ON THE ALMOST SURE CENTRAL LIMIT THEOREM FOR A CLASS OF $\mathbb{Z}^d$ -ACTIONS

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ABSTRACT. As an extension of earlier papers on stationary sequences, a concept of weak dependence for strictly stationary random fields is introduced in terms of so-called homoclinic transformations. Under assumptions made within the framework of this concept a form of the almost sure central limit theorem (ASCLT) is established for random fields arising from a class of algebraic $\mathbb{Z}^d$ -actions on compact abelian groups. As an auxiliary result, the central limit theorem is proved via Ch. Stein's method. The next stage of the proof includes some estimates which are specific for ASCLT. Both steps are based on making use of homoclinic transformations.

1. INTRODUCTION

Let $(\xi_{\mathbf{n}})_{\mathbf{n} \in \mathbb{Z}^d}$ be an array of random variables defined on a probability space $(X, \mathcal{F}, P)$. Much attention was attracted by the problem under which weak dependence assumptions about this array one may expect the asymptotic normality of the sums $\sum_{\mathbf{n} \in \mathbf{D}} \xi_{\mathbf{n}}$ (under appropriate norming) as the number of elements $|\mathbf{D}|$ of a finite set $\mathbf{D} \subset \mathbb{Z}^d$ goes to infinity. Various types of mixing conditions were discussed in this connection. Also some conditions of a different type are known. Namely, J. Dedecker proved in [De] a form of the central limit theorem (CLT) under certain assumptions formulated in terms of some bilinear operators applied to the array $(\xi_{\mathbf{n}})_{\mathbf{n} \in \mathbb{Z}^d}$. Anyway, every type of the above mentioned conditions requires a certain system of σ -fields to be fixed. Usually the σ -fields correspond to a collection of finite or infinite subsets of $\mathbb{Z}^d$. These subsets are defined in terms of some distance, product structure or even lexicographic ordering on $\mathbb{Z}^d$, and corresponding σ -fields may be thought of as generated by a stationary random field (generally speaking, different from $(\xi_{\mathbf{n}})_{\mathbf{n} \in \mathbb{Z}^d}$) on the corresponding subset of $\mathbb{Z}^d$. The reader is referred to the introduction in [De] for the further survey information and references.

We follow here a different approach. We consider such form of weak dependence which permits to change considerably the values of random variables in a finite domain $\mathbf{D} \subset \mathbb{Z}^d$, leaving these values almost unchanged at sites situated very far from this domain. Moreover, the joint distribution of all random variables of which consists our random field is supposed to be preserved by this transformation. The simplest example can be given by the random field of independent identically distributed symmetric binary random variables; the transformation under discussion

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is the flip on some finite set $\mathbf{D}$ and the identity outside $\mathbf{D}$. Independence and symmetry are both crucial in this example to assure that such localized flip preserves the joint distribution. However, there are some cases when similar transformations exist even for dependent random fields. As was noted in [Go2] (see also some terminological and historical comments and further references there) it is the situation described above where one can apply quite naturally the well-known Ch. Stein's method to prove the CLT for random fields with one- or multidimensional parameter set. Originally Stein's method was applied under some mixing conditions stated in terms of σ -fields to prove a form of the CLT (with certain convergence rate) for stationary random sequences. Stein's method was used by Bolthausen [Bo] to establish a variant of the CLT for random fields under some mixing conditions (these results were improved in [De] on the base of Lindeberg's method). As to the transformations which can be used to change the values of random variables, they seemed to be not discussed in the literature in a general form until a remarkable paper [Ca] of Cappocaccia. More precisely, in [Ca] partial deterministic transformations are considered as a suitable structure to introduce the concept of Gibbs measure (Cappocaccia considered in [Ca] an action of $\mathbb{Z}^d$ on a compact space).

In the present paper we deal with a certain class of measure preserving $\mathbb{Z}^d$ -actions supplied with some additional structure called *homoclinic transformations*. Every measure preserving $\mathbb{Z}^d$ -action on a probability space $(X, \mathcal{F}, P)$, together with arbitrary measurable function $f : X \rightarrow \mathbb{R}$, give rise to a real valued random field on $\mathbb{Z}^d$. We prove for such fields, under certain additional assumptions made in terms of homoclinic transformations, a form of the almost sure central limit theorem (ASCLT). As an auxiliary result we also prove the usual CLT.

Our premises include, in particular, the requirement that every homoclinic transformation is globally defined (we have only germs of such transformations in general hyperbolic dynamics), and that they commute each with other. In view of these assumptions, all our considerations could be put into the framework of algebraic $\mathbb{Z}^d$ -actions on compact abelian groups. The main reason to carry out them in a formally more abstract setting is that thus we avoid many unnecessary details and make the proof more transparent.

The statements of the main results, together with required definitions, are presented in Section 2. The proof of the CLT in Section 3 follows very closely to the version of Stein's method in [Go2], including the application of the *homoclinic Laplace operator* introduced in this paper. In the proof of the ASCLT in the Section 3 a familiar approach proposed in [LPh] and an application of the concept of quasiorthogonality from [We1] are combined with a substantial new ingredient in the proof of Lemma 4.1 based again on exploiting the homoclinic Laplace operator.

Passing to the applications we restrict ourselves within this paper by those expansive algebraic $\mathbb{Z}^d$ -actions on compact abelian groups which have dense homoclinic groups. In a recent paper [LiSch] a rather complete study of homoclinic groups for algebraic $\mathbb{Z}^d$ -actions on compact abelian groups was carried out, and we make use of the results of this paper (remark that the algebraic description of the homoclinic group and its properties for a hyperbolic action of $\mathbb{Z}^1$ by algebraic automorphisms of the torus may be found in [Go2]). It follows from the results in [LiSch] that an expansive action of $\mathbb{Z}^d$ by automorphisms of a nontrivial compact abelian second countable group has a dense homoclinic group if and only if this action has completely positive entropy. As in [Go2], the desirable spectral properties of the homoclinic Laplace operator are deduced from expansiveness of the dual action.

The CLT and ASCLT phenomena may also take place for some $\mathbb{Z}^d$ -actions on compact abelian groups which do not enjoy at least one of two just mentioned properties (expansiveness and completely positive entropy). Many of such actions can be treated in the spirit of this paper. However, this requires a modification of our tools and will be done elsewhere.

2. STATEMENT OF RESULTS

2.1. CLT and ASCLT for an $\mathbb{Z}^d$ -action. Let $(T^{\mathbf{n}})_{\mathbf{n} \in \mathbb{Z}^d}$ be a measure preserving action of $\mathbb{Z}^d$ on a probability space $(X, \mathcal{F}, P)$. By definition, it means that T is a homomorphism $\mathbf{n} \mapsto T^{\mathbf{n}}$ of $\mathbb{Z}^d$ to the group of invertible P -preserving measurable transformations of $(X, \mathcal{F}, P)$.

For every function f with domain X we write $U^{\mathbf{n}}f$ instead of $f \circ T^{\mathbf{n}}$ for $\mathbf{n} \in \mathbb{Z}^d$. If B is a Banach space and $p \in [1, \infty]$ then $(U^{\mathbf{n}})_{\mathbf{n} \in \mathbb{Z}^d}$ acts in an invertible and isometric way on the space $L_p(B)$ of measurable functions taking their values in B and having finite L_p -norm $|\cdot|_{p,B}$. Let $L_p^0(B) \subset L_p(B)$ be the subspace of functions with integral zero. The symbol B may be omitted when $B = \mathbb{R}^1$ or $B = \mathbb{C}^1$. Brackets $(\cdot, \cdot)$ denote the inner product on L_2 . We set $\|\mathbf{n}\| = \max_{1 \leq i \leq d} |n_i|$ for $\mathbf{n} = (n_1, \dots, n_d) \in \mathbb{Z}^d$ and $\text{dist}(\mathbf{m}, \mathbf{n}) = \|\mathbf{m} - \mathbf{n}\|$ for $\mathbf{m}, \mathbf{n} \in \mathbb{Z}^d$. For any $\mathbf{D} \subset \mathbb{Z}^d$ we define its boundary $\partial \mathbf{D}$ by $\partial \mathbf{D} = \{\mathbf{n} : \mathbf{n} \in \mathbf{D}, \text{dist}(\mathbf{n}, \mathbb{Z}^d \setminus \mathbf{D}) = 1\}$. Also we set

$$S_{\mathbf{D}} = \sum_{\mathbf{n} \in \mathbf{D}} U^{\mathbf{n}}, \quad \mathbf{D} \in \mathbb{Z}^d, |\mathbf{D}| < \infty,$$

where $|\mathbf{D}|$ denotes the cardinality of a set $\mathbf{D}$. Denoting by $\delta(x)$ the unit point mass at $x \in \mathbb{R}$, we mean by the normal distribution $N(m, 0)$ with mean value $m \in \mathbb{R}$ and variance 0 the unit point mass $\delta(m)$ at $m \in \mathbb{R}$.

Definition 2.1. A measurable real-valued function f on X is said to satisfy the central limit theorem (CLT) with respect to an action $(T^{\mathbf{n}})_{\mathbf{n} \in \mathbb{Z}^d}$ if there exists such $\sigma_f^2 \geq 0$ that for any sequence $(\mathbf{D}_n)_{n \in \mathbb{Z}_+}$ of finite subsets of $\mathbb{Z}^d$ satisfying the relations

$$(2.1) \quad |\mathbf{D}_n| \rightarrow \infty, \quad |\partial \mathbf{D}_n|/|\mathbf{D}_n| \rightarrow 0,$$

the sequence

$$(|\mathbf{D}_n|^{-\frac{1}{2}} S_{\mathbf{D}_n} f)_{n \geq 0}$$

converges in distribution to the normal law $N(0, \sigma_f^2)$.

Let $\mathcal{S}$ be a certain class of sequences $(\mathbf{D}_n)_{n \in \mathbb{Z}_+}$ of finite subsets of $\mathbb{Z}^d$ satisfying the condition (2.1).

