to α and β_1 to β up to a nontrivial constant in $\mathbb{W}/3$.

Proof. a) The definition of Δ implies immediately that its reduction modulo $(3, u_1)$ is equal to that of u^{-12}. On the other hand the reduction of u^{-24} is equal to v_2^3.

b) It is clear from our calculation of $H^*(G_{12}, E_*)$ and the short exact sequence of G_{12}-modules

$$(*) \qquad 0 \to E_* \to E_* \to E_*/(3) \to 0$$

that v_1^2 and v_1^3 are in the image of mod-3 reduction. Furthermore they generate the invariants in degree 8 resp. 12 as module over $H^0(G_{12}, E_0/(3))$. On the other hand the G_{12}-invariants in degree 8 resp. 12 of E_* are freely generated (as modules over $H^0(G_{12}, E_0)$) by c_4 resp. c_6. This proves the statement on c_4 and c_6 and also gives the result for $H^0(G_{12}, E_*/(3))$. We define $\widetilde{\alpha}$ such that $\delta^0(\widetilde{\alpha}) = \beta$ where δ^0 denotes the boundary homomorphism associated to the exact sequence (*). Then everything else except perhaps the relation $v_1\beta = \alpha\widetilde{\alpha}$ is straightforward to check. This relation is obtained by calculating

$$\delta^0(v_1\beta + \alpha\widetilde{\alpha}) = \delta^0(v_1)\beta - \alpha\delta^0(\widetilde{\alpha}) = \alpha\beta - \alpha\beta = 0$$

and by noting that δ^0 is a monomorphism in the relevant bidegree.

c) This is a consequence of the compatibility (with respect to the maps λ_X) of the Greek letter construction for $Ext_{BP_*BP}(BP_*, -)$ and an analogous Greek letter construction for $H^*(G_{12}, -)$.

In fact, the image of $v_1 \in Ext^0_{BP_*BP_*}(BP_*, BP_*/(3))$ (which is the class u_1u^{-2}) defines an element in $H^0(G_{12}, E_*/(3))$. The short exact sequence

$$0 \to E_* \to E_* \to E_*/(3) \to 0$$

shows that this class has a nontrivial image $\delta^0(v_1) \in H^1(G_{12}, E_4) \cong \mathbb{W}/3$ and this latter group is generated by α.

Similarly, for β_1 we just need to check that the result of the Greek letter construction on $u_2^{-8} \in H^0(G_{12}, E_{16}/(3, u_1))$ yields a nontrivial element in $H^2(G_{12}, E_{12})$. First we note that the boundary map δ^1 associated to the exact sequence

$$0 \to \Sigma^4 E_*/(3) \xrightarrow{v_1} E_*/(3) \to E_*/(3, u_1) \to 0$$

maps u^{-8} nontrivially and hence to $\widetilde{\alpha}$, up to a nonconstant multiple. In the proof of (b) we have seen that $\delta^0(\widetilde{\alpha}) = \beta$ and hence we are done. $\square$

In the sequel we redefine α resp. β such that $\alpha = \lambda_{S^0}(\alpha_1)$ and $\beta = \lambda_{S^0}(\beta_1)$. We are now ready to describe the differentials in our SS.

Theorem 8. *In the spectral sequence*

$$H^s(G_{12}, E_t) \Longrightarrow \pi_{t-s}(EO_2)$$

we have an inclusion of subrings

$$E_\infty^{0,*} \cong \mathbb{W}[[c_4^3\Delta^{-1}]][c_4, c_6, c_4\Delta^{\pm1}, c_6\Delta^{\pm1}, 3\Delta^{\pm1}, \Delta^{\pm3}]/(c_4^3 - c_6^2 = 27\Delta)\ .$$

In positive filtration $E_\infty^{s,t}$ is additively generated by the elements α, $\alpha\beta$, $\Delta\alpha$, $\Delta\alpha\beta$, β^j, $j = 1, 2, 3, 4$ and their multiples by powers of $\Delta^{\pm3}$. All these elements are of order 3 and c_4 and c_6 act trivially on elements in positive filtration.

Proof. First we observe that every element in the image of the transfer is a permanent cycle. The last proposition implies furthermore that the elements α and β are permanent cycles detecting homotopy classes with the same name. Next we use Toda's relation $\alpha_1\beta_1^3 = 0$ in $\pi_*(S^0)$. This implies that $\alpha\beta^3 = 0$ in $\pi_*(EO_2)$ and this can only happen if $d_5(\Delta) = a_1\alpha\beta^2$ for some $a_1 \in \mathbb{F}_9^\times$.

Then we use the Toda bracket relation $\beta_1 \in \pm < \alpha_1, \alpha_1, \alpha_1 >$ in $\pi_*(S^0)$. Consequently we have $\beta \in \pm < \alpha, \alpha, \alpha >$ in $\pi_*(EO_2)$. This and $\alpha\beta^2 = 0$ imply that β^3 is in the indeterminacy of the bracket $< \alpha\beta^2, \alpha, \alpha >$. This is only possible if $\alpha\Delta$ is a permanent cycle and $\alpha(\alpha\Delta) = \beta^3$, up to a nontrivial constant.

The next possible differential is d_9. Up to nontrivial constants we have $\beta^5 = \beta^2\beta^3 = \beta^2\alpha(\alpha\Delta) = 0$ in $\pi_*(EO_2)$ and this forces $d_9(\Delta^2\alpha) = a_2\beta^5$ for some $a_2 \in \mathbb{F}_9^\times$. Then there is no more room for further differentials and $E_\infty \cong E_{10}$ is a stated in the theorem. $\square$

Remark With the same reasoning we can also compute the homotopy of $E_2^{hC_3}$ and $E_2^{hC_6}$. In the case of C_3 we obtain an inclusion of subrings

$$E_\infty^{0,*} \cong \mathbb{W}[[c_4^3\Delta^{-1}]][c_4, c_6, c_4\Delta^{\pm1/4}, c_6\Delta^{\pm1/4}, 3\Delta^{\pm1/4}, \Delta^{\pm3/4}]/(c_4^3 - c_6^2 = 27\Delta)$$

and in positive filtration $E_\infty^{s,t}$ is additively generated by the elements α, $\alpha\beta$, $\Delta\alpha$, $\Delta\alpha\beta$, β^j, $j = 1, 2, 3, 4$ and their multiples by powers of $\Delta^{\pm 3/4}$. These elements are of order 3 and c_4 and c_6 act trivially on elements in positive filtration.

In the case of C_6 we obtain an inclusion of subrings

$$E_\infty^{0,*} \cong \mathbb{W}[[c_4^3\Delta^{-1}]][c_4, c_6, c_4\Delta^{\pm 1/2}, c_6\Delta^{\pm 1/2}, 3\Delta^{\pm 1/2}, \Delta^{\pm 3/2}]/(c_4^3 - c_6^2 = 27\Delta)$$

and in positive filtration $E_\infty^{s,t}$ is additively generated by the elements α, $\alpha\beta$, $\Delta\alpha$, $\Delta\alpha\beta$, β^j, $j = 1, 2, 3, 4$ and their multiples by powers of $\Delta^{\pm 3/2}$. Again these elements are of order 3 and c_4 and c_6 act trivially on elements in positive filtration.

