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On an estimation of polynomial roots by Lagrange

by

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Abstract: Lagrange stated the following inequality: An upper bound for the positive real roots of a monic polynomial over \(\mathbb{R}\) is equal to \(R + \rho\), where \(R\) and \(\rho\) are the two largest numbers in the set \(\{|\sqrt{|a_j|}; j \in J\}\) and \(\{a_j; j \in J\}\) are the negative coefficients.

Since Lagrange does not provide a complete proof, we give one following Cauchy’s method. We also present a slight generalization of this theorem of Lagrange, using a result of Kojiima.

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Keywords and phrases: Polynomial roots, bound of Lagrange.

Introduction

Let \(F(X) = X^n + a_1 X^{n-1} + \cdots + a_{n-1} X + a_n\) be a nonconstant monic polynomial over \(\mathbb{C}\). In many problems it is useful to know upper bounds for the absolute values of the roots. These estimates may be used, for example, to give bounds for linear factors of univariate polynomials. Other applications are in stability problems, mechanics and statistics.

The first results on bounds for polynomial (real) roots were obtained in the XVIIth and XVIIIth centuries. We mention the works of Descartes (1637), Newton (1673), Rolle (1690), MacLaurin (1729), Lagrange (1767) and Cauchy (1829).

The result of Lagrange was first published in his memory on the resolution of polynomial equations (1769) and further included in his lectures [6] at École Normale Supérieure. It states that an upper bound for the positive real roots of a polynomial over \(\mathbb{R}\) is given by the sum of the two largest numbers in the set \(\{|\sqrt{|a_j|}; j \in J\}\), where \(a_j; j \in J\) are the negative coefficients. By the method of Cauchy (1829, [1]) it obviously gives an upper bound for polynomials with complex coefficients. Note that this bound is stronger than the “2\(R\)” bound of Fujiwara [3].
Apparently this result was completely forgotten. We found two modern references to the work [6] of Lagrange in the book of L. Derwidué [2] and the recent historical study [7] of R. Laubenbacher, G. McGrath and D. Pengelley. The bound \( R + \rho \) is not mentioned in any of them.

We studied the original work of Lagrange and observed that he gives only a "vague" hint of how to obtain the \( R + \rho \) bound. We give here a proof based on the method of Cauchy. More precisely, if \( F(X) = X^n + a_1 X^{n-1} + \cdots + a_n \) is a nonconstant polynomial with complex coefficients, we prove that \( R + \rho \) is an upper bound for the unique root of the polynomial \( X^n - |a_1| X^{n-1} - \cdots - |a_n| \), where \( R \) and \( \rho \) are the two largest numbers in the set \( \{ |a_k|^{1/k} \mid 1 \leq k \leq n \} \). This implies (see Theorem 1.4 of Cauchy below) the same bound for the roots of \( F \).

We also give a new proof of a result of Kojima on bounds of polynomial roots. Instead of the theorem of Hadamard invoked by Kojima [5], we use again the method of Cauchy. Then we obtain an extension of the inequality of Lagrange.

Finally we test by experimentation the bound that we develop, and we compare it to those of Lagrange, Fujiwara and the exact values of the roots.

1 The bound of Lagrange

1.1 Prerequisites for Lagrange

Lagrange mentions the following result and attributes it to Newton and MacLaurin:

**Theorem 1.1 (MacLaurin–Newton)** Let \( P(X) = X^n + a_1 X^{n-1} + \cdots + a_n \in \mathbb{R}[X] \) be nonconstant. If \( P^{(k)}(l) > 0 \) for all \( k = 0, 1, 2, \ldots, n-1 \), then \( l \) is an upper bound for the positive real roots of \( P \).

Another useful result for the estimation of polynomial roots is Descartes’ rule of signs:

**Theorem 1.2 (Descartes)** The number of positive real roots of a polynomial over \( \mathbb{R} \) is at most equal to the number of changes of signs in the sequence of the coefficients.

1.2 The statement of Lagrange

Lagrange’s treatise of 1798 is an extended version of his memory from 1767 published in *Recueil des Mémoires de l’Académie de Berin*.

In his Note VIII, Lagrange says that “the search of bounds of the roots is the first problem which is to be considered in the theory of equations, after that of their general resolution”.

Then he mentions the previous work on numerical solutions of polynomial equations by Hudde, Rolle, Stirling, Euler and De Gua.

The original result of Lagrange is the following:
Theorem 1.3 (Lagrange, 1767) Let $F$ be a nonconstant monic polynomial of degree $n$ over $\mathbb{R}$ and let $\{a_j; j \in J\}$ be the set of its negative coefficients. Then an upper bound for the positive real roots of $F$ is given by the sum of the largest and the second largest numbers in the set
$$\left\{ \sqrt[|a_j|]{a_j}; j \in J \right\}.$$  

Lagrange gives no proof of Theorem 1.3. He only mentions that the verification of this bound is similar with that of the bound $B = 1 + \max\{|a_j|; j \in J\}$ and that it gives values much closer to the true absolute values of the roots that the bound $B$.