Under the same premises as above a function f is said to satisfy the almost sure central limit theorem (ASCLT) for a class $\mathcal{S}$ if there exists such $\sigma_f^2 \geq 0$ that for any sequence $(\mathbf{D}_n)_{n \in \mathbb{Z}_+} \in \mathcal{S}$ the sequence of random measures

$$((\log n)^{-1} \sum_{k \leq n} k^{-1} \delta(|\mathbf{D}_k|^{-\frac{1}{2}} S_{\mathbf{D}_k} f))_{n \geq 0}$$

with probability 1 tends weakly as $n \rightarrow \infty$ to the normal law $N(0, \sigma_f^2)$.

We see that the ASCLT (at least in the form as above) may need some additional requirements about $(\mathbf{D}_n)_{n \in \mathbb{Z}_+}$. These requirements will be made precise in the next subsection.

2.2. Algebraic $\mathbb{Z}^d$ -actions on compact abelian groups: Theorem 2.2.

Let X be in this section a compact second countable abelian group with Haar probability measure P on it, and the action $(T^{\mathbf{n}})_{\mathbf{n} \in \mathbb{Z}^d}$ is now a homomorphism of $\mathbb{Z}^d$ to the group of (continuous) automorphisms of X (we call such an action *algebraic*). Then each automorphism $T^{\mathbf{n}}$ preserves P . Let ρ be a metric on X compatible with its topology. An action $(T^{\mathbf{n}})_{\mathbf{n} \in \mathbb{Z}^d}$ on X is said to be *expansive* if there exists $\epsilon > 0$ such that for any two points $x, x' \in X$ from $\sup_{\mathbf{n} \in \mathbb{Z}^d} \rho(T^{\mathbf{n}}x, T^{\mathbf{n}}x') < \epsilon$ follows $x = x'$. Any such number ϵ , if exists, is called an *expansive constant* for the action $(T^{\mathbf{n}})_{\mathbf{n} \in \mathbb{Z}^d}$.

In view of compactness of X , whether an action $(T^{\mathbf{n}})_{\mathbf{n} \in \mathbb{Z}^d}$ enjoys expansiveness or not does not depend on the choice of the metric ρ . Moreover, the expansiveness property of an action may be correctly defined in the presence of a uniform structure, in particular, for actions on non-metrizable compacts. However, the (maximal possible) value of the expansive constant for an action on a metrizable space does depend on the choice of the metric.

The *homoclinic group* for an action $(T^{\mathbf{n}})_{\mathbf{n} \in \mathbb{Z}^d}$ is, by definition, a subgroup $\Gamma \subset X$ defined by $\Gamma = \{\gamma : \gamma \in X, T^{\mathbf{n}}\gamma \rightarrow 0 \text{ as } \|\mathbf{n}\| \rightarrow \infty\}$.

The homoclinic group of an algebraic $\mathbb{Z}^d$ -action on a compact abelian group is studied in detail in [LiSch], at least in the case of an expansive action. It is shown there that Γ is at most countable in this case. Further, it is proved in [LiSch] that, for an expansive algebraic action on a group $X \neq \{0\}$, Γ is dense in X if and only if the action has completely positive entropy with respect to Haar measure on X . The subgroup Γ is obviously stable under the action $(T^{\mathbf{n}})_{\mathbf{n} \in \mathbb{Z}^d}$. For $x \in X$ let H_x be the associated translation operator so that $(H_x f)(\cdot) = f(\cdot + x)$, $f \in L_2$. Remark that the following relation holds for $x \in X$, $\mathbf{n} \in \mathbb{Z}^d$:

$$U^{-\mathbf{n}} H_x U^{\mathbf{n}} = H_{T^{\mathbf{n}}x}.$$

Unlike the CLT, the ASCLT to hold true in that standard form as stated in Definition 2.1, we need to impose some regularity condition on the growth of cardinality of $|\mathbf{D}_n|$. By definition, a sequence $(\mathbf{D}_n)_{n \in \mathbb{Z}_+}$ belongs to the class $\mathcal{R}$ if and only if it satisfies the condition (2.1) and for certain $\alpha \in [1, \infty)$, A_1, A_2 ($0 < A_1 \leq A_2 < \infty$) we have, for every $n \in \mathbb{Z}_+$, that $A_1 n^\alpha \leq |\mathbf{D}_n| \leq A_2 n^\alpha$.

Theorem 2.2. *Let $(T^{\mathbf{n}})_{\mathbf{n} \in \mathbb{Z}^d}$ be an expansive algebraic action with completely positive entropy on a compact second countable abelian group X , and let $f \in L_2^0$ be a real-valued function satisfying, for every $\gamma \in \Gamma$,*

$$\sum_{\mathbf{p} \in \mathbb{Z}^d} |f - H_{T^{\mathbf{p}}\gamma} f|_2 < \infty.$$

Then the CLT holds for f and

$$\sigma_f^2 = \sum_{\mathbf{p} \in \mathbb{Z}^d} (U^{\mathbf{p}} f, f),$$

where the series converges absolutely. Moreover, the function f satisfies the ASCLT for the class $\mathcal{R}$ with the same value of σ_f^2 .

2.3. Homoclinic Laplace operator: Propositions 2.4 and 2.5. Our consideration in this subsection are carried out in a more abstract setting than that of the previous subsection. The proofs will be given in the Sections 3 and 4, applications to the algebraic actions on compact abelian groups are contained in the Section 5.

Suppose that H is a unitary operator associated with an automorphism (that is a measure preserving invertible transformation) R of $(X, \mathcal{F}, P)$ by the relation $Hf = f \circ R$, $f \in L_2$.

Definition 2.3. ([Ca],[Go1]) R is said to be a (measure preserving) *homoclinic transformation* for $(T^n)_{n \in \mathbb{Z}^d}$ and H is said to be a *homoclinic operator* for $(U^n)_{n \in \mathbb{Z}^d}$ (or for $(T^n)_{n \in \mathbb{Z}^d}$) if they satisfy the relation

$$U^{-n} H U^n \rightarrow I$$

as $\|n\| \rightarrow \infty$.

All operators are considered here as acting on L_2 , I denotes the identity operator and convergence is meant in the sense of the strong operator topology.

Now we need to fix some objects and notations to state our main results. Let $H_1, \dots, H_r$ be a finite set of homoclinic operators corresponding to P -preserving invertible measurable homoclinic transformations $R_1, \dots, R_r$. Let us define some operators on L_2 by the following relations:

$$\begin{aligned} D_{\mathbf{p},l} &= U^{\mathbf{p}}(H_l - I)U^{-\mathbf{p}}, \quad \mathbf{p} \in \mathbb{Z}^d, \quad l \in \{1, \dots, r\}, \\ M_{\mathbf{p}} &= \sum_{l=1}^r D_{\mathbf{p},l}^* D_{\mathbf{p},l}, \quad \mathbf{p} \in \mathbb{Z}^d, \\ \Delta &= \sum_{\mathbf{p} \in \mathbb{Z}^d} M_{\mathbf{p}}. \end{aligned}$$

All $D_{\mathbf{p},l}$ and $M_{\mathbf{p}}$ are bounded operators but Δ (which is called, following [Go2], the *homoclinic Laplace operator*) is not, because its definition includes an infinite sum. Moreover, at the moment Δ is just a formal expression. In order to give a precise meaning to Δ we need some assumptions which will imply, in particular, that Δ is a densely defined selfadjoint operator. Moreover, we impose a strong condition of commutativity on the family $(D_{\mathbf{p},l}, D_{\mathbf{p},l}^*)_{\mathbf{p} \in \mathbb{Z}^d, l \in \{1, \dots, r\}}$.

Assumption 2.4

- i) there exists such a dense linear subspace $\mathcal{D}'$ of L_2^0 and such a constant $c > 0$, that for all $f \in \mathcal{D}'$ the series $\sum_{\mathbf{p} \in \mathbb{Z}^d} M_{\mathbf{p}} f$ strongly converges to an element $\Delta' f \in L_2^0$ such that we have

$$(\Delta' f, f) \geq c(f, f);$$

- ii) operators in the family $(D_{\mathbf{p},l}, D_{\mathbf{p},l}^*)_{\mathbf{p} \in \mathbb{Z}^d, l \in \{1, \dots, r\}}$ pairwise commute.

Remark 2.5.

i) It is wellknown (see, for instance [RiN]), that, assuming i) in 2.4, the completion of $\mathcal{D}'$ with respect to the norm $(\Delta' \cdot, \cdot)^{1/2}$ can be realized as a subspace $\mathcal{D}$ of L_2^0 such that $\mathcal{D} \supseteq \mathcal{D}'$. Further, there exists a unique selfadjoint extension Δ of Δ' (known as Friedrichs' extension) to a subspace $\text{Dom}(\Delta)$ of $\mathcal{D}$ so that

$\mathcal{D}' \subseteq \text{Dom}(\Delta) \subseteq \mathcal{D} \subset L_2^0$. More precisely, $\text{Dom}(\Delta) = \mathcal{D} \cap \text{Dom}((\Delta')^*)$. Then we also have that $\Delta(\text{Dom}(\Delta)) = L_2^0$, every $f \in \text{Dom}(\Delta)$ satisfies the inequality

$$(\Delta f, f) \geq c(f, f),$$

and the quadratic form $(\Delta \cdot, \cdot)$ admits for every $f \in \text{Dom}(\Delta)$ the representation

$$(\Delta f, f) = \sum_{\mathbf{p} \in \mathbb{Z}^d} (M_{\mathbf{p}} f, f).$$

ii) Let us also note that Δ as an operator acting from $\mathcal{D} \subset L_2^0$ to L_2^0 has a right inverse Δ^{-1} with the norm bounded by c^{-1} . Further, all $D_{p,l}$, $D_{p,l}^*$ map L_2 into L_2^0 and can be also viewed as bounded operators in L_2^0 . Then they commute with Δ on $\text{Dom}(\Delta)$ (because they commute on $\mathcal{D}'$). This implies that they commute on L_2^0 with Δ^{-1} . Let us set $F = \Delta_0^{-1} f$ for a certain $f \in L_2^0$. Then we see that

$$|D_{p,l} F|_2 = |D_{p,l} \Delta_0^{-1} f|_2 \leq c^{-1} |D_{p,l} f|_2.$$

iii) It can be easily verified that $f \in \text{Dom}(\Delta)$ whenever $f \in L_2^0$ satisfies the relation

$$\sum_{\mathbf{p} \in \mathbb{Z}^d} \sum_{l=1}^r |D_{\mathbf{p},l} f|_2 < \infty.$$

□

Now we formulate our version of the CLT for a random field produced from an $\mathbb{Z}^d$ -action and a function.