In particular we see that EO_2 is 72 periodic with periodicity genertor Δ^3, $E_2^{hC_3}$ is 18 periodic with periodicity generator $\Delta^{3/4}$ and $E_2^{hC_6}$ is 36 periodic with periodicity generator $\Delta^{3/2}$.

1.4. The ANSS converging to $\pi_*(EO_2 \wedge V(1))$.

In this section we calculate $\pi_*(EO_2 \wedge V(1))$. We can do this by using Theorem 8 and the long exact sequences associated to the defining cofibre sequences of $V(0)$ and $V(1)$. However, later on we will make use of the structure of the ANSS for $L_2V(1)$ and so we choose to give a presentation in terms of the ANSS

$$E_2^{s,t}(V(1)) = H^s(G_{12}, E_*V(1)) \Longrightarrow \pi_{t-s}(EO_2 \wedge V(1)) \ .$$

First we note that $E_*(V(1))$ is given as $\mathbb{F}_9[u^{\pm 1}]$. The element $s \in G_{12}$ acts necessarily trivially on this ring while t acts via $t_*(u) = \omega^2 u$ (cf. the proof of Lemma 1). This gives us the following E_2-term

$$E_2^{*,*} \cong \left(\mathbb{F}_9[u^{\pm 1}, y] \otimes \Lambda(x)\right)^{C_4}$$

in which the (s,t)-bidegree of u is $(0, -2)$, that of y is $(2, 0)$ and that of x is $(1, 0)$. The invariants can then be identified with $\mathbb{F}_9[u^{\pm 4}, \beta] \otimes \Lambda(\alpha)$ where $\beta := u^{-6}y$ is a permanent cycle detecting $\beta \in \pi_{10}(V(1))$ and $\alpha := u^{-2}x$ is a permanent cycle detecting $\alpha \in \pi_3(V(1))$ and where β and α are the images of the classical elements β_1 and α_1 in $\pi_*(S^0)$. (This can easily be checked via the long exact sequences of homotopy groups mentioned above). The element u^{-8} is the image of v_2 in the E_2-term of the ANSS for $\pi_*V(1)$ with respect to the localization map (cp. the proof of Proposition 7 above).

If k is an integer than we will write from now on $v_2^{k/2}$ instead of u^{-4k}. If x is an element of E_2 we will denote $u^{-4k}x$ by $v_2^{k/2}x$. We note that the periodicity generator Δ^3 of $\pi_{72}EO_2$ projects to $v_2^{9/2}$.

Theorem 9. *There are elements $v_2^{k/2} \in \pi_{8k}(EO_2 \wedge V(1))$, $k = 0, 1, 2$, and $v_2^{k/2}\alpha \in \pi_{8k+3}(EO_2 \wedge V(1))$, $k = 0, 1, 2, 3, 4, 5$, such that as a module over $\mathbb{F}_9[v_2^{\pm 9/2}, \beta]$ there is an isomorphism*

$$\pi_*(EO_2 \wedge V(1)) \cong \mathbb{F}_9[v_2^{\pm 9/2}] \otimes \Big(\mathbb{F}_9[\beta]/(\beta^5)\{1, v_2^{1/2}, v_2\} \oplus \mathbb{F}_9[\beta]/(\beta^2)\{\alpha, \cdots, v_2^{5/2}\alpha\}\Big) \ .$$

<u>Remark</u> We remark that additively $\pi_k(EO_2)$ is nontrivial and of dimension 1 over $\mathbb{F}_9$ if $k \equiv 10m + 8k \bmod (72)$ with $0 \le m \le 4$, $0 \le k \le 2$, or if $k = 10m + 8k + 3$ if $0 \le m \le 1$ and $0 \le k \le 5$. For all other k the homotopy group is trivial.

Proof. Because α and β are permanent cycles, the first possible non-trivial differential is d_5 and it is determined by its value on the powers of $v_2^{1/2}$. By using the long exact homotopy sequences associated to the defining cofibre sequences of $V0)$ and $V(1)$ together with Theorem 8 and Proposition 7b it is easy to verify that the elements 1, $v_2^{1/2}$ and v_2 are permanent cycles.

Now $E_r^{*,*}(V(1))$ is a differential graded module over $E_r^{*,*}(S^0)$. This implies

$$d_5(v_2^{k/2}) = \begin{cases} 0 & \text{if } k = 0,1,2 \bmod 9 \\ c_k v_2^{k-3/2}\alpha\beta^2 & \text{if } k = 3,4,5,6,7,8 \bmod 9 \end{cases}$$

for suitable nontrivial constants c_k and therefore

$$\begin{aligned} E_6 \cong\ & \mathbb{F}_9[v_2^{\pm 9/2},\beta]\{1, v_{1/2}, v_2\} \oplus \mathbb{F}_9[v_2^{\pm 9/2},\beta]\{v_2^{6/2}\alpha, v_2^{7/2}\alpha, v_2^{8/2}\alpha\} \\ & \oplus \mathbb{F}_9[v_2^{\pm 9/2},\beta]/(\beta^2)\{\alpha, v_2^{1/2}\alpha, v_2\alpha, v_2^{3/2}\alpha, v_2^2\alpha, v_2^{5/2}\alpha\}\ . \end{aligned}$$

The next possible differential is d_9 and by using the module structure again we obtain

$$d_9(v_2^{k/2}\alpha) = \begin{cases} 0 & \text{if } k = 0,1,2,3,4,5 \bmod 9 \\ c_k' v_2^{k/2-3}\beta^5 & \text{if } k = 6,7,8 \bmod 9 \end{cases}$$

for nontrivial constants c_k'. The resulting E_{10}-term is isomorphic to the stated result, and in fact, there is no room for further differentials. $\square$

2. The homotopy fixed point spectrum E^{hN}

2.1. The subgroups N and N^1.

Next we introduce certain infinite closed subgroups of $\mathbb{S}_2$ which are closely related to the subgroup G_{12} which is used to define EO_2. We refer to [H, section 3] for more details on the following discussion. The centralizer $C := C_{\mathbb{S}_2}(C_3)$ of the subgroup $C_3 \subset G_{12}$ generated by s can be identified with the maximal order of the units in the cyclotomic extension $\mathbb{Q}_3(\zeta_3)$ of $\mathbb{Q}_3$ generated by a third of unity ζ_3, and is hence isomorphic to $C_3 \times C_2 \times \mathbb{Z}_3^2$. Furthermore C is of index 2 in its normalizer $N := N_{\mathbb{S}_2}(C_3)$. The action of the element n of order 2 in N/C on C is via the Galois automorphism of the cyclotomic extension. The action can be diagonalized, i.e. the splitting of $C_3 \times C_2 \times \mathbb{Z}_3^2$. can be chosen to be invariant with respect to the action of n, and n acts trivially on C_2 and on one copy of $\mathbb{Z}_3$ while it acts by multiplication by -1 on the other copy and on C_3.

