General Adaptive Neighborhood Image Processing

Part I: Introduction and Theoretical Aspects

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Abstract. The so-called General Adaptive Neighborhood Image Processing (GANIP) approach is presented in a two parts paper dealing respectively with its theoretical and practical aspects.
The Adaptive Neighborhood (AN) paradigm allows the building of new image processing transformations using context-dependent analysis. Such operators are no longer spatially invariant, but vary over the whole image with ANs as adaptive operational windows, taking *intrinsically* into account the local image features. This AN concept is here largely extended, using well-defined mathematical concepts, to that General Adaptive Neighborhood (GAN) in two main ways. Firstly, an *analyzing criterion* is added within the definition of the ANs in order to consider the radiometric, morphological or geometrical characteristics of the image, allowing a more significant spatial analysis to be addressed. Secondly, *general linear image processing* frameworks are introduced in the GAN approach, using concepts of abstract linear algebra, so as to develop operators that are consistent with the physical and/or physiological settings of the image to be processed.
In this paper, the GANIP approach is more particularly studied in the context of Mathematical Morphology (MM). The structuring elements, required for MM, are substituted by GAN-based structuring elements, fitting to the local contextual details of the studied image. The resulting transforms perform a relevant spatially-adaptive image processing, in an *intrinsic manner*, that is to say without a priori knowledge needed about the image structures. Moreover, in several important and practical cases, the adaptive morphological operators are connected, which is an overwhelming advantage compared to the usual ones that fail to this property.

Keywords: General Adaptive Neighborhoods, Image Processing Frameworks, Intrinsic Spatially-Adaptive Analysis, Mathematical Morphology, Nonlinear Image Representation

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Part II: Practical Application Examples

Abbreviations

AN	:	Adaptive Neighborhood
ANIP	:	Adaptive Neighborhood Image Processing
ASE	:	Adaptive Structuring Element
ASF	:	Alternating Sequential Filter
CLIP	:	Classical Linear Image Processing
IP	:	Image Processing
GAN	:	General Adaptive Neighborhood
GANIP	:	General Adaptive Neighborhood Image Processing
GANMM	:	General Adaptive Neighborhood Mathematical Morphology
GLIP	:	General Linear Image Processing
LIP	:	Logarithmic Image Processing
LRIP	:	Log-Ratio Image Processing
MHIP	:	Multiplicative Homomorphic Image Processing
MM	:	Mathematical Morphology
SE	:	Structuring Element

This paper deals with intensity images, that is to say image mappings defined on a spatial support D in the Euclidean space $\mathbb{R}^2$ and valued into a gray tone range, which is a positive real numbers interval.
The first occurrence of a specific and/or important term will appear in italics.

1. Introduction

1.1. Intensity-based Image Processing Frameworks

In order to develop powerful image processing operators, it's necessary to represent images within mathematical frameworks (most of the time of a vectorial nature) based on a *physically and/or psychophysically relevant image formation process* [100, 44]. In addition, their mathematical structures and operations (the vector addition and then the scalar multiplication) have to be consistent with the physical nature of the images and/or the human visual system [39, 33], and computationally effective [58]. At last, it must enable to develop successful practical applications [87].
Such considerations have been initiated with the generalization of linear systems [64, 65, 99], using concepts and structures coming from *abstract linear algebra* [48, 36, 101]. It allows to include situations in which signals or images are combined by operations other than the usual vector addition [66]. Indeed, it was shown [41] that the usual addition is not a satisfying solution in some non-linear physical settings, such as that based on multiplicative or convolutive image formation model [66]. The reasons are that the classical addition operation and consequently the usual scalar multiplication are not consistent with the combination and amplification laws to which such physical settings obey [72, 99]. Regarding digital images, the problem [84] lies in the fact that a direct usual addition of two intensity values may be out of the range where such images are valued, resulting in an unwanted out-of-range [27].
Consequently, operators based on such *intensity-based image processing frameworks* should be consistent with the physical and/or physiological settings of the images to be processed.

1.2. Spatially-Adaptive Image Processing

The image processing techniques using spatially invariant transformations, with *fixed operational windows*, give efficient and compact computing structures, with the conventional separation between data and operations. However, those operators have several strong drawbacks, such as removing significant details, changing the detailed parts of large objects and creating artificial patterns [2].
Alternative approaches towards context-dependent processing have been proposed with the introduction of adaptive operators which are subdivided in two main classes : the *adaptive-weighted operators* and the *spatially-adaptive operators*. The adaptive concept results respectively from the adjustment of the weights upon the operational window [50, 83] and from the spatial adjustment of the window [63, 98, 85, 107].
A *spatially-adaptive image processing* approach implies that operators are no longer spatially invariant, but must vary over the whole image with adaptive windows, taking locally into account the image context. Some authors [82, 80] have introduced 'Image Algebra' so as to develop a comprehensive and unified algebraic structure for the representation of all image-to-image operations [81, 37], including spatially-adaptive operators. Nevertheless, the general operational windows (called templates) of such operators have a linear behavior and do not take explicitly into account physical and/or psychophysical settings.
Usually, the spatially-adaptive operators possess some limitations concerning their adaptive templates. In fact, these transformations are generally extrinsically defined using a priori knowledge on the image, contrary to those intrinsic ones that provide a more significant spatial analysis, such as operators based on the paradigm of adaptive neighborhood [32].