Proposition 2.6. *Suppose that the action $(T^{\mathbf{n}})_{\mathbf{n} \in \mathbb{Z}^d}$ is ergodic, assumption 2.4 is fulfilled and $f \in L_2^0$ is such that*

$$\sum_{\mathbf{p} \in \mathbb{Z}^d} \sum_{l=1}^r |D_{\mathbf{p},l} f|_2 < \infty.$$

Let $(\mathbf{D}_n)_{n \in \mathbb{Z}_+}$ be a sequence of finite subsets of $\mathbb{Z}^d$ satisfying $|\mathbf{D}_n| \rightarrow \infty$, $|\partial \mathbf{D}_n|/|\mathbf{D}_n| \rightarrow 0$.

Then the sequence $(|\mathbf{D}_n|^{-\frac{1}{2}} S_{\mathbf{D}_n} f)_{n \in \mathbb{Z}_+}$ converges in distribution to the normal law $N(0, \sigma_f^2)$, where

$$\sigma_f^2 = \sum_{l=1}^r \left(\sum_{\mathbf{p} \in \mathbb{Z}^d} D_{0,l} U^{\mathbf{p}} F, \sum_{\mathbf{p} \in \mathbb{Z}^d} D_{0,l} U^{\mathbf{p}} f \right).$$

Proposition 2.7. *Let us assume that the action $(T^{\mathbf{n}})_{\mathbf{n} \in \mathbb{Z}^d}$, a function f and a sequence $(\mathbf{D}_n)_{n \in \mathbb{Z}}$ satisfy the assumptions of Proposition 2.6. Denote by $\delta(x)$ the point mass at $x \in \mathbb{R}$. Then with probability 1 the random measure*

$$(\log n)^{-1} \sum_{k \leq n} k^{-1} \delta(|\mathbf{D}_k|^{-\frac{1}{2}} S_{\mathbf{D}_k} f)$$

tends weakly as $n \rightarrow \infty$ to the normal law $N(0, \sigma_f^2)$, where σ_f^2 is the same as in Theorem 2.3.

3. PROOF OF PROPOSITION 2.6

First we need some notations and lemmas. For $\Lambda = \mathbb{Z}^d \times \{1, \dots, r\}$ supplied with the counting measure let us introduce the Hilbert space (over $\mathbb{R}^1$) $E = \ell_2(\Lambda)$ of functions on Λ taking values in $\mathbb{R}^1$. E is equipped with the inner product $[\cdot, \cdot]$ and the norm $\|\cdot\|_2$. We need also the Hilbert space $L_2(E)$ of functions on X with the values in E . $L_2(E)$ has the inner product $(\cdot, \cdot)_{2,E}$ and the norm $\|\cdot\|_{2,E}$. We can describe a function $h \in L_2(E)$ as a family $(h_{\mathbf{p},l})_{(\mathbf{p},l) \in \Lambda}$ where every $h_{\mathbf{p},l}$ is a function from L_2 . We have for such a function h

$$\|h\|_{2,E}^2 = \sum_{(\mathbf{p},l) \in \Lambda} \|h_{\mathbf{p},l}\|_2^2.$$

Let **grad** and **div** be partial mappings from L_2 to $L_2(E)$ and from $L_2(E)$ to L_2 , correspondingly, defined by the formulas:

$$\begin{aligned} (\mathbf{grad} f) &= (D_{\mathbf{p},l} f)_{(\mathbf{p},l) \in \Lambda}, \\ (\mathbf{div} h) &= \sum_{\mathbf{p} \in \mathbb{Z}} \sum_{l=1}^r D_{\mathbf{p},l}^* h_{\mathbf{p},l} \end{aligned}$$

for $f \in L_2^0$ and $h = (h_{\mathbf{p},l})_{(\mathbf{p},l) \in \Lambda} \in L_2(E)$ such that

$$(3.1) \quad \sum_{\mathbf{p} \in \mathbb{Z}} \sum_{l=1}^r \|D_{\mathbf{p},l} f\|_2 < \infty,$$

$$(3.2) \quad \sum_{\mathbf{p} \in \mathbb{Z}} \sum_{l=1}^r \|h_{\mathbf{p},l}\|_2 < \infty.$$

The following formulas are verified easily:

$$\begin{aligned} \Delta f &= \mathbf{div} \mathbf{grad} f, \\ (\Delta f, g) &= (\mathbf{grad} f, \mathbf{grad} g)_{2,E} = ([\mathbf{grad} f, \mathbf{grad} g], 1), \\ \mathbf{grad} Uf &= V \mathbf{grad} f = U(s \mathbf{grad} f), \\ U(\mathbf{div} h) &= \mathbf{div}(Vh) = \mathbf{div}(sUh), \end{aligned}$$

where $h = (h_{\mathbf{p},l})_{(\mathbf{p},l) \in \Lambda}$ and the operators $U^{\mathbf{k}}$, $s^{\mathbf{k}}$ (the shift operator) and $V^{\mathbf{k}}$ are defined by the following relations:

$$\begin{aligned} U^{\mathbf{k}} h &= (U^{\mathbf{k}} h_{\mathbf{p},l})_{(\mathbf{p},l) \in \Lambda} \\ s^{\mathbf{k}} h &= (h_{\mathbf{p}-\mathbf{k},l})_{(\mathbf{p},l) \in \Lambda} \\ V^{\mathbf{k}} h &= s^{\mathbf{k}} U^{\mathbf{k}} h = (U^{\mathbf{k}} h_{\mathbf{p}-\mathbf{k},l})_{(\mathbf{p},l) \in \Lambda}. \end{aligned}$$

Let us denote A the subspace of $L_2(E)$ consisting of all such $h \in L_2(E)$, which satisfy the condition

$$\sum_{(\mathbf{p},l) \in \Lambda} \|h_{\mathbf{p},l}\|_2 < \infty.$$

Lemma 3.1. *If $h \in A$ then the limit*

$$\lim |\mathbf{D}_n|^{-1} \left| \sum_{\mathbf{k} \in \mathbf{D}_n} V^{\mathbf{k}} h \right|_{2,E}^2 = v(h)$$

exists and

$$v(h) = \sum_{l=1}^r \left| \sum_{\mathbf{p} \in \mathbb{Z}^d} U^{-\mathbf{p}} h_{\mathbf{p},l} \right|_2^2,$$

where the series converges absolutely.

Proof. We follow here a reasoning in [De]. Let $\mathbf{D} - \mathbf{k}$ denotes the translate of $\mathbf{D} \subset \mathbb{Z}^d$ by a vector $-\mathbf{k} \in \mathbb{Z}^d$. Remark that, in view of the relations $|\mathbf{D}_n| \rightarrow \infty$, $|\partial \mathbf{D}_n|/|\mathbf{D}_n| \rightarrow 0$, we have for any fixed $\mathbf{k} \in \mathbb{Z}^d$

$$|\mathbf{D}_n|^{-1} |\mathbf{D}_n \cap (\mathbf{D}_n - \mathbf{k})| \rightarrow 1$$

as $n \rightarrow \infty$. Now we have

$$\begin{aligned} \sum_{\mathbf{k} \in \mathbb{Z}^d} |(V^{\mathbf{k}} h, h)_{2,E}| &= \sum_{\mathbf{k} \in \mathbb{Z}^d} \left| \sum_{(\mathbf{p},l) \in \Lambda} (U^{\mathbf{k}} h_{\mathbf{p}-\mathbf{k},l}, h_{\mathbf{p},l}) \right| \\ &\leq \sum_{\mathbf{k} \in \mathbb{Z}^d} \sum_{(\mathbf{p},l) \in \Lambda} |h_{\mathbf{p}-\mathbf{k},l}|_2 |h_{\mathbf{p},l}|_2 \\ &\leq \left(\sum_{(\mathbf{p},l) \in \Lambda} |h_{\mathbf{p},l}|_2 \right)^2 < \infty. \end{aligned}$$

Therefore, the series

$$\sum_{\mathbf{k} \in \mathbb{Z}^d} (V^{\mathbf{k}} h, h)_{2,E} = \sum_{l=1}^r \sum_{\mathbf{p} \in \mathbb{Z}^d} \sum_{\mathbf{q} \in \mathbb{Z}^d} (U^{-\mathbf{p}} h_{\mathbf{p},l}, U^{-\mathbf{q}} h_{(\mathbf{q},l)})$$

converges absolutely, and the limit

$$\lim_{n \rightarrow \infty} |\mathbf{D}_n|^{-1} \left| \sum_{\mathbf{k} \in \mathbf{D}_n} V^{\mathbf{k}} h \right|_{2,E}^2 = \lim_{n \rightarrow \infty} \sum_{\mathbf{k} \in \mathbb{Z}^d} |\mathbf{D}_n|^{-1} |\mathbf{D}_n \cap (\mathbf{D}_n - \mathbf{k})| (V^{\mathbf{k}} h, h)_{2,E}$$

exists and equals to $v(h)$. $\square$

Lemma 3.2. *Let $F \in L_2^0$ and satisfy the condition (3.1). Let us define a function $B(F) \in L_2(E)$ by the expression*

$$\begin{aligned} B_{(\mathbf{0},l)}(F) &= \sum_{\mathbf{p} \in \mathbb{Z}^d} U^{-\mathbf{p}} D_{\mathbf{p},l} F, \\ B_{(\mathbf{s},l)}(F) &= 0, \quad \mathbf{s} \in \mathbb{Z}^d, \mathbf{s} \neq \mathbf{0}, \quad l = 1, \dots, r. \end{aligned}$$

