On the heights of algebraic points on curves over number fields

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Let X be a semi-stable regular curve over the spectrum S of the integers in a number field F, and $\bar{L} = (L, h)$ an hermitian line bundle on X, i.e. L is an algebraic line bundle on X and h is a smooth hermitian metric (invariant by complex conjugation) on the restriction of L to the set $X(\mathbb{C})$ of complex points of X. In this paper we are interested in the height $h_{\bar{L}}(D)$ of irreducible divisors D on X which are flat over S, i.e. the arithmetic degree of the restriction of $\bar{L}$ to D.

First we assume that the degree $\deg(L)$ of L on the generic fiber X_F is positive and we denote by $\bar{L} \cdot \bar{L} \in \mathbb{R}$ the self-intersection of the first arithmetic Chern class of $\bar{L}$. Define

$$e(\bar{L}, d) = \inf_{\deg(D)=d} \frac{h_{\bar{L}}(D)}{d} \,.$$

Our first result (Theorem 2) is that

$$\liminf_{d} e(\bar{L}, d) \geq \frac{\bar{L} \cdot \bar{L}}{2 \deg(L)} \,.$$

This is a generalization of an inequality of S. Zhang ([13] , Th. 6.3).

Next, when X_F has genus at least two and $\bar{\omega}$ denotes the relative dualizing sheaf of X over S with its Arakelov metric [1], we obtain in Theorem 3 explicit lower bounds for $e(\bar{\omega}, d)$.

We prove also some upper bounds. Assume that $\deg(L) > 0$ and that $\deg(L_{|E}) \geq 0$ for every vertical irreducible divisor E on X. For any integer $d_0 > 0$ we define

$$e'(\bar{L}, d_0) = \sup_{D_0} \inf_{D \pitchfork D_0} \frac{h_{\bar{L}}(D)}{\deg(D)} \,,$$

where D_0 runs over all irreducible horizontal divisors of degree d_0, and D runs over all such divisors which meet D_0 properly. We prove in Theorem 4 that

$$\limsup_{d_0} e'(\bar{L}, d_0) \leq \frac{\bar{L} \cdot \bar{L}}{2 \deg(L)} \,,$$

and, when X_F has genus at least two, we give in Theorem 5 explicit upper bounds for $e'(\bar{\omega}, d_0)$.

The main tool in the proof of these inequalities is the lower bounds for successive minima of the lattice $H^1(X, M^{-1})$ with its L^2-metric which we obtained in previous papers [9] [10] [11]. From these lower bounds we deduce upper bounds for the successive minima of $H^0(X, M \otimes \omega)$ by using a transference theorem relating the successive minima of a lattice with those of its dual (Theorem 1).

1 Duality and successive minima :

1.1

Let F be a number field, $\mathcal{O}_F$ its ring of integers and $S = \mathrm{Spec}(\mathcal{O}_F)$. Consider an hermitian vector bundle $\bar{E} = (E, h)$ on S, i.e. E is a finitely generated projective $\mathcal{O}_F$-module and, for every complex embedding $\sigma : F \to \mathbb{C}$, the corresponding extension $E_\sigma = E \underset{\mathcal{O}_F}{\otimes} \mathbb{C}$ of E from $\mathcal{O}_F$ to $\mathbb{C}$ is equipped with an hermitian scalar product h_σ. Furthermore, we assume that $h = (h_\sigma)$ is invariant under complex conjugation.

We are interested in (the logarithm of) the successive minima of $\bar{E}$. Namely, for any positive integer $k \leq N$, where N is the rank of E, we let $\mu_k(\bar{E})$ be the infimum of the set of real numbers μ such that there exist k vectors $e_1, \ldots, e_k$ in E which are linearly independent in $E \otimes F$ and such that, for every complex embedding $\sigma : F \to \mathbb{C}$ and for all $i = 1, \ldots, k$,

$$\|e_i\|_\sigma \leq \exp(\mu)\,,$$

where $\|\cdot\|_\sigma$ is the norm defined by h_σ. We shall compare the successive minima of $\bar{E}$ with those of its dual $\bar{E}^*$.

Let r_1 (resp. r_2) be the number of real (resp. complex) places of F, $r = [F : \mathbb{Q}]$ the degree of F over $\mathbb{Q}$, and Δ_F its absolute discriminant. We define

$$C(N, F) = \frac{1}{r} \log|\Delta_F| + \frac{3}{2}\log(N) + \frac{5}{2}\log(r) - \frac{r_2}{r}\log(\pi)\,. \tag{1}$$

Theorem 1. *For every $k \leq N$ the following inequalities hold:*

$$0 \leq \mu_k(\bar{E}) + \mu_{N+1-k}(\bar{E}^*) \leq C(N, F)\,.$$

1.2

To prove the first inequality in Theorem 1 we use a result of Borek [3] which compares the successive minima and the slopes of hermitian vector bundles over S. Namely, according to [3], Th. 1, if $\sigma_k(\bar{E})$ is the k-th slope of $\bar{E}$, the following inequality holds :

$$0 \leq \mu_k(\bar{E}) + \sigma_k(\bar{E})\,.$$

Similarly

$$0 \leq \mu_{N+1-k}(\bar{E}^*) + \sigma_{N+1-k}(\bar{E}^*)\,.$$

On the other hand, we know that

$$\sigma_k(\bar{E}) + \sigma_{N+1-k}(\bar{E}^*) = 0$$

(see [6], 5.15(2)). So, by adding up, we get

$$0 \leq \mu_k(\bar{E}) + \mu_{N+1-k}(\bar{E}^*)\,.$$

1.3

The second inequality in Theorem 1 will be proved by reducing it to the case $F = \mathbb{Q}$. For every positive integer $k \leq Nr$ let λ_k be the infimum of the set of real numbers λ such that there exist k vectors $e_1, \ldots, e_k \in E$ which are $\mathbb{Q}$-linearly independent in $E \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$ and such that, for every $\sigma \in \Sigma$ and every $i = 1, \ldots, k$,

$$\|e_i\|_\sigma \leq \exp(\lambda)\,.$$

The following lemma is used in [12].