Furthermore, there is a homomorphism from $\mathbb{S}_2 \to \mathbb{Z}_3^\times \to (\mathbb{Z}_3^\times)/\{\pm 1\}$ given as the composition of the reduced norm and the canonical projection. Its kernel is denoted by $\mathbb{S}_2^1$, and $\mathbb{S}_2$ decomposes as $\mathbb{S}_2^1 \times \mathbb{Z}_3$ where the complimentary factor $\mathbb{Z}_3$ comes from the center of the division algebra. There is a corresponding splitting $N \cong N^1 \times \mathbb{Z}_3$ and this splitting is preserved by the action of n (where n acts trivially on the complementary factor $\mathbb{Z}_3$). We observe that the subgroup G_{12} is contained in N^1 and that N^1 is a (nonsplit) extension of C_2 by $C^1 := C_3 \times C_2 \times \mathbb{Z}_3$ where n preserves the splitting of C^1 and acts non-trivially on the factors C_3 and $\mathbb{Z}_3$.

In the sequel we will make use of the homotopy fixed point spectra E^{hN} and E^{hN^1}. These spectra are closer to $L_{K(2)}S^0$ but we will see that they are also closely related to EO_2.

2.2. E^{hN}, E^{hN^1} and EO_2.

The splitting $N \cong N^1 \times \mathbb{Z}_3$ implies the following result.

Theorem 10. *There is a cofibration sequence*

$$E^{hN} \to E^{hN^1} \to E^{hN^1}$$

where the map $E^{hN^1} \to E^{hN^1}$ is given by $id - k$ if k denotes a topological generator of the central $\mathbb{Z}_3$.

Proof. There is a canonical map $f : E^{hN} \to E^{hN^1}$ induced by the inclusion $N^1 \subset N$. Furthermore k induces a self map of E^{hN^1} and $(id - k) \circ f$ is null. This gives us a map g from E^{hN} to the fibre F of $id - k$. The one shows that g, or equivalently $L_{K(2)}(g \wedge id_E)$ is an equivalence. In fact, a slight modification of the argument in [DH2, Prop. 7.1] allows to identify $\pi_* L_{K(2)}(E^{hN} \wedge E)$ with $\mathrm{map}_{cts}(\mathbb{G}_2/N, E_*)$, the continuous maps from the coset space $\mathbb{G}_2/N$ to E_*, and likewise $\pi_* L_{K(2)}(E^{hN^1} \wedge E)$ with $\mathrm{map}_{cts}(\mathbb{G}_2/N_1, E_*)$. Then one sees that the map $id - k$ induces a surjection on $\pi_*(L_{K(2)}(- \wedge E))$ and g induces an isomorphism between $\mathrm{map}_{cts}(\mathbb{G}_2/N, E_*)$ and the kernel. $\square$

The spectrum E^{hN^1} itself can be obtained from EO_2 in a slightly more sophisticated fashion. For this we consider the ring spectrum $E^{hC_6} = E^{h(C_3 \times C_2)}$. The group C_6 is normal in G_{12} and we obtain an induced action of the quotient $G_{12}/C_6 \cong C_2$ on the ring spectrum E^{hC_6}. The spectrum E^{hC_6} splits with respect to this action as $E^+ \vee E^-$ where E^+ and E^- are the "eigenspectra" of E^{hC_6} with respect to the two non-trivial characters of C_2. Furthermore $E^+ \simeq (E^{hC_6})^{hC_2}$ can be identified with EO_2 and thus E^{hC_6} and E^- become EO_2-module spectra. The following elementary observation introduces a suspension which may seem surprising at first but which becomes very important for the sequel.

Proposition 11. *There is an equivalence of EO_2-module spectra*

$$E^- \simeq \Sigma^{36} EO_2 \ .$$

Proof. We have seen in section 1.3 above that E^{hC_6} is a periodic ring spectrum with periodicity generator $\Delta^{3/2}$ of degree 36. Furthermore, the periodicity generator is in the -1 eigenspace of the action of C_2 on $\pi_*(E^{hC_6})$ and represents an element in $\pi_{36}(E^-)$. Using the structure of E^- as a module spectrum it defines an equivalence between $\Sigma^{36}EO_2$ and E^-. $\square$

We have other elements of order 2 acting on E^{hC_6}, e.g. all elements of order 2 in the group $N^1/C_6 \cong \mathbb{Z}_3 \rtimes C_2$. In particular, if k_1 is a topological generator of $\mathbb{Z}_3$ and if we choose the image of $t \in G_{12}$ as generator of C_2, then $k_1 t$ is such an element. We refer to the corresponding eigenspectra of any element τ of order 2 as $E_\tau^\pm$. In particular we have $E_t^+ = EO_2$, $E_t^- = \Sigma^{36} EO_2$.

Theorem 12.
a) There is a cofibration sequence

$$E^{hN^1} \longrightarrow E_t^+ \longrightarrow E_{k_1 t}^-$$

and the map between the eigenspectra is induced by $id - k_1$ (on the level of E^{hC_6}).

b) There is an equivalence

$$E^{-}_{k_1t} \simeq \Sigma^{36} EO_2 \ .$$

Proof. a) This follows the same strategy as the proof of Theorem 10. The map $(id - k_1)$ induces a self map of E^{hC_6} which we can easily check to induce a map $E^{+}_{t} \longrightarrow E^{-}_{k_1t}$. Let F be the fibre of this map. As before the canonical map $f : E^{hN^1} \to E^{+}_{t}$ becomes null after composing it with $id - k_1$ so that we obtain a map $g : E^{hN^1} \to F$. This time we get an identification

$$\pi_* L_{K(2)}(E^{hC_6} \wedge E) \cong \ \mathrm{map}_{cts}(\mathbb{G}_2/C_6, E_*) \cong \mathrm{Hom}_{cts}(\mathbb{Z}_3[[\mathbb{G}_2/C_6]], E_*)$$

where for a profinite $\mathbb{G}_2$-set $X = \lim_\alpha X_\alpha$ with finite $\mathbb{G}_2$-sets X_α we write $\mathbb{Z}_3[[X]]$ for $\lim_{\alpha,n} \mathbb{Z}_3/(3^n)[X_\alpha]$ and where Hom_{cts} denotes continuous homomomorphisms. The elements t and k_1t act on the coset space and after linearization we can pass to the corresponding $\pm$ eigenspaces which we denote $\mathrm{Hom}^{t,\pm}_{cts}$ etc.