Then

$$|\mathbf{D}_n|^{-1} \left| \mathbf{grad} S_{\mathbf{D}_n} F - \sum_{\mathbf{k} \in \mathbf{D}_n} V^{\mathbf{k}} B(F) \right|_{2,E}^2 \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Proof. If h is defined by the expression $h = \mathbf{grad} F - B(F) \in A$, then we have for $l \in \{1, \dots, r\}$ that

$$\begin{aligned} h_{\mathbf{0},l} &= D_{\mathbf{0},l} F - \sum_{\mathbf{q} \in \mathbb{Z}^d} U^{-\mathbf{q}} D_{\mathbf{q},l} F, \\ h_{\mathbf{p},l} &= D_{\mathbf{p},l} F, \quad \mathbf{p} \in \mathbb{Z}^d \setminus \{0\}, \end{aligned}$$

and

$$\sum_{\mathbf{p} \in \mathbb{Z}^d} U^{-\mathbf{p}} h_{\mathbf{p},l} = 0.$$

Taking in view the identity

$$\mathbf{grad} S_{\mathbf{D}_n} F - \sum_{\mathbf{k} \in \mathbf{D}_n} V^{\mathbf{k}} B(F) = \sum_{\mathbf{k} \in \mathbf{D}_n} V^{\mathbf{k}} (\mathbf{grad} F - B(F))$$

we obtain by Lemma 3.1 that

$$\lim_{n \rightarrow \infty} |\mathbf{D}_n|^{-1} \left| \mathbf{grad} S_{\mathbf{D}_n} F - \sum_{\mathbf{k} \in \mathbf{D}_n} V^{\mathbf{k}} B(F) \right|_{2,E}^2 = \sum_{l=1}^r \left| \sum_{\mathbf{p} \in \mathbb{Z}^d} U^{-\mathbf{p}} h_{\mathbf{p},l} \right|_2^2 = 0.$$

□

Lemma 3.3. *If $f, g \in L_2^0$ are such that*

$$\sum_{l=1}^r \sum_{\mathbf{p} \in \mathbb{Z}^d} |D_{\mathbf{p},l} f|_2 < \infty, \quad \sum_{l=1}^r \sum_{\mathbf{p} \in \mathbb{Z}^d} |D_{\mathbf{p},l} g|_2 < \infty$$

and, by definition,

$$\begin{aligned} \langle f, g \rangle &= \sum_{l=1}^r \left(\sum_{\mathbf{p} \in \mathbb{Z}^d} U^{-\mathbf{p}} D_{\mathbf{p},l} f \right) \left(\sum_{\mathbf{q} \in \mathbb{Z}^d} U^{-\mathbf{q}} D_{\mathbf{q},l} g \right) \\ &= \sum_{l=1}^r \sum_{\mathbf{p} \in \mathbb{Z}^d} \sum_{\mathbf{q} \in \mathbb{Z}^d} \left(D_{\mathbf{0},l} U^{-\mathbf{p}} f \cdot D_{\mathbf{0},l} U^{-\mathbf{q}} g \right), \end{aligned}$$

then the series representing $\langle f, g \rangle$ converges in the L_1 -norm and

$$|\mathbf{D}_n|^{-1} \left| [\mathbf{grad} S_{\mathbf{D}_n} f, \mathbf{grad} S_{\mathbf{D}_n} g] - S_{\mathbf{D}_n} \langle f, g \rangle \right|_1 \rightarrow 0$$

as $n \rightarrow \infty$.

Proof. The L_1 -convergence of the series is a consequence of the estimate

$$|U^{-\mathbf{p}} D_{\mathbf{p},l} f \cdot U^{-\mathbf{q}} D_{\mathbf{q},l} g|_1 \leq |D_{\mathbf{p},l} f|_2 |D_{\mathbf{q},l} g|_2.$$

By polarisation argument it suffices to consider the case $f = g$. Then we see that $S_{\mathbf{D}_n}\langle f, f \rangle = [\sum_{\mathbf{k} \in \mathbf{D}_n} V^{\mathbf{k}} B(f), \sum_{\mathbf{k} \in \mathbf{D}_n} V^{\mathbf{k}} B(f)]$ where $B(f)$ is a function defined in the statement of Lemma 3.2.

It follows from Lemmas 3.1 and 3.2 that

$$\lim_{n \rightarrow \infty} |\mathbf{D}_n|^{-\frac{1}{2}} \left| \mathbf{grad} S_{\mathbf{D}_n} f - \sum_{\mathbf{k} \in \mathbf{D}_n} V^{\mathbf{k}} B(f) \right|_{2,E} = 0$$

and

$$\lim_{n \rightarrow \infty} |\mathbf{D}_n|^{-\frac{1}{2}} \left| \mathbf{grad} S_{\mathbf{D}_n} f \right|_{2,E} = \lim_{n \rightarrow \infty} |\mathbf{D}_n|^{-\frac{1}{2}} \left| \sum_{\mathbf{k} \in \mathbf{D}_n} V^{\mathbf{k}} B(f) \right|_{2,E} < \infty.$$

Then

$$\begin{aligned} & |\mathbf{D}_n|^{-1} \left| [\mathbf{grad} S_{\mathbf{D}_n} f, \mathbf{grad} S_{\mathbf{D}_n} f] - S_{\mathbf{D}_n} \langle f, f \rangle \right|_1 \\ &= |\mathbf{D}_n|^{-1} \left| \left\| \mathbf{grad} S_{\mathbf{D}_n} f \right\|^2 - \left\| \sum_{\mathbf{k} \in \mathbf{D}_n} V^{\mathbf{k}} B(f) \right\|^2 \right|_1 \\ &\leq |\mathbf{D}_n|^{-1} \left| \left\| \mathbf{grad} S_{\mathbf{D}_n} f - \sum_{\mathbf{k} \in \mathbf{D}_n} V^{\mathbf{k}} B(f) \right\| \cdot \left(\left\| \mathbf{grad} S_{\mathbf{D}_n} f \right\| + \left\| \sum_{\mathbf{k} \in \mathbf{D}_n} V^{\mathbf{k}} B(f) \right\| \right) \right|_1 \\ &\leq |\mathbf{D}_n|^{-1} \left| \mathbf{grad} S_{\mathbf{D}_n} f - \sum_{\mathbf{k} \in \mathbf{D}_n} V^{\mathbf{k}} B(f) \right|_{2,E} \cdot \left| 2 \left\| \mathbf{grad} S_{\mathbf{D}_n} f \right\|^2 + 2 \left\| \sum_{\mathbf{k} \in \mathbf{D}_n} V^{\mathbf{k}} B(f) \right\|^2 \right|_1^{\frac{1}{2}} \\ &\leq 2^{\frac{1}{2}} |\mathbf{D}_n|^{-\frac{1}{2}} \left| \mathbf{grad} S_{\mathbf{D}_n} f - \sum_{\mathbf{k} \in \mathbf{D}_n} V^{\mathbf{k}} B(f) \right|_{2,E} \\ &\quad \times |\mathbf{D}_n|^{-\frac{1}{2}} \left(\left\| \mathbf{grad} S_{\mathbf{D}_n} f \right\|_{2,E}^2 + \left\| \sum_{\mathbf{k} \in \mathbf{D}_n} V^{\mathbf{k}} B(f) \right\|_{2,E}^2 \right)^{\frac{1}{2}} \rightarrow 0 \end{aligned}$$

as $n \rightarrow \infty$. $\square$

Lemma 3.4. *Let $\varphi: \mathbb{R}^1 \rightarrow \mathbb{R}^1$ be a bounded continuously differentiable function with continuous derivative φ' such that $\varphi'(u)$ tends to finite limits as $u \rightarrow -\infty$ and $u \rightarrow +\infty$. If $f \in L_2^0$ satisfies the condition (1) then we have that*

$$\left| \mathbf{grad} \varphi(|\mathbf{D}_n|^{-\frac{1}{2}} S_{\mathbf{D}_n} f) - \varphi'(|\mathbf{D}_n|^{-\frac{1}{2}} S_{\mathbf{D}_n} f) \mathbf{grad}(|\mathbf{D}_n|^{-\frac{1}{2}} S_{\mathbf{D}_n} f) \right|_{2,E} \rightarrow 0$$

as $n \rightarrow \infty$.

Proof. We have

$$\begin{aligned} & \left| \mathbf{grad} \varphi(|\mathbf{D}_n|^{-\frac{1}{2}} S_{\mathbf{D}_n} f) - \varphi'(|\mathbf{D}_n|^{-\frac{1}{2}} S_{\mathbf{D}_n} f) \mathbf{grad}(|\mathbf{D}_n|^{-\frac{1}{2}} S_{\mathbf{D}_n} f) \right|_{2,E}^2 \\ &= \sum_{(\mathbf{p}, l) \in \Lambda} \left| D_{\mathbf{p}, l} \varphi(|\mathbf{D}_n|^{-\frac{1}{2}} S_{\mathbf{D}_n} f) - \varphi'(|\mathbf{D}_n|^{-\frac{1}{2}} S_{\mathbf{D}_n} f) |\mathbf{D}_n|^{-\frac{1}{2}} D_{\mathbf{p}, l} S_{\mathbf{D}_n} f \right|_2^2 \\ &= \sum_{(\mathbf{p}, l) \in \Lambda} \left| D_{\mathbf{0}, l} \varphi(|\mathbf{D}_n|^{-\frac{1}{2}} U^{-\mathbf{p}} S_{\mathbf{D}_n} f) - \varphi'(|\mathbf{D}_n|^{-\frac{1}{2}} U^{-\mathbf{p}} S_{\mathbf{D}_n} f) |\mathbf{D}_n|^{-\frac{1}{2}} D_{\mathbf{0}, l} U^{-\mathbf{p}} S_{\mathbf{D}_n} f \right|_2^2. \end{aligned}$$