Lemma 1. *For every positive integer $k \leq N$, the following inequality holds :*

$$\mu_{k+1}(\bar{E}) \leq \lambda_{kr+1}\,.$$

Proof. Let $e_1, \ldots, e_{kr+1} \in E$ be vectors which are $\mathbb{Q}$-linearly independent, and V (resp. W) the F-vector space (resp. the $\mathbb{Q}$-vector space) spanned by these vectors. Since $W \subset V$ and $\dim_{\mathbb{Q}}(V) = r \dim_F(V)$ we get

$$r \dim_F(V) \geq kr + 1\,,$$

hence $\dim_F(V) \geq k + 1$. The lemma follows from this inequality and the definition of successive minima.

1.4

Let $E^\vee = \mathrm{Hom}_{\mathbb{Z}}(E, \mathbb{Z})$ and $\omega = \mathrm{Hom}_{\mathbb{Z}}(\mathcal{O}_F, \mathbb{Z})$. The morphism

$$\alpha : E^* \otimes_{\mathcal{O}_F} \omega \to E^\vee$$

mapping $u \otimes T$ to $u \circ T$ is an isomorphism of $\mathcal{O}_F$-modules. If $\mathrm{Tr} \in \omega$ is the trace morphism, we endow ω with the hermitian metric such that $|\mathrm{Tr}|_\sigma = 1$ (resp. $|\mathrm{Tr}|_\sigma = 2$) if $\sigma = \bar{\sigma}$ (resp. $\sigma \neq \bar{\sigma}$). For every $\sigma \in \Sigma$, the morphism

$$E^\vee_\sigma \to E^*_\sigma$$

induced by α is an isometry ([7], p. 354). For any positive integer $k \leq Nr$, let $\lambda_k^\vee$ be the infimum of the set of real numbers λ such that there exist k vectors $e_1, \ldots, e_k \in E^\vee$ which are linearly independent over $\mathbb{Q}$ and such that, for every $i = 1, \ldots, k$,

$$\sum_{\sigma \in \Sigma} \|e_i\|_\sigma \leq \exp(\lambda)\,.$$

According to [2] Theorem 2.1 and section 3, we have, for $k = 1, \ldots, Nr$,

$$\lambda_k + \lambda_{Nr+1-k}^\vee \leq \frac{3}{2} \log(Nr)\,. \tag{2}$$

1.5

Since ω is invertible we have

$$E^* \simeq E^\vee \otimes \omega^{-1}$$

and, for any $v \in \omega^{-1}$, $v \neq 0$,

$$\mu_k(\bar{E}^*) \leq \mu_k(\bar{E}^\vee) + \sup_{\sigma \in \Sigma} \log \|v\|_\sigma\,. \tag{3}$$

By Minkowski theorem we can choose v such that, for every $\sigma \in \Sigma$,

$$r \log \|v\|_\sigma \leq r \log(2) + \log \operatorname{covol}(\omega^{-1}) - \log \operatorname{vol}(B)\,,$$

where $\operatorname{vol}(B)$ is the volume of the unit ball in the real vector space $\omega_{\mathbb{R}}^{-1}$ and $\operatorname{covol}(\omega^{-1})$ is the covolume of the lattice ω^{-1}. We have

$$\operatorname{vol}(B) = 2^{r_1}\, \pi^{r_2}$$

and, according to [7] p. 355,

$$\log \operatorname{covol}(\omega^{-1}) = \log |\Delta_F| - 2r_2 \log(2)\,.$$

So we can choose $v \in \omega^{-1}$, $v \neq 0$, such that

$$\sup_{\sigma \in \Sigma} \log \|v\|_\sigma \leq \frac{1}{r} \log |\Delta_F| - \frac{r_2}{r} \log(\pi)\,. \tag{4}$$

1.6

¿From Lemma 1 and the fact that

$$\sum_{\sigma \in \Sigma} \|x\|_\sigma \leq r \sup_\sigma \|x\|_\sigma$$

we get, for every $k \leq N$,

$$\mu_{k+1}(\bar{E}^\vee) \leq \lambda_{kr+1}^\vee + \log(r)\,. \tag{5}$$

Therefore, using (3) and (4), we get

$$\begin{aligned}
& \mu_k(\bar{E}) + \mu_{N+1-k}(\bar{E}^*) \\
\leq \quad & \lambda_{(k-1)r+1} + \mu_{N+1-k}(\bar{E}^\vee) + \frac{1}{r}\log|\Delta_F| - \frac{r_2}{r}\log(\pi) \\
\leq \quad & \lambda_{k+1-r} + \lambda^\vee_{(N-k)r+1} + \log(r) + \frac{1}{r}\log|\Delta_F| - \frac{r_2}{r}\log(\pi)\,.
\end{aligned}$$

Since, by (2),

$$\lambda_{k+1-r} + \lambda^\vee_{(N-k)r+1} \leq \lambda_{kr} + \lambda^\vee_{Nr-kr+1} \leq \frac{3}{2}\log(Nr)\,,$$

Theorem 1 follows.