Now $id - k_1$ induces as before a surjective map

$$\mathrm{Hom}^{t,+}_{cts}(\mathbb{Z}_3[[\mathbb{G}_2/C_6]], E_*) \to \mathrm{Hom}^{k_1t,-}_{cts}(\mathbb{Z}_3[[\mathbb{G}_2/C_6]], E_*)$$

whose kernel gets identified via g with $\mathrm{Hom}_{cts}(\mathbb{Z}_3[[\mathbb{G}_2/N^1]], E_*)$. This finishes the proof of (a).

b) This is an immediate consequence of Proposition 11 and the fact that the elements t and k_1t are conjugate in N^1 because 2 is a unit in $\mathbb{Z}_3$. Hence they have equivalent eigenspectra. $\square$

<u>Remark</u> Theorem 10 and Theorem 12 have the following discrete analogues which may, in particular in the case of Theorem 12, help to explain the situation.

It is well known that in the case of an action a discrete infinite cyclic group C_∞ on a spectrum (or suspension) X one can obtain X^{hC_∞}, up to homotopy as the fibre of the map $X \to X$, given by $id - k$ and this fibration may be thought of as being induced from the equivariant skeletal filtration of the real line $\mathbb{R}$ thought of as universal C_∞-space EC_∞ with C_∞ acting via translations. If we consider $C_\infty \rtimes C_2$ as acting on $\mathbb{R}$ via translation and reflections and if we ignore 2-primary phenomena then $\mathbb{R}$ is still a "reasonable model" for the universal $C_\infty \rtimes C_2$ space $E(C_\infty \rtimes C_2)$. This time the 0 - simplices are the integral points on the real line with isotropy isomorphic to C_2 and the isotropy group of a 1 - simplex is C_2 with C_2 acting nontrivially on the 1 - simplex. By using the skeletal filtration once again the homotopy fixed points $X^{h(C_\infty \rtimes C_2)}$ can, under suitable assumptions, be obtained as the fibre of a map as in Theorem 12.

Corollary 13. *There is a cofibration sequence*

$$E^{hN^1} \longrightarrow EO_2 \longrightarrow \Sigma^{36} EO_2 \ . \quad \square$$

2.3. The homotopy groups of $E^{hN} \wedge V(1)$ and of $E^{hN^1} \wedge V(1)$.

It is not hard to see that the spectrum E^{hN^1} is no longer periodic. Nevertheless the following lemma shows that the homotopy groups $\pi_k(E^{hN^1} \wedge V(1))$ remain 72-periodic.

Lemma 14. *The map*

$$\pi_*(EO_2 \wedge V(1)) \to \pi_*(\Sigma^{36}EO_2 \wedge V(1))$$

which is induced by $id - k_1$ is trivial.

Proof. Theorem 9 implies that if $\pi_n(EO_2 \wedge V(1))$ and $\pi_n(\Sigma^{36}EO_2 \wedge V(1))$ are both nonzero then

$$n \in \{0, 10, 20, 36, 46, 56\} \bmod 72 .$$

Because $id - k_1$ commutes with the action of β we see that the map is trivial if $n \equiv 36, 46, 56$ and that it suffices to concentrate on the case $n \equiv 0$. Now the periodicity generator $v_2^{9/2} \in \pi_{72}(EO_2 \wedge V(1))$ comes from $\Delta^3 \in \pi_{72}(EO_2)$ and therefore it suffices to show that the composition

$$\pi_{72}(EO_2) \to \pi_{72}(\Sigma^{36}EO_2) \to \pi_{72}(\Sigma^{36}EO_2 \wedge V(1)))$$

annihilates Δ^3. In fact, we see from Proposition 7b and Theorem 8 that the image of $\pi_{72}(\Sigma^{36}EO_2)$ in $\pi_{72}(\Sigma^{36}EO_2 \wedge V(0))$ is divisible by v_1 and becomes therefore trivial in $\pi_{72}(\Sigma^{36}EO_2 \wedge V(1)))$. □

The lemma allows us to analyze the ANSS

$$E_2^{s,t} \cong H^s(N^1, E_*V(1)) \Longrightarrow \pi_{t-s}(E^{hN^1} \wedge V(1)) .$$

Its E_2-term is easily calculated to be

$$H^*(N^1, E_*V(1)) \cong \left(\mathbb{F}_9[u^{\pm 1}, y] \otimes \Lambda(x, a')\right)^{C_4} \cong \mathbb{F}_9[v_2^{\pm 1/2}, \beta] \otimes \Lambda(\alpha, u^{-18}a')$$

where as before $\alpha = u^{-2}x$, $\beta = u^{-6}y$, $v_2^{1/2} = u^{-4}$ and the exterior generator a' is the contribution of the factor $\mathbb{Z}_3$ in the centralizer $C^1 \cong C_3 \times C_2 \times \mathbb{Z}_3$. Its bidegree is $(1, 0)$ and the generator t of C_4 acts on it by multiplication by -1. Therefore $a_{35} := u^{-18}a'$ is a new invariant class. We note that the elements x and y are a priori not canonically defined, not even up to a nontrivial constant because the corresponding groups are of rank 2 over $\mathbb{F}_9$. We can and will choose them such that α and β detect the images of α_1 and β_1 in $\pi_*(E_2^{hN^1} \wedge V(1)$ for example by defining them via Greek letter constructions in $H^*(N, -)$.

Proposition 15.

a) The ANSS

$$E_2^{s,t} \cong H^s(N^1, E_*V(1)) \cong \mathbb{F}_9[v_2^{\pm 1/2}, \beta] \otimes \Lambda(\alpha)\{1, a_{35}\} \Longrightarrow \pi_{t-s}(E^{hN^1} \wedge V(1))$$

splits as the direct sum of the ANSS for $EO_2 \wedge V(1)$ and that of $\Sigma^{35}EO_2 \wedge V(1)$ (where the summand indexed by 1 corresponds to $EO_2 \wedge V(1)$ and that by a_{35} to $\Sigma^{35}EO_2 \wedge V(1)$).

b) As modules over $\mathbb{F}_9[v_2^{\pm 9/2}, \beta]$ there is an isomorphism

$$\pi_*(E^{hN_1} \wedge V(1)) \cong \pi_*(EO_2 \wedge V(1))\{1, a_{35}\} .$$

Remark We emphasize that the module structure over $v_2^{\pm 9/2}$ is (at least at this point) a purely algebraic accident induced by an algebraic module structure on the level of E_2-terms.