Since for any $l = 1, \dots, r$ and all $n \in \mathbb{Z}_+$

$$|D_{0,l}U^{-\mathbf{p}}S_{\mathbf{D}_n}f| \leq \sum_{\mathbf{q} \in \mathbb{Z}^d} |D_{0,l}(U^{\mathbf{q}}f)| \in L_2$$

and φ' is a uniformly continuous function then

$$\begin{aligned} \delta_n &= \sup_{(\mathbf{p},l) \in \Lambda} \frac{|D_{0,l}\varphi(|\mathbf{D}_n|^{-\frac{1}{2}}U^{-\mathbf{p}}S_{\mathbf{D}_n}f) - \varphi'(|\mathbf{D}_n|^{-\frac{1}{2}}U^{-\mathbf{p}}S_{\mathbf{D}_n}f)|\mathbf{D}_n|^{-\frac{1}{2}}D_{0,l}U^{-\mathbf{p}}S_{\mathbf{D}_n}f|}{|D_{0,l}(|\mathbf{D}_n|^{-\frac{1}{2}}U^{-\mathbf{p}}S_{\mathbf{D}_n}f)|} \\ &= \sup_{(\mathbf{p},l) \in \Lambda} \left| \varphi'(|\mathbf{D}_n|^{-\frac{1}{2}}U^{-\mathbf{p}}S_{\mathbf{D}_n}f + \theta_{\mathbf{p},l,n}|\mathbf{D}_n|^{-\frac{1}{2}}D_{0,l}U^{-\mathbf{p}}S_{\mathbf{D}_n}f) - \varphi'(|\mathbf{D}_n|^{-\frac{1}{2}}U^{-\mathbf{p}}S_{\mathbf{D}_n}f) \right| \xrightarrow[n \rightarrow \infty]{} 0, \end{aligned}$$

where $\theta_{\mathbf{p},l,n}$, $|\theta_{\mathbf{p},l,n}| \leq 1$, are multipliers from the Lagrange formula, and $0/0 = 0$ by definition. As a consequence of this we have that $\delta_n \rightarrow 0$ in probability as $n \rightarrow \infty$. Besides we have $|\delta_n|_\infty \leq 2 \sup |\varphi'|$. Furthermore,

$$\begin{aligned} \sum_{(\mathbf{p},l) \in \Lambda} (|\mathbf{D}_n|^{-\frac{1}{2}}D_{0,l}U^{-\mathbf{p}}S_{\mathbf{D}_n}f)^2 &= \frac{1}{|\mathbf{D}_n|} \sum_{l=0}^r \sum_{\mathbf{m} \in \mathbf{D}_n} \sum_{\mathbf{k} \in \mathbf{D}_n} \sum_{\mathbf{p} \in \mathbb{Z}^d} (D_{0,l}U^{\mathbf{p}+\mathbf{m}}f)(D_{0,l}U^{\mathbf{p}+\mathbf{k}}f) \\ &= \sum_{l=0}^r \sum_{\mathbf{k} \in \mathbf{D}_n} \frac{1}{|\mathbf{D}_n|} |\mathbf{D}_n \cap (\mathbf{D}_n - \mathbf{k})| \sum_{\mathbf{p} \in \mathbb{Z}^d} (D_{0,l}U^{\mathbf{p}+\mathbf{k}}f)(D_{0,l}U^{\mathbf{p}}f) \\ &\leq \left(\sum_{(\mathbf{p},l) \in \Lambda} |D_{0,l}U^{\mathbf{p}}f| \right)^2 \in L_1 \end{aligned}$$

because

$$\sum_{(\mathbf{p},l) \in \Lambda} |D_{0,l}U^{\mathbf{p}}f|_2 = \sum_{(\mathbf{p},l) \in \Lambda} |D_{\mathbf{p},l}f|_2 < \infty.$$

Together with convergence and boundedness properties of $\{\delta_n\}$ it gives us that

$$\begin{aligned} &|\mathbf{grad} \varphi(|\mathbf{D}_n|^{-\frac{1}{2}}S_{\mathbf{D}_n}f) - \varphi'(|\mathbf{D}_n|^{-\frac{1}{2}}S_{\mathbf{D}_n}f) \mathbf{grad}(|\mathbf{D}_n|^{-\frac{1}{2}}S_{\mathbf{D}_n}f)|_{2,E}^2 \\ &\leq \left| \delta_n \sum_{(\mathbf{p},l) \in \Lambda} |D_{0,l}U^{\mathbf{p}}f| \right|_2^2 \rightarrow 0 \end{aligned}$$

as $n \rightarrow \infty$, and Lemma 3.4 is proved. $\square$

Proof of Proposition 2.6. We will use Ch. Stein's approach (see [St] or [Go1] for details). Set $W_n = |\mathbf{D}_n|^{-\frac{1}{2}}S_{\mathbf{D}_n}f$. According to this approach, to prove the convergence in distribution of a sequence $\{W_n\}$ of random variables to the normal law $N(0, \sigma_f^2)$ it suffices to establish that $(W_n, \varphi(W_n)) - \sigma_f^2(\varphi'(W_n), 1) \rightarrow 0$ as $n \rightarrow \infty$ for every continuously differentiable function $\varphi: \mathbb{R}^1 \rightarrow \mathbb{R}^1$ such that $\varphi(x)$ and $\varphi'(x)$ converges to finite limits as x goes to $+\infty$ or to $-\infty$. Set $F = \Delta^{-1}f$. It follows from

Remark 2.5, ii), that F satisfies (3.1) because so is f . We have

$$\begin{aligned}
(W_n, \varphi(W_n)) &= (|\mathbf{D}_n|^{-\frac{1}{2}} S_{\mathbf{D}_n} f, \varphi(W_n)) \\
&= (|\mathbf{D}_n|^{-\frac{1}{2}} S_{\mathbf{D}_n} \Delta F, \varphi(W_n)) \\
&= (|\mathbf{D}_n|^{-\frac{1}{2}} \Delta S_{\mathbf{D}_n} F, \varphi(W_n)) \\
&= (|\mathbf{D}_n|^{-\frac{1}{2}} \mathbf{div} \mathbf{grad}(S_{\mathbf{D}_n} F), \varphi(W_n)) \\
&= (|\mathbf{D}_n|^{-\frac{1}{2}} \mathbf{grad} S_{\mathbf{D}_n} F, \mathbf{grad} \varphi(W_n))_{2,E} \\
&= (|\mathbf{D}_n|^{-\frac{1}{2}} \mathbf{grad} S_{\mathbf{D}_n} F, \varphi'(W_n) \mathbf{grad} W_n)_{2,E} + C_{1,n} \\
&= |\mathbf{D}_n|^{-1} ([\mathbf{grad} S_{\mathbf{D}_n} F, \mathbf{grad} S_{\mathbf{D}_n} f], \varphi'(W_n))_2 + C_{1,n} \\
&= |\mathbf{D}_n|^{-1} (S_{\mathbf{D}_n} \langle F, f \rangle, \varphi'(W_n))_2 + C_{1,n} + C_{2,n} \\
&= (\langle F, f \rangle, 1)(\varphi'(W_n), 1) + C_{1,n} + C_{2,n} + C_{3,n},
\end{aligned}$$

where

$$\begin{aligned}
C_{1,n} &= (|\mathbf{D}_n|^{-\frac{1}{2}} \mathbf{grad} S_{\mathbf{D}_n} F, \mathbf{grad} \varphi(W_n) - \varphi'(W_n) \mathbf{grad} W_n)_{2,E}, \\
C_{2,n} &= (|\mathbf{D}_n|^{-1} [\mathbf{grad} S_{\mathbf{D}_n} F, \mathbf{grad} S_{\mathbf{D}_n} f] - |\mathbf{D}_n|^{-1} S_{\mathbf{D}_n} \langle F, f \rangle, \varphi'(W_n)), \\
C_{3,n} &= (|\mathbf{D}_n|^{-1} S_{\mathbf{D}_n} \langle F, f \rangle, \varphi'(W_n)) - (\langle F, f \rangle, 1)(\varphi'(W_n), 1).
\end{aligned}$$

To complete the proof we have to verify that $C_{i,n} \rightarrow 0$ as $n \rightarrow \infty$ for $i = 1, 2, 3$.

It follows from Lemma 3.1 and the identity $\mathbf{grad} S_{\mathbf{D}_n} F = \sum_{\mathbf{k} \in \mathbb{Z}^d} V^{\mathbf{k}} \mathbf{grad} F$ that the left component of the inner product in the above formula for $(C_{1,n})$ is bounded in the $L_{2,E}$ -norm. The right component of the same product tends to 0 by Lemma 3.4. Thus, the sequence $(C_{1,n})$ goes to 0. It follows from Lemma 3.3 and boundedness of φ' that $C_{2,n} \rightarrow 0$. Finally, $(C_{3,n})$ goes to 0 by the ergodic theorem for stationary random fields([Ge]) applied to the function $\langle F, f \rangle$.

The relation

$$(\langle F, f \rangle, 1) = \sum_{l=1}^r \left(\sum_{p \in \mathbb{Z}^d} D_{0,l} U^p F, \sum_{q \in \mathbb{Z}^d} D_{0,l} U^q f \right)$$

is a consequence of the definition of $\langle \cdot, \cdot \rangle$ in the statement of Lemma 3.3. $\square$ $\square$

4. PROOF OF PROPOSITION 2.7

We set for a function $\varphi : \mathbb{R} \mapsto \mathbb{R}$

$$\|\varphi\|_L = \sup \{ |\varphi(x) - \varphi(y)| / |x - y| : x, y \in \mathbb{R}, x \neq y \}.$$

Let $BL = BL(\mathbb{R})$ be the space of all such functions $\varphi : \mathbb{R} \mapsto \mathbb{R}$ that the norm $\|\varphi\|_{BL} = \|\varphi\|_L + \sup_{x \in \mathbb{R}} |\varphi(x)|$ is finite. We suppose that $(\mathbf{D}_n)_{n \geq 0}$ is a sequence of finite subsets of $\mathbb{Z}^d$ and $f \in L_2$ is a function both satisfying the assumptions of the Proposition 2.7. Let random variables ξ_k be defined for $k > 0$ by the relation

$$\xi_k = \varphi(|\mathbf{D}_k|^{-\frac{1}{2}} S_{\mathbf{D}_k} f) - \mathbf{E} \varphi(|\mathbf{D}_k|^{-\frac{1}{2}} S_{\mathbf{D}_k} f).$$

In the sequel we use the denotations introduced in the section 2.