1.3. Extrinsic vs Intrinsic Approaches

Indeed, a priori constraints, defined extrinsically to the local features of the image, are generally imposed upon the size and/or the shape of the operational windows, which is not the most appropriate, especially in the context of multiscale image analysis. In such cases, the analyzing scales are a priori determined independently of the image structures. Thus, the size and/or shape of the operational windows are *extrinsically* defined with regard to the specified scales (wavelets [55], morphological pyramids [102, 49], scale-spaces [53, 38], . . .).
Alternative pathways were proposed (anisotropic scale-spaces [68, 1], adaptive neighborhood-based alternating sequential filtering [6]) for which the scales depend *intrinsically* on the analyzing operational windows and consequently on the local structures of the image. Therefore, a priori information is not required and there is no limitation to the operational window pattern, except for the connectivity in order to take into account the local topological characteristics.

1.4. General Adaptive-Neighborhood Image Processing

In this way, the paradigm of *Adaptive Neighborhood* (AN), proposed by Gordon and Rangayyan [32], was used in various image filtering processes [67, 76, 78, 79, 15, 8, 14]. In *Adaptive Neighborhood Image Processing* (ANIP), a *set of adaptive neighborhoods* (ANs set) is defined for each point of the studied image. The spatial extent of an AN depends on the local characteristics of the image where the seed point is situated. So, an image becomes represented as a collection of homogeneous regions, rather than a priori defined collection of points or neighboring points. Thus, for each point to be processed, its associated AN is used as *adaptive operational window* of the image to image transformation.
Thereafter, the AN paradigm can be largely generalized, as shown in this paper. In the so-called *General Adaptive Neighborhood Image Processing* (GANIP) approach, local neighborhoods are identified in the image to be analyzed as sets of connected points. Their gray tones are also within a specified homogeneity tolerance in relation with a selected *analyzing criterion* such as luminance, contrast, curvature, . . . They are called general for two main reasons. Firstly, the addition of a radiometric, morphological, or geometrical criterion in the definition of the usual AN sets allows a more significant spatial analysis to be performed. Secondly, both image and criterion mappings are represented in *General Linear Image Processing* (GLIP) frameworks [64, 65] allowing to choose a relevant structure consistent with the application to be addressed.

1.5. Application to Mathematical Morphology

Mathematical Morphology (MM) [59, 89] is an important and nowadays a traditional theory in image processing [96]. A morphological transformation consists in determining whether a template pattern, called *Structuring Element* (SE), fits or does not fit the image objects or structures. In this paper, the *General Adaptive Neighborhood* (GAN) paradigm is more particularly applied to MM. The basic idea in the proposed approach is to substitute the fixed-shape, fixed-size SEs generally used for morphological operators, by *Adaptive Structuring Elements* (ASEs). Those last ones are adjusted to the *General Adaptive Neighborhoods* (GANs), leading to the *General Adaptive Neighborhood Mathematical Morphology* (GANMM). The resulting operators perform a really spatially-adaptive image processing and, in several important and practical cases (see Subsection 4.3), are *connected*. This is a great advantage contrary to the usual MM operators which fail to this property.

1.6. Summary of the paper

First, in Section 2, the paper describes the main requirements for an intensity-based Image Processing (IP) framework. Four reported General Linear Image Processing (GLIP) frameworks

[64, 65] are briefly exposed: the *Classical Linear Image Processing* (CLIP), the *Multiplicative Homomorphic Image Processing* (MHIP), the *Log-Ratio Image Processing* (LRIP) and the *Logarithmic Image Processing* (LIP) frameworks. Secondly, in Section 3, the benefits of spatially-adaptive image processing are discussed, and more particularly those of morphological operators that are intrinsically defined according to the local features of the image. Then, in Section 4, the General Adaptive Neighborhood Image Processing (GANIP) approach is introduced, studied, and afterwards more particularly applied to mathematical morphology. Finally, in Section 5, the conclusion highlights some promising prospects about the GANIP approach, notably the application to other fields (than the mathematical morphology).

2. Intensity-based Image Processing Frameworks

2.1. Fundamental Requirements for an Image Processing Framework

To efficiently handle and process intensity images, it's necessary to represent image mappings, in a mathematically rigorous and pertinent way, so as to develop operators defined within relevant frameworks. In order to represent the *superposition* and *amplification* physical and/or psychophysical *processes*, an image processing framework consists of a vector space for the image mappings with its operations of *vector addition* and *scalar multiplication*.
In developing image processing techniques, Stockham [99], Jain [39], Marr [58] and Granrath [33] have recognized that it is of central importance that an image processing framework must satisfy to the following fundamental requirements:

- it is based on a physically and/or psychophysically relevant image formation model,
- its mathematical structures and operations are both powerful and consistent with the physical nature of the images and/or the human visual system,
- its operations are computationally effective, or at least tractable,
- it is practically fruitful in the sense that it enables to develop successful applications in real situations.

