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Vincent Blanloeil, Osamu Saeki. A theory of concordance for non-spherical 3-knots. 2001. hal-00129670

HAL Id: hal-00129670
https://hal.science/hal-00129670

Preprint submitted on 8 Feb 2007

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A THEORY OF CONCORDANCE FOR NON-SPHERICAL 3-KNOTS

VINCENT BLANLŒIL AND OSAMU SAEKI

ABSTRACT. Consider a closed connected oriented 3-manifold embedded in the 5-sphere, which is called a 3-*knot* in this paper. For two such knots, we say that their Seifert forms are *spin concordant*, if they are algebraically concordant with respect to a diffeomorphism between the 3-manifolds which preserves their spin structures. Then we show that two simple fibered 3-knots are geometrically concordant if and only if they have spin concordant Seifert forms, provided that they have torsion free first homology groups. Some related results are also obtained.

1. INTRODUCTION

In this paper, a closed $(n-1)$-connected oriented $(2n-1)$-dimensional manifold embedded in the $(2n+1)$-sphere S^{2n+1} will be called a $(2n-1)$-*knot*, or simply a *knot*. A $(2n-1)$-knot is *spherical*, if the embedded manifold is abstractly homeomorphic to S^{2n-1}. Throughout the paper, we shall work in the smooth category. All the homology and cohomology groups are understood to be with integer coefficients unless otherwise specified.

In the 1960's Kervaire [9] and Levine [11] gave a classification of concordance classes of spherical $(2n-1)$-knots with $n \geq 2$. They have proved that there is a bijective correspondence between the concordance classes of such spherical knots and the Witt-equivalence classes of certain Seifert forms.

More recently, for simple fibered $(2n-1)$-knots with $n \geq 3$, which are not necessarily spherical, a complete characterization of concordance has been given by the first author and Michel in [1]. This has been done by using Seifert forms as in the case of spherical knots: more precisely, two simple fibered $(2n-1)$-knots are concordant if and only if their Seifert forms with respect to the $(n-1)$-connected fibers are algebraically concordant (for details, see §2).

The case of 3-knots is much more difficult to study, since the dimension of the Seifert manifold is equal to four and the standard higher dimensional techniques, such as Whitney's trick, are not available. In [17], the second author has developed a theory of concordance for 3-knots which are homology 3-spheres. In fact, such a case is very similar to the case of spherical 3-knots, which had been done by Levine [11].

In this paper, we shall try to develop a theory of concordance for 3-knots which are not necessarily homology 3-spheres.

We shall first observe that each 3-knot carries a canonical spin structure induced from a Seifert manifold. For two 3-knots, we say that their Seifert forms are *spin concordant*, if they are algebraically concordant with respect to a diffemorphism

2000 *Mathematics Subject Classification.* Primary 57Q45; Secondary 57R40.

Key words: Concordance, 3-knot, Seifert form, algebraic concordance, spin structure, fibered knot.

The second author has been supported in part by Grant-in-Aid for Scientific Research (No. 11440022), Ministry of Education, Science and Culture, Japan, and was supported in part by Louis Pasteur University, France, during his stay there in September 2000.

which preserves their spin structures. Then we shall prove that two simple fibered 3-knots with torsion free first homology groups are concordant if and only if they have spin concordant Seifert forms (see Theorem 2.15).

The paper is organized as follows. In §2, we shall give more precise definitions to the notions introduced above, and state our main theorem.

In §3, we shall show that the spin concordance is a necessary condition for concordance. This will follow from the results already known in other dimensions together with a slight modification involving the spin structures. As corollaries, we give some interesting results which relate the concordance of 3-knots to their (image) regular homotopy classes, and to the isotopy classes of their punctured 3-knots.

In §4, we shall show that the spin concordance is a sufficient condition for concordance. In fact, we shall prove this fact for simple 3-knots in general which may not necessarily be fibered. The main techniques used in the proof are mainly from 4-dimensional topology. In order to perform appropriate embedded surgeries as in the higher dimensional case, we shall have to find a set of embedded 2-spheres in a certain 4-manifold. The existence of such embedded spheres follows from Whitney's trick in higher dimensions, while in our dimension, the same argument fails. Thus we shall use a different method: we shall first construct embedded 2-spheres and then find a diffeomorphism which sends the embedded 2-spheres to others that represent correct homology classes.

In §5, we shall extend the class of simple 3-knots for which our characterization of concordance continues to be valid. We also show that a similar extention is valid in higher dimensions as well.

In §6, we shall restrict ourselves to embeddings of specific 3-manifolds. For example, we shall show that two embeddings of the connected sum of some copies of $S^1 \times S^2$ into S^5 as simple fibered knots are concordant if and only if their Seifert forms are algebraically concordant; i.e., in this case, the spin strucutures are not necessary.

Finally in §7, we shall give some examples of simple 3-knots which have interesting properties.

The second author would like to thank the people at the Louis Pasteur University of Strasbourg, France, for their hospitality during his visit there.

2. Preliminaries

In this section, we shall give necessary definitions and then state our main theorem of this paper.

Definition 2.1. We say that two $(2n-1)$-knots K_0 and K_1, abstractly diffeomorphic to the same $(2n-1)$-dimensional manifold $\mathcal{K}$, are *concordant*, if there exists an embedding $\Phi : \mathcal{K} \times [0,1] \to S^{2n+1} \times [0,1]$ such that

$$\begin{aligned} \Phi(\mathcal{K} \times \{0\}) &= K_0 \subset S^{2n+1} \times \{0\}, \text{ and} \\ \Phi(\mathcal{K} \times \{1\}) &= -K_1 \subset S^{2n+1} \times \{1\}, \end{aligned}$$

where $-K_1$ is the mirror image of the knot K_1 with the orientation reversed (see Fig. 1).

It is well-known that, for every $(2n-1)$-knot K, there exists a compact oriented $2n$-dimensional submanifold F of S^{2n+1} having K as the boundary. Such a manifold F is called a *Seifert manifold* for K.

Definition 2.2. Suppose that F is a compact oriented $2n$-dimensional submanifold of S^{2n+1}, and let G be the quotient of $H_n(F)$ by its $\mathbf{Z}$-torsion. The *Seifert form* associated with F is the bilinear form $A : G \times G \to \mathbf{Z}$ defined as follows. For

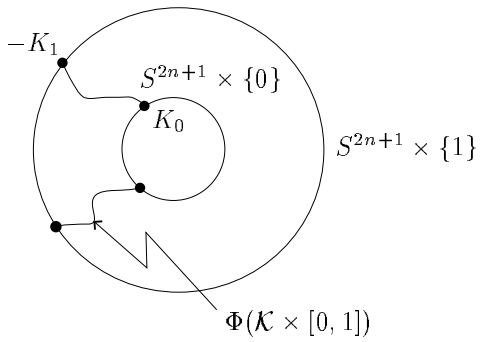


FIGURE 1. Concordance between K_0 and K_1

$(x, y) \in G \times G$, we define $A(x, y)$ to be the linking number in S^{2n+1} of ξ and η_+, where ξ and η are n-cycles in F representing x and y respectively, and η_+ is the n-cycle η pushed off F into the positive normal direction to F in S^{2n+1}.