2 Lower bounds for the height of irreducible divisors

2.1

Let $S = \mathrm{Spec}(\mathcal{O}_F)$ be as above. Consider a semi-stable curve X over S such that X is regular and its generic fiber X_F is geometrically irreducible of genus g. Let h_X be an hermitian metric, invariant under complex conjugation, on the variety $X(\mathbb{C})$ of complex points of X. Let ω_0 be the associated Kähler form, defined by the formula

$$\omega_0 = \frac{i}{2\pi}\, h_X\left(\frac{\partial}{\partial z}, \frac{\partial}{\partial z}\right) dz\, d\bar{z}$$

if z is any local holomorphic coordinate on $X(\mathbb{C})$. Let $\bar{L} = (L, h)$ be an hermitian line bundle over X (with h invariant under complex conjugation). If $L_{\mathbb{C}}$ is the restriction of L to $X(\mathbb{C})$, the vector space $H^0(X(\mathbb{C}), L_{\mathbb{C}})$ of holomorphic sections of $L_{\mathbb{C}}$ on $X(\mathbb{C})$ is equipped with the sup norm

$$\|s\|_{\sup} = \sup_{x\in X(\mathbb{C})} \|s(x)\|\,,$$

where $\|\cdot\|$ is the norm defined by h, and with the L^2-norm

$$\|s\|^2_{L^2} = \sup_\sigma \int_{X_\sigma} \|s(x)\|^2\, \omega_0\,,$$

where σ runs over all complex embeddings of F and $X_\sigma = X \underset{\mathcal{O}_F}{\otimes} \mathbb{C}$ is the corresponding complex variety. We let

$$A(\bar{L}_{\mathbb{C}}) = \sup_s \log(\|s\|_{\sup}/\|s\|_{L^2})\,,$$

where s runs over all sections of $L_{\mathbb{C}}$.

Consider the relative dualizing sheaf $\bar{\omega}_{X/S}$ of X over S, equipped with the metric dual to h_X, and let $\bar{M} = \bar{L} \otimes \bar{\omega}^*_{X/S}$. We endow the $\mathcal{O}_F$-module

$$H^1 = H^1(X, M^{-1})$$

with the L^2-metric and we denote by $\mu_k(H^1)$ its successive minima, $k = 1, \ldots, N = \dim_F H^1(X_F, M^{-1})$.

Let now D be an irreducible divisor on X, flat over S, of degree d on X_F. We are interested in the Faltings height $h_{\bar{L}}(D)$ of D with respect to $\bar{L}$. Recall [4] that $h_{\bar{L}}(D) \in \mathbb{R}$ is the arithmetic degree of the restriction of $\bar{L}$ to D. Let $t = \dim_F H^0(X_F, L(-D))$ and assume that $N > t$.

Proposition 1. *The following inequality holds :*

$$\frac{h_{\bar{L}}(D)}{dr} \geq \mu_{N-t}(H^1) - A(\bar{L}_{\mathbb{C}}) - C(N, F)\,.$$

Proof. To prove Proposition 1, let $s \in H^0(X, L)$ be a section of L which does not belong to the vector space $H^0(X_F, L(-D))$. The restriction of s to $D(\mathbb{C})$ does not vanish hence, since D is irreducible, for any point P in $D(\mathbb{C})$ we have $s(P) \neq 0$. The height of D can be computed using s ([4] (3.2.2))

$$h_{\bar{L}}(D) = h_{\bar{L}}(\mathrm{div}(s_{|D})) - \sum_{\alpha} \log \|s(P_\alpha)\| \geq - \sum_{\alpha} \log \|s(P_\alpha)\|\,,$$

where $D(\mathbb{C}) = \sum\limits_{\alpha} P_\alpha$. Next we have

$$\sum_{\alpha} \log \|s(P_\alpha)\| \leq dr \log \|s\|_{\sup} \leq dr(\log \|s\|_{L^2} + A(\bar{L}_{\mathbb{C}}))\,.$$

Let $\bar{E} = (H^0(X, L), h_{L^2})$. If t is the rank of $H^0(X, L(-D))$ we can choose s such that

$$\log \|s\|_{L^2} \leq \mu_{t+1}(\bar{E})\,. \tag{6}$$

By Theorem 1

$$\mu_{t+1}(\bar{E}) \leq -\mu_{N-t}(\bar{E}^*) + C(N, F)\,, \tag{7}$$

and, by Serre duality, $\bar{E}^* = H^1(X, M^{-1})$ with the L^2-metric. Therefore Proposition 1 follows from (6) and (7).

2.2

We keep the hypotheses of Proposition 1 and we denote by $\bar{M} \cdot \bar{M} \in \mathbb{R}$ the self-intersection of the first arithmetic Chern class $\hat{c}_1(\bar{M}) \in \widehat{\mathrm{CH}}^1(X)$. Let $\delta = \deg(L)$ be the degree of L on X_F and $m = \deg(M) = \delta - 2g + 2$.

Proposition 2. *Assume that δ is even and that*

$$2g+1 \leq d \leq \delta \leq 2d-2\,.$$

Then

$$\frac{h_{\bar{L}}(D)}{dr} \geq \frac{\bar{M}\cdot\bar{M}}{2mr} - A(\bar{L}_{\mathbb{C}}) - C(N,F) - \log(\delta(\delta-g+1))\,.$$

Proof. According to [11] Th. 2 and [11] 2.3.1, the inequality

$$\mu_k(\bar{E}^*) \geq \frac{\bar{M}\cdot\bar{M}}{2mr} - \log(\delta(\delta-g+1)) \tag{8}$$

holds

$$k \geq \frac{m}{2} + g = \frac{\delta}{2} + 1\,.$$

Consider the exact sequence of cohomology groups

$$\begin{aligned}&0 \to H^0(X_F, L(-D)) \to H^0(X_F, L) \to H^0(D_F, L_{|D})\\ &\to H^1(X_F, L(-D)) \to H^1(X_F, L)\,.\end{aligned} \tag{9}$$

We first assume that $\delta > d+2g-2$ i.e.

$$\deg(L(-D)) > 2g-2\,.$$

This implies $H^1(X_F, L(-D)) = 0$ and

$$N - t = \dim_F H^0(D_F, L_{|D}) = d\,.$$

Since $d \geq \frac{\delta}{2}+1$, the proposition follows from Proposition 1 and (8).