2.2. Need and Usefulness of Abstract Linear Mathematics

When studying *non-linear images or imaging systems*, such as images formed by transmitted light or the human brightness perception system, it is not rigorous to stick to the usual definition of linearity. Therefore, the usual addition $+$ and scalar multiplication $\times$ operations are incongruous, as noted by Jourlin and Pinoli [41]. Indeed, the superposition of such images does not obey to the classical additive law. Consequently, it is pointed out that the Classical Linear Image Processing (CLIP) [52] framework is not adapted to non-linear images or imaging systems. Moreover, intensity images being valued within a given bounded range, due to the way they are digitized and stored, the result of many classical linear image processing transformations is not accurate. For example, the simple sum of two images, using the usual addition $+$, may be out of this bounded range where it must be in for physical reasons or should be in for practical reasons [84].
Thus, although the Classical Linear Image Processing (CLIP) framework has played a central role in image processing, it is not necessarily the best choice [26, 58, 69, 42]. However, using the power of *abstract linear algebra* [48, 36, 101], it is possible to go up to the abstract level and explore operations other than the usual addition and scalar multiplication for a specific setting or

a particular problem. It leaded to General Linear Image Processing (GLIP) frameworks [64, 65], such as those exposed in Subsection 2.4.

2.3. Importance of the Ordered Sets Theory

Nevertheless, a vector space representing a GLIP framework is a too poor mathematical structure. Indeed, it only enables to describe how images are combined and amplified. In addition to abstract algebra, it is then also necessary to resort to other mathematical fields, such as topology, functional analysis, ...
Particularly, the *ordered sets theory* [54, 46] offers powerful and useful notions for image processing. Indeed, from an image processing viewpoint, images being positively-valued signals, the *positivity* notion is thus of fundamental importance. An *ordered vector space* S is a vector space structured by its vectorial operations $\oplus$, $\ominus$ and $\otimes$ and an *order relation*, denoted $\succeq$, which obeys the reflexive, antisymmetric and transitive laws [54, 46].
Any vector s of S can then be expressed as:

$$s = s_{\oplus} \ominus s_{\ominus} \tag{1}$$

where $s_{\oplus}$ and $s_{\ominus}$ are called the positive part and negative part of s, respectively.
The positive part and negative part of s are defined as:

Definition 1 (Positive and negative part of a vector).

$$s_{\oplus} = \max_{\succeq}(s, 0_{\bigcirc}) \tag{2}$$
$$s_{\ominus} = \max_{\succeq}(\ominus s, 0_{\bigcirc}) \tag{3}$$

where $\max_{\succeq}(.,.)$ *denotes the maximum in the sense of the order relation* $\succeq$*, and* $0_{\bigcirc}$ *is the zero vector (i.e. the neutral element for the vector addition* $\oplus$*).*

From this point, the *modulus* of a vector s, denoted $|s|_{\bigcirc}$, is defined as:

Definition 2 (Vector Modulus).
$\forall s \in (S, \oplus, \otimes, \succeq)$

$$|s|_{\bigcirc} = s_{\oplus} \oplus s_{\ominus} \tag{4}$$

Note that the positive part, negative part and modulus, of a vector s belonging to an ordered vector space S are positive elements:

$$s_{\oplus} \succeq 0_{\bigcirc} \tag{5}$$
$$s_{\ominus} \succeq 0_{\bigcirc} \tag{6}$$
$$|s|_{\bigcirc} \succeq 0_{\bigcirc} \tag{7}$$

The ordered sets theory has played a fundamental role within some GLIP approaches, and has allowed mathematically-justified powerful image processing techniques to be developed [72].

From this point, a GLIP framework will be represented by an ordered vector space structure.

2.4. The CLIP, MHIP, LRIP and LIP Frameworks

According to these abstract algebraic concepts (Subsection 2.2), the Multiplicative Homomorphic Image Processing (MHIP), the Log-Ratio Image Processing (LRIP) and the Logarithmic Image Processing (LIP) have been respectively introduced by Oppenheim and Stockham [64], Shvayster and Peleg [94, 95], and Jourlin and Pinoli [41, 42, 69, 71, 73, 44]. The MHIP approach was introduced to define homomorphically a vector space structure on the set of images valued in the unbounded real number range $(0, +\infty)$, in a consistent way with the physical laws of concrete image settings. The LRIP approach was developed to set up a topological vector space structure on the set of images valued in the bounded range $(0, M)$, where M denotes the upper bound of the range where images are digitized and stored, by resorting to a homeomorphism between this range and the real number space $\mathbb{R}$. The LIP approach was introduced to define an additive operation closed in the bounded real number range $(0, M)$, which is mathematically well defined, and also physically consistent with concrete physical and/or practical image settings. It allows [71, 73] then the introduction of an abstract ordered linear topological and functional framework [47, 12, 40, 58].