Not that Theorem 1.3 of Lagrange gives more explicit bounds than $B$. A similar result for complex polynomials can be obtained:

Theorem 1.4 (Cauchy) Let $F(X) = X^n + a_1 X^{n-1} + \cdots + a_{n-1} X + a_n \in \mathbb{C}[X]$ be nonconstant and let $\sigma$ be the unique positive real root of the polynomial
$$G(X) = X^n - |a_1| X^{n-1} - \cdots - |a_{n-1}| X - |a_n|.$$  

Then any number surpassing $\sigma$ is a bound for the absolute values of the roots of $F$.

Then we have the corresponding result of Lagrange for polynomials with complex coefficients:

Theorem 1.5 ("Lagrange" over $\mathbb{C}$) If $F(X) = X^n + a_1 X^{n-1} + \cdots + a_{n-1} X + a_n \in \mathbb{C}[X] \setminus \mathbb{C}$, an upper bound for the absolute values of the roots of $F$ is given by $R + \rho$, where $R \geq \rho$ are the largest numbers in the set
$$\left\{ |a_k|^{1/k}; 1 \leq k \leq n \right\}.$$  

In particular this gives:

Corollary 1.6 (M. Fujiwara) An upper bound for the absolute values of the roots of $F$ is
$$2 \cdot \max_{a_n} \{|a_n|^{1/n}\}.$$  

The aim of this paper is to prove Theorem 1.5 and to determine whether this result is optimal.

1.3 A proof à la Cauchy

In fact, Lagrange only stated that if
$$-\mu y^{r-m} - \nu y^{r-n} - \mu y^{r-p} - \cdots$$
represents the "negative terms" of a polynomial $F$, then an upper bound for the roots of $F$ is given by the sum of the first two largest of the "quantities"
$$\sqrt[\mu]{{\mu}}, \sqrt[\nu]{{\nu}}, \sqrt[\mu]{\nu}, \ldots$$

or "a number larger than this sum".

This statement implies the following:
Proposition 1.7 (Lagrange) Let $a_1, a_2, \ldots, a_n$ be positive real numbers. If $R = a_{i/j}^{1/j} \geq a_{i}^{1/i} = \rho \geq a_{k}^{1/k}$ for all $k \neq i, j$, then $R + \rho$ is an upper bound for the roots of the polynomial

$$F(X) = X^n - a_1 X^{n-1} - \cdots - a_{n-1} X - a_n.$$ 

Proof:
By hypotheses we have $a_k \leq \rho^k$ for all $k \neq i$. Also,

$$F(x) \geq x^n - R^j x^{n-j} + \rho^j x^{n-j} - \sum_{k=1}^{n} \rho^k x^{n-k} \quad \text{for all } x > 0.$$ 

Then we obtain

$$\frac{F(R+\rho)}{(R+\rho)^n} \geq 1 - \left( \frac{R}{R+\rho} \right)^j + \left( \frac{\rho}{R+\rho} \right)^j - \sum_{k=1}^{n} \left( \frac{\rho}{R+\rho} \right)^k$$

$$= 1 - \left( \frac{R}{R+\rho} \right)^j + \left( \frac{\rho}{R+\rho} \right)^j - \frac{\rho}{R+\rho} \cdot \frac{1 - \left( \frac{\rho}{R+\rho} \right)^n}{1 - \frac{\rho}{R+\rho}}$$

$$= 1 + \left( \frac{\rho}{R+\rho} \right)^j + \frac{\rho}{R \cdot \left( \frac{\rho}{R+\rho} \right)^j - \rho - \left( \frac{R}{R+\rho} \right)^j}$$

$$= \frac{R - \rho}{R} - \frac{R^j - \rho^j}{(R+\rho)^j} + \frac{\rho^{n+1}}{R(R+\rho)^n}$$

$$= \frac{R - \rho}{R(R+\rho)^j} \cdot \left( (R+\rho)^j - (R^j + R^{j-1} \rho + \cdots + R \rho^{j-1}) \right) + \frac{\rho^{n+1}}{R(R+\rho)^n}$$

$$= \frac{R - \rho}{R(R+\rho)^j} \cdot \left( \rho^j + \sum_{s=1}^{j} \left( \begin{pmatrix} j \\ s \end{pmatrix} - 1 \right) R^s \rho^{j-s} \right) + \frac{\rho^{n+1}}{R(R+\rho)^n}$$

$$> 0.$$ 

It follows that $R + \rho$ is an upper bound for the roots of $F$. 

\[ \square \]

Remark: Theorem 1.3 can be proved in a similar way.

1.4 A proof à la Fujiwara

Theorem 1.5 may be obtained by a suitable choice of the $\lambda_k$’s in the method of Fujiwara. It gives more explicit bounds than the estimate of Cauchy. Fujiwara ([3]) obtained in the upper bound

$$\max_{k=1}^{n} \lambda_k^{1/k},$$

where $\lambda_k > 0$ and $\sum_{k=1}^{n} \frac{1}{\lambda_k} \leq 1$. 