By definition a *Seifert form* for a $(2n-1)$-knot K is the Seifert form associated with a Seifert manifold for K.

Definition 2.3. A $(2n-1)$-knot K is *fibered*, if there exists a fibration $\phi : S^{2n+1} \setminus K \to S^1$ with ϕ being trivial on $U \setminus K$, where U is a small open tubular neighborhood of K, such that the closure of each fiber is a Seifert manifold for K. We shall often call the closure of each fiber simply a *fiber*. The Seifert form defined on a fiber is always unimodular (for example, see [6]). In the following, for a fibered $(2n-1)$-knot, we use the Seifert form defined on a fiber unless otherwise specified.

Definition 2.4. A $(2n-1)$-knot is *simple*, if it admits an $(n-1)$-connected Seifert manifold. Furthermore, a fibered $(2n-1)$-knot is *simple*, if its fiber is $(n-1)$-connected.

Recall that for higher dimensional simple fibered knots we have a complete characterization of concordance using their Seifert forms as follows (see [1]).

Theorem 2.5. *For $n \geq 3$, two simple fibered $(2n-1)$-knots are concordant if and only if their Seifert forms with respect to the $(n-1)$-connected fibers are algebraically concordant.*

Here, the algebraic concordance is defined as follows.

Definition 2.6. Let $\mathcal{A}$ be the set of all bilinear forms defined on free $\mathbf{Z}$-modules G of finite rank. Set $\varepsilon = (-1)^n$. For $A \in \mathcal{A}$, let us denote by A^T the transpose of A, by S the ε-symmetric form $A + \varepsilon A^T$ associated with A, by $S^* : G \to G^*$ the adjoint of S with G^* being the dual $\mathrm{Hom}_{\mathbf{Z}}(G, \mathbf{Z})$ of G, and by $\overline{S} : \overline{G} \times \overline{G} \to \mathbf{Z}$ the ε-symmetric nondegenerate form induced by S on $\overline{G} = G/\mathrm{Ker}\, S^*$. A submodule M of G is said to be *pure*, if G/M is torsion free, or equivalently if M is a direct summand of G. For a submodule M of G, let us denote by $M^\wedge$ the smallest pure submodule of G which contains M. We denote by $\overline{M}$ the image of M in $\overline{G}$ by the natural projection map.

Definition 2.7. Let $A : G \times G \to \mathbf{Z}$ be a bilinear form in $\mathcal{A}$. The form A is *Witt associated* to 0, if the rank m of G is even and there exists a pure submodule M of rank $m/2$ in G such that A vanishes on M. Such a submodule M is called a *metabolizer* for A.

Definition 2.8. Let $A_i : G_i \times G_i \to \mathbf{Z}$, $i = 0, 1$, be two bilinear forms in $\mathcal{A}$. Set $G = G_0 \oplus G_1$ and $A = A_0 \oplus -A_1$. The form A_0 is said to be *algebraically concordant*

to A_1, if there exist a metabolizer M for A such that $\overline{M}$ is pure in $\overline{G}$, an isomorphism $\psi : \operatorname{Ker} S_0^* \to \operatorname{Ker} S_1^*$, and an isomorphism $\theta : \operatorname{Tors}(\operatorname{Coker} S_0^*) \to \operatorname{Tors}(\operatorname{Coker} S_1^*)$ which satisfy the following two conditions:

(c1) $$M \cap \operatorname{Ker} S^* = \{(x, \psi(x)); x \in \operatorname{Ker} S_0^*\} \subset \operatorname{Ker} S_0^* \oplus \operatorname{Ker} S_1^* = \operatorname{Ker} S^*,$$

(c2) $$d(S^*(M)^\wedge) = \{(y, \theta(y)); y \in \operatorname{Tors}(\operatorname{Coker} S_0^*)\}$$
$$\subset \operatorname{Tors}(\operatorname{Coker} S_0^*) \oplus \operatorname{Tors}(\operatorname{Coker} S_1^*) = \operatorname{Tors}(\operatorname{Coker} S^*),$$

where d is the quotient map $G^* \to \operatorname{Coker} S^*$ and "Tors" means the torsion subgroup. In the above situation, we also say that A_0 and A_1 are *algebraically concordant with respect to ψ and θ*. Note that the algebraic concordance is an equivalence relation as has been shown in [1, Theorem 1].

Remark 2.9. In the above definition, if A_i are Seifert forms associated with $(n-1)$-connected Seifert manifolds F_i of *simple* $(2n-1)$-knots K_i, $i = 0, 1$, then $\operatorname{Ker} S_i^*$ and $\operatorname{Coker} S_i^*$ are naturally identified with $H_n(K_i)$ and $H_{n-1}(K_i)$ respectively, since we have the exact sequence

$$0 = H_{n+1}(F_i, K_i) \to H_n(K_i) \to H_n(F_i) \xrightarrow{S_i^*} H_n(F_i, K_i)$$
$$\to H_{n-1}(K_i) \to H_{n-1}(F_i) = 0,$$

where we identify $H_n(F_i, K_i)$ with the dual of $H_n(F_i)$ by Poincaré-Lefschetz duality.

We know that algebraic concordance is necessary for concordance of simple fibered knots in any dimension (see [1]). In the case of 3-knots, we have to define a new equivalence relation for Seifert forms in order to characterize concordance classes of simple fibered 3-knots.

In the following, a *spin structure* on a manifold X means the homotopy class of a trivialization of its tangent bundle over the 2-skeleton $X^{(2)}$. Note that X admits a spin structure if and only if its second Stiefel-Whitney class $w_2(X)$ vanishes and that if it admits, then the set of all spin structures on X is in one-to-one correspondence with $H^1(X; \mathbf{Z}_2)$.

Let K be a 3-knot with Seifert manifold F. Recall that everything is oriented. Then K has a natural normal 2-framing $\nu = (\nu_1, \nu_1)$ in S^5 such that the first normal vector field ν_1 is obtained as the inward normal vector field of $K = \partial F$ in F. It is easy to show that the homotopy class of this 2-framing does not depend on the choice of the Seifert manifold F. Then K carries a tangent 3-framing on its 2-skeleton $K^{(2)}$ such that the juxtaposition with the above 2-framing gives the standard framing of S^5 restricted to $K^{(2)}$ up to homotopy. This means that K carries a natural spin structure, which is determined uniquely up to homotopy. Furthermore, this spin structure coincides with that induced from the Seifert manifold F, which is endowed with the natural spin structure induced from S^5.