Physically, it is well-known that images have positive intensity values. Intensity images are then represented by mappings defined on a *spatial support* $D \subseteq \mathbb{R}^2$ and valued in a positive real number set, called the *initial intensity value range*.
In the CLIP or LIP approach, the linear space representing images is the *positive vector cone* [30, 104] constituted by the set of these mappings structured with a *vector addition* (denoted $+$ or ⨹, respectively) and a *scalar (positive) multiplication* (denoted $\times$ or ⨻, respectively). Therefore, in order to enlarge this positive vector cone into a vector space, it is necessary to give a mathematical meaning to the *opposite* operation (denoted $-$ or ⨺, respectively), and to extent the scalar multiplication to any real number (still named $\times$ or ⨻, respectively). Since these operations can be valued in a real number range, the set of intensity images defined on the spatial support D and valued in an *extended intensity value range* is introduced. Structured with its linear operations of vector addition and scalar multiplication, this images set becomes a real vector space.
Regarding the MHIP or LRIP approach, the linear space representing images is the vector space constituted by the set of the intensity images structured with a vector addition (denoted $\boxplus$ or ◇+, respectively) and a scalar multiplication (denoted $\boxtimes$ or ◇×, respectively). These operations are defined homomorphically ([66, 100] and [94, 95], respectively). However, the direct expressions of the MHIP and LRIP operations may be easily formulated ([72] and [28], respectively). On the contrary, the operations structuring the LIP framework have been directly introduced [41, 69, 42]. Afterwards, it has been shown that the LIP vector space is isomorphically related to the CLIP one [41, 69, 42] (ie the vector space representing the intensity images valued in the unbounded real number set).
Finally, the CLIP, MHIP, LRIP and LIP frameworks possess direct expressions of their linear operations (vector addition, scalar multiplication, opposite and *vector subtraction*), and they are homomorphically related to the CLIP one (Table 1).

Thereafter, the vector spaces representing the CLIP, MHIP, LRIP and LIP frameworks are structured into ordered vector spaces using their linear operations and the classical order relation $\geq$. It allows the modulus in the CLIP, MHIP, LRIP or LIP sense to be defined. Such an operation is required in practical applications, such as differentiation-based edge detection, for the calculation of the gradient vector magnitude [27]. Likewise, the modulus enables the introduction of mathematically well-defined physical and/or psychophysical notions, such as the contrast in the CLIP, MHIP, LRIP or LIP sense [73, 72].

Table I calls back the structures and operations of these four image processing frameworks. For each one, its initial intensity value range, its extended intensity value range (required in the vector space representing images), its homomorphism in relation with the CLIP vector space, its linear operations rules (vector addition, scalar multiplication, opposite and vector subtraction), its *neutral element* for addition, its positive intensity value range (defining the positive vector cone) and its vector modulus, are summarized.

Table I. Structures and operations of the CLIP, MHIP, LRIP, and LIP frameworks [72].

CLIP	**MHIP**	**LRIP**	**LIP**
initial intensity value range			
$(0,+\infty)$	$(0,+\infty)$	$(0,M)$	$(0,M)$
extended intensity value range (defining the vector space)			
$(-\infty,+\infty)$	$(0,+\infty)$	$(0,M)$	$(-\infty,M)$
homomorphism related to the CLIP vector space			
$f \mapsto f$	$f \mapsto \ln(f)$	$f \mapsto \ln\left(\frac{f}{M-f}\right)$	$f \mapsto -M \times \ln\left(\frac{M-f}{M}\right)$
vector addition			
usual $+$	$f \boxplus g = fg$	$f \mathbin{\Diamond\!\!\!\!+} g = \frac{M}{\left(\frac{M-f}{f}\right)\left(\frac{M-g}{g}\right)+1}$	$f \mathbin{\triangle\!\!\!\!+} g = f + g - \frac{fg}{M}$
scalar multiplication			
usual $\times$	$\alpha \boxtimes f = \exp(\alpha \times \ln(f))$	$\alpha \mathbin{\Diamond\!\!\!\!\times} f = \frac{M}{\left(\frac{M-f}{f}\right)^{\alpha}+1}$	$\alpha \mathbin{\triangle\!\!\!\!\times} f = M - M\left(1-\frac{f}{M}\right)^{\alpha}$
opposite			
usual $-$	$\boxminus f = \frac{1}{f}$	$\mathbin{\Diamond\!\!\!\!-} f = M - f$	$\mathbin{\triangle\!\!\!\!-} f = \frac{-Mf}{M-f}$
vector subtraction			
usual $-$	$f \boxminus g = \frac{f}{g}$	$f \mathbin{\Diamond\!\!\!\!-} g = \frac{M}{\left(\frac{M-f}{f}\right)\left(\frac{g}{M-g}\right)+1}$	$f \mathbin{\triangle\!\!\!\!-} g = M\left(\frac{f-g}{M-g}\right)$
zero vector (neutral element for vector addition)			
usual 0	$0_{\square} \equiv 1$	$0_{\Diamond} \equiv \frac{M}{2}$	$0_{\triangle} \equiv 0$
positive intensity value range (defining the positive vector cone)			
$(0,+\infty)$	$(1,+\infty)$	$\left(\frac{M}{2}, M\right)$	$(0,M)$
vector modulus			
usual $\lvert . \rvert$	$\lvert f\rvert_{\square} = \max_{\geq}(f,1) \boxplus \max_{\geq}\left(\frac{1}{f},1\right)$	$\lvert f\rvert_{\Diamond} = \max_{\geq}\left(f,\frac{M}{2}\right) \mathbin{\Diamond\!\!\!\!+} \max_{\geq}\left(M-f,\frac{M}{2}\right)$	$\lvert f\rvert_{\triangle} = \max_{\geq}(f,0) \mathbin{\triangle\!\!\!\!+} \max_{\geq}\left(\frac{-Mf}{M-f},0\right)$