Next, we assume that

$$d \leq \delta \leq d+2g-2\,,$$

and we apply Clifford's theorem to the Serre dual of $L(-D)$ on X_F. It is special unless $H^0(X_F, L(-D)) = 0$, in which case $t = 0$ hence

$$N - t = \delta - g + 1 \geq \frac{\delta}{2} + 1$$

since $\delta \geq 2g$, and we can conclude as above.

When $H^0(X_F, L(-D))$ does not vanish, Clifford's theorem says that

$$\dim_F\ H^1(X_F, L(-D)) - 1 \leq \frac{1}{2}\deg(\omega_{X/S} \otimes L^{-1}(D)) = g - 1 - \frac{\delta}{2} + \frac{d}{2}\,.$$

¿From (9) it follows that

$$N - t \geq d - \dim H^1(X_F, L(-D))$$

and therefore

$$N - t \geq \frac{d}{2} + \frac{\delta}{2} - g \, .$$

Since $d \geq 2g + 1$ this implies

$$N - t \geq \frac{\delta}{2} + \frac{1}{2}$$

and, since δ is even, we get

$$N - t \geq \frac{\delta}{2} + 1$$

and the proposition follows from Proposition 1 and (8).

2.3

For any hermitian line bundle $\bar{L}$ on X, and any integer d, we define

$$e(\bar{L}, d) = \inf_{\deg(D)=d} \frac{h_{\bar{L}}(D)}{d}$$

and

$$e(\bar{L}, \infty) = \liminf_{d} e(\bar{L}, d) \, .$$

Theorem 2. *If* $\deg(L)$ *is positive we have :*

$$e(\bar{L}, \infty) \geq \frac{\bar{L} \cdot \bar{L}}{2 \deg(L)} \, .$$

Proof. By definition

$$e(\bar{L}, \infty) = \lim_{n \to \infty} \inf_{\deg(D)=d \geq n} \frac{h_{\bar{L}}(D)}{d} \, .$$

Assume that $n \geq 2g + 1$ and $n \geq \deg(L) + 3$. Then, for any $d \geq n$, there exists an even integer k such that, if $\delta = k \deg(L)$, the inequalities

$$2g + 1 \leq d \leq \delta \leq 2d - 2$$

hold. Fix a Kähler metric h_X on $X(\mathbb{C})$ (invariant by complex conjugation) and let

$$\bar{M} = \bar{L}^{\otimes k} \otimes \bar{\omega}^* \, .$$

From Proposition 2 applied to $\bar{L}^{\otimes k}$ we get, for any irreducible horizontal divisor D of degree d,

$$k \frac{h_{\bar{L}}(D)}{d\, r} \geq \frac{\bar{M} \cdot \bar{M}}{2 \deg(M)\, r} - A(\bar{L}_{\mathbb{C}}^{\otimes k}) - C(N, F) - \log(\delta(\delta - g + 1)) \, . \qquad (10)$$

When n tends to infinity, the same is true for d and k. Therefore

$$\lim_{n\to\infty} \frac{\log(\delta(\delta - g + 1))}{k} = 0\,. \tag{11}$$

The rank N of $H^0(X_F, L^{\otimes k})$ is $\delta - g + 1$ so, by (1), we have

$$\lim_{n\to\infty} \frac{C(N,F)}{k} = 0\,. \tag{12}$$

According to a result of Gromov ([8] Lemma 30) the quantity $\exp A(\bar{L}_{\mathbb{C}}^{\otimes k})$ is bounded from above by a polynomial in k. Therefore

$$\lim_{n\to\infty} \frac{A(\bar{L}_{\mathbb{C}}^{\otimes k})}{k} = 0\,. \tag{13}$$

Finally

$$\deg(M) = k\deg(L) - 2g + 2$$

and

$$\bar{M}\cdot\bar{M} = (k\,\bar{L} - \bar{\omega})^2\,,$$

therefore

$$\lim_{n\to\infty} \frac{\bar{M}\cdot\bar{M}}{k\deg(M)} = \frac{\bar{L}\cdot\bar{L}}{\deg(L)}\,. \tag{14}$$

The theorem follows from (10)–(14).

2.4

In [13] S. Zhang defines

$$e_{\bar{L}} = \inf_D \frac{h_{\bar{L}}(D)}{r\deg(D)}$$

and

$$e'_{\bar{L}} = \liminf_D \frac{h_{\bar{L}}(D)}{r\deg(D)}\,,$$

where D runs over all irreducible horizontal divisors on X.

Lemma 2. *When* $\deg(L)$ *is positive we have*

$$e(\bar{L},\infty) = r\,e'_{\bar{L}}\,.$$

Proof. By definition

$$e(\bar{L},\infty) = \lim_n \inf_{\deg(D)\geq n} \frac{h_{\bar{L}}(D)}{\deg(D)}\,. \tag{15}$$

For any positive integer n let $X(n)$ be the set of horizontal irreducible divisors D such that

$$\deg(D) < n \quad \text{and} \quad h_{\bar{L}}(D) \leq (e(\bar{L},\infty)+1)\,n\,.$$

¿From [4], Cor. 3.2.5, we know that $X(n)$ is finite and we get

$$r\, e'(\bar{L}) = \lim_{n} \inf_{D \notin X(n)} \frac{h_{\bar{L}}(D)}{\deg(D)} \,. \tag{16}$$

The complement of $X(n)$ consists of those D such that either $\deg(D) \geq n$ or $\deg(D) \leq n$ and $h_{\bar{L}}(D) > (e(\bar{L}, \infty) + 1)n$. In the second case we have

$$\frac{h_{\bar{L}}(D)}{\deg(D)} > e(\bar{L}, \infty) + 1 \,.$$

Therefore (16) and (17) imply

$$r\, e'(\bar{L}) = \mathrm{Inf}(e(\bar{L}, \infty), e(\bar{L}, \infty) + 1) = e(\bar{L}, \infty) \,.$$

q.e.d.

