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On Magid’s Approach to the Inverse Problem in Differential Galois Theory

J. Kovacic* C. Mitschi† M.F. Singer‡

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The aim of this note is to expand on a footnote contained in our paper [4]. This footnote alluded to a counterexample (due to Kovacic) to the following Theorem (Theorem 7.13 of [3]):

Let $D_1, \ldots, D_n$ be a $C$-basis of $\text{Lie}(G)$ and let $f_1, \ldots, f_n \in F = C(x)$ be linearly independent over $C$. With $D_F = \frac{d}{dx}$ let $D = D_F \otimes 1 + \sum f_i \otimes D_i$ be the corresponding derivation of $F[G_F]$. Let $P$ be a maximal $D$-stable ideal of $F[G_F]$ and let $E$ be the fraction field of $F[G_F]/P$. Then $E \supset F$ is a Picard-Vessiot extension with $G(E/F) \leq G$, where $\text{codim}_G(G(E/F)) \leq 1$.

In particular, if $G$ has no algebraic subgroups of codimension 1, then $G(E/F) = G$ and $F(G_F) \supset F$ is a Picard-Vessiot extension with $G(F(G_F)/F) = G$.

This statement is incorrect as the following examples show [2]. Note that Theorem 7.3 of [3] is incorrect as well since the ideals presented in these examples have height larger than 1.

**Example 1:** Let $G$ be $C \times C$, under addition. If we wish, we may consider $G$ to be a subgroup of $\text{GL}(3)$ by:

$$(a, b) \mapsto \begin{pmatrix} 1 & a & b \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The ring of regular functions on $G$ is $C[G] = C[X, Y]$ where $X$ and $Y$ are the coordinate functions: $X(a, b) = a$ and $Y(a, b) = b$. It is nothing more than the ring of polynomials in the indeterminates $X$ and $Y$. The Lie algebra of right-invariant derivations (over $C$) has

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*JYACC, 116 John Street, New York, N.Y. 10038
†Institut de Recherche Mathématique Avancée, Université Louis Pasteur et C.N.R.S., 7 rue René Descartes, 67084 Strasbourg Cedex, France.
‡North Carolina State University, Department of Mathematics, Box 8205, Raleigh, NC 27695-8205.
basis $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$, the usual partial derivatives with respect to the variables.

Define $f_1 = 1$ and $f_2 = 2x$. These are in $F$ and are linearly independent over $C$. Let $D$ be a derivation on $F[G_F] = F \otimes_C C[G]$ defined as follows:

$$D = \frac{d}{dx} \otimes 1 + 1 \otimes \frac{\partial}{\partial X} + 2x \otimes \frac{\partial}{\partial Y}$$

Consider the ideal $I = (x \otimes 1 - 1 \otimes X, x^2 \otimes 1 - 1 \otimes Y)$ of $F[G]$. Then

$$D(x \otimes 1 - 1 \otimes X) = 1 \otimes 1 - 1 \otimes 1 = 0$$

and

$$D(x^2 \otimes 1 - 1 \otimes Y) = 2x \otimes 1 - 2x \otimes 1 = 0$$

Thus $I$ is a $D$-differential ideal of $F[G]$. Note that if we identify $F[G]$ with $F[X, Y]$ then $I = (X - x, Y - x^2)$.

Now consider the substitution homomorphism $\pi : X \mapsto x, Y \mapsto x^2$ over $F$. In other words, $\pi : F[G] = F \otimes C[G] \to F$ is given by $\sum(k_i \otimes h_i) \mapsto \sum(k_i h_i(x, x^2))$. One can show that this is a $D$ to $d/dx$ differential homomorphism whose kernel is $I$. Since the image of $\pi$ is a field (i.e., $F$), $I$ is a maximal ideal, and therefore a maximal differential ideal. But $E = F[G]/I$ is isomorphic to $F$. The Galois group $G(E/F)$ is the identity, which is of codimension 2 in $G$. We note that arbitrary codimension can be achieved by using $G = C^n$ and elements $f_i = i x^{i-1}, i = 1, \ldots, n$.

Analyzing this example further shows where the argument in the proof of Theorem 7.3 is not valid. With notations as before, let $V = \text{Spec } C[x] = A^1_C$ as in the proof of Theorem 7.3 in [3] (the $f_i$ here have no denominators). Then $V \times G = \text{Spec } C[x, X, Y]$ and the irreducible variety defined by the ideal $I$ is the rational curve $W = \text{Spec } C[x, X, Y]/(X - x, Y - x^2)$.

The generic point $\text{Spec } F$ of $V$ (as before, $F = C(x)$) becomes an $F$-point of $V \times G$ via the natural homomorphism $C[x, X, Y] \to C[x, X, Y]/(X, Y)] \to F$. It extends to an $\overline{F}$-point of $V \times G$ which is not an $\overline{F}$-point of $W$, since the ideal $I$ is not contained in the kernel $(X, Y)$ of $C[x, X, Y] \to \overline{F}$. On line 24 of [3], an implicit assumption is made that (using the notation of [3]) $(\overline{t}, e)$ is a point of $W_{\overline{t}}$ that is, $(t, e)$ is a point of $W(\overline{F})$ for all $t \in \overline{F}$. In the above example $(t, e) = (t, 0, 0)$ and $W(\overline{F}) = \{(t, t^2), t \in \overline{F}\}$ and so we see that this is not the case.

One can also construct a counterexample for a reductive group.