Definition 2.10. A 3-knot K is said to be *free*, if $H_1(K)$ is torsion free over $\mathbf{Z}$.

Note that since $\operatorname{Coker} S^*$ is torsion free for free knots, we do not need to consider the condition (c2) in the definition of the algebraic concordance (see Definition 2.8).

Definition 2.11. Consider two simple 3-knots K_0 and K_1. Let A_0 and A_1 be the Seifert forms of K_0 and K_1 respectively with respect to 1-connected Seifert manifolds. We say that A_0 and A_1 are *spin concordant*, if there exists an orientation preserving diffeomorphism $h : K_0 \to K_1$ such that

(1) h preserves their spin structures,

(2) A_0 and A_1 are algebraically concordant with respect to $h_* : H_2(K_0) \to H_2(K_1)$ and $h_*|\mathrm{Tors}\, H_1(K_0) : \mathrm{Tors}\, H_1(K_0) \to \mathrm{Tors}\, H_1(K_1)$, where we identify $H_2(K_i)$ and $H_1(K_i)$ with $\mathrm{Ker}\, S_i^*$ and $\mathrm{Coker}\, S_i^*$ respectively for $i = 0, 1$ (see Remark 2.9).

Note that if K_0 and K_1 are free 3-knots, then we do not need to consider the condition (c2) of Definition 2.8 and hence the isomorphism $h_*|\mathrm{Tors}\, H_1(K_0)$ in the above definition.

Remark 2.12. Spin concordance of Seifert forms with respect to 1-connected Seifert manifolds is an equivalence relation. More precisely, it defines an equivalence relation on the set of pairs of a Seifert manifold of a simple 3-knot and its Seifert form. This fact can be shown by using [1, Theorem 1] together with an additional argument concerning spin structures.

Remark 2.13. Algebraic concordance of Seifert forms does not necessarily imply spin concordance. See Examples 7.1 and 7.2 in §7.

Remark 2.14. It is essential that we use 1-*connected* Seifert manifolds in Definition 2.11. See Example 7.5.

The main theorem of this paper is the following.

Theorem 2.15. *Two simple fibered free* 3*-knots are concordant if and only if their Seifert forms with respect to* 1*-connected fibers are spin concordant.*

We shall restrict ourselves to free 3-knots for two reasons. The first one is that algebraic concordance of Seifert forms is easier for free knots, since the condition (c2) is not necessary in the definition of the algebraic concordance. The second, and the most important one, is the dimension of the fiber. We shall use some results from 4-dimensional topology which require the condition that the boundary 3-manifolds should have torsion free first homology groups.

3. Spin concordance as a necessary condition for knot concordance

In this section, we shall prove the following, which is valid for simple fibered 3-knots in general which may not be free.

Proposition 3.1. *If two simple fibered* 3*-knots are concordant, then their Seifert forms with respect to* 1*-connected fibers are spin concordant.*

Proof. Let K_0 and K_1 be two concordant simple fibered 3-knots. We denote by F_0 and F_1 their 1-connected fibers, and by A_0 and A_1 the associated Seifert forms, respectively. By [1], we know that A_0 and A_1 are algebraically concordant, and the metabolizer M for $A_0 \oplus -A_1$ is constructed as follows.

Let us denote by $\mathcal{S}$ the product $S^5 \times [0, 1]$. The definition of concordance (see Definition 2.1) gives a submanifold $C = \Phi(\mathcal{K} \times [0, 1])$ of $\mathcal{S}$ such that $C \cap (S^5 \times \{0\}) = K_0$ and $C \cap (S^5 \times \{1\}) = -K_1$. Set $N = F_0 \cup C \cup (-F_1)$. By construction N is a closed oriented 4-dimensional submanifold of $\mathcal{S}$. By classical obstruction theory, we see that there exists a compact oriented 5-dimensional submanifold W of $\mathcal{S}$ such that $N = \partial W$. Let $\lambda_i : H_2(F_i) \to H_2(N)$, $i = 0, 1$, be the homomorphisms induced by the inclusions $F_i \hookrightarrow N$, and let $j : H_2(N) \to H_2(W)$ be the homomorphism induced by the inclusion $N \hookrightarrow W$. Then the submodule $M = \lambda^{-1}(\mathrm{Ker}\, j^\wedge)$ is a metabolizer for $A_0 \oplus -A_1$, where $\mathrm{Ker}\, j^\wedge$ is the smallest pure submodule containing $\mathrm{Ker}\, j$ and $\lambda = \lambda_0 - \lambda_1 : G = H_2(F_0) \oplus H_2(F_1) \to H_2(N)$. In [1], it has been shown that $\overline{M}$ is pure in $\overline{G}$.

Using a normal 2-framing of C in $\mathcal{S}$ induced from the inward normal vector field along $N = \partial W$ in W, we see that the diffeomorphism h between K_0 and K_1 induced by C preserves their spin structures.

Moreover, in [1], it has been shown that A_0 and A_1 are algebraically concordant with respect to $h_* : H_2(K_0) \to H_2(K_1)$ and $h_*|\mathrm{Tors}\, H_1(K_0) : \mathrm{Tors}\, H_1(K_0) \to \mathrm{Tors}\, H_1(K_1)$ by using the fact that A_0 and A_1 are unimodular.

Thus the spin concordance is a necessary condition for concordance in the case of simple fibered 3-knots. □

Remark 3.2. For a 3-knot K, a Seifert manifold is called an *almost fiber* of K, if its Seifert form is unimodular. A simple 3-knot is called a *simple almost fibered* 3-knot, if it admits a 1-connected almost fiber as a Seifert manifold (see [18, Remark 2.8 (1)], [4]). From the above proof, we easily see that Proposition 3.1 holds for simple almost fibered 3-knots in general.

As a corollary to Proposition 3.1, we have the following.

Corollary 3.3. *Let K be a simple almost fibered 3-knot. Then the spin concordance class of a Seifert form with respect to a 1-connected almost fiber does not depend on the choice of the almost fiber of K.*

Note that a 1-connected (almost) fiber is not necessarily unique for a simple 3-knot (see [18, §5]).

As a corollary to the proof of Proposition 3.1, we have the following.

Corollary 3.4. *If two 3-knots (which may not necessarily be fibered or almost fibered) are concordant, then they are (image) regularly homotopic as immersed manifolds in S^5.*

Compare the above corollary with [13, Theorem 7].