Lemma 4.1. *For some constant $K = K(\varphi, f)$ every pair ξ_j, ξ_k $i, j \in \mathbb{Z}_+$ satisfies the inequality*

$$|\mathbf{E}(\xi_j, \xi_k)| \leq K |\mathbf{D}_j|^{\frac{1}{2}} |\mathbf{D}_k|^{-\frac{1}{2}}.$$

Proof. Set for $k > 0$

$$t_k = |\mathbf{D}_k|^{-\frac{1}{2}} S_{\mathbf{D}_k} f,$$

so that

$$\xi_k = \varphi(t_k), \quad k > 0.$$

Let us remark first that for any $(\mathbf{p}, l) \in \Lambda$ we have

$$|D_{\mathbf{p}, l} t_k|_2 \leq |\mathbf{D}_k|^{-\frac{1}{2}} \sum_{\mathbf{m} \in \mathbf{D}_k} |D_{\mathbf{p}, l} U^{\mathbf{m}} f|_2 = |\mathbf{D}_k|^{-\frac{1}{2}} \sum_{\mathbf{m} \in \mathbf{D}_k} |D_{\mathbf{p}-\mathbf{m}, l} f|_2.$$

Furthermore,

$$\begin{aligned} |D_{\mathbf{p}, l} \xi_k|_2 &= |\varphi(t_k) - \varphi(t_k \circ (T^{-\mathbf{p}} \circ R_l \circ T^{\mathbf{p}}))|_2 \\ &\leq \|\varphi\|_{BL} |t_k - t_k \circ T^{-\mathbf{p}} \circ R_l \circ T^{\mathbf{p}}|_2 = \|\varphi\|_{BL} |D_{\mathbf{p}, l} t_k|_2, \end{aligned}$$

which implies that

$$|D_{\mathbf{p}, l} \xi_k|_2 \leq \|\varphi\|_{BL} |\mathbf{D}_k|^{-\frac{1}{2}} \sum_{\mathbf{m} \in \mathbf{D}_k} |D_{\mathbf{p}-\mathbf{m}, l} f|_2.$$

Finally, $D_{\mathbf{p}, l} \xi_k \in L_2^0$, because T and R_l preserves the measure P . Under assumptions of Proposition 2.7 the operator Δ maps its domain onto the space L_2^0 and has on this space a uniquely determined right inverse operator Δ^{-1} bounded by c^{-1} . In particular, we have for each $k > 0$

$$\xi_k = \Delta \Xi_k, \quad \Xi_k \in L^0, \quad |\Xi_k|_2 \leq c^{-1} |\xi_k|_2.$$

It follows from our commutativity assumptions that

$$\Delta D_{\mathbf{p}, l} \Xi_k = D_{\mathbf{p}, l} \Delta \Xi_k = D_{\mathbf{p}, l} \xi_k.$$

Now we can conclude from the above inequalities that

$$|D_{\mathbf{p}, l} \Xi_k|_2 \leq c^{-1} |D_{\mathbf{p}, l} \xi_k|_2 \leq c^{-1} \|\varphi\|_{BL} |D_{\mathbf{p}, l} t_k|_2 \leq c^{-1} \|\varphi\|_{BL} |\mathbf{D}_k|^{-\frac{1}{2}} \sum_{\mathbf{m} \in \mathbf{D}_k} |D_{\mathbf{p}-\mathbf{m}, l} f|_2.$$

Then we obtain from our estimates for $|D_{\mathbf{p}, l} \Xi_k|_2$ and $|D_{\mathbf{p}, l} \xi_k|_2$ that

$$\begin{aligned} |(\xi_k, \xi_j)| &= |(\Delta \Xi_k, \xi_j)| \leq \sum_{(\mathbf{p}, l) \in \Lambda} |(D_{\mathbf{p}, l} \Xi_k, D_{\mathbf{p}, l} \xi_j)| \\ &\leq c^{-1} \|\varphi\|_{BL} |\mathbf{D}_k|^{-\frac{1}{2}} |\mathbf{D}_j|^{-\frac{1}{2}} \sum_{(\mathbf{p}, l) \in \Lambda} \sum_{\mathbf{m} \in \mathbf{D}_j} \sum_{\mathbf{n} \in \mathbf{D}_k} |D_{\mathbf{p}-\mathbf{m}, l} f|_2 |D_{\mathbf{p}-\mathbf{n}, l} f|_2 \\ &\leq c^{-1} \|\varphi\|_{BL} |\mathbf{D}_k|^{-\frac{1}{2}} |\mathbf{D}_j|^{-\frac{1}{2}} \sum_{l=0}^r \sum_{(\mathbf{p}, l) \in \Lambda} \sum_{\mathbf{n} \in \mathbb{Z}^d} |D_{\mathbf{n}, l} f|_2 \sum_{\mathbf{m} \in \mathbf{D}_j} |D_{\mathbf{p}-\mathbf{m}, l} f|_2 \end{aligned}$$

$$\begin{aligned}
&\leq c^{-1} \|\varphi\|_{BL} |\mathbf{D}_k|^{-\frac{1}{2}} |\mathbf{D}_j|^{-\frac{1}{2}} \sum_{l=0}^r \sum_{\mathbf{n} \in \mathbb{Z}^d} |D_{\mathbf{n},l} f|_2 \sum_{\mathbf{p} \in \mathbb{Z}^d} |\mathbf{D}_j| |D_{\mathbf{p},l} f|_2 \\
&\leq c^{-1} \|\varphi\|_{BL} |\mathbf{D}_j|^{\frac{1}{2}} |\mathbf{D}_k|^{-\frac{1}{2}} \left(\sum_{(\mathbf{p},l) \in \Lambda} |D_{\mathbf{p},l} f|_2 \right)^2,
\end{aligned}$$

which proofs the lemma. $\square$

Set for $n > 0$

$$Z_n = \sum_{2^n \leq k < 2^{n+1}} k^{-1} \xi_k,$$

and for $k \geq 0, m \geq 0$

$$a_{k,m} = \mathbf{E} Z_k Z_m.$$

Lemma 4.2. *There exists such a constant C_1 that for any $k > 0, m > 0$ we have*

$$|a_{k,m}| \leq C_1 2^{-\alpha|m-k|/2}.$$

Proof. Recall that $(|\mathbf{D}_n|)_{n \in \mathbb{Z}_+}$ satisfies $A_1 n^\alpha \leq |\mathbf{D}_n| \leq A_2 n^\alpha$. By Lemma 4.1 we have for $k \leq m$ that

$$\begin{aligned}
|a_{k,m}| &= |\mathbf{E} Z_k Z_m| \leq K \sum_{2^k \leq I < 2^{k+1}} \sum_{2^m \leq J < 2^{m+1}} \frac{1}{IJ} (|\mathbf{D}_I|/|\mathbf{D}_J|)^{\frac{1}{2}} \\
&\leq K (A_2/A_1)^{\frac{1}{2}} \sum_{2^k \leq I < 2^{k+1}} I^{\frac{\alpha}{2}-1} \sum_{2^m \leq J < 2^{m+1}} 1/J^{\frac{\alpha}{2}+1},
\end{aligned}$$

and we get

$$|a_{k,m}| \leq C_1 2^{-\alpha(m-k)/2},$$

and the lemma follows from $a_{k,m} = a_{m,k}$. $\square$

The following lemma asserts that $(Z_k)_{k \geq 1}$ is a *quasiorthogonal sequence*.

Lemma 4.3. *There exists such a constant C that for any real sequence $(x_i)_{i \leq 1} \in l_2$ we have*

$$\sum_{k,m \geq 1} a_{k,m} x_k x_m \leq C \sum_{k \geq 1} x_k^2.$$

Proof. It follows from Lemma 4.2 that $\sup_{k \geq 1} \sum_{m \geq 1} |a_{k,m}| \leq 3C_1(\alpha-1)^{-1} < \infty$. The latter inequality implies quasiorthogonality of $(Z_k)_{k \geq 1}$. Indeed, making use the relation $a_{k,m} = a_{m,k}$, we get

$$\begin{aligned}
&\left| \sum_{1 \leq k, m < \infty} a_{k,m} x_k x_m \right| \leq \sum_{1 \leq k, m < \infty} |a_{k,m}| |x_k| |x_m| \\
&\leq \sum_{1 \leq k, m < \infty} |a_{k,m}| \left(\frac{x_k^2 + x_m^2}{2} \right) \leq 6C_1 \sum_{k \geq 1} x_k^2 = C \sum_{k \geq 1} x_k^2.
\end{aligned}$$

$\square$

Proof of Proposition 2.7. It is known ([We1]) that quasiorthogonality of $(Z_k)_{k \geq 1}$ implies the almost sure convergence of the series $\sum c_n Z_n$ provided the relation $\sum c_n^2 \log^2 n < \infty$ holds (we assume that $c_n \in \mathbb{R}$ for all $n \geq 1$). Choose $b > 3/2$ and set $c_n = n^{-1/2}(\log n)^{-b}$. Then the series

$$\sum_n n^{-1/2}(\log n)^{-b} \left(\sum_{2^n \leq k < 2^{n+1}} k^{-1} \xi_k \right)$$

converges almost surely. By Kronecker lemma, with probability 1

$$\lim_{n \rightarrow \infty} \frac{1}{(\log 2^n)^{1/2} (\log \log 2^n)^b} \sum_{k=1}^{2^n} k^{-1} \xi_k = 0.$$

To deduce from this relation that in fact even

$$\lim_{m \rightarrow \infty} \frac{1}{(\log m)^{1/2} (\log \log m)^b} \sum_{k=1}^m k^{-1} \xi_k = 0$$

almost surely it suffices to remark that for $2^n \leq m < 2^{n+1}$

$$\begin{aligned} \left| \frac{1}{(\log m)^{1/2} (\log \log m)^b} \sum_{k=1}^m k^{-1} \xi_k \right| &\leq \left| \frac{1}{(\log 2^n)^{1/2} (\log \log 2^n)^b} \sum_{k=1}^{2^n} k^{-1} \xi_k \right| \\ &+ \frac{2}{(\log 2^n)^{1/2} (\log \log 2^n)^b} \|\varphi\|_{BL}. \end{aligned}$$