When the first Chern form of $\bar{L}_{\mathbb{C}}$ is semi-positive and $\deg(L_{|E}) \geq 0$ for any vertical irreducible divisor E on X, Theorem 6.3 in [13] states that

$$r\, e'_{\bar{L}} \geq \frac{\bar{L} \cdot \bar{L}}{2 \deg(L)} \,.$$

Therefore Theorem 2 is not new in that case.

2.5

We come back to the situation of § 2.1 and 2.2, and we fix an integer $k \geq 1$. Furthermore we assume that the first Chern form of $\bar{M}_{\mathbb{C}}$ is positive and that $\deg(M_{|E}) \geq 0$ for any vertical irreducible divisor E on X. If $k > 1$ define

$$D(m,k) = (m+g) \sum_{\alpha=0}^{\mathrm{Inf}(k-1,g)} \binom{m+g-k-\alpha}{k-1-\alpha} \binom{g}{\alpha} \,,$$

and let $D(m,1) = 1$.

Proposition 3. *Assume that $\delta \geq d \geq k$ and that either $m > 2k > 2$ or $m > k = 1$. Then the following inequality holds :*

$$\frac{h_{\bar{L}}(D)}{dr} \geq \frac{k}{m^2\, r} \bar{M}^2 - \frac{2k}{m} e_{\bar{M}} + e_{\bar{M}} - A(\bar{L}_{\mathbb{C}}) - C(N,F) - \frac{\log D(m,k)}{m^2} - 1 \,.$$

Proof. According to [10] Th. 4 i) (resp. [9] Th. 2) we have

$$1 + \mu_k(H^1) \geq \frac{k}{m^2\, r} \bar{M} \cdot \bar{M} - \frac{2k}{m} e_{\bar{M}} + e_{\bar{M}} - \frac{\log D(m,k)}{m^2} \tag{17}$$

as soon as $m > 2k > 2$ [1](resp. $k = 1$ and $m > 1$). If we assume that $\delta > d+2g-2$ we have $H^1(X_F, L(-D)) = 0$ hence $N - t = d \geq k$. Therefore

$$\mu_{N-t}(H^1) \geq \mu_k(H^1)$$

and the proposition follows from (18) and Proposition 1. When $d \leq \delta \leq d+2g-2$ we consider the Serre dual of $L(-D)$ over X_F. It is special unless $t = 0$, in which case

$$N - t = \delta - g + 1 = m + g - 1 \geq k\,.$$

When $t \neq 0$, Clifford's theorem says that

$$\dim H^1(X_F, L(-D)) - 1 \leq \frac{1}{2}\deg(\omega \otimes L^{-1}(D)) = g - 1 - \frac{\delta}{2} + \frac{d}{2}\,,$$

and

$$N - t \geq \frac{\delta}{2} + \frac{d}{2} - g\,.$$

But

$$\frac{\delta}{2} - g = \frac{m}{2} - 1 \geq k - 1\,,$$

hence

$$N - t \geq k + \frac{d}{2} - 1$$

and $N - t \geq k$ since $d \geq 1$.

Again, the proposition follows from (18) and Proposition 1.

2.6

We now assume that $g \geq 2$ and we let $\bar{\omega}$ be the relative dualizing sheaf $\omega_{X/S}$ of X over S, equipped with its Arakelov metric [1]. As in 2.3 above we consider

$$e(\bar{\omega}, d) = \inf_{\deg(D)=d} \frac{h_{\bar{\omega}}(D)}{d}\,. \tag{18}$$

Theorem 3. *There is a constant $C = C(g, r)$ such that the following inequalities hold:*

$$e(\bar{\omega}, d) \geq \frac{\bar{\omega}\cdot\bar{\omega}}{4g(g-1)}\,\frac{dg+g-1}{d+2g-2} - \frac{g-1}{d+2g-2}\log|\Delta_F| - C\,\frac{\log(d)}{d}\,, \tag{19}$$

and, if $d \geq 2g+1$,

$$e(\bar{\omega}, d) \geq \frac{\bar{\omega}\cdot\bar{\omega}}{4(g-1)}\,\frac{d-2g+1}{d-g} - \frac{g-1}{d-g}\log|\Delta_F| - C\,\frac{\log(d)}{d}\,. \tag{20}$$

[1] Theorem 4, i) in [10] assumes that $g \geq 2$ and the metric on $L_{\mathbb{C}}$ is admissible in the sense of Arakelov [1], but these extra hypotheses are not used in the proof of that statement.