Proof. The fibration sequence

$$E^{hN^1} \wedge V(1) \to EO_2 \wedge V(1) \to \Sigma^{36} EO_2 \wedge V(1)$$

induces an exact sequence

$$0 \to E_*(E^{hN^1} \wedge V(1)) \to E_*(EO_2 \wedge V(1)) \to E_*(\Sigma^{36} EO_2 \wedge V(1)) \to 0$$

where E_*X has to be interpreted as $\pi_*(L_{K(2)}(E \wedge X))$. In fact, as in the proof of Theorem 12 this sequence can be identified with the sequence

$$\begin{aligned} 0 \to \mathrm{Hom}_{cts}(\mathbb{Z}_3[[\mathbb{G}_2/N^1]], E_*V(1)) &\to \mathrm{Hom}^{t,+}_{cts}(\mathbb{Z}_3[[\mathbb{G}_2/C_6]], E_*V(1)) \\ &\to \mathrm{Hom}^{k_1 t,-}_{cts}(\mathbb{Z}_3[[\mathbb{G}_2/C_6]], E_*V(1)) \to 0 \ . \end{aligned}$$

In cohomology (i.e. on E_2-terms of the relevant ANSS) this sequence induces short exact sequences for all $s \geq 0$

$$0 \to H^{s-1}(G_{12}, E_t(\Sigma^{36} V(1))) \to H^s(N^1, E_t V(1)) \to H^s(G_{12}, E_t(V(1)) \to 0$$

where the monomorphism converges towards the map

$$\pi_{t-s}(\Sigma^{35} EO_2 \wedge V(1)) \to \pi_{t-s}(E^{hN^1} \wedge V(1))$$

by the geometric boundary theorem and the epimorphism converges towards the map

$$\pi_{t-s}(E^{hN^1} \wedge V(1)) \to \pi_{t-s}(EO_2 \wedge V(1)) \ .$$

by naturality. The proposition follows. □

Now we turn towards $E^{hN} \wedge V(1)$ and consider the ANSS spectral sequence

$$E_2^{s,t} \cong H^s(N, E_t V(1)) \Longrightarrow \pi_{t-s}(E^{hN} \wedge V(1)) \ .$$

The E_2 -term of the SS is easily calculated to be

$$H^*(N, E_*V(1)) \cong \ \big(\mathbb{F}_9[u^{\pm 1}, y] \otimes \Lambda(x, a', \zeta)\big)^{C_4} \cong \mathbb{F}_9[v_2^{\pm 1/2}, \beta] \otimes \Lambda(\alpha, a_{35}, \zeta)$$

where the new exterior generator ζ is the contribution of the central factor $\mathbb{Z}_3$ in the centralizer C. Its bidegree is $(1, 0)$ and it is fixed by the action of t.

Proposition 16.
a) The ANSS

$$E_2^{s,t} \cong H^s(N, E_t V(1)) \cong \mathbb{F}_9[v_2^{\pm 1/2}, \beta] \otimes \Lambda(\alpha, \zeta)\{1, a_{35}\} \Longrightarrow \pi_{t-s}(E^{hN} \wedge V(1))$$

splits as the direct sum of the ANSS of $E^{hN^1} \wedge V(1)$ and of $\Sigma^{-1}E^{hN^1} \wedge V(1)$ (where the summand indexed by 1 corresponds to $E_2^{hN_1} \wedge V(1)$ and that by ζ to $\Sigma^{-1}E_2^{hN_1} \wedge V(1)$.)

b) As modules over $\mathbb{F}_9[v_2^{\pm 9/2}, \beta]$ there is an isomorphism

$$\pi_*(E_2^{hN^1} \wedge V(1)) \cong \pi_*(EO_2 \wedge V(1))\{1, a_{35}, \zeta, \zeta a_{35}\} \ . \quad \square$$

<u>Remark</u> We emphasize that as before the module structure over $v_2^{\pm 9/2}$ is (at least at this point) a purely algebraic accident on the level of E_∞ - terms.

Proof. As before the proof will be an easy consequence of the following result (which is analogous to Lemma 14) whose proof will, however, make use of the structure of the SS considered in Proposition 16. □

Lemma 17. *The map*

$$(id-k)_* : \pi_*(E^{hN^1} \wedge V(1)) \to \pi_*(E^{hN^1} \wedge V(1))$$

is trivial.

Proof. The action of k on $H^s(N^1, E_tV(1))$ is trivial except perhaps on u. The generator k of the central $\mathbb{Z}_3 \subset \mathbb{D}_2^\times$ is necessarily congruent 1 mod 3 (for example, one can take $k=4$). Then $u \in \pi_{-2}E$ gets multiplied with $k \equiv 1$ mod 3. Therefore the action of $id-k$ is trivial on the E_2 - term of the ANSS. This shows that the action on $\pi_k(E^{hN^1} \wedge V(1))$ is trivial except possibly in degrees

$$n \equiv 0, 3, 8, 11, 16, 19, 35, 38, 43, 45, 46, 48, 53, 56 \bmod 72$$

where the total degree n of the E_∞ - term of the SS is made of two copies of $\mathbb{F}_9$. Degrees 35 and 43 resp. 45 and 53 can be excluded from the list because both copies have the same filtration (= 1 resp. 3). Next the action of α and β and the compatibility of k with the fibration sequence of Theorem 12 resp. Corollary 13 imply that degrees 38, 46, 48 and 56 can also be excluded. Similarly the action of α and β show that it is enough to consider the cases $n \equiv 0, 8, 16$ mod 72. If $id-k$ acts nontrivially in one of these dimensions then there exists some integer p and there exists $q \in \{0,1,2\}$ such that

$$(id-k)_*(v_2^{9p+q/2}) = cv_2^{(9(p-1)+q+3)/2}\beta\alpha a_{35}$$

for some nontrivial constant c. This implies then that in the ANSS for $E^{hN} \wedge V(1)$ the element $v_2^{(9(p-1)+q+3)/2}\beta\alpha\zeta a_{35}$ (which is a permanent cycle by Proposition 14 and the fact that ζ detects a homotopy class which comes from $L_{K(2)}S^0$) does not survive and hence that it is in the image of a differential. At this point we turn attention towards the analysis of the SS to show that htis cannot happen.

In the ANSS converging to $\pi_*(E^{hN} \wedge V(1))$ the elements α, β and ζ come from the sphere (or at least from the $K(2)$-local sphere). Furthermore we know from [Ra2, Table A3.4] that the Greek letter element $\beta_{6/3} \in Ext^{2,84}_{BP_*BP}(BP*, BP_*)$ is a permanent cycle in the ANSS for S^0. By [Sh1, Lemma 2.4] and Corollary 19 below this element is detected in $E_2^{2,84}$ in our SS and therefore agrees with $v_2^{9/2}\beta$, up to a nontrivial constant. Because E_2 is free over $\mathbb{F}_9[\beta]$ we deduce that the first differential is linear with respect to $v_2^{9/2}$.

So for the first differential we need to study the elements

$$v_2^{r/2}, \quad a_{35}v_2^{r/2} \text{ if } r \equiv 0, 1, \ldots, 8 \bmod 9 \ .$$

Degree reasons (i.e. calculating modulo total degree 8) shows that the first possibility for a differential is d_5. The possible targets are as follows:

- $d_5(v_2^{r/2})$ is a linear combination of $v_2^{(r-3)/2}\alpha\beta^2$, $v_2^{(r-7)/2}\beta^2 a_{35}$ and $v_2^{(r-6)/2}\alpha\beta\zeta a_{35}$.
- $d_5(a_{35}v_2^{r/2})$ is a multiple of $v_2^{(r-3)/2}\alpha\beta^2 a_{35}$.