4
**Proof of Theorem 1.5:**

We suppose that $R = a_j^{1/j} \geq |a_i|^{1/i} = \rho \geq |a_k|^{1/k}$ for all $k \neq i,j$ and search $\lambda_1, \ldots, \lambda_n > 0$ such that

$$\lambda_k |a_k| \leq (R + \rho)^k \text{ for all } k = 1, 2, \ldots, n.$$  

It is sufficient to have

$$\lambda_k \rho^k \leq (R + \rho)^k \text{ for all } k \neq j$$

and

$$\lambda_j R^j \leq (R + \rho)^j.$$  

Choose

$$\lambda_j = \left(\frac{R + \rho}{R}\right)^j, \quad \lambda_k = \left(\frac{R + \rho}{\rho}\right)^k \text{ for } k \neq j \quad (1 \leq k \leq n).$$  

Then

$$\sum_{k=1}^{n} \frac{1}{\lambda_k} = \sum_{k \neq j} \frac{1}{\lambda_k} + \frac{1}{\lambda_j}$$

$$= \sum_{k=1}^{n} \left(\frac{\rho}{R + \rho}\right)^k + \left(\frac{R}{R + \rho}\right)^j$$

$$= \frac{\rho}{R} \left(1 - \left(\frac{\rho}{R + \rho}\right)^n\right) + \frac{R^j - \rho^j}{(R + \rho)^j} = \frac{\rho}{R} + \frac{R^j - \rho^j}{(R + \rho)^j}.$$  

Let $y = R/\rho$ and observe that $y \geq 1$ and

$$\frac{\rho}{R} + \frac{R^j - \rho^j}{(R + \rho)^j} = \frac{1}{y} + \frac{y^j - 1}{(y + 1)^j} = \frac{y^{j+1} + (y + 1)^j - y}{y(y + 1)^j}.$$  

The right hand side of the previous equality is $\leq 1$ if and only if

$$g(y) = y(y + 1)^j - y^{j+1} - (y + 1)^j + y \geq 1.$$  

In fact, we have

$$\begin{align*}
g(y) &= \binom{j}{1}y^j + \binom{j}{2}y^{j-1} + \binom{j}{3}y^{j-2} + \cdots + \binom{j}{j-2}y^3 + \binom{j}{j-1}y^2 + y \\
&\quad - y^j - \left(\frac{j}{1}\right)y^{j-1} - \left(\frac{j}{2}\right)y^{j-2} - \cdots - \left(\frac{j}{j-3}\right)y^3 - \left(\frac{j}{j-2}\right)y^2 - \left(\frac{j}{j-1}\right)y - 1 \\
&\quad + y \\
&= \left(\binom{j}{1} - 1\right) \cdot (y^j - y) + \left(\binom{j}{2} - \binom{j}{1}\right) \cdot (y^{j-1} - y^2) + \cdots + (y - 1) \\
&\geq 1.
\end{align*}$$
because all parantheses on the last line are positive.

Therefore \[ \sum_{k=1}^{n} \frac{1}{\lambda_k} \leq 1 \], so \( R + \rho \) is an upper bound for the roots of \( F \).

\[ \square \]

2 Remarks on the optimization of the results

Theorem 1.5 of Lagrange is, in some cases, almost optimal. However, it does not give the right answer to the following:

**Problem** Let \( F(X) = X^n + a_1 X^{n-1} + \cdots + a_{n-1} X + a_n \) be a nonconstant complex polynomial such that

\[ |a_i|^{1/i} = |a_j|^{1/j} = R > \rho = |a_s|^{1/s} \geq |a_k| \quad \text{for all} \quad k \neq i, j, s \quad (1 \leq k \leq n). \]

Find an “optimal” upper bound for the roots of \( F \) as a function of \( R \) and \( \rho \).

**Example 1.** Let \( R, \rho \in \mathbb{R} \), with \( R > \rho > 0 \) and

\[
F(X) = X^n - \rho X^{n-1} - \cdots - R^i X^{n-i} - \cdots - \rho^n X - \rho^n,
\]

\[
G(X) = X^n - \rho X^{n-1} - \cdots - R^i X^{n-i} - \cdots - R^j X^{n-j} - \cdots - \rho^n X - \rho^n.
\]

In \( F \) all coefficients involve \( \rho \) with the exception of that of \( X^{n-i} \), in \( G \) there are two coefficients involving \( R \).

For the polynomial \( F \) we consider the bounds of Lagrange, respectively of Fujiwara. They are \( R + \rho \), respectively \( 2R \), so that of Lagrange is sharper.

For \( G \) the limits of Lagrange and Fujiwara give the same bound \( 2R \).

**Example 2.** Let \( F(X) = X^n - X^{n-1} - \cdots - X - 1 \). Since

\[
F(X) = X^n - \frac{X^n - 1}{X - 1} = \frac{X^{n+1} - 2X^n + 1}{X - 1}
\]

we look to the zeros of the polynomial \( Q(X) = X^{n+1} - 2X^n + 1 \) and observe that the absolute value of the dominant root approaches 2 quickly as \( n \) grows.

Indeed, we have \( Q(2) = 1 \). For \( \varepsilon \in (0, 2) \) we have

\[
Q(2 - \varepsilon) = (2 - \varepsilon)^{n+1} - 2(2 - \varepsilon)^n + 1 = -\varepsilon(2 - \varepsilon)^n + 1 = 0 \iff \varepsilon \approx 2^{-n}.
\]

This is equivalent to

\[
\frac{1}{2^n} \cdot \left(2 - \frac{1}{2^n}\right)^n = \left(1 - \frac{1}{2^{n+1}}\right)^n \approx 1,
\]

therefore the upper bound is smaller than 2 and \( \approx 2 - 2^{-n} \). This proves that the bound of Lagrange is almost optimal.
With the notation of Corollary 1.6 of Fujiwara we have $R = 1$, so the upper bound for the dominant root of $F$ is $2R = 2$.