In fact we only need much weaker relation

$$\lim_{m \rightarrow \infty} \frac{1}{\log m} \sum_{k=1}^m k^{-1/2} \xi_k$$

than was established above. Now we notice that by the Proposition 2.6 we have

$$\lim_{n \rightarrow \infty} (\log n)^{-1} \sum_{k \leq n} k^{-1} \mathbf{E} \varphi(|\mathbf{D}_k|^{-\frac{1}{2}} S_{\mathbf{D}_k} f) = \int \varphi(t) n_{0, \sigma_f^2}(t) dt,$$

where $n_{0, \sigma_f^2}(\cdot)$ is the density of $N(0, \sigma_f^2)$ (we mean that $\int \varphi(t) n_{0, \sigma_f^2}(t) dt = \varphi(0)$). Thus, we proved that for each $\varphi \in BL$ the integral of φ with respect to the "empiric measure"

$$(\log n)^{-1} \sum_{k \leq n} k^{-1} \delta(|\mathbf{D}_k|^{-\frac{1}{2}} S_{\mathbf{D}_k} f)$$

tends almost surely to $\int \varphi(u) n_{0, \sigma_f^2}(u) du$. Functions from BL form in the uniform norm a dense subspace in the space of bounded continuous functions on $\mathbb{R}$. The proof is completed. $\square$

5. APPLICATION TO ALGEBRAIC $\mathbb{Z}^d$ -ACTIONS ON COMPACT ABELIAN GROUPS: PROOF OF THEOREM 2.2

An *algebraic* action of a discrete group is an action of this group by continuous group automorphisms of a locally compact abelian group. In this section, as in subsection 2.2, X is a compact abelian group with Haar probability measure P on it, and $(T^n)_{n \in \mathbb{Z}^d}$ is an algebraic action of $\mathbb{Z}^d$ on X . Let M be the dual group of X (that is the discrete abelian group of all complex characters of X) with the dual action $(\hat{T}^n)_{n \in \mathbb{Z}^d}$, and let R_d denote the group ring of the group $\mathbb{Z}^d$ over $\mathbb{Z}$. The ring R_d may be identified with the ring of Laurent polynomials in d variables over integers. The action $(\hat{T}^n)_{n \in \mathbb{Z}^d}$ supplies M with a structure of an R_d -module. Conversely, if M is a discrete R_d -module (or, equivalently, M is a discrete abelian group with an algebraic action of $\mathbb{Z}^d$), then we also have a dual algebraic action of $\mathbb{Z}^d$ on a compact abelian group X , X being the dual group of M . As a rule, X is supposed to be second countable which takes place if and only if M is countable. A (discrete) R_d -module is called *coexpansive* if its dual action is expansive.

Let Γ be the *homoclinic group* (see subsection 2.2 for the definition) of the action $(T^n)_{n \in \mathbb{Z}^d}$. The group Γ is obviously stable under this action and thus possess a structure of an R_d -module which will be called the *homoclinic module* of the action $(T^n)_{n \in \mathbb{Z}^d}$. Analogously, if $\tilde{\Gamma}$ is a subgroup of Γ which is stable respective to the action $(T^n)_{n \in \mathbb{Z}^d}$, it gives rise to an R_d -module which will be called a *homoclinic submodule* (or *homoclinic subgroup*).

In the sequel we are mostly interested in those algebraic actions which are *expansive* and have *dense* homoclinic groups. One of related results proved in [LiSch] is that the homoclinic group of an expansive algebraic action on a nontrivial compact abelian group is dense if and only if the action has completely positive entropy. However, we need more detailed results which require some portion of commutative algebra.

The most treatable algebraic $\mathbb{Z}^d$ -actions on a compact abelian groups are those which give rise to *Noetherian* dual R_d -modules. In particular, every expansive action fall into this category ([LiSch], [Sch]). It is known that for a Noetherian module there are only finitely many prime ideals of R_d associated with M (a prime ideal $J \subset R_d$ is *associated with* an R_d -module M if $J = \{p \in R_d : p \cdot a = 0_M\}$ for some $a \in M$). If M is a Noetherian R_d -module (which is necessarily at most countable), it defines an algebraic action of $\mathbb{Z}^d$ on a compact second countable abelian group X , the dual group of M (M is considered as a discrete abelian group), and many dynamical properties of this action can be expressed in terms of prime ideals associated with M ([Sch], [LiSch].)

According to Lemma 2.1 and Theorems 4.1 and 4.2 in [LiSch], if an algebraic $\mathbb{Z}^d$ -action on a nontrivial compact abelian group X with the dual R_d -module M is expansive and its homoclinic group is dense in X (or, equivalently, has completely positive entropy) then every prime ideal of R_d associated with M is principal, and the corresponding Laurent polynomial p satisfies the condition $V_{\mathbb{C}} \cap \mathbb{S}^d = \emptyset$, where $V_{\mathbb{C}} = \{\mathbf{z} \in (\mathbb{C}^\times)^d : p(\mathbf{z}) = 0\}$, $\mathbb{C}^\times = \mathbb{C} \setminus \{0\}$ and $\mathbb{S} = \{z \in \mathbb{C} : |z| = 1\}$.

Conversely, let $(T^n)_{n \in \mathbb{Z}^d}$ be an algebraic action of $\mathbb{Z}^d$ on a compact abelian group X with a Noetherian dual R_d -module M . Then this action is expansive and its homoclinic group is dense in X whenever every prime ideal of R_d associated with M is principal and the corresponding Laurent polynomial p satisfies the condition $V_{\mathbb{C}} \cap \mathbb{S}^d = \emptyset$.

The authors do not know whether it is always true that the homoclinic R_d -module Γ is Noetherian (or even finitely generated) whenever the algebraic action $(T^n)_{n \in \mathbb{Z}^d}$ is expansive. However, for a certain class of actions it has been shown in [LiSch] that M and Γ are (noncanonically) isomorphic as R_d -modules, and we shall reduce our consideration to this case.

Under the assumption that a homoclinic submodule $\tilde{\Gamma}$ is dense in X , M can be canonically embedded as a dense subgroup into a compact group $\tilde{G}$, the character group of $\tilde{\Gamma}$. Moreover, the dual action $(\hat{T}^n)_{n \in \mathbb{Z}^d}$ on M can be canonically extended to $\tilde{G}$. We use the same denotation $(\hat{T}^n)_{n \in \mathbb{Z}^d}$ both for the action on M and for the extended action on $\tilde{G}$.

Proposition 5.1. *Let $(T^n)_{n \in \mathbb{Z}^d}$ be an algebraic action of $\mathbb{Z}^d$ on a compact second countable abelian group X . Suppose that a homoclinic submodule $\tilde{\Gamma}$ of the action $(T^n)_{n \in \mathbb{Z}^d}$ is dense in X and coexpansive. Let $f \in L_2^0(X, P)$ be a real-valued function satisfying for every $\gamma \in \tilde{\Gamma}$ the relation*

$$\sum_{\mathbf{p} \in \mathbb{Z}^d} |f - H_{T^{\mathbf{p}}\gamma} f|_2 < \infty.$$

Then both CLT and ASCLT hold for f and

$$\sigma_f^2 = \sum_{\mathbf{p} \in \mathbb{Z}^d} (U^{\mathbf{p}} f, f),$$

where the series converges absolutely.

Proof. According to [Sch], it follows from the expansiveness of the action of $\mathbb{Z}^d$ on $\tilde{G}$ that $\tilde{\Gamma}$ is a Noetherian R_d -module. Let $\{\gamma_1, \dots, \gamma_r\}$ be a set generating $\tilde{\Gamma}$ as a R_d -module. Set $H_l = H_{\gamma_l}$ for $l = 1, \dots, r$. In the sequel we shall use the denotations introduced in subsection 2.3. We are going to deduce Proposition 5.1 from Propositions 2.6 and 2.7. In its turn this requires verifying Assumption 2.4. It follows from ([LiSch], Lemma 4.3) applied to the expansive action $(\hat{T}^n)_{n \in \mathbb{Z}^d}$ on $\tilde{G}$ that for any $\gamma \in \tilde{\Gamma}$ and any complex character $\chi \in M$ of the group X

$$|U^{-\mathbf{p}}(H_\gamma - I)U^{\mathbf{p}}\chi|_2^2 = |\chi(T^{\mathbf{p}}\gamma) - 1|^2 = |\langle \gamma, \tilde{T}^{\mathbf{p}}\chi \rangle - 1| \rightarrow 0$$

exponentially fast as $\mathbf{p} \rightarrow \infty$. Here $\langle \cdot, \cdot \rangle$ is the coupling map $\tilde{\Gamma} \times \tilde{G} \rightarrow \{z \in \mathbb{C} : |z| = 1\}$ between the (countable discrete) homoclinic subgroup $\tilde{\Gamma}$ and its (compact) character group $\tilde{G} \supset M$. Finite linear combinations of complex characters compose a dense linear subspace of $L_2 = L_2(X, P)$, hence the operator Δ is defined on a dense subset of L_2 . The point ii) of Assumption 2.4 follows from the fact that the group X is abelian, so all we need to prove is to establish the inequality in the point i) there.