Now we use that 1, v_2 and v_2^5 are permanent cycles coming from $V(1)$ (see [Sh1, Thm 2.6]). This implies that for $r \equiv 0, 1, 2$ mod 9 we have

$$d_5(v_2^{r/2}) = 0 \ ,$$

in particular $v_2^{(9(p-1)+l+3)/2}\beta\alpha\zeta a_{35}$ is not in the image of d_5, hence it survives and the proof of the lemma is complete. □

3. The homotopy of $L_2V(1)$

In this section we will calculate the homotopy of $L_2V(1) \simeq E^{h\mathbb{G}_2} \wedge V(1)$ by comparing it to that of $E_2^{hN} \wedge V(1)$. The E_2-term of the ANSS converging to $\pi_* L_2V(1)$ is isomorphic to

$$H^*(\mathbb{G}_2, \mathbb{F}_9[u^{\pm 1}]) \cong (H^*(\mathbb{S}_2, \mathbb{F}_9[u^{\pm 1}])^{C_2}$$

where C_2 acts via conjugation on $\mathbb{S}_2$ and via Frobenius on $\mathbb{F}_9$ (hence the action is free and the spectral sequence of the extension $\mathbb{S}_2 \to \mathbb{G}_2 \to C_2$ degenerates into the stated isomorphism). Furthermore there is a canonical isomorphism

$$H^*(\mathbb{S}_2, \mathbb{F}_9[u^{\pm 1}]) \cong (H^*(S_2, \mathbb{F}_9) \otimes_{\mathbb{F}_9} \mathbb{F}_9[u^{\pm 1}])^{\mathbb{F}_9^\times}$$

where S_2 is the 3-Sylow subgroup of $\mathbb{S}_2$ and acts trivially, and the invariants are taken with respect to the action of the quotient $\mathbb{S}_2/S_2$ which can be naturally identified with $\mathbb{F}_9^\times$ generated by ω. The generator ω of $\mathbb{F}_9^\times$ acts diagonally on this tensor product, via conjugation on S_2 and via multiplication with ω on u, so that taking invariants amounts to taking the eigenspace decomposition of $H^*(S_2, \mathbb{F}_9)$ with respect to the action of ω (determined implicitly by Theorem 18 below) and tensoring the eigenspace E_{ω^i} (with eigenvalue ω^i) with powers u^{-i+8k} to get invariants.

In [H, Prop. 3.4 and Thm. 4.2] $H^*(S_2, \mathbb{F}_3)$ was studied via the restriction map to the centralizers $C_{S_2}(E_i) \cong C_3 \times \mathbb{Z}_3^2$, $i = 1, 2$, where the E_i are representatives of the two different conjugacy classes of C_3's in S_2. We can choose E_1 to be the subgroup generated by $s \in G_{12}$ and then E_2 can be choosen to be $\omega^{-1}E_1\omega$ so that the restriction map

$$H^*(S_2, \mathbb{F}_3) \to \prod_{i=1}^{2} H^*(C_{S_2}(E_i), \mathbb{F}_3) \cong \prod_{i=1}^{2} \mathbb{F}_3[y_i] \otimes \Lambda(x_i, \zeta_i, a_i{}')$$

becomes $\mathbb{F}_9^\times$-equivariant where the $\mathbb{F}_9^\times$-action on the right is induced from the conjugation action of $N_{\mathbb{S}_2}(E_i)/C_{S_2}(E_i) \cong C_4 \subset \mathbb{F}_9^\times$. We note that $t \in G_{12} \subset N_{\mathbb{S}_2}(E_1)$ projects to a generator in C_4. We can choose the cohomology classes such that y_i and x_i correspond to the generators of the cohomology of the cyclic subgroup, ζ_i to the cohomology of the central factor $\mathbb{Z}_3$, and $a_i{}'$ to that of the noncentral factor $\mathbb{Z}_3$ on which t acts by multiplication by -1. This notation differs somewhat from that in [H] but is consistent with our notation in section 2.

Theorem 18 [H].
a) The restriction map

$$H^*(S_2, \mathbb{F}_3) \to \prod_{i=1}^{2} H^*(C_{S_2}(E_i), \mathbb{F}_3) \cong \prod_{i=1}^{2} \mathbb{F}_3[y_i] \otimes \Lambda(x_i, \zeta_i, a_i{}')$$

is an $\mathbb{F}_9^\times$-invariant monomorphism whose image is the $\mathbb{F}_3$-subalgebra generated by x_1, x_2, y_1, y_2, $\zeta_1 + \zeta_2$, $x_1a_1{}' - x_2a_2{}'$, $y_1a_1{}'$ and $y_2a_2{}'$.
b) In particular $H^*(S_2, \mathbb{F}_3)$ is a free module over $\mathbb{F}_3[y_1 + y_2] \otimes \Lambda(\zeta_1 + \zeta_2)$ generated by 1, x_1, x_2, y_1, $x_1a_1{}' - x_2a_2{}'$, $y_1a_1{}'$, $y_2a_2{}'$ and $y_1x_1a_1{}'$. □

We note that a priori the elements x_i, y_i and $a_i{}'$ are not canonical (because they depend on the chosen decomposition of $C_{S_2}(E_1)$). The theorem implies, however, that x_i and y_i (as the Bockstein of x_i) are distinguished (at least up to nontrivial constants).

Next we describe the invariants of $H^*(S_2, \mathbb{F}_9[u^{\pm1}])$ with respect to the action of $\mathbb{F}_9^\times$ (which is determined by $\omega_*(u) = \omega u$). By using the $\mathbb{F}_9^\times$-linear monomorphism these invariants can be identified with a subring of

$$(\prod_{i=1}^{2} H^*(C_{S_2}(E_i), \mathbb{F}_9[u^{\pm1}])^{\mathbb{F}_9^\times} \cong (H^*(C_{S_2}(E_1), \mathbb{F}_9[u^{\pm1}])^{C_4} \cong H^*(N_{\mathbb{S}_2}(E_1), \mathbb{F}_9[u^{\pm1}])$$

where C_4 is as before generated by $t \in G_{12}$. Its action on $H^*(C_{S_2}(E_1), \mathbb{F}_3)$ is given by

$$t_*(y) = -y, \quad t_*(x) = -x, \quad t_*(a') = -a', \quad t_*(\zeta) = \zeta$$

(where we have omitted the indices for simplicity). The following corollary is now straightforward to verify. As in section 1.4. and chapter 2 we write $v_2^{k/2}$ for u^{-4k}.