Proof. To prove (19) we apply Proposition 3 to a power $\bar{L} = \bar{\omega}^{\otimes n}$ of $\bar{\omega}$. We take $k = d$. When $d = 1$, (19) follows from the inequalities

$$e(\bar{\omega}, 1) \geq r\, e_{\bar{\omega}}$$

and

$$r\, e_{\bar{\omega}} \geq \frac{\bar{\omega} \cdot \bar{\omega}}{4g(g-1)} \tag{21}$$

(cf. [5]). When $d > 1$, the condition $m > 2k$ in Proposition 3 becomes

$$(n-1)(g-1) > d\,,$$

i.e.

$$n > \frac{d}{g-1} + 1\,.$$

We take

$$n = \left[\frac{d}{g-1}\right] + 2\,.$$

According to Proposition 3, for any irreducible horizontal divisor D of degree d,

$$\begin{aligned} \frac{h_{\bar{L}}(D)}{d} \quad &\geq \quad k\, \frac{\bar{\omega} \cdot \bar{\omega}}{4(g-1)^2} + r\, e_{\bar{\omega}} \left(n - 1 - \frac{k}{g-1}\right) \\ &- \quad r \left(A(\bar{L}_{\mathbb{C}}) + C(N,F) + \frac{\log D(m,k)}{m^2} + 1\right). \end{aligned}$$

Using the lower bound (21) for $e_{\bar{\omega}}$ and the fact that

$$h_{\bar{L}}(D) = n\, h_{\bar{\omega}}(D)$$

we get

$$\begin{aligned} e(\bar{\omega}, d) \quad &\geq \quad \frac{\bar{\omega} \cdot \bar{\omega}}{4g(g-1)}\, \frac{k+n-1}{n} \\ &- \quad \frac{r}{n} \left(A(\bar{L}_{\mathbb{C}}) + C(N,F) + \frac{\log D(m,k)}{m^2} + 1\right). \end{aligned} \tag{22}$$

Since

$$n \leq 2 + \frac{d}{g-1}$$

we get

$$\frac{k+n-1}{n} \geq \frac{dg+g-1}{d+2g-2}\,. \tag{23}$$

Gromov's estimate for $A(\bar{\omega}^{\otimes n})$ implies

$$\frac{A(\omega^{\otimes n})}{n} = O\left(\frac{\log(n)}{n}\right) = O\left(\frac{\log(d)}{d}\right). \tag{24}$$

From (1) we deduce that

$$\frac{r}{n}\, C(N,F) = \frac{1}{n} \log |\Delta_F| + O\left(\frac{\log(n)}{n}\right) . \tag{25}$$

Finally, according to [10] § 3.8,

$$\log D(m,k) = O(m \log(m)) = O(d \log(d)) . \tag{26}$$

The inequality (19) follows from (22)–(26).

To prove (20) we apply Proposition 2 to a power $\bar{L} = \bar{\omega}^{\otimes n}$ of $\bar{\omega}$. We get

$$e(\bar{\omega}, d) \geq \frac{n-1}{n} \, \frac{\bar{\omega} \cdot \bar{\omega}}{4(g-1)} - \frac{r}{n} (A(\bar{L}_{\mathbb{C}}) + C(N,F) + \log(\delta(\delta - g + 1))) \tag{27}$$

as soon as

$$2g + 1 \leq d \leq (2g-2)\, n \leq 2d - 2 .$$

We choose

$$n = \left[\frac{d-1}{g-1}\right] \geq \frac{d-g}{g-1}$$

in which case

$$\frac{n-1}{n} \geq \frac{d - 2g + 1}{d-g} .$$

The second summand of the right-hand side of (27) is estimated as above. This proves (20).

3 Upper bounds for the height of irreducible divisors

3.1

Let X and h_X be as in § 2.1. Let $\bar{L}$ and $\bar{M}$ be two hermitian line bundles on X. We assume that $\deg(L) > 0$ and $\deg(L_{|E}) \geq 0$ for every vertical irreducible divisor E on X. Let D_0 be an irreducible horizontal divisor,

$$N = \dim_F H^0(X_F, M)$$

and

$$t = \dim_F H^0(X_F, M(-D_0)) .$$

We assume that $N > t$. Denote by $\mu_k(H^1)$, $k = 1, \ldots, N$, the successive minima of $H^1 = H^1(X, \omega_{X/S} \otimes M^{-1})$ equipped with its L^2-metric. We write $\bar{L} \cdot \bar{M} \in \mathbb{R}$ for the arithmetic intersection of $\hat{c}_1(\bar{L})$ with $\hat{c}_1(\bar{M})$, and we write $D \pitchfork D_0$ to mean that D is an irreducible horizontal divisor meeting D_0 properly.

Proposition 4. *The following inequality holds :*

$$\begin{aligned}\inf_{D \pitchfork D_0} \frac{h_{\bar{L}}(D)}{r\deg(D)} &\leq \frac{\bar{L}\cdot\bar{M}}{r\deg(M)} - \mu_{N-t}(H^1)\frac{\deg(L)}{\deg(M)} \\ &+ \frac{\deg(L)}{\deg(M)}\left(A(\bar{M}_{\mathbb{C}}) + C(N,F)\right).\end{aligned}$$

Proof. Let $\bar{E} = (H^0(X,M), h_{L^2})$ and choose a section $s \in H^0(X,M)$ such that $s \notin H^0(X_F, M(-D_0))$ and

$$\log \|s\|_{L^2} \leq \mu_{t+1}(\bar{E}).$$

If $\mathrm{div}(s)$ is the divisor of s we get ([4] (3.2.2))

$$\begin{aligned}\bar{L}\cdot\bar{M} &= h_{\bar{L}}(\mathrm{div}(s)) - \int_{X(\mathbb{C})} \log\|s\|\, c_1(\bar{L}_{\mathbb{C}}) \\ &\geq h_{\bar{L}}(\mathrm{div}(s)) - r\deg(L)(\mu_{t+1}(\bar{E}) + A(\bar{M}_{\mathbb{C}})).\end{aligned} \tag{28}$$

We can write

$$\mathrm{div}(s) = \sum_\alpha D_\alpha + V$$

where each D_α is irreducible and flat over S, and V is effective and vertical on X. Therefore, by our assumption on L, we have

$$h_{\bar{L}}(\mathrm{div}(s)) \geq \sum_\alpha h_{\bar{L}}(D_\alpha)$$

and

$$\deg(\mathrm{div}(s)) = \sum_\alpha \deg(D_\alpha).$$

Therefore, since each D_α is transverse to D_0,

$$\frac{h_{\bar{L}}(\mathrm{div}(s))}{\deg(M)} \geq \inf_\alpha \frac{h_{\bar{L}}(D_\alpha)}{\deg(D_\alpha)} \geq \inf_{D\pitchfork D_0} \frac{h_{\bar{L}}(D)}{\deg(D)}. \tag{29}$$

From Theorem 1 we get

$$\mu_{t+1}(\bar{E}) \leq -\mu_{N-t}(H^1) + C(N,F) \tag{30}$$

and the proposition follows from (28), (29) and (30).