Proof of Corollary 3.4. By [20], the regular homotopy class of an embedding of an oriented 3-manifold into S^5 is completely determined by two invariants: the Wu invariant and the signature of a Seifert manifold. If two 3-knots are concordant, then by using the argument in the proof of Proposition 3.1, we can show that their spin structures coincide with each other and that their Seifert manifolds have the same signature. Since the Wu invariant is determined by the spin structure, the two embeddings have the same Wu invariant. Hence they are (image) regularly homotopic as immersions into S^5. □

Note that the converse of the above corollary does not hold, since there exist a lot of pairs of forms with the same signature but with distinct algebraic concordance class. For example, consider the simple spherical 3-knots with Seifert forms A_k, $k \geq 1$, defined in [11, p. 243]. Then they are all (image) regularly homotopic, but are pairwise nonconcordant.

Let K be a 3-knot. We shall denote by K° the punctured 3-manifold $K \setminus \mathrm{Int}\, D^3$ embedded in S^5. Then we have the following corollary to the proof of Proposition 3.1 as well.

Corollary 3.5. *If two free 3-knots K_0 and K_1 are concordant, then K_0° and K_1° are isotopic in S^5.*

Proof. By using the manifold C in the proof of Proposition 3.1, we can show that the Seifert forms of K_0° and K_1° with respect to their normal framings determined by their Seifert manifolds coincide with each other (see [19]). Then by [19], we see that the punctured 3-manifolds are isotopic. □

4. Spin concordance as a sufficient condition for knot concordance

In this section, we shall prove the following, which is valid for simple free 3-knots in general, which may not be fibered.

Theorem 4.1. *Consider two simple free 3-knots. If their Seifert forms with respect to 1-connected Seifert manifolds are spin concordant, then the 3-knots are concordant.*

Proof. Let K_0 and K_1 be simple free 3-knots such that the Seifert forms A_0 and A_1 with respect to their 1-connected Seifert manifolds F_0 and F_1, respectively, are spin concordant. Let M be the metabolizer and $h : K_0 \to K_1$ the diffeomorphism as in Definitions 2.8 and 2.11 respectively. Set $F = F_0 \natural (-F_1)$ and $V = (K_0 \setminus \operatorname{Int} D^3) \times [0,1]$, where the symbol "$\natural$" means a boundary connected sum. Note that $\partial F = K_0 \sharp (-K_1)$ and $\partial V = K_0 \sharp (-K_0)$, where the symbol "$\sharp$" means a usual connected sum (see Fig. 2 and Fig. 3).

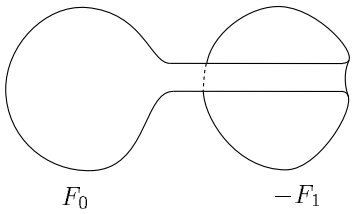


FIGURE 2. $F = F_0 \natural (-F_1)$

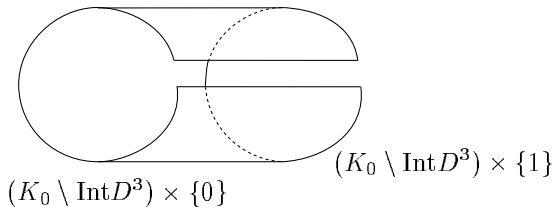


FIGURE 3. $V = (K_0 \setminus \mathrm{Int} D^3) \times [0,1]$

Note also that the compact 4-manifold V is spin, where the spin structure is induced from K_0. In the following, a *spin surgery* along a simple closed curve c in a spin 4-manifold is a process of taking off the tubular neighborhood $N(c) \cong S^1 \times D^3$ of c and replacing it with $D^2 \times S^2$ by gluing it along the boundary so that the resulting 4-manifold is spin and that the spin structure on the exterior of c coincides with that of the original one.

Lemma 4.2. *For some integer $k \geq 0$, there exists a compact 4-manifold $\widetilde{V}$ and a diffeomorphism $\tilde{h} : F \sharp k(S^2 \times S^2) \to \widetilde{V}$ such that*

(1) *$\widetilde{V}$ is obtained from V by spin surgeries along simple closed curves, and*
(2) *$\tilde{h}|\partial(F \sharp k(S^2 \times S^2)) = \mathrm{id}_{K_0} \sharp h^{-1} : K_0 \sharp (-K_1) \to K_0 \sharp (-K_0)$.*

Proof. Step 1. Since $H_1(V) \cong H_1(K_0)$ is a finitely generated free abelian group, we can obtain a 4-manifold V_1 with $H_1(V_1) = 0$ from V by spin surgeries along a finite set of simple closed curves c_i, $1 \leq i \leq r = \operatorname{rank} H_1(K_0)$, representing a basis of $H_1(V)$.

Step 2. Since $\pi_1(V_1)$ is finitely generated, we can obtain a simply connected 4-manifold V_2 from V_1 by some spin surgeries.

Step 3. Since we have assumed that $H_1(K_i)$ is a free abelian group, the intersection forms of F and V_2 are direct sums of a unimodular form and a zero form, where the dimensions of the null spaces are equal to the rank of $H_1(K_0\sharp(-K_1)) \cong H_1(K_0\sharp(-K_0))$. Furthermore, since they are spin, their intersection forms are of even type. Finally, since the Seifert forms of F_0 and F_1 are algebraically concordant, the signature of $F = F_0\sharp(-F_1)$ vanishes, and that of V_2 is equal to that of V, which is zero. Thus, by the algebraic classification of unimodular forms (see, for example, [12]), by repeating some spin surgeries along trivial simple closed curves in V_2 if necessary, we may assume that there exists an isometry $\Lambda : H_2(F^{(k)}) \to H_2(V_2)$ for some integer $k \geq 0$ such that the diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & H_2(\partial F^{(k)}) & \longrightarrow & H_2(F^{(k)}) & \longrightarrow & H_2(F^{(k)}, \partial F^{(k)}) & & \\
 & & \downarrow{\scriptstyle (\mathrm{id}_{K_0}\sharp h^{-1})_*} & & \downarrow{\scriptstyle \Lambda} & & \uparrow{\scriptstyle \Lambda^*} & & \\
0 & \longrightarrow & H_2(\partial V_2) & \longrightarrow & H_2(V_2) & \longrightarrow & H_2(V_2, \partial V_2) & & \\
 & & & & & & \longrightarrow H_1(\partial F^{(k)}) & \longrightarrow & 0 \\
 & & & & & & \downarrow{\scriptstyle (\mathrm{id}_{K_0}\sharp h^{-1})_*} & & \\
 & & & & & & \longrightarrow H_1(\partial V_2) & \longrightarrow & 0
\end{array}
$$

is commutative, where $F^{(k)} = F\sharp k(S^2 \times S^2)$, we use Poincaré-Lefschetz duality to identify $H_2(F^{(k)}, \partial F^{(k)})$ and $H_2(V_2, \partial V_2)$ with the duals of $H_2(F^{(k)})$ and $H_2(V_2)$ respectively, and Λ^* is the adjoint of Λ.