The CLIP framework clearly presents too much drawbacks, already exposed in Subsection 2.1. Moreover, the LRIP one has not been yet rigorously connected to a physical image setting [72]. Thus, it does not satisfy to one of the four fundamental requirements for an image processing framework claimed in Subsection 2.1. On the contrary, the MHIP and LIP frameworks follow the physical, mathematical, computational and practical requirements [72, 73]. However, the MHIP is surpassed for physical, mathematical, physiological and computational reasons [72].

The theoretical advantages of the LIP approach [73, 72, 27, 44] have been practically confirmed and illustrated through successful applications examples such as image background removing [61], illumination correction [61], image interpolation [34], image enhancement (dynamic range and sharpness modification) [25, 26, 43, 61], image 3D-reconstruction [34], contrast estimation [45, 7], image restoration [7], edge detection and image segmentation [45, 27], image filtering [26], and so on.

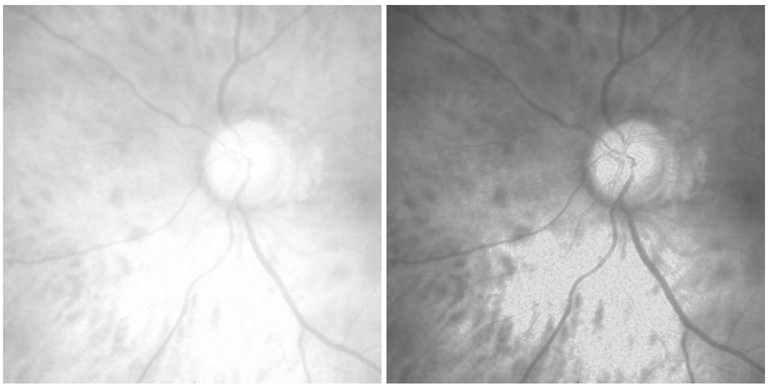
2.5. Application Example to Image Enhancement

The unboundedness of the positive intensity value range within the CLIP and MHIP frameworks makes impossible the introduction of a rigorous image enhancement technique that only uses the vectorial operations [72]. On contrary, the LRIP and LIP approaches allow optimal dynamic range expansions to be mathematically and computationally defined. Nevertheless, the LIP enhancement performs well and far better than the LRIP one, confirming on the one hand, the physical and physiological connections of the LIP approach, and on the other hand the lack of physical basis of the LRIP approach [72]. In this way, the image enhancement problem is only illustrated (Fig. 1) within the LIP framework.

The LIP framework enables an image transformation to be defined that maximally enlarges the dynamic range of an image f while preserving a physical meaning. It has been proved [43] that there exists a positive real number, denoted by $\lambda_0(f)$ and called the *optimal logarithmic gain*, by which the image f has to be multiplied in order to get a new image $\lambda_0(f) \triangle\!\!\!\!\!\times f$ that possesses the *maximal dynamic range* among the image class $(\lambda \triangle\!\!\!\!\!\times f)_{\lambda>0}$. Therefore, the image transformation, called *image dynamic range maximization* and denoted Enh, is then defined as following:

$$Enh(f) = \lambda_0(f) \triangle\!\!\!\!\!\times f \tag{8}$$

An illustration of image enhancement by dynamic range maximization is given in Figure 1, on a real image acquired on the retina of a human eye.



a. original image f b. enhanced image $Enh(f)$

Figure 1. LIP-based image dynamic range maximization [100] applied on a real human retina image. Original 8-bits (a) image: intensity value range $[151, 254]$; enhanced (b) image: intensity value range $[75, 225]$. The LIP enhancement allows both the structures of the blind spot and of some blood vessels to be more easily distinguished, which is rather hard on the original image.