**Example 2:** Let $G = \text{GL}(n)$. Let $\xi_{ij}$ be the coordinate functions on $G$. Thus $\xi_{ij}(g) = g_{ij}$ whenever $g \in \text{GL}(n)$. The ring of regular functions $C[G]$ is $C[\xi_{ij}, \frac{d}{d \xi_{ij}}]$. The derivations $D_{ij} = \sum_k \xi_{jk} \frac{\partial}{\partial \xi_{ik}}$ form a basis of $\text{Lie}(G)$ (cf. [1], p. 329). As before, let $F = C(x)$.

One wishes to find $f_{11}, \ldots, f_{nn} \in F$ and $p_{11}, \ldots, p_{nn} \in F$ such that:

1) the $f_{ij}$ are linearly independent over $C$,
2) the ideal $P$ generated by $p_{ij} \otimes 1 - 1 \otimes \xi_{ij}$ is a maximal ideal of $F[G_F]$, \\
3) if $D = \frac{d}{dx} \otimes 1 + \sum_{ij} f_{ij} \otimes D_{ij}$ then $P$ is a $D$-stable ideal. \\
We will then have a counterexample to the theorem. \\

Leave condition 1 aside for the moment. Condition 2 is satisfied so long as det($p$) is not 0. This is equivalent to saying that $p \in G_F$. Condition 3 is more interesting. \\
\[
D(p_{\mu\nu} \otimes 1 - 1 \otimes \xi_{\mu\nu}) = p'_{\mu\nu} \otimes 1 - \sum_{ij} f_{ij} \otimes D_{ij} \xi_{\mu\nu} \\
= p'_{\mu\nu} \otimes 1 - \sum_{ijk} f_{ij} \otimes \xi_{jk} \frac{\partial \xi_{\mu\nu}}{\partial \xi_{ik}} \\
= p'_{\mu\nu} \otimes 1 - \sum_{j} f_{\mu j} \otimes \xi_{j\nu} \\
\equiv p'_{\mu\nu} \otimes 1 - \sum_{j} f_{\mu j} P_{j\nu} \otimes 1 \pmod{P}.
\]
So it suffices to find a matrix $p = (p_{\mu\nu})$ such that $p'_{\mu\nu} = \sum_{j} f_{\mu j} P_{j\nu}$, or, in matrix form, $f = p' p^{-1}$.

In order that all the conditions be satisfied (including condition 1), we need to find examples of $p \in G_F$ such that the coordinates of its logarithmic derivative are linearly independent over $C$. We now do this for GL(2). Let \\
\[
a = 2^u \\
b = 2^v \\
c = 2^w \\
d = 2^x
\]
where the $u, v, w, x$ are positive integers chosen so that with $u < v < w < x$ (in fact $a = 2, b = 2^2, c = 2^3, d = 2^4$ will work). Let $p = (p_{\mu\nu}) = \\
\[
\begin{pmatrix}
\frac{1}{x^u} & \frac{1}{x^v} \\
\frac{1}{x^w} & \frac{1}{x^x}
\end{pmatrix}
\]
We then have that $p' p^{-1} = \\
\[
\begin{pmatrix}
\frac{1}{x^u} & \frac{1}{x^v} \\
\frac{1}{x^w} & \frac{1}{x^x}
\end{pmatrix}' \\
\begin{pmatrix}
\frac{1}{x^u} & \frac{1}{x^v} \\
\frac{1}{x^w} & \frac{1}{x^x}
\end{pmatrix} = \frac{x^{a+b+c+d}}{x^{b+c} - x^{a+d}}
\]
To show that the entries of this matrix are linearly independent over the constants, it is enough to show that the entries of the following matrix are linearly independent \\
\[
\begin{pmatrix}
\frac{1}{x^u} & \frac{1}{x^v} \\
\frac{1}{x^w} & \frac{1}{x^x}
\end{pmatrix}' \\
\begin{pmatrix}
\frac{1}{x^u} & \frac{1}{x^v} \\
\frac{1}{x^w} & \frac{1}{x^x}
\end{pmatrix} = \begin{pmatrix}
\frac{-a}{x^{u+1}} & \frac{-b}{x^{v+1}} \\
\frac{-c}{x^{c+1}} & \frac{-d}{x^{d+1}}
\end{pmatrix} \\
\begin{pmatrix}
\frac{1}{x^u} & \frac{1}{x^v} \\
\frac{1}{x^w} & \frac{1}{x^x}
\end{pmatrix} = \begin{pmatrix}
\frac{-a}{x^{u+d+1}} + \frac{b}{x^{c+d+1}} & \frac{a}{x^{a+b+d+1}} - \frac{b}{x^{a+b+c+1}} \\
\frac{-c}{x^{c+d+1}} & \frac{-d}{x^{c+d+1}}
\end{pmatrix}
\]

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Assume we have a relation

\[
A\left(\frac{-a}{x^{a+d+1}} + \frac{b}{x^{c+d+1}}\right) + B\left(\frac{a}{x^{a+b+1}} - \frac{b}{x^{a+b+1}}\right) + C\left(\frac{-c}{x^{c+d+1}} + \frac{d}{x^{c+d+1}}\right) + D\left(\frac{c}{x^{a+b+c+1}} - \frac{d}{x^{a+d+1}}\right)
\]

for some constants \(A, B, C, D\). Since \(d + c + 1 > d + a + 1 > c + b + 1 > a + b + 1\) (these are 2-adic expansions) we must have that

\[
\begin{align*}
C(d - c) &= 0 \\
A(-a) + D(-d) &= 0 \\
A(b) + D(c) &= 0 \\
B(a - b) &= 0
\end{align*}
\]

Since \(d > c\) and \(b > a\), we have that \(C = B = 0\). Since \(ac > bd\) we have that \(-ac + bd \neq 0\) so \(A = D = 0\). Therefore the entries of \(p/p^{-1}\) are linearly independent. A similar idea should work for \(\text{GL}(n), n > 2\) as well.

References


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