Step 4. Note that the spin structures of $K_0\sharp(-K_1)$ and $K_0\sharp(-K_0)$ coincide with those induced from $F^{(k)}$ and V_2 respectively. Thus $F^{(k)} \cup_{\mathrm{id}_{K_0}\sharp h^{-1}} (-V_2)$ is a closed spin 4-manifold, since $\mathrm{id}_{K_0}\sharp h^{-1}$ preserves the spin structures by our hypothesis. Furthermore, $K_0\sharp(-K_1)$ and $K_0\sharp(-K_0)$ are connected. Then by an argument of Boyer [2, p. 347], we see that there exists a smooth h-cobordism relative to boundary between $F^{(k)}$ and V_2 such that the induced diffeomorphism between the boundaries of $F^{(k)}$ and V_2 coincides with $\mathrm{id}_{K_0}\sharp h^{-1}$, and that the induced isomorphism between $H_2(F^{(k)})$ and $H_2(V_2)$ coincides with Λ above.

Step 5. Finally, by the 5-dimensional stable h-cobordism theorem due to Lawson [10] and Quinn [14], we see that there exists a diffeomorphism between $F^{(k+k')} = F^{(k)}\sharp k'(S^2 \times S^2)$ and $\widetilde{V} = V_2\sharp k'(S^2 \times S^2)$ extending $\mathrm{id}_{K_0}\sharp h^{-1} : \partial F^{(k)} \to \partial V_2$. Since $\widetilde{V}$ can be obtained from V_2 by repeating k' times the spin surgeries along trivial simple closed curves, we get the result. This completes the proof of Lemma 4.2. □

Remark 4.3. In Step 1, we can choose the curves c_i, $1 \leq i \leq r$, inside $(K_0 \setminus \mathrm{Int}\, D^3) \times \{1/2\}$. After the surgeries, the embedded 2-sphere Σ_i in V_1 corresponding to the center sphere $\{0\} \times S^2$ of the piece $D^2 \times S^2$ replacing $N(c_i)$ is homologous to the boundary of a meridian 3-disk of c_i in V. Let γ_i^*, $1 \leq i \leq r$, be a basis of $H_2(K_0 \setminus \mathrm{Int}\, D^3) \cong H_2(K_0)$ which is Poincaré dual to the basis $[c_i]$, $1 \leq i \leq r$, of $H_1(K_0)$, where $[*]$ denotes the homology class represented by $*$. Then, by the above observation, we have $[\Sigma_i] = i_{0*}\gamma_i^* - i_{1*}\gamma_i^*$, where $i_0 : K_0 \to K_0 \times \{0\} \subset V_1$ and $i_1 : K_0 \to K_0 \times \{1\} \subset V_1$ denote the inclusions.

Lemma 4.4. *For some integer $k \geq 0$, there exist a compact 4-manifold $\widetilde{V'}$ and a diffeomorphism $\tilde{h}' : F_0\sharp(-F_1)\sharp k(S^2 \times S^2) \to \widetilde{V'}$ such that*

(1) $\widetilde{V'}$ *is obtained from* $V' = K_0 \times [0,1]$ *by spin surgeries along simple closed curves, and*
(2) $\tilde{h'}|\partial(F_0\sharp(-F_1)\sharp k(S^2\times S^2)) = \mathrm{id}_{K_0} \coprod h^{-1} : K_0 \coprod (-K_1) \to K_0 \coprod (-K_0)$.

Proof. Just glue $D^3 \times [0,1]$ to $F\sharp k(S^2\times S^2)$ and $\widetilde{V}$ in Lemma 4.2 along $\partial D^3 \times [0,1]$ to obtain $F_0\sharp(-F_1)\sharp k(S^2\times S^2)$ and $\widetilde{V'}$ respectively (see Fig. 2 and Fig. 4). □

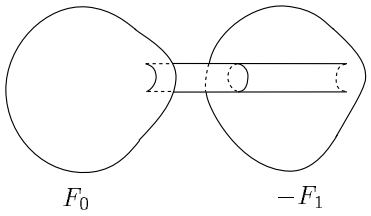


FIGURE 4. $F = F_0\sharp(-F_1)$

Let $\widetilde{V'}$ be as in the above lemma, which is obtained from $K_0 \times [0,1]$ by some spin surgeries. Let $\Sigma_1, \Sigma_2, \ldots, \Sigma_r$ be the embedded 2-spheres in $\widetilde{V'}$ which have been created in the course of the surgeries in Step 1 of the proof of Lemma 4.2 (for details, see Remark 4.3). Furthermore, let $\Sigma_{r+1}, \Sigma_{r+2}, \ldots, \Sigma_{r+s}$ be the 2-spheres in $\widetilde{V'}$ created in Steps 2–5 in the proof of Lemma 4.2. For the latter spheres, since the surgery curves are all null homologous, we see that there exist homology classes $\sigma^*_{r+1}, \sigma^*_{r+2}, \ldots, \sigma^*_{r+s} \in H_2(\widetilde{V'})$ such that

$$[\Sigma_i] \cdot \sigma^*_j = \begin{cases} 1, & i = j, \\ 0, & i \neq j, \end{cases}$$

for $r+1 \le i, j \le r+s$. Modifying σ^*_i, $r+1 \le i \le r+s$, appropriately, we may further assume that the s submodules $\langle [\Sigma_i], \sigma^*_i \rangle$ are orthogonal to each other with respect to the intersection form $\widetilde{S}$ of $\widetilde{V'}$ and that the intersection matrix of $\langle [\Sigma_i], \sigma^*_i \rangle$ is equal to

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

where for a subset X of a module, $\langle X \rangle$ denotes the submodule generated by X. Note also that

$$H_2(\widetilde{V'}) = \mathrm{Ker}\, \widetilde{S}^* \oplus \left(\bigoplus_{i=r+1}^{r+s}{}^{\perp} \langle [\Sigma_i], \sigma^*_i \rangle \right),$$

where the symbol "$\oplus^{\perp}$" denotes an orthogonal direct sum.

By taking the connected sum of k copies of $S^2 \times S^2$ with F_0 inside S^5, we may assume that A_0 is the Seifert form with respect to $F_0\sharp k(S^2\times S^2)$. Let A_1 be the Seifert form with respect to F_1. Furthermore, let S be the symmetric form associated with $A_0 \oplus (-A_1)$ (see Definition 2.6). Note that S can be naturally identified with the intersection form of $F_0\sharp(-F_1)\sharp k(S^2\times S^2)$ and hence with that of $\widetilde{V'}$. In the following, we shall identify $F_0\sharp(-F_1)\sharp k(S^2\times S^2)$ with $\widetilde{V'}$ by using $\tilde{h'}$ in Lemma 4.4.