Corollary 19.
a) The restriction map

$$H^*(\mathbb{S}_2, \mathbb{F}_9[u^{\pm1}]) \to H^*(N, \mathbb{F}_9[u^{\pm1}]) \cong \big(\mathbb{F}_9[u^{\pm1}, y] \otimes \Lambda(x, \zeta, a')\big)^{C_4}$$

is a monomorphism. Its target is isomorphic to

$$\mathbb{F}_9[v_2^{\pm1/2}, \beta] \otimes \Lambda(\alpha, \zeta, a_{35})$$

(with $\beta = u^{-6}y$, $\alpha = u^{-2}x$, ζ and $a_{35} = u^{-18}a'$) and its image is the $\mathbb{F}_9$-subalgebra of $\mathbb{F}_9[v_2^{\pm1/2}, \beta] \otimes \Lambda(\alpha, \zeta, a_{35})$ generated by $v_2^{\pm1}$, α, $v_2^{1/2}\alpha$, β, $v_2^{1/2}\beta$, ζ, αa_{35}, βa_{35} and $v_2^{1/2}\beta a_{35}$.
b) In particular $H^*(\mathbb{S}_2; \mathbb{F}_9[u^{\pm1}])$ is the free $\mathbb{F}_9[v_2^{\pm1}, \beta] \otimes \Lambda(\zeta)$-submodule of $\mathbb{F}_9[v_2^{\pm1/2}, \beta] \otimes \Lambda(\alpha, \zeta, a_{35})$ generated by 1, α, $v_2^{1/2}\alpha$, $v_2^{1/2}\beta$, αa_{35} βa_{35}, $v_2^{1/2}\beta a_{35}$, and $v_2^{1/2}\beta\alpha a_{35}$. $\square$

The restriction map above is the comparison map at the E_2-level between the two ANSS converging to $\pi_{t-s}(E^{h\mathbb{S}_2} \wedge V(1))$ resp. $\pi_{t-s}(E^{hN} \wedge V(1))$. We still have to deal with the Galois action of C_2 if we want to get at $L_2V(1)$.

Now the Galois generator $\phi \in C_2$ acts on $\mathbb{S}_2 \subset \mathbb{D}_2^\times$ by conjugation by S, hence it is clear that $\omega\phi$ centralizes $s = -\frac{1}{2}(1 + \omega S)$ and thus everything in the commutative subfield of $\mathbb{D}_2$ generated by s. In particular $\omega\phi$ commutes with the units in the maximal order of this subfield, i.e. with $C_{\mathbb{S}_2}(E_1)$. Therefore the group $C_{\mathbb{G}_2}(E_1)$ (which is generated by $C_{\mathbb{S}_2}(E_1)$ and $\omega\phi$) is an abelian group and $\omega\phi$ acts trivially on $H^*(C_{S_2}(E_1), \mathbb{F}_3)$. The action on the coefficient ring $\mathbb{F}_9[u^{\pm1}]$ is given by $(\omega\phi)_*(cu^k) = \phi(c)\omega^k u^k$ if $c \in \mathbb{F}_9$.

The monomorphism of Theorem 18 (with coefficients extended to $\mathbb{F}_9[u^{\pm1}]$) is actually linear even with respect to $\mathbb{F}_9^\times \rtimes C_2$ where the Galois generator ϕ acts on the target on the level of groups by conjugation by S while it acts on $\mathbb{F}_9[u^{\pm1}]$ by Frobenius again. The $\mathbb{F}_9^\times \rtimes C_2$-invariants in the target of this monomorphism can therefore be identified with

$$\big(\prod_{i=1}^{2} H^*(C_{S_2}(E_i), \mathbb{F}_9[u^{\pm1}])\big)^{\mathbb{F}_9^\times \rtimes C_2} \cong \big(H^*(C_{S_2}(E_1), \mathbb{F}_9[u^{\pm1}])^{C_4}\big)^{<\omega\phi>} .$$

The preceding two paragraphs show that we can modify (if necessary) the elements $v_2^{1/2}$, β, α, ζ and a_{35} of Corollary 19 by scalars in $\mathbb{F}_9$ so that they become invariant with respect to the action of $\omega\phi$. After having done this $H^*(\mathbb{G}_2,\mathbb{F}_9[u^{\pm 1}])$ can be identified with the $\mathbb{F}_3$-subalgebra of $\mathbb{F}_9[v_2^{1/2},\beta]\otimes\Lambda(\alpha,\zeta,a_{35})$ generated by elements with the same name as in Corollary 19a and $H^*(\mathbb{G}_2,\mathbb{F}_9[u^{\pm 1}])$ is a free module over $\mathbb{F}_3[v_2^{\pm 1},\beta]\otimes\Lambda(\zeta)$ on elements with the same name as in Corollary 19b.

Now we are ready to compare the differentials in the two ANSS converging to $L_2V(1)\simeq E_2^{h\mathbb{G}_2}\wedge V(1)$ resp. $E_2^{hN}\wedge V(1)$. We refer to them as the source SS resp. the target SS.

For the target SS we deduce from section 2.3 that $v_2^{\pm 9/2}$, β, α, ζ and a_{35} are permanent cycles and that the differentials are linear with respect to the algebra

$$R := \mathbb{F}_9[v_2^{\pm 9/2},\beta]\otimes\Lambda(\alpha,\zeta,a_{35})\ .$$

To better compare with the spectral sequence of the source we should consider this SS as one of modules over $S:=\mathbb{F}_9[v_2^{\pm 9},\beta]\otimes\Lambda(\alpha,\zeta,a_{35})$. In fact, the E_2-term of the target SS is a free module over S on generators $v_2^{k/2}$, $k=0,\dots,17$.

We consider the E_2-term of the ANSS of the source as a free module over

$$P := \mathbb{F}_3[v_2^{\pm 9},\beta]\otimes\Lambda(\zeta)$$

(note that here we have taken the prime field) on the following generators (where $l = 0,1,\dots,8$):

- v_2^l, $v_2^{l+1/2}\beta$, $v_2^l\beta a_{35}$, $v_2^{l+1/2}\beta a_{35}$
- $v_2^l\alpha$, $v_2^{l+1/2}\alpha$, $v_2^l\alpha a_{35}$, $v_2^{l+1/2}\beta\alpha a_{35}$

We have seen in section 1 and 2 that the first differential for the target SS is d_5. It is determined by

$$d_5(v_2^{k/2}) = \begin{cases} 0 & \text{if } k\equiv 0,1,2 \bmod 9\\ c_k v_2^{(k-3)/2}\alpha\beta^2 & \text{if } k\equiv 3,\dots,8 \bmod 9\ . \end{cases}$$