**Example 3.** We change a little the previous polynomial by introducing a $\rho < R$. Let $F(X) = X^n - 0.5 \cdot X^{n-1} - X^{n-2} - 0.5^3 \cdot X^{n-3} - \cdots - 0.5^{n-1} \cdot X - 0.5^n$.

Then $\rho = 0.5$, $R = 1$, so the upper bound of Lagrange is $1 + \sqrt{0.5} \approx 1.7071067$. The following table gives the true upper bound for various values of $n$:

<table>
<thead>
<tr>
<th>n</th>
<th>upper bound</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>1.3554670</td>
</tr>
<tr>
<td>4</td>
<td>1.3410818</td>
</tr>
<tr>
<td>5</td>
<td>1.3466254</td>
</tr>
<tr>
<td>11</td>
<td>1.3498059</td>
</tr>
<tr>
<td>51</td>
<td>1.3498141</td>
</tr>
<tr>
<td>101</td>
<td>1.3498141</td>
</tr>
</tbody>
</table>

Note that $R + \rho = 1.5$ is larger than the true upper bound for all $n \geq 3$. Even though not optimal, it gives better bounds than the bound of Fujiwara which is $2 \cdot R = 2 \cdot 1 = 2$.

**Example 4.** If we consider

$$F(X) = X^n - 3X^{n-1} - 4^2 X^{n-2} - 3^3 X^{n-3} - \cdots - 3^{n-1} X - 3^n,$$

we have $R = 4$, $\rho = 3$, so $R + \rho = 7$. For $n \geq 8$ the true upper bound lies between 6.573249 and 6.5789.

**Remark:** The optimal bound given by Theorem 1.5 can be obtained for polynomials of the form

$$F(X) = X^n - R^i X^{n-i} - \sum_{1 \leq k \leq n} \rho^k X^{n-k} \quad \text{where} \quad 0 < \rho < R,$$

for a suitable exponent $i = i(n)$, $i \in \{2, 3, \ldots, n\}$.

We consider the case $i = 2$ and associate to $F$ the polynomials

$$F_d(X) = X^d - \rho X^{d-1} - R^2 X^{d-2} - \cdots - \rho^{d-1} X - \rho^d \quad \text{for} \quad d = 2, 3, \ldots, n.$$

Let $\xi_d := \xi$ be the unique positive root of $F_d$. Then

$$F(\xi) = \xi^{n-d} \cdot F_d(\xi) - \sum_{k=d+1}^{n} \rho^k \xi^{n-k} = - \sum_{k=d+1}^{n} \rho^k \xi^{n-k} > 0.$$

By Theorem 1.4 of Cauchy it follows that $\xi_d$ is a lower bound for the absolute values of the dominant roots of $F$. Therefore the dominant roots of $F$ lie in the strip $\xi_d \leq |z| \leq R + \rho$. 

7
Also, $F_{d+1}(\xi_d) = \xi_d \cdot F_d(\xi_d) - \rho^{d+1} = -\rho^{d+1} < 0$, therefore $\xi_d < \xi_{d+1}$, so

$$0 < \xi_1 < \xi_2 < \cdots < \xi_n.$$ 

Note that $\xi_n$ is the positive root of the polynomial $F_n = F$.

**Lemma 2.1** Let $\xi_n$ be the positive root of the polynomial

$$F_n(X) = X^n - R^i X^{n-i} - \sum_{\frac{n-i}{i} \neq i} \rho^k X^{n-k},$$

with the position $i$ "fixed". Then the sequence $(\xi_n)_{n \geq 2}$ is convergent.

**Proof:**

We have $F_{n+1} = X F_n(X) - \rho^{n+1}$, so by the previous argument $F_{n+1}(\xi_n) = -\rho^{n+1} < 0$ and the sequence $(\xi_n)_n$ is increasing. Also, by the result of Fujiwara (our Corollary 1.6) we know that $2R$ is an upper bound, so the sequence converges.

Note that $(\xi_n)$ converges quickly.

**Example 5.** Suppose $R = 1$, $\rho = 0.75$ and $i = 3$. The following table gives values of $\xi_n$:

<table>
<thead>
<tr>
<th>$n$</th>
<th>$\xi_n$</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>1.538273216333878241921</td>
</tr>
<tr>
<td>4</td>
<td>1.583617827600907072964</td>
</tr>
<tr>
<td>5</td>
<td>1.6029068699501609492581</td>
</tr>
<tr>
<td>6</td>
<td>1.61139489469626850932</td>
</tr>
<tr>
<td>7</td>
<td>1.615210962767615892929</td>
</tr>
<tr>
<td>8</td>
<td>1.616949830499490340520</td>
</tr>
<tr>
<td>9</td>
<td>1.617748250082844027496</td>
</tr>
<tr>
<td>10</td>
<td>1.61816421740096767111</td>
</tr>
<tr>
<td>11</td>
<td>1.618286591736488890089</td>
</tr>
<tr>
<td>51</td>
<td>1.618433292610718049459</td>
</tr>
<tr>
<td>64</td>
<td>1.618433292610718055836</td>
</tr>
<tr>
<td>65</td>
<td>1.618433292610718055837</td>
</tr>
<tr>
<td>151</td>
<td>1.618433292610718055837</td>
</tr>
<tr>
<td>191</td>
<td>1.618433292610718055837</td>
</tr>
</tbody>
</table>