3. Spatially-Adaptive Image Processing and Mathematical Morphology

The nonlinear filtering community has responded to the well-known shortcomings of linear filters. Several classes of nonlinear filters (homomorphic filters [66, 100, 74], order statistic filters [74, 75, 3], morphological filters [56, 57, 29], ...) have been developed, and have found numerous applications in the areas of image processing and analysis.
The early type of those nonlinear operators uses a spatial operational window with fixed shape and size. Later, the development of new techniques allows to build more efficient image processing transformations, using spatially-adaptive operational windows. A particular attention has been turned to such operators based on Mathematical Morphology (MM) [59, 89] which is a well-defined approach to analyze spatial structures within images. The output of a MM operator generally describes how well an a priori selected shape, called Structuring Element (SE), either fits or does not fit inside a local image feature, known as the hit or miss transform [89, 88]. In most of the applications of MM, the SE used in morphological operations has a fixed shape and size. This kind of nonlinear operators presents several drawbacks such as creating artificial patterns and removing significant details, because of the fixed operational window [2]. However, the spatially-adaptive mathematical morphology deals with this problem using SEs that change their size and/or shape as they probe different parts of an image, fitting to the local features of the image. Those adaptive morphological transformations can be subdivided in two main classes where the size and/or shape change of spatially-adaptive SEs is determined either *extrinsically* or *intrinsically* for each point within the image.

3.1. Extrinsic Approaches

In the first case of extrinsic approaches, some MM operators have been described [107] with Structuring Elements (SEs) assigning a natural size of the SE for each point within the image, such as the morphological operator causes the largest change in its value. Nevertheless, the SE pattern is still a priori fixed and shapely identical for each point of the studied image and its size depends on the choice of the morphological operator. Other morphological operators have been built with such constraints on the size and/or shape of the SEs [85]. For instance in [105], the shape of SEs that automatically adjust the gray tones in a range image is rectangular or ellipsoidal. Consequently, those approaches require a priori knowledge of the image, which is not completely satisfying.

3.2. Intrinsic Approaches

In the other case of intrinsic approaches, the Structuring Elements (SEs) of morphological operators are assigned *intrinsically* for each point within the image without any constraints, excepting the connectivity of the pattern. Their shape and size are determined according to the local geometrical features of the image. Those SEs are based on the paradigm of Adaptive Neighborhood (AN) that was proposed by Gordon and Rangayyan [32] and used in varied image filtering processes [67, 76, 78, 79, 15, 8, 14]. For instance, Braga Neto [6] tackled to apply the AN paradigm to MM, but the approach was overlooked so far.

In this way, extended ANs sets, taking into account a criterion mapping and a selected general image processing framework, are built in the next section. They will be later used in the context of MM so as to define the so-called General Adaptive Neighborhood Mathematical Morphology (GANMM).

4. General Adaptive Neighborhood Image Processing

In Adaptive Neighborhood Image Processing (ANIP), a set of adaptive neighborhoods (ANs set) is defined for each point within the image. Their spatial extent depend on the local characteristics of the image where the seed point is situated. Then, for each point to be processed, its associates AN is used as adaptive operational window of the considered transformation. [67, 76, 78, 79, 15, 8, 14]. Furthermore, the AN paradigm can be largely extended, as shown in Subsection 4.1.

4.1. GAN paradigm

In the so-called General Adaptive Neighborhood Image Processing (GANIP) approach, a set of General Adaptive Neighborhoods (GANs set) is identified according to each point in the image to be analyzed. A GAN is a subset of the spatial support D constituted by connected points whose measurement values, in relation to a selected criterion (such as luminance, contrast, thickness, curvature, ...), fit within a specified homogeneity tolerance.
They are called general for two main reasons. Firstly, the addition of a radiometric, morphological, or geometrical criterion in the definition of the usual AN sets allows a more significant and specific spatial analysis to be performed. Secondly both image and criterion mappings are represented in General Linear Image Processing (GLIP) frameworks allowing to choose the most appropriate structure compatible with the application to be processed.
Thus, two GLIP frameworks will be introduced, with formal definitions, representing the space of image and criterion mappings, respectively.

4.2. GANs Sets

The space of *image* (resp. *criterion*) *mappings*, defined on the spatial support D and valued in a real numbers interval $\tilde{E}$ (resp. E), is represented in a GLIP framework (Section 2), denoted $\mathcal{I}$ (resp. $\mathcal{C}$).

The GLIP framework $\mathcal{I}$ (resp. $\mathcal{C}$) is then supplied with an ordered vectorial structure, using the formal vector addition $\tilde{\oplus}$ (resp. $\oplus$), the formal scalar multiplication $\tilde{\otimes}$ (resp. $\otimes$) and the classical total order relation $\geq$ defined directly from those of real numbers:

$$\forall (f,g) \in \mathcal{I}^2, \mathcal{C}^2 \quad f \geq g \Leftrightarrow (\forall x \in D \quad f(x) \geq g(x)) \tag{9}$$

There are several GANs sets. Each collection satisfies specific properties. The present paper presents two kinds of GANs sets: the *weak GANs* and the *strong GANs*. They are mainly differentiated by a symmetry property, which is of great importance for the application of the GANIP approach to Mathematical Morphology (Subsection 4.3), or to build relevant metrics [19].