Complex characters form an orthonormal basis in L_2 which diagonalises the operator Δ . Further, nonconstant characters span L_2^0 . Thus it suffices to prove the following assertion: there exists such a constant $c > 0$ that for every character χ of X distinct from χ_0 (where χ_0 is the unity character) we have

$$(\Delta\chi, \chi) = \sum_{\mathbf{p} \in \mathbb{Z}^d} \sum_{l=1}^r |D_{\mathbf{p}, l}\chi|_2^2 = \sum_{\mathbf{p} \in \mathbb{Z}^d} \sum_{l=1}^r |\chi(T^{\mathbf{p}}\gamma_l) - 1|^2 \geq c.$$

An equivalent form of this inequality is

$$\sum_{\mathbf{p} \in \mathbb{Z}^d} \sum_{l=1}^r |\langle \gamma_l, \widehat{T}^{\mathbf{p}} \chi - \widehat{T}^{\mathbf{p}} \chi_0 \rangle|^2 \geq c.$$

Now we see that there exists such $c > 0$ that for every $\chi \in M, \chi \neq \chi_0$, we have that

$$\sum_{\mathbf{p} \in \mathbb{Z}^d} \sum_{l=1}^r |\langle \gamma_l, \widehat{T}^{\mathbf{p}} \chi - \widehat{T}^{\mathbf{p}} \chi_0 \rangle|^2 \geq \sup_{\mathbf{p} \in \mathbb{Z}^d} \sum_{l=1}^r |\langle \gamma_l, \widehat{T}^{\mathbf{p}} \chi - \widehat{T}^{\mathbf{p}} \chi_0 \rangle|^2 \geq c.$$

Indeed, existence of such $c > 0$ is a consequence of the expansiveness of the action $(\widehat{T}^{\mathbf{n}})_{\mathbf{n} \in \mathbb{Z}^d}$ on $\widetilde{G}$ with respect to the distance

$$\rho(g_1, g_2) = \sum_{\mathbf{p} \in \mathbb{Z}^d} \alpha_{\mathbf{p}} \left(\sum_{l=1}^r |\langle \gamma_l, \widehat{T}^{\mathbf{p}} g_1 - \widehat{T}^{\mathbf{p}} g_2 \rangle|^2 \right)^{1/2}, g_1, g_2 \in \widetilde{G}.$$

Here $(\alpha_{\mathbf{p}})_{\mathbf{p} \in \mathbb{Z}^d}$ is a certain family of positive real numbers with $\sum_{\mathbf{p} \in \mathbb{Z}^d} \alpha_{\mathbf{p}} < \infty$. $\square$

Lemma 5.2. *Let $(T^{\mathbf{n}})_{\mathbf{n} \in \mathbb{Z}^d}$ be an algebraic action of $\mathbb{Z}^d$ on a compact abelian group X , and let M be its dual $\mathbb{Z}^d$ -module. Suppose that $(T^{\mathbf{n}})_{\mathbf{n} \in \mathbb{Z}^d}$ is expansive and its homoclinic group Γ is dense in X . Then there exist an algebraic action $(T_0^{\mathbf{n}})_{\mathbf{n} \in \mathbb{Z}^d}$ of $\mathbb{Z}^d$ on a compact abelian group X_0 and a surjective homomorphism $\pi : X_0 \rightarrow X$ such that:*

- (1) $\pi \circ T_0^{\mathbf{n}} = T^{\mathbf{n}} \circ \pi$ for every $\mathbf{n} \in \mathbb{Z}^d$;
- (2) the action $(T_0^{\mathbf{n}})_{\mathbf{n} \in \mathbb{Z}^d}$ is expansive;
- (3) the homoclinic group Γ_0 of the action $(T_0^{\mathbf{n}})_{\mathbf{n} \in \mathbb{Z}^d}$ is dense in X_0 ;
- (4) the homoclinic R_d -module Γ_0 of the action $(T_0^{\mathbf{n}})_{\mathbf{n} \in \mathbb{Z}^d}$ is coexpansive.

Proof of Lemma 5.2. We shall deduce from the results in [Sch] and [LiSch] that M can be embedded into another R_d -module M_0 such that the corresponding $\mathbb{Z}^d$ -action on the dual group X_0 enjoys the properties proclaimed in the statement of the lemma. Then π can be constructed as the dual of the embedding $M \subseteq M_0$.

In the case $M = \{0\}$ we set $M_0 = \{0\}$ which defines $(T_0^{\mathbf{n}})_{\mathbf{n} \in \mathbb{Z}^d}$ and π in a unique way, and the assertions of the lemma follow readily. Now let $M \neq \{0\}$. Then M is a Noetherian R_d -module with a nonempty set $\{\mathfrak{p}_1, \dots, \mathfrak{p}_m\}$ of all prime ideals of R_d associated with M . According to Lemma 2.1 in [LiSch], there exists such a Noetherian R_d -module $M_0 \supseteq M$ that $M_0 = M_0^{(1)} \oplus \dots \oplus M_0^{(m)}$, where $M_0^{(j)} = M_{0,s_j}^{(j)} \supset M_{0,s_j-1}^{(j)} \supset \dots \supset M_{0,0}^{(j)} = \{0\}$ with $M_{0,k}^{(j)} / M_{0,k-1}^{(j)} \cong R_d / \mathfrak{p}_j$ for $k = 1, \dots, s_j$ and $j = 1, \dots, m$. This implies that $\{\mathfrak{p}_1, \dots, \mathfrak{p}_m\}$ is also the list of all prime ideals of R_d associated with M_0 . Consequently, the R_d -module M_0 and its dual action $(T_0^{\mathbf{n}})_{\mathbf{n} \in \mathbb{Z}^d}$ on X_0 share those properties of the R_d -module M and its dual action $(T^{\mathbf{n}})_{\mathbf{n} \in \mathbb{Z}^d}$ on X which are determined (within the class of Noetherian modules) by the properties of the associated prime ideals. By Theorem 4.2 in [LiSch] the action $(T^{\mathbf{n}})_{\mathbf{n} \in \mathbb{Z}^d}$ on X is an action with completely positive entropy because it is expansive, has the dense homoclinic group and X is nontrivial. This implies that $(T_0^{\mathbf{n}})_{\mathbf{n} \in \mathbb{Z}^d}$ is expansive and has completely positive entropy. Indeed, according to Lemma 2.1 in [LiSch], these properties are determined by the associated prime ideals

of a Noetherian modules in question (the precise description of these properties in algebraic terms is quoted in the beginning of this section). By Theorem 4.2 in [LiSch] we conclude that the homoclinic group of the action $(T_0^n)_{n \in \mathbb{Z}^d}$ is dense.

To complete the proof we need to show that the R_d -module Γ_0 is coexpansive. The correspondence "an algebraic $\mathbb{Z}^d$ -action $\mapsto$ its homoclinic module" takes every finite direct product of actions to the finite direct product (or, equivalently, the finite direct sum) of the corresponding homoclinic modules. Further, a finite direct sum of coexpansive R_d -modules is coexpansive. So, it suffices to consider a generic summand in the representation $M_0 = M_0^{(1)} \oplus \cdots \oplus M_0^{(m)}$.

Thus, let $M_0 = M_{0,s} \supset M_{0,s-1} \supset \cdots \supset M_{0,0} = \{0\}$ with $M_{0,k}/M_{0,k-1} \cong R_d/\mathfrak{p}$ for $k = 1, \dots, s$, where $\mathfrak{p}$ is a prime ideal of the ring R_d and an R_d -module M_0 corresponds to an expansive algebraic action with the dense homoclinic group. Remark that $\mathfrak{p}$ is the only prime ideal associated with M_0 . Let $(T_0^n)_{n \in \mathbb{Z}^d}$ be the algebraic action on X_0 corresponding to the R_d -module M_0 .

The prime ideal $\mathfrak{p}$ corresponds to an expansive action with completely positive entropy which, in particular, implies by Lemma 2.1 in [LiSch] that $\mathfrak{p} = pR_d$ with some irreducible $p \in R_d$. Setting $R = R_d/p^s R_d$ we obtain a ring R which may be viewed as a (Noetherian) R_d -module. It follows from the elementary theory of Noetherian modules that the prime ideal $\mathfrak{p}$ is the only prime ideal associated with the R_d -module R as well as with an R_d -module R^n , where $n > 0$ is an integer. According to Lemma 4.9 in [LiSch] there exists such integer $n > 0$ that the homoclinic module Γ_0 of the action $(T_0^n)_{n \in \mathbb{Z}^d}$ is isomorphic to a submodule of the R_d -module R^n . This submodule is nontrivial as Γ is dense in the nontrivial group X_0 . Hence, the only prime ideal associated with the homoclinic R_d -module Γ_0 is $\mathfrak{p}$ which implies that Γ is a coexpansive R_d -module. $\square$

Proof of Theorem 2.2. The assumptions of the theorem are the same as those of Lemma 5.2, hence we may apply the lemma to show that the action $(T^n)_{n \in \mathbb{Z}^d}$ meets the requirements of Proposition 5.1. Set $\tilde{\Gamma} = \pi(\Gamma_0)$ where Γ_0 and π are given by Lemma 5.2. It is apparent that $\tilde{\Gamma}$ is a homoclinic submodule for $(T^n)_{n \in \mathbb{Z}^d}$ acting on X . Further, $\tilde{\Gamma}$ is dense in X because of surjectivity of π and density of Γ_0 in X_0 . Finally, the R_d -module $\tilde{\Gamma}$ is coexpansive as the image of the coexpansive R_d -module Γ_0 : indeed, a map which is dual to $\pi|_{\Gamma_0}$ embeds $\tilde{G}$, the dual of $\tilde{\Gamma}$, into G_0 , the dual of Γ_0 , in the way compatible with the action of $\mathbb{Z}^d$; so, the action of $\mathbb{Z}^d$ on $\tilde{G}$ is expansive as a restriction of an expansive action of the same group. Now the theorem follows from proposition 5.1. $\square$

Remark 5.3. The assumption $\sum_{\mathbf{p} \in \mathbb{Z}^d} |f - H_{T^{\mathbf{p}}\gamma} f|_2 < \infty$ of Theorem 2.2 is fulfilled for every $\gamma \in \Gamma$ if there exists such a distance ρ on X that for every $\gamma \in \Gamma$ $\rho(T^{\mathbf{p}}\gamma, 0) \rightarrow 0$ exponentially fast in $\|\mathbf{p}\|$ as $\|\mathbf{p}\| \rightarrow \infty$, and the function $f \in L_2^0$ satisfies the Dini condition $|f(x + \cdot) - f(\cdot)|_2 \leq C/|\log(\rho(x, 0))|^{1+\delta}$ for certain $C > 0$ and $\delta > 0$. The first property is known to hold in many cases for various distances (see, for example, the proof of Lemma 5.1 above, where a distance was tailored to fit the prescribed action). This property can be verified for any Riemannian translation invariant metric on tori. Then some conditions in terms of Fourier expansion of f may be considered instead of the Dini condition (see, for example, [Le].)

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