Lemma 4.5. *There exists an isometry* Φ *of* $H_2(F_0\sharp(-F_1)\sharp k(S^2\times S^2))$ *with respect to* S *such that*

(1) $\Phi|\mathrm{Ker}\, S^* = \mathrm{id}$,

(2) $\Phi_*[\Sigma_1], \Phi_*[\Sigma_2], \ldots, \Phi_*[\Sigma_{r+s}]$ *are generators of the metabolizer* M.

Proof. First recall that $[\Sigma_1], [\Sigma_2], \ldots, [\Sigma_r]$ lie in $\operatorname{Ker} S^*$ by Remark 4.3.

As has been shown in [1, Proposition 2.1], there exist a basis $\{m_i, m_i^*; i = 1, 2, \ldots, r+s\}$ of $G = H_2(F_0\sharp(-F_1)\sharp k(S^2 \times S^2))$ such that

(a) $\{m_i; i = 1, 2, \ldots, r+s\}$ is a basis of M,

(b) $\{m_i, m_i^*; i = 1, 2, \ldots, r\}$ is a basis of $\operatorname{Ker} S^*$ and $\{m_i^*; i = 1, 2, \ldots, r\}$ is a basis of $\operatorname{Ker} S_0^*$, where S_0 is the symmetric form associated with A_0,

(c) the submodules $\langle m_i, m_i^* \rangle$, $i = 1, 2, \ldots, r+s$, are orthogonal for S; i.e.,

$$G = \bigoplus_{i=1}^{r+s}{}^{\perp} \langle m_i, m_i^* \rangle.$$

We may further assume that

$$S(m_i, m_i) = 0, \quad S(m_i, m_i^*) = 1, \quad S(m_i^*, m_i^*) = 0$$

for $r+1 \leq i \leq r+s$, since $\operatorname{Coker} S^*$ is torsion free. Then define the isometry $\Phi : G \to G$ by $\Phi|\operatorname{Ker} S^* = \mathrm{id}$, $\Phi([\Sigma_i]) = m_i$ and $\Phi(\sigma_i^*) = m_i^*$ for $i = r+1, r+2, \ldots, r+s$. This completes the proof of Lemma 4.5. □

Lemma 4.6. *For some integer* $k \geq 0$, *there exists an orientation preserving self-diffeomorphism* φ *of the* 4*-manifold* $F_0\sharp(-F_1)\sharp k(S^2 \times S^2)$ *which is the identity on the boundary such that* $\varphi_* = \Phi$ *on the second homology group.*

Proof. Let J be the submodule of $G = H_2(F_0\sharp(-F_1)\sharp k(S^2 \times S^2))$ generated by $[\Sigma_i]$ and σ_i^* with $r+1 \leq i \leq r+s$. Note that $G = \operatorname{Ker} S^* \oplus J$ and that the intersection matrix with respect to this decomposition is of the form $0 \oplus Q$, where Q is a unimodular symmetric matrix of even type and zero signature.

Then it is not difficult to see that an arbitrary isometry of $(\operatorname{Ker} S^* \oplus J; 0 \oplus Q)$ which is the identity on $\operatorname{Ker} S^*$ is a composition of the following isometries:

(a) $\mathrm{id} \oplus \Lambda$, where Λ is an isometry of $(J; Q)$,

(b) an isometry represented by the matrix of the form

$$\begin{pmatrix} \mathrm{id} & * \\ 0 & \mathrm{id} \end{pmatrix}$$

with respect to the decomposition $\operatorname{Ker} S^* \oplus J$.

We can easily realize isometries of type (a) by diffeomorphisms which are the identity on the boundary, by using Wall's argument [21], since we may assume $k \geq 1$.

In order to realize isometries of type (b), we need the following lemma.

Lemma 4.7. *Increasing* k *if necessary, we may assume that* $F_0\sharp(-F_1)\sharp k(S^2 \times S^2) \setminus \Sigma_i$ *is simply connected for* $1 \leq i \leq r+s$.

Proof. Since $F_0\sharp(-F_1)\sharp k(S^2 \times S^2)$ is simply connected, $\pi_1(F_0\sharp(-F_1)\sharp k(S^2 \times S^2) \setminus \Sigma_i)$ is normally generated by a meridian μ_i of Σ_i, where μ_i is the boundary of a fiber of the 2-disk bundle neighborhood of Σ_i. Then, performing the spin surgeries along μ_i, we get a desired situation. □

Now let us go back to the proof of Lemma 4.6. By Lemma 4.7, the spin surgery creating each Σ_i corresponds to the connected sum operation with $S^2 \times S^2$. Thus by [21, Theorem 1], we get a diffeomorphism realizing an isometry of type (b) corresponding to a matrix of the form

$$\begin{pmatrix} \mathrm{id} & E \\ 0 & \mathrm{id} \end{pmatrix},$$

where E is a matrix having one entry equal to 1 and all the others equal to zero. Using this type of diffeomorphisms (sometimes, we have to interchange the two factors of $S^2 \times S^2$, or use the inverse diffeomorphism), we get a desired diffeomorphism. This completes the proof of Lemma 4.6. □

Thus we have proved that the embedded 2-spheres $\varphi(\Sigma_1), \varphi(\Sigma_2), \ldots, \varphi(\Sigma_{r+s})$ in $F_0 \sharp (-F_1) \sharp k(S^2 \times S^2)$ constitute a set of generators for the metabolizer M.

Recall that $F_0 \sharp (-F_1) \sharp k(S^2 \times S^2)$ is embedded in $\mathcal{S} = S^5 \times [0,1]$. Then we can perform appropriate surgeries along these embedded 2-spheres inside $\mathcal{S}$ as in [1, §4]. Since each surgery process is exactly the inverse operation of each spin surgery performed in the construction of $\widetilde{V'}$ (modified by the diffeomorphism φ), the resulting 4-manifold is diffeomorphic to $K_0 \times [0,1]$, which is embedded in $\mathcal{S}$. Thus K_0 and K_1 are concordant. This completes the proof of Theorem 4.1 and hence Theorem 2.15. □

Remark 4.8. As has been noted in Remark 2.13, algebraic concordance does not necessarily imply spin concordance. Hence, Theorem 4.1 does not hold if we replace the spin concordance with the algebraic concordance, even if we add the assumption that the 3-knots are abstractly diffeomorphic.

5. Extension to a larger class of 3-knots

As the arguments of §4 show, the sufficiency of Theorem 2.15 holds for simple free 3-knots: i.e., the 3-knots in question may not be fibered. However, for the proof of necessity in §2, we have used the hypothesis that the 3-knots are fibered. In this section, we shall try to extend the class of simple fibered free 3-knots in such a way that the necessity continues to hold for a larger class of 3-knots.