**Remark:** There exist polynomials as in our Problem such that the true upper bound for the absolute values of the roots is between $R + \rho$ and $2R$. For example:

$$F_1(X) = X^{18} - 3.1 \ X^{17} - (3.1)^8 \ X^{10} - \sum_{\frac{n-i}{i} \neq i} 3^k X^{18-k}$$

$$F_2(X) = X^{25} - 7.1 \ X^{24} - (7.1)^2 \ X^{23} - \sum_{\frac{n-i}{i} \neq i} 3^k X^{25-k}$$
In fact we have

<table>
<thead>
<tr>
<th></th>
<th></th>
<th>true u. bound</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>$F$</td>
<td>$R + \rho$</td>
<td>$2R$</td>
<td></td>
</tr>
<tr>
<td>$F_1$</td>
<td>6.1</td>
<td>6.10164</td>
<td>6.2</td>
</tr>
<tr>
<td>$F_2$</td>
<td>10.1</td>
<td>11.68157</td>
<td>14.2</td>
</tr>
</tbody>
</table>

For such polynomials none of the bounds of Lagrange and Fujiwara is optimal. In the next section we obtain a convenient bound using a result of Kojima [5].

3 Theorem of Kojima

3.1 The result of Kojima

We state the result of Kojima ([5], 1917) for monic polynomials. His original proof is based on Hadamard’s theorem on invertible matrices. We give here another proof, based on Theorems 1.2 of Descartes and 1.4 of Cauchy.

We first establish a result with weaker hypotheses than that of Kojima:

**Theorem 3.1** Let $F(X) = X^n + a_1 X^{n-1} + \cdots + a_{n-1} X + a_n$ be a nonconstant polynomial with complex coefficients. If $\lambda_1, \lambda_2, \ldots, \lambda_{n-1}$ are arbitrary nonnegative numbers, $b_1 = |a_1|$ and $b_2, b_3, \ldots, b_n$ are arbitrary nonnegative numbers, then $B_1 > 0$ and $B_2 > 0$ are upper bounds for the absolute values of the roots of $F$, provided the following conditions are fulfilled:

1. 
   \[ |a_1| + \lambda_1 \leq B_1 \]
   \[ |a_k| + \lambda_1 \cdots \lambda_k \leq \lambda_1 \cdots \lambda_{k-1} B_1 \quad \text{for} \quad k = 2, 3, \ldots, n-1, \]
   \[ |a_n| \leq \lambda_1 \cdots \lambda_{n-1} B_1. \]

2. 
   \[ |a_1| + \sum_{i=2}^{n} b_i \leq B_2, \]
   \[ |a_k| \leq b_k \cdot B_2^{k-1} \quad \text{for} \quad k = 2, 3, \ldots, n. \]

**Proof:**

[À la Cauchy]

We follow the classical argument of Cauchy (1829) given by Theorem 1.4.

Denote by $\xi$ the unique positive root of the polynomial

\[ G(X) = X^n - |a_1|X^{n-1} - \cdots - |a_{n-1}|X - |a_n|. \]
i. By the definition of $B_1$ we have

$$ |a_1| \leq B_1 - \lambda_1,$$

$$ |a_k| \leq \lambda_1 \cdots \lambda_{k-1} B_1 - \lambda_1 \cdots \lambda_{k-1} \lambda_k \quad (k = 2, \ldots, n - 1),$$

$$ |a_n| \leq \lambda_1 \cdots \lambda_{n-1} B_1.$$ 

Then

$$ 0 = -G(\xi) = -\xi^n + |a_1|\xi^{n-1} + \cdots + |a_{n-1}|\xi + |a_n| $$

$$ \leq -\xi^n + (B_1 - \lambda_1)\xi^{n-1} + (\lambda_1 B_1 - \lambda_1 \lambda_2)\xi^{n-2} + \cdots + (\lambda_1 \cdots \lambda_{n-2} B_1 - \lambda_1 \cdots \lambda_{n-1})\xi $$

$$ + \lambda_1 \cdots \lambda_{n-1} B_1 $$

$$ = (B_1 \xi^{n-1} - \xi^n) + (B_1 \lambda_1 \xi^{n-2} - \lambda_1 \xi^{n-1}) + \cdots + (\lambda_1 \cdots \lambda_{n-2} B_1 \xi - \lambda_1 \cdots \lambda_{n-2} \xi^n) $$

$$ + (\lambda_1 \cdots \lambda_{n-1} B_1 - \lambda_1 \cdots \lambda_{n-1})$$

$$ = (B_1 - \xi) \cdot (\xi^{n-1} + \lambda_1 \xi^{n-2} + \cdots + \lambda_1 \cdots \lambda_{n-2} \xi + \lambda_1 \cdots \lambda_{n-1}). $$