3.2

We keep the notation of the previous section and we let

$$\bar{K} = \bar{M}\otimes\bar{\omega}^*_{X/S}, \; m = \deg(M) \quad \text{and} \quad d_0 = \deg(D_0).$$

Proposition 5. *Assume that m is even and*

$$2g+1 \leq d_0 \leq m \leq 2d_0 - 2\,.$$

The following inequality holds :

$$\begin{aligned}\inf_{D \pitchfork D_0} \frac{h_{\bar{L}}(D)}{r\deg(D)} &\leq \frac{\bar{L}\cdot\bar{M}}{rm} - \frac{\bar{K}\cdot\bar{K}}{2r\deg(K)}\frac{\deg(L)}{m} \\ &+ \frac{\deg(L)}{m}(A(\bar{M}_{\mathbb{C}}) + C(N,F) + \log(m(m-g+1)))\,.\end{aligned}$$

Proof. The number $\mu_{N-t}(H^1)$ can be estimated from below using [11] exactly as in the proof of Proposition 2. Therefore the proposition follows from Proposition 4.

3.3

Let $\bar{L}$ be an hermitian line bundle on X such that $\deg(L) > 0$ and $\deg(L_{|E}) \geq 0$ for any irreducible vertical divisor E on X. For any integer $d_0 \geq 1$ consider

$$e'(\bar{L}, d_0) = \sup_{D_0} \inf_{D \pitchfork D_0} \frac{h_{\bar{L}}(D)}{\deg(D)}\,,$$

where D_0 runs over all irreducible horizontal divisors of degree d_0. Let

$$e'(\bar{L}, \infty) = \limsup_{d_0} e'(\bar{L}, d_0)\,.$$

Theorem 4. *The following inequality holds :*

$$e'(\bar{L}, \infty) \leq \frac{\bar{L}\cdot\bar{L}}{2\deg(L)}\,.$$

Proof. As in the proof of Theorem 2, when the integer n is big enough, for any $d_0 \geq n$ we can choose an even power $\bar{M}$ of $\bar{L}$ such that, if $m = \deg(M)$, the following inequalities hold :

$$2g+1 \leq d_0 \leq m \leq 2d_0 - 2\,.$$

Then we apply Proposition 5 to $\bar{L}$ and $\bar{M}$. If $\bar{K} = \bar{M} \otimes \bar{\omega}^*_{X/S}$ we get

$$\lim_{n\to\infty} \frac{\bar{K}\cdot\bar{K}}{\deg(K)}\frac{\deg(L)}{m} = \frac{\bar{L}\cdot\bar{L}}{\deg(L)} \tag{31}$$

and

$$\lim_{n\to\infty} \frac{\bar{L}\cdot\bar{M}}{m} = \frac{\bar{L}\cdot\bar{L}}{\deg(L)}\,. \tag{32}$$

By the same estimates as in the proof of Theorem 2 we get

$$\lim_{n\to\infty} (A(\bar{M}_{\mathbb{C}}) + C(N,F) + \log(m(m-g+1)))/m = 0\,. \tag{33}$$

The theorem follows from (31), (32), (33) and Proposition 5.

Remark. For any d_0 we have

$$r\, e_{\bar{L}} \leq e'(\bar{L}, d_0)\,.$$

Therefore Theorem 3 implies

$$r\, e_{\bar{L}} \leq \frac{\bar{L}\cdot\bar{L}}{2\deg(L)}\,.$$

But it does not follow from [13], Th. 6.3.

3.4

We come back to the notation of 3.2 and we let

$$k = \deg(K) = m - 2g + 2\,.$$

We fix an integer $h \geq 1$. We assume that the first Chern form of $\bar{K}_{\mathbb{C}}$ is positive and that $\deg(K_{|E}) \geq 0$ for every irreducible vertical divisor E on X.

Proposition 6. *Assume that $m \geq d_0 \geq h$ and that either $k > 2h > 2$ or $k > h = 1$. Then the following inequality :*

$$\begin{aligned} \inf_{D \pitchfork D_0} \frac{h_{\bar{L}}(D)}{r\deg(D)} \quad &\leq \quad \frac{\bar{L}\cdot\bar{M}}{rm} - \frac{\deg(L)}{m}\left(\frac{h}{k^2 r}\bar{K}^2 - \frac{2h}{k}e_{\bar{K}} + e_{\bar{K}}\right) \qquad (34)\\ &+ \quad \frac{\deg(L)}{m}\left(A(\bar{M}_{\mathbb{C}}) + C(N,F) + \frac{\log D(k,h)}{h^2} + 1\right)\,. \end{aligned}$$

Proof. This inequality follows from Proposition 4 by bounding $\mu_{N-t}(H^1)$ from below in the same way as in the proof of Proposition 3.