First we give a definition which is valid for any dimension.

Definition 5.1. We say that a simple $(2n-1)$-knot K is *C-algebraically fibered*, if the Seifert form of K with respect to an $(n-1)$-connected Seifert manifold is algebraically concordant to a unimodular form, where the zero form is also considered to be unimodular. In the following, for a C-algebraically fibered $(2n-1)$-knot, we use the Seifert form defined on an $(n-1)$-connected Seifert manifold which is algebraically concordant to a unimodular form, unless otherwise specified. Note that simple fibered knots are always C-algebraically fibered.

Remark 5.2. A simple $(2n-1)$-knot is said to be *algebraically fibered*, if the Seifert form with respect to an $(n-1)$-connected Seifert manifold is S-equivalent to a unimodular matrix (see [7], [16, §4]). Then we see easily that for a simple $(2n-1)$-knot, we have

$$\text{simple fibered} \Longrightarrow \text{algebraically fibered} \Longrightarrow C\text{-algebraically fibered}.$$

Note that the reverse implications do not hold in general. See [7] and Example 7.3.

Now, let us consider the case of 3-knots. Recall that a *stabilizer* K_S is a simple fibered spherical 3-knot whose fiber is diffeomorphic to $(S^2 \times S^2) \sharp (S^2 \times S^2) \setminus \operatorname{Int} D^4$. Such a stabilizer does exist. For details, see [16, §4]. Furthermore, there also exists a stabilizer which is null cobordant (see [15, p. 600] or [18, §10]). In the following, K_S will denote such a null cobordant stabilizer.

Proposition 5.3. *Let K be a simple free 3-knot. If K is C-algebraically fibered, then there exists a simple fibered free 3-knot K' such that*

(1) *K and K' are concordant,*
(2) *the Seifert form of K with respect to a 1-connected Seifert manifold and that of K' with respect to a 1-connected fiber are spin concordant.*

Compare the above proposition with [16, Proposition 4.4].

Proof of Proposition 5.3. Let F be a 1-connected Seifert manifold of K and A the Seifert form for F. Note that A is algebraically concordant to a unimodular form L by our assumption. Let $\psi : H_2(K) = \mathrm{Ker}\,(A + A^T) \to \mathrm{Ker}\,(L + L^T)$ be the isomorphism with respect to which A and L are algebraically concordant.

Let us first show that there exists a compact 1-connected oriented spin 4-manifold F' with boundary diffeomorphic to K such that the spin structures induced from F and F' on K coincide with each other and that the intersection form of F' is isomorphic to

$$(L + L^T) \oplus 2k \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{5.1}$$

for some $k \geq 0$. We can construct such a 4-manifold F' as follows.

We first construct a 4-dimensional special handlebody F_1 consisting of one 0-handle and some 2-handles attached to the 0-handle simultaneously such that ∂F_1 is diffeomorphic to K, that F_1 is spin, and that the spin structure induced from F_1 coincides with the given spin structure on K (for details, see [5]). Then by Rohlin's theorem together with Novikov additivity for signature, the difference of the signatures of F and F_1 must be divisible by 16. Hence, by using some copies of a spin 4-dimensional special handlebody with boundary S^3 and with signature ± 16 (see [5]), we may assume that F and F_1 have the same signature. Note that the signature of F is equal to that of $L + L^T$. Then by the classification of symmetric unimodular forms, we see that the intersection form of $F' = F_1 \sharp k'(S^2 \times S^2)$ is isomorphic to the form (5.1) for some $k, k' \geq 0$. Here, we need the assumption that $H_1(K)$ is free.

Note that the above isomorphism between the intersection form of F' and the form (5.1) induces an isomorphism $H_2(K) = \mathrm{Ker}\,(A+A^T) \to \mathrm{Ker}\,(L+L^T)$. Changing the isomorphism between the intersection form of F' and (5.1) if necessary, we may assume that the induced isomorphism coincides with ψ.

Recall that F' has a handlebody decomposition consisting of one 0-handle and some 2-handles. Thus, by using Kervaire's argument [8, pp. 255–257], we can embed F' into S^5 so that its Seifert form is given by $L \oplus kL_S$, where L_S is the Seifert form of a null cobordant stabilizer K_S with respect to the 1-connected fiber. Set $K' = \partial F'$.

Since L is unimodular, by using the stabilization technique developed in [16, §4], we may assume that K' is a simple fibered 3-knot, increasing k if necessary.

Note that the Seifert form $L \oplus kL_S$ for K' is algebraically concordant to L, which is algebraically concordant to the Seifert form A for K by our assumption. Furthermore, by the above construction, we see easily that $L \oplus kL_S$ and A are spin concordant. Thus we have proved the item (2) in the proposition. The item (1) then follows from Theorem 4.1. This completes the proof of Proposition 5.3. □

Corollary 5.4. *If two C-algebraically fibered simple free 3-knots are concordant, then their Seifert forms with respect to 1-connected Seifert manifolds are spin concordant.*

Proof. Let K_0 and K_1 be the simple free 3-knots as above. Then by Proposition 5.3, K_0 and K_1 are concordant to simple fibered free 3-knots K_0' and K_1' with spin concordant Seifert forms respectively. Then, since K_0' and K_1' are concordant, they have spin concordant Seifert forms by Proposition 3.1. Thus K_0 and K_1 have spin concordant Seifert forms, since spin concordance is an equivalence relation (see Remark 2.12). This completes the proof. □

Combining the above corollary with Theorem 4.1, we get the following.

Theorem 5.5. *Two C-algebraically fibered simple free 3-knots are concordant if and only if their Seifert forms with respect to 1-connected Seifert manifolds are spin concordant.*

Note that there are a lot of C-algebraically fibered simple free 3-knots which are not fibered (see Example 7.3 of §7).

We can prove a similar theorem for higher dimensions as well as follows.

Theorem 5.6. *For $n \geq 3$, two C-algebraically fibered simple $(2n-1)$-knots are concordant if and only if their Seifert forms with respect to $(n-1)$-connected Seifert manifolds are algebraically concordant.*

Proof. Replacing Proposition 3.1 in the argument for the 3-dimensional case with [1, Theorem 2'], we see that we have only to show the following: if a simple $(2n-1)$-knot K with $n \geq 3$ is C-algebraically fibered, then K is concordant to a simple fibered $(2n-1)$-knot K' such that the Seifert form A of an $(n-1)$-connected Seifert manifold for K is algebraically concordant to the Seifert form of an $(n-1)$-connected fiber of K'.

Since K is C-algebraically fibered, A is algebraically concordant to a unimodular form L. By Durfee [3], such a form is realized as the Seifert form of a simple fibered $(2n-1)$-knot K'. Then by [1, Theorem 3], K is concordant to K'. This completes the proof. □

6. Special cases

So far, we had to consider spin concordance of Seifert forms instead of the usual algebraic concordance for 3-knots. In this section, we shall show that in some special cases, the algebraic concordance is sufficient.