Since the last parenthesis in the right hand side is positive, it follows that $B_1 \geq \xi$. By Theorem 1.4 of Cauchy, $B_1$ is an upper bound for the absolute values of the roots of $F$.

ii. Since

$$ |a_1| \leq B_2 - \sum_{i=2}^{n} b_i,$$

$$ |a_2| \leq B_2 \cdot b_2,$$

$$ |a_3| \leq B_2^2 \cdot b_3$$

$$ \vdots $$

$$ |a_n| \leq B_2^{n-1} b_n.$$
we have

\[
0 = -G(\xi) = -\xi^n + |a_1|\xi^{n-1} + \cdots + |a_{n-1}|\xi + |a_n| \\
\leq -\xi^n + (B_2 - b_2 - b_3 - \cdots - b_n)\xi^{n-1} + B_2 \cdot b_2 \cdot \xi^{n-2} \\
+ B_2^2 \cdot b_3 \cdot \xi^{n-3} + \cdots + B_2^{n-2} \cdot b_{n-1} \cdot \xi + B_2^{n-1} \cdot b_n \\
= (B_2 \xi^{n-1} - \xi^n) + (B_2 b_2 \xi^{n-2} - b_2 \xi^{n-1}) + (B_2^2 b_3 \xi^{n-3} - b_3 \xi^{n-2}) \\
+ \cdots + (B_2^{n-2} b_{n-1} \xi - b_{n-1} \xi^{n-1}) + (B_2^{n-1} b_n - b_n \xi^{n-1}) \\
= (B_2 - \xi)\xi^{n-1} + (B_2 - \xi) b_2 \xi^{n-2} + (B_2^2 - \xi \xi) b_3 \xi^{n-3} \\
+ \cdots + (B_2^{n-2} - \xi^{n-2}) b_{n-1} \xi + (B_2^{n-1} - \xi^{n-1}) b_n \\
= (B_2 - \xi) \cdot U(B_2, \xi),
\]

where

\[
U(B_2, \xi) = \xi^{n-1} + b_2 \xi^{n-2} + (B_2 + \xi) b_3 \xi^{n-3} \\
+ \cdots + (B_2^{n-3} + B_2^{n-4} \xi + \cdots + \xi^{n-3}) b_{n-1} \xi \\
+ (B_2^{n-2} + B_2^{n-3} \xi + \cdots + B_2 \xi^{n-3} + \xi^{n-2}) b_n \\
> 0.
\]

It follows that \( B_2 - \xi \geq 0 \), so \( B_2 \) is also an upper bound for the dominant root of the polynomial \( F \).

\[ \square \]

**Corollary 3.2 (T. Kojima)** Let \( F(X) = X^n + a_1 X^{n-1} + \cdots + a_{n-1} X + a_n \) be a nonconstant polynomial with complex coefficients. If \( \lambda_1, \lambda_2, \ldots, \lambda_n \) are arbitrary positive numbers, \( b_1 = |a_1| \) and \( b_2, b_3, \ldots, b_n \) are arbitrary real numbers, we have the following bounds for the absolute values of the roots of \( F \):

\[
i. \quad M_1 = \max \left\{ \lambda_1 + |a_1|, \lambda_k + \frac{|a_k|}{\lambda_1 \lambda_2 \cdots \lambda_{k-1}} (k = 2, \ldots, n-1), \frac{|a_n|}{\lambda_1 \lambda_2 \cdots \lambda_{n-1}} \right\}, \\
ii. \quad M_2 = \max \left\{ |a_1| + \frac{|a_2|}{b_2}, \left| \frac{a_{k+1} b_k}{a_k b_{k+1}} \right| (k = 2, \ldots, n-1), |b_n| \right\}, \\
iii. \quad M_3 = \max \left\{ |a_1| + \sum_{i=2}^{n} b_i \left| \frac{a_2}{b_2} \right| \left| \frac{a_{k+1} b_k}{a_k b_{k+1}} \right| (k = 2, \ldots, n-1) \right\}.
\]

**Proof:**

i. Let \( M_1 = B_1 \) and observe that it verifies conditions i. from Theorem 3.1.
ii. In i., let \( \lambda_k = \left| \frac{a_k b_{k+1}}{a_k b_{k+1}} \right| \) for \( k = 1, 2, \ldots, n - 1 \). Then

\[
\lambda_1 = \left| \frac{a_2}{b_2} \right| \quad \text{and} \quad \lambda_1 \cdots \lambda_{k-1} = \left| \frac{a_k}{b_k} \right| \quad \text{for} \quad k = 1, 2, \ldots n - 1,
\]

which gives the bound \( M_2 \).

iii. We have

\[
|a_1| \leq M_3 - \sum_{i=2}^{n} |b_i|,
\]

\[
|a_2| \leq M_3 \cdot |b_2|,
\]

\[
|a_3| \leq M_3 \cdot \left| \frac{a_2 b_3}{b_2} \right| \leq M_3^2 \cdot |b_3|
\]

\[
\vdots
\]

\[
|a_n| \leq M_3 \cdot \left| \frac{a_{n-1} b_n}{b_{n-1}} \right| \leq \cdots \leq M_3^{n-1} |b_n|.
\]

It follows that \( M_3 \) satisfies conditions ii. in Theorem 3.1, therefore it is an upper bound for the absolute values of the roots.

\[ \square \]

3.2 Applications of the bounds of Kojima

**Proposition 3.3** Let \( F(X) = X^n + a_1 X^{n-1} + \cdots + a_{n-1} X + a_n \in \mathbb{C}[X] \), of degree \( n \geq 3 \). Let \( R = \max_{k=1}^{n} |a_k|^{1/k} \) and suppose that \( R \) is realized for exactly two indices \( 1 < i < j \leq n \).