3.5

Assume now that $g \geq 2$ and let $\bar{\omega}$ be $\omega_{X/S}$ with its Arakelov metric. Recall that

$$e'(\bar{\omega}, d_0) = \sup_{\deg(D_0)=d_0} \inf_{D \pitchfork D_0} \frac{h_{\bar{L}}(D)}{\deg(D)}\,.$$

Theorem 5. *There exists a constant $C = C(g,r)$ such that the following inequalities hold :*

$$e'(\bar{\omega}, d_0) \leq \frac{\bar{\omega}\cdot\bar{\omega}}{4(g-1)} + \frac{2g-1}{4g(d_0+2g-2)}\,\bar{\omega}\cdot\bar{\omega} + \frac{g-1}{d_0+g-1}\log|\Delta_F| + C\,\frac{\log(d_0)}{d_0}\,, \tag{35}$$

and, when $d_0 \geq 2g+1$,

$$e'(\bar{\omega}, d_0) \leq \frac{\bar{\omega}\cdot\bar{\omega}}{4(g-1)} + \frac{\bar{\omega}\cdot\bar{\omega}}{4(d_0-g)} + \frac{g-1}{d_0-g}\log|\Delta_F| + C\,\frac{\log(d_0)}{d_0}\,. \tag{36}$$

Proof. To prove (35) we apply Proposition 6 with $\bar{L} = \bar{\omega}$, $\bar{M} = \bar{\omega}^{\otimes n}$ and $h = d_0$. When $d_0 = 1 < k$ we have $n(g-1) \geq g$. When $d_0 > 1$ and

$$k = n(2g-2) - 2g + 2 > 2\,d_0$$

we get $n(g-1) > d_0 + g - 1$.

In both cases we choose

$$n = 2 + \left[\frac{d_0}{g-1}\right].$$

The right hand side of (34) (Proposition 6) becomes $X_1 + X_2$, with

$$X_1 = \frac{n\,\bar{\omega}\cdot\bar{\omega}}{rn(2g-2)} - \frac{1}{n}\left(d_0\,\frac{\bar{\omega}\cdot\bar{\omega}}{(2g-2)^2 r} + \left(1 - \frac{2\,d_0}{(n-1)(2g-2)}\right)(n-1)\,e_{\bar{\omega}}\right)$$

and

$$X_2 = \frac{\deg(L)}{m}\left(A(\bar{M}_{\mathbb{C}}) + C(N,F) + \frac{\log D(k,h)}{h^2} + 1\right).$$

As in the proof of Theorem 3 we get

$$X_2 \leq C\,\frac{\log(d_0)}{d_0} + \frac{1}{nr}\log|\Delta_F|$$

and

$$\frac{1}{n} \leq \frac{g-1}{d_0+g-1}\,.$$

On the other hand, since

$$re_{\bar{\omega}} \geq \frac{\bar{\omega}\cdot\bar{\omega}}{4g(g-1)}\,,$$

we get

$$\begin{aligned} r\,X_1 &\leq \bar{\omega}\cdot\bar{\omega}\left(\frac{1}{2g-2} - \frac{d_0}{n(2g-2)^2} - \frac{n-1}{4g(g-1)n} + \frac{d_0}{4ng(g-1)^2}\right) \\ &= \frac{\bar{\omega}\cdot\bar{\omega}}{4g(g-1)}\left(2g-1-\frac{d_0-1}{n}\right). \end{aligned}$$

Since $n \leq 2 + \frac{d_0}{g-1}$ we get

$$\begin{aligned} r\,X_1 &\leq \frac{\bar{\omega}\cdot\bar{\omega}}{4g(g-1)}\left(2g-1-\frac{(d_0-1)(g-1)}{2g-2+d_0}\right) \\ &= \frac{\bar{\omega}\cdot\bar{\omega}}{4(g-1)} + \frac{2g-1}{4g(d_0+2g-2)}\,\bar{\omega}\cdot\bar{\omega}\,. \end{aligned}$$

This proves (35).

To prove (36) we apply Proposition 5 when $\bar{L} = \bar{\omega}$ and $\bar{M} = \bar{\omega}^{\otimes n}$. If $d_0 \leq m \leq 2d_0 - 2$ we get

$$e(\bar{L}, d_0) \leq rY_1 + rY_2$$

where

$$\begin{aligned} Y_2 &= \frac{\deg(L)}{m} \left(A(\bar{M}_{\mathbb{C}}) + C(N,F) + \log(m(m-g+1))\right) \\ &\leq C\,\frac{\log(d_0)}{d_0} + \frac{1}{nr}\log|\Delta_F| \end{aligned}$$

as in the proof of Theorem 3, and

$$\begin{aligned} r\,Y_1 &= \frac{\bar{L}\cdot\bar{M}}{m} - \frac{\bar{K}\cdot\bar{K}}{2\deg(K)}\,\frac{\deg(L)}{m} \\ &= \frac{\bar{\omega}\cdot\bar{\omega}}{2g-2} - \frac{n-1}{4n(g-1)}\,\bar{\omega}\cdot\bar{\omega} \\ &= \frac{\bar{\omega}\cdot\bar{\omega}}{4(g-1)} + \frac{\bar{\omega}\cdot\bar{\omega}}{4n(g-1)}\,. \end{aligned}$$

Since $n(g-1) \leq d_0 - 1$ we can assume that

$$n = \left[\frac{d_0 - 1}{g-1}\right],$$

hence $n \geq \frac{d_0-1}{g-1} - 1$. This implies

$$\frac{1}{n}\log|\Delta_F| \leq \frac{g-1}{d_0 - g}\log|\Delta_F|$$

and

$$r\,Y_1 \leq \frac{\bar{\omega}\cdot\bar{\omega}}{4(g-1)} + \frac{\bar{\omega}\cdot\bar{\omega}}{4(d_0-g)}\,,$$

from which (36) follows.

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