4.2.1. ***Weak GANs***

For each point $x \in D$ and for an image $f \in \mathcal{I}$, the *Weak General Adaptive Neighborhoods* (W-GANs), denoted $V^h_{m_\bigcirc}(x)$, are subsets of D. They are built upon a *criterion mapping* $h \in \mathcal{C}$ (based on a local measurement such as luminance, contrast, thickness, ... related to f), in relation with an *homogeneity tolerance* $m_\bigcirc$ belonging to the positive intensity value range (Tab. I), $E^{\oplus} = \{t \in E | t \geq 0_\bigcirc\}$.
More precisely, the W-GAN $V^h_{m_\bigcirc}(x)$ is a subset of D that fulfills two conditions:

- its points have a criterion measurement value closed to the one of the seed (the point x to be analized):

$$\forall y \in V^h_{m_\bigcirc}(x) \quad |h(y) \ominus h(x)|_\bigcirc \leq m_\bigcirc$$

- it is a path-connected set [13] (according to the usual Euclidean topology on $D \subseteq \mathbb{R}^2$)

The Weak General Adaptive Neighborhoods (W-GANs) are then defined as:

Definition 3 (Weak General Adaptive Neighborhoods).
$\forall (m_{\triangle}, h, x) \in E^{\oplus} \times \mathcal{C} \times D$

$$V^h_{m_{\triangle}}(x) \;=\; C_{h^{-1}([h(x) \ominus m_{\triangle}, h(x) \oplus m_{\triangle}])}(x) \tag{10}$$

where $C_X(x)$ denotes the path-connected component [13] (according to the usual Euclidean topology on $D \subseteq \mathbb{R}^2$) of $X \subseteq D$ containing $x \in D$.

Remark 4. *Other GANs sets may be introduced and studied [19], using different conditions for the GANs homogeneity, such as:*

$$V^h_{m^1_{\triangle}, m^2_{\triangle}}(x) \;=\; C_{h^{-1}([h(x) \ominus m^1_{\triangle}, h(x) \oplus m^2_{\triangle}])}(x)$$

To visualize the W-GANs (Eq. 10), a one-dimensional example is presented in Figure 2, with the CLIP framework (Subsection 2.4) selected for the space of criterion mappings.

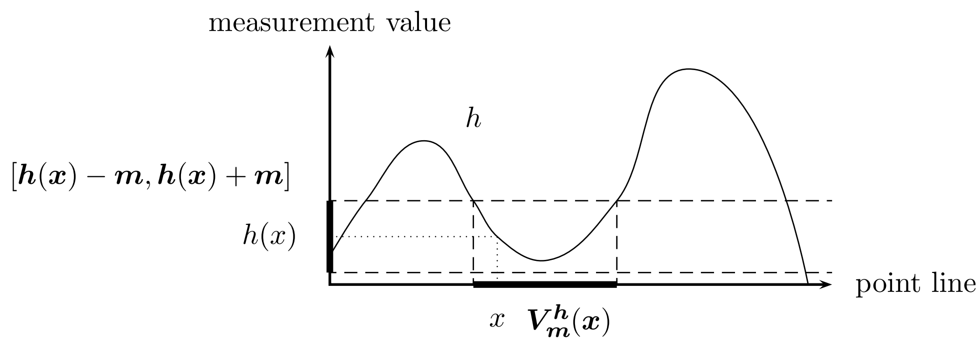


Figure 2. One-dimensional representation of a W-GAN in the CLIP framework selected for the space of criterion mappings: for a point $x \in D$, its associated W-GAN, $V^h_m(x)$, is computed in relation with the considered criterion mapping $h \in \mathcal{C}$ and a specified homogeneity tolerance $m \in \mathbb{R}^+$.