If \( \rho = \max \{|a_k|^{1/k} ; k \neq i, j\} \) then

\[
\rho + \max \left\{ R + \varepsilon, \frac{R^2}{R + \varepsilon} \left( \frac{R}{\rho} \right)^{i-2} \right\}
\]

is an upper bound for the absolute values of the roots of \( F \) for all \( \varepsilon \) such that \( 0 < \varepsilon < R - \rho \).

If \( j = n \) such a bound is

\[
\max \left\{ \rho + R + \varepsilon, \rho + \frac{R^2}{R + \varepsilon} \left( \frac{R}{\rho} \right)^{i-2}, \frac{R^2}{R + \varepsilon} \left( \frac{R}{\rho} \right)^{n-2} \right\}.
\]

**Proof:**

We use Corollary 3.2 i. Let

\[
\lambda_1 = R + \varepsilon \quad \text{and} \quad \lambda_2 = \cdots = \lambda_{n-1} = \rho.
\]

Then \( \lambda_1 + |a_1| \leq R + \varepsilon + |a_1| \leq R + \rho + \varepsilon \).

Assume first \( j < n \). For \( k \neq 1, i, j, n \) we obtain

\[
\lambda_k + \frac{|a_k|}{\lambda_1 \cdots \lambda_{k-1}} \leq \rho + \frac{\rho^k}{(R + \varepsilon) \rho^{i-2}} = \rho + \frac{\rho^2}{R + \varepsilon} < 2\rho.
\]
For $k = n$, the corresponding bound is
\[
\frac{|a_n|}{\lambda_1 \cdots \lambda_{n-1}} \leq \frac{\rho^n}{(R + \varepsilon)\rho^{n-2}} = \frac{\rho^2}{R + \varepsilon} < \rho. 
\]
For $k = i$ or $j$, we have
\[
\lambda_k + \frac{|a_k|}{\lambda_1 \cdots \lambda_{k-1}} \leq \rho + \frac{R^k}{(R + \varepsilon)\rho^{k-2}} = \rho + \frac{R^2}{R + \varepsilon} \cdot \left(\frac{R}{\rho}\right)^{k-2},
\]
hence the first bound.
To compute the maximum we need also to examine the case $j = n$. We have
\[
\frac{|a_n|}{\lambda_1 \cdots \lambda_{n-1}} = \frac{R^n}{(R + \varepsilon)\rho^{n-2}} = \frac{R^2}{R + \varepsilon} \cdot \left(\frac{R}{\rho}\right)^{n-2}.
\]
The second bound now follows.

\[\square\]

**Remark:** Let $B$ be one of the upper bounds in Proposition 3.3. Since $R + \rho + \varepsilon \leq B$, we have $R + \rho < B$.

We compare now the previous bound with those of Lagrange and Fujiwara. Note that for the polynomials $F$ in Proposition 3.3 the bounds of Lagrange and Fujiwara are both equal to $2R$. In the two cases ($j < n$ and $j = n$) we would like to have $B < 2R$. For this purpose we establish first several inequalities for some special polynomials.

**Lemma 3.4** Let $k > 1$ and
\[
g_1(X) = kX^3 - 2(k + 1)X^2 + (k + 3)X - 1.
\]
There exists $\eta > 1$ such that $g_1(\eta) < 0$.

**Proof:**
Observe that $g_1$ has only positive real roots:
\[
\frac{k + 2 - \sqrt{k^2 + 4}}{2k}, \quad \frac{k + 2 + \sqrt{k^2 + 4}}{2k}.
\]
Thus $g_1(\eta) < 0$ for all $\eta \in \left(1, \frac{k + 2 + \sqrt{k^2 + 4}}{2k}\right)$.

\[\square\]

**Corollary 3.5** Let $k > 1$ and $\rho > 0$. There exists $R = R(\rho, k) > \rho$, such that
\[
\rho + \frac{R^2}{R + \varepsilon} \left(\frac{R}{\rho}\right)^{i-2} < 2R \quad \text{for all} \quad i \in \left(\frac{R - \rho}{k}, R - \rho\right) \quad \text{and } \quad i \in \{2, 3\}.
\]
Proof:
Since
\[ \frac{R^2}{R + \varepsilon} < \frac{R^2}{R + \frac{R - \rho}{k}} \quad \text{for all} \quad \varepsilon \in \left( \frac{R - \rho}{k}, R - \rho \right) \]
it is sufficient to consider \( \varepsilon = (R - \rho)/k \).