This implies that the first differential of the source SS is also d_5. Furthermore by the discussion above the nontrivial constants c_k have to be in $\mathbb{F}_3$ and d_5 in the source SS is given by

$$d_5(v_2^l) = \begin{cases} 0 & \text{if } l\equiv 0,1,5 \bmod 9\\ \pm v_2^{l-2+1/2}\alpha\beta^2 & \text{if } l\equiv 2,3,4,6,7,8 \bmod 9 \end{cases}$$

$$d_5(v_2^{l+1/2}\beta) = \begin{cases} 0 & \text{if } l\equiv 0,4,5 \bmod 9\\ \pm v_2^{l-1}\alpha\beta^3 & \text{if } l\equiv 1,2,3,6,7,8 \bmod 9 \end{cases}$$

$$d_5(v_2^l\beta a_{35}) = \begin{cases} 0 & \text{if } l\equiv 0,1,5 \bmod 9\\ \pm v_2^{l-2+1/2}\alpha\beta^3 a_{35} & \text{if } l\equiv 2,3,4,6,7,8 \bmod 9 \end{cases}$$

$$d_5(v_2^{l+1/2}\beta a_{35}) = \begin{cases} 0 & \text{if } l\equiv 0,4,5 \bmod 9\\ \pm v_2^{l-1}\alpha\beta^3 a_{35} & \text{if } l\equiv 1,2,3,6,7,8 \bmod 9 \end{cases}$$

$$d_5(v_2^l\alpha) = d_5(v_2^{l+1/2}\alpha) = 0 \quad \text{if } l=0,\dots,8 \bmod 9$$
$$d_5(v_2^l\alpha a_{35}) = d_5(v_2^{l+1/2}\beta\alpha a_{35}) = 0 \quad \text{if } l=0,\dots,8 \bmod 9\ .$$

This yields the following E_6-term (which is already presented in a form which is adapted to the discussion of the next differential):

$$\begin{aligned} E_6 \cong\ & P\{v_2^l\}_{l=0,1,5} \oplus P\{v_2^l\alpha\}_{l=3,4,8} \oplus P/(\beta^3)\{v_2^l\alpha\}_{l=0,1,2,5,6,7} \\ \oplus\ & P\{v_2^{l+1/2}\beta\}_{l=0,4,5} \oplus P\{v_2^{l+1/2}\alpha\}_{l=3,7,8} \oplus\ P/(\beta^2)\{v_2^{l+1/2}\alpha\}_{l=0,1,2,4,5,6} \\ \oplus\ & P\{v_2^l\beta a_{35}\}_{l=0,1,5} \oplus P\{v_2^l\alpha a_{35}\}_{l=3,4,8} \oplus\ P/(\beta^3)\{v_2^l\alpha a_{35}\}_{l=0,1,2,5,6,7} \\ \oplus\ & P\{v_2^{l+1/2}\beta a_{35}\}_{l=0,4,5} \oplus P\{v_2^{l+1/2}\beta\alpha a_{35}\}_{l=3,7,8} \oplus\ P/(\beta^2)\{v_2^{l+1/2}\beta\alpha a_{35}\}_{l=0,1,2,4,5,6} \end{aligned}$$

We know that the next differential in the target SS is d_9 and is determined by

$$d_9(v_2^{k/2}\alpha) = \begin{cases} 0 & \text{if } k = 0,1,2,3,4,5 \bmod 9 \\ c'_k v_2^{k/2-3}\beta^5 & \text{if } k = 6,7,8 \bmod 9 \end{cases}$$

for suitable nontrivial constants c'_k. Again by comparing with the spectral sequence of the target we deduce that the next differential in the source SS is also d_9, the constants have to be in $\mathbb{F}_3$ and d_9 in the source SS is given by

$$d_9(v_2^l\alpha) = \begin{cases} 0 & \text{if } l \equiv 0,1,2,5,6,7 \bmod 9 \\ \pm v_2^{l-3}\beta^5 & \text{if } l \equiv 3,4,8 \bmod 9 \end{cases}$$

$$d_9(v_2^{l+1/2}\alpha) = \begin{cases} 0 & \text{if } l \equiv 0,1,2,4,5,6 \bmod 9 \\ \pm v_2^{l-3+1/2}\beta^5 & \text{if } l \equiv 3,7,8 \bmod 9 \end{cases}$$

$$d_9(v_2^l\alpha a_{35}) = \begin{cases} 0 & \text{if } l \equiv 0,1,2,5,6,7 \bmod 9 \\ \pm v_2^{l-3}\beta^5 a_{35} & \text{if } l \equiv 3,4,8 \bmod 9 \end{cases}$$

$$d_9(v_2^{l+1/2}\beta\alpha a_{35}) = \begin{cases} 0 & \text{if } l \equiv 0,1,2,4,5,6 \bmod 9 \\ \pm v_2^{l-3+1/2}\beta^6 a_{35} & \text{if } l \equiv 3,7,8 \bmod 9 \end{cases}$$

and we obtain the following E_{10}-term

$$\begin{aligned} E_{10} \cong\ & P/(\beta^5)\{v_2^l\}_{l=0,1,5} \oplus P/(\beta^3)\{v_2^l\alpha\}_{l=0,1,2,5,6,7} \\ \oplus\ & P/(\beta^4)\{v_2^{l+1/2}\beta\}_{l=0,4,5} \oplus\ P/(\beta^2)\{v_2^{l+1/2}\alpha\}_{l=0,1,2,4,5,6} \\ \oplus\ & P/(\beta^4)\{v_2^l\beta a_{35}\}_{l=0,1,5} \oplus\ P/(\beta^3)\{v_2^l\alpha a_{35}\}_{l=0,1,2,5,6,7} \\ \oplus\ & P/(\beta^5)\{v_2^{l+1/2}\beta a_{35}\}_{l=0,4,5} \oplus\ P/(\beta^2)\{v_2^{l+1/2}\beta\alpha a_{35}\}_{l=0,1,2,4,5,6}\ . \end{aligned}$$

At this point there is no more room for further nontrivial differentials and we arrive at the desired result below in which the names of the generators are chosen so as to describe their image in the E_2-term of the ANSS for $\pi_*(E_2^{hN} \wedge V(1))$.

Theorem 20. *As a module over $P = \mathbb{F}_3[v_2^{\pm 9}, \beta] \otimes \Lambda(\zeta)$ there is an isomorphism*

$$\begin{aligned} \pi_* L_2V(1) \cong\ & P/(\beta^5)\{v_2^l\}_{l=0,1,5} \oplus P/(\beta^3)\{v_2^l\alpha\}_{l=0,1,2,5,6,7} \\ \oplus\ & P/(\beta^4)\{v_2^{l+1/2}\beta\}_{l=0,4,5} \oplus\ P/(\beta^2)\{v_2^{l+1/2}\alpha\}_{l=0,1,2,4,5,6} \\ \oplus\ & P/(\beta^4)\{v_2^l\beta a_{35}\}_{l=0,1,5} \oplus\ P/(\beta^3)\{v_2^l\alpha a_{35}\}_{l=0,1,2,5,6,7} \\ \oplus\ & P/(\beta^5)\{v_2^{l+1/2}\beta a_{35}\}_{l=0,4,5} \oplus\ P/(\beta^2)\{v_2^{l+1/2}\beta\alpha a_{35}\}_{l=0,1,2,4,5,6}\ . \qquad \square \end{aligned}$$

We finish by observing that this matches with Shimomura's result (if his parameter is taken to be $k = 1!$).

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