Let us begin by the following definition.

Definition 6.1. Let $\mathcal{M}$ be a closed connected oriented 3-manifold. A 3-knot K is called an $\mathcal{M}$-knot, if K is abstractly diffeomorphic to $\mathcal{M}$, orientation preservingly.

For a closed connected oriented 3-manifold $\mathcal{M}$, let us consider the following conditions.

(6.1) For any isomorphism $\psi : H_2(\mathcal{M}) \to H_2(\mathcal{M})$, there exists an orientation preserving diffeomorphism $h_1 : \mathcal{M} \to \mathcal{M}$ such that $h_{1*} = \psi$.

(6.2) For any two spin structures of $\mathcal{M}$, there exists an orientation preserving diffeomorphism $h_2 : \mathcal{M} \to \mathcal{M}$ which sends one spin structure to the other such that $h_{2*} : H_2(\mathcal{M}) \to H_2(\mathcal{M})$ is the identity.

Then we have the following.

Proposition 6.2. *Let $\mathcal{M}$ be a closed connected oriented 3-manifold with torsion free first homology group. Suppose that the above conditions (6.1) and (6.2) are satisfied for $\mathcal{M}$. Then two C-algebraically fibered simple $\mathcal{M}$-knots are concordant if and only if their Seifert forms with respect to 1-connected Seifert manifolds are algebraically concordant.*

Proof. The necessity follows from Corollary 5.4.

Now, suppose that K_0 and K_1 are C-algebraically fibered simple $\mathcal{M}$-knots whose Seifert forms A_0 and A_1 with respect to 1-connected Seifert manifolds F_0 and F_1, respectively, are algebraically concordant. We suppose that A_0 and A_1 are algebraically concordant with respect to the isomorphism $\psi : H_2(K_0) \to H_2(K_1)$. By the conditions (6.1) and (6.2), we see that there exists an orientation preserving diffeomorphism $h : K_0 \to K_1$ such that $h_* = \psi$ and h sends the spin structure of K_0 to that of K_1. Hence, A_0 and A_1 are spin concordant with respect to h. Thus by Theorem 4.1, K_0 and K_1 are concordant. This completes the proof. □

For example, if $\mathcal{M}$ is a $\mathbf{Z}$-homology 3-sphere, i.e., if $H_1(\mathcal{M}) = 0$, then $H_2(\mathcal{M}) = 0$ and $\mathcal{M}$ admits a unique spin structure. Thus the conditions (6.1) and (6.2) are automatically satisfied. As another example, consider $\mathcal{M} = \sharp^k(S^1 \times S^2)$, the connected sum of k copies of $S^1 \times S^2$ with $k \geq 1$. Then it is well known that the conditions (6.1) and (6.2) are satisfied also in this case. Thus we have the following.

Corollary 6.3. *Suppose $\mathcal{M}$ is a $\mathbf{Z}$-homology 3-sphere, or $\mathcal{M} = \sharp^k(S^1 \times S^2)$, $k \geq 1$. Then two C-algebraically fibered simple $\mathcal{M}$-knots are concordant if and only if their Seifert forms with respect to 1-connected Seifert manifolds are algebraically concordant.*

In fact, when $\mathcal{M}$ is a $\mathbf{Z}$-homology 3-sphere, a stronger result has been known. For details, see [17].

7. Examples

In this section, we give various examples with interesting properties.

Example 7.1. Set $\mathcal{M} = S^1 \times \Sigma_g$, where Σ_g is the closed connected orientable surface of genus $g \geq 2$. Note that $H_1(\mathcal{M})$ is torsion free. Let K_0 and K_1 be the simple fibered $\mathcal{M}$-knots constructed in [18, Proposition 3.8]. They have the property that their Seifert forms are isomorphic, but that there exists no diffeomorphism between K_0 and K_1 which preserves their spin structures. Thus, the Seifert forms of K_0 and K_1 are algebraically concordant, but are not spin concordant. Hence they are not concordant by Proposition 3.1.

This example shows that the spin structure plays an essential role in the theory of concordance for 3-knots.

Example 7.2. Let $\mathcal{M}$ be a nontrivial orientable S^1-bundle over the closed connected orientable surface of genus $g \geq 2$. Note that $H_1(\mathcal{M})$ is not torsion free in general. Let $K_1, K_2, \ldots, K_n$ be the simple fibered $\mathcal{M}$-knots constructed in [18, Theorem 3.1]. They have the property that their Seifert forms are isomorphic to each other, but that any such isomorphism restricted to $H_2(K_i)$ cannot be realized by a diffeomorphism. Thus, the Seifert forms of K_i are algebraically concordant to each other, but are not spin concordant. Hence they are not concordant by Proposition 3.1, which is valid also for non-free simple fibered 3-knots.

Example 7.3. We can construct simple 3-knots which are C-algebraically fibered, but are not fibered, as follows. Let K be a simple fibered 3-knot. It is easy to see that there are a lot of simple algebraically non-fibered spherical 3-knots K' which are null cobordant. For example, consider the boundary of $S^2 \times S^2 - \operatorname{Int} D^4$ embedded in S^5 so that its Seifert form is isomorphic to

$$\begin{pmatrix} 0 & k \\ 1-k & 0 \end{pmatrix}, \quad k \geq 1.$$

Then the simple 3-knot $K \sharp K'$ is C-algebraically fibered, since its Seifert form is algebraically concordant to that of K. However, $K \sharp K'$ is not (algebraically) fibered.

Example 7.4. We can construct concordant, but not isotopic non-spherical 3-knots as follows. Let K_S be a null cobordant stabilizer as in §5. Note that K_S is a non-trivial 3-knot which is concordant to the trivial 3-knot. Then consider any simple fibered 3-knot K which is not spherical. Then the two simple fibered 3-knots $K \sharp K_S$ and K are not isotopic, since the ranks of the second homology groups of their fibers are distinct. However, these knots are concordant.

Example 7.5. For a simple 3-knot, the algebraic concordance class of a Seifert form depends on the choice of the Seifert manifold in general. For example, let K be the trivially embedded $S^1 \times S^2$ in S^5; i.e., S^4 is trivially embedded in S^5 and S^4

decomposes into $F_0 = D^2 \times S^2$ and $F_1 = S^1 \times D^3$ along K. Note that F_0 is a 1-connected Seifert manifold of K, while F_1 is a Seifert manifold which is not 1-connected. Then the Seifert forms with respect to F_0 and F_1 are not algebraically concordant, since the ranks of $H_2(F_i)$, $i = 0, 1$, do not have the same parity.

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