By Lemma 3.4 there exists \( \eta > 1 \) such that
\[ k\eta^3 - 2(k + 1)\eta^2 + (k + 3)\eta - 1 < 0. \]

Choose \( R = \eta \rho \). Then we have (after some computation)
\[ (k + 1)R - \rho + kR \cdot \left( \frac{R}{\rho} \right)^2 < 2 \left( (k + 1)(R - \rho) \right) \frac{R}{\rho} \]
that is
\[ 1 + \frac{kR}{(k + 1)R - \rho} \cdot \left( \frac{R}{\rho} \right)^2 < 2 \frac{R}{\rho}, \]
which gives
\[ \rho + \frac{R^2}{R + \frac{R - \rho}{k}} \cdot \frac{R}{\rho} < 2 R \]
and this proves the result for \( i = 3 \).

If \( i = 2 \), consider \( \theta > 1 \) such that
\[ g_2(\theta) = (k + 2)\theta^2 - (k + 3)\theta + 1 > 0. \]

Take \( R = \theta \rho \) and obtain
\[ (k + 1) \frac{R}{\rho} - 1 + k \left( \frac{R}{\rho} \right)^2 < 2 \frac{R}{\rho} \left( (k + 1) \frac{R}{\rho} - 1 \right), \]
that is
\[ 1 + \frac{kR}{(k + 1)R - \rho} \cdot \frac{R}{\rho} < 2 \frac{R}{\rho}, \]
which gives
\[ \rho + \frac{R^2}{R + \frac{R - \rho}{k}} < 2 R. \]

\[ \square \]

Lemma 3.6 Let \( n \in \mathbb{N}, \ n \geq 3, \ k > 1 \) and \( \rho > 0 \). There exists \( R = R(n, k, \rho) > \rho \) such that
\[ \frac{R}{R + \varepsilon} \left( \frac{R}{\rho} \right)^{n-2} < 2 \quad \text{for all} \quad \varepsilon \in \left( \frac{R - \rho}{k}, R - \rho \right). \]
Proof:
As in the proof of Lemma 3.5 it is sufficient to consider \( \varepsilon = (R - \rho)/k \).
Let \( g_3(X) = kX^{n-1} - 2(k + 1)X + 2 \). Since \( g_3(1) = -k < 0 \) there exists \( \eta > 1 \) such that
\( g_3(\eta) < 0 \). Taking \( R = \eta \rho \) we obtain
\[
k \left( \frac{R}{\rho} \right)^{n-1} < 2 \left( \frac{(k + 1)R}{\rho} - 1 \right),
\]
hence
\[
\frac{k}{(k + 1)R - \rho} \cdot \left( \frac{R}{\rho} \right)^{n-1} < \frac{2}{\rho},
\]
which gives
\[
\frac{R}{R + \frac{R}{k}} \cdot \left( \frac{R}{\rho} \right)^{n-2} < 2.
\]
\[\square\]

4 Conclusions

4.1 Preliminary remarks

Notation: Let
\[
B_1(\rho, R) = R + \rho + \varepsilon,
\]
\[
B_2(\rho, R, i) = \rho + \frac{R^2}{R + \varepsilon} \cdot \left( \frac{R}{\rho} \right)^{i-2},
\]
\[
B_3(R, n) = \frac{R^2}{R + \varepsilon} \cdot \left( \frac{R}{\rho} \right)^{n-2}.
\]

Remark: Observe that if \( i = 2 \), then
\[
B_2 = \rho + \frac{R^2}{R + \varepsilon} < \rho + R < B_1 < 2R.
\]
In this case we have to compare only \( B_3 \) with \( 2R \).
For \( i = 3 \) we need a \( k > 1 \) such that \( g_1(\eta) < 1 \), with \( \eta = R/\rho \). This gives
\[
1 < k < \frac{2\eta^2 - 3\eta + 1}{\eta^2 - 2\eta + 1},
\]
therefore \( \eta^3 - 4\eta^2 + 4\eta - 1 < 0 \).
Since the equation \( \eta^3 - 4\eta^2 + 4\eta - 1 = 0 \) has the roots
\[
\frac{3 - \sqrt{5}}{2}, 1, \frac{3 + \sqrt{5}}{2},
\]
we find \( \eta \in \left( 1, \frac{3 + \sqrt{5}}{2} \right) \).
4.2 Examples

Let

\[ F(X) = X^n - R^i X^{n-i} - X^j X^{n-j} - \sum_{k \neq i,j} \rho^k X^{n-k}, \]

with \( 0 < \rho < R \) and \( 2 \leq i < j \leq n \).

We consider \( \varepsilon(k) = \frac{k+1}{2k} (R - \rho) \in \left( \frac{R - \rho}{k}, R - \rho \right) \) and \( B = \max\{B_1, B_2, B_3\} \), where

\[
B_1 = R + \rho + \frac{k+1}{2k} (R - \rho),
\]

\[
B_2 = \rho + \frac{2kR^2}{(3k+1)R - (k+1)\rho} \left( \frac{R}{\rho} \right)^{k-2},
\]

\[
B_3 = \frac{2kR^2}{(3k+1)R - (k+1)\rho} \left( \frac{R}{\rho} \right)^{k-2}.
\]

We obtained:

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In the column for $B$ we indicated the cases when the bound is not realized through $B_1$. By the previous remark for $i = 2$, the bound $B_2$ is never reached.

The computations were done using the package pari.

References