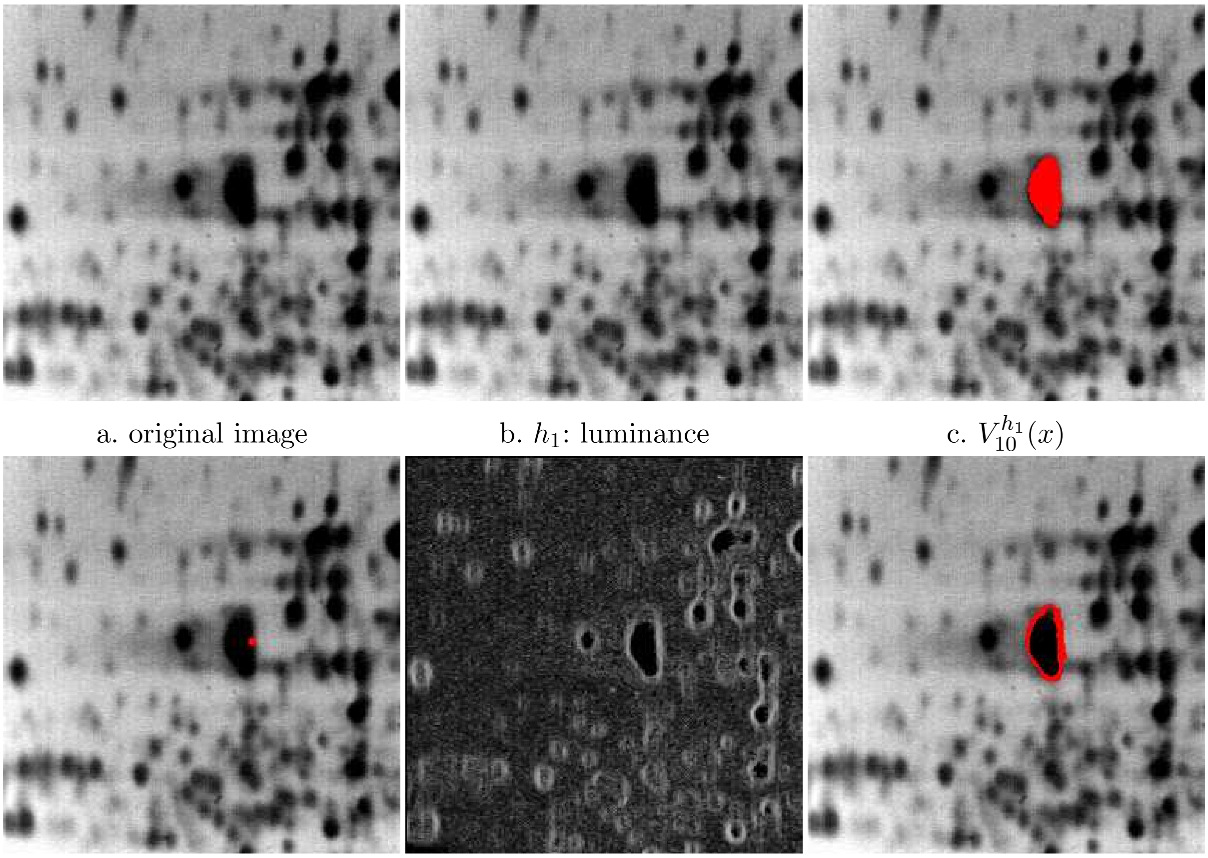
Figure 3 illustrates the W-GAN of a point x computed with the luminance criterion in the CLIP framework or the contrast (defined in the sense of [45, 70]) criterion in the LIP framework, on an electrophoresis gel image provided by the software Micromorph®. In practice, the choice of the appropriate criterion results from kind of the considered application.

In the following, the notion of *path* (Def. 5 below) is defined so as to get a practical equivalent definition of the W-GANs (Def. 6 below), involving computing interests.

Definition 5 (Path).
A path of extremities $x \in D$ and $y \in D$ respectively, denoted P_x^y, is a continuous mapping (with the usual Euclidean topologies on $[0,1]$ and D) [13]:

$$\mathrm{P}_x^y : \begin{cases} [0,1] & \rightarrow \; D \\ 0 & \mapsto \; x \\ 1 & \mapsto \; y \end{cases} \tag{11}$$

So, the W-GANs $V^h_{m_{\triangle}}(x)$ are defined by of a *region growing process* where the aggregating condition is given by: $|h(.) \ominus h(x)|_{\triangle} \leq m_{\triangle}$, that is of great computing importance.



a. original image   b. h_1: luminance   c. $V_{10}^{h_1}(x)$

d. seed point x e. h_2: contrast   f. $V_{30_\triangle}^{h_2}(x)$

Figure 3. Original electrophoresis gel image (a). The weak general adaptive neighborhood set for the seed point highlighted in (d) is respectively homogeneous (c,f), with respect to the tolerance $m = 10$ and $m_\triangle = 30_\triangle$, in relation to the luminance criterion (b) in the CLIP framework or to the contrast criterion (e) in the LIP framework.

Definition 6 (Weak General Adaptive Neighborhoods - equivalent definition).
$\forall (m_{\hexagon}, h, x) \in E^{\oplus} \times \mathcal{C} \times D$

$$V_{m_{\hexagon}}^{h}(x) \;=\; \{y \in D | y \overset{h,m_{\hexagon}}{\longrightarrow} x\} \tag{12}$$

where $\overset{h,m_{\hexagon}}{\longrightarrow}$ *denotes the path-connectivity relationship:*

$$y \overset{h,m_{\hexagon}}{\longrightarrow} x \;\Leftrightarrow\; \exists\, \mathrm{P}_x^y | \forall z \in \mathrm{P}_x^y([0,1]) \quad |h(z) \ominus h(x)|_{\hexagon} \leq m_{\hexagon} \tag{13}$$

These sets satisfy several properties as stated and proved in the following.

Proposition 7 (Weak General Adaptive Neighborhoods).
Let $(m_{\hexagon}, h, x) \in E^{\oplus} \times \mathcal{C} \times D$

1. *reflexivity:*

$$x \in V_{m_{\hexagon}}^{h}(x) \tag{14}$$

2. *increasing with respect to* $m_{\hexagon}$*:*

$$\begin{pmatrix} (m_{\hexagon}^1, m_{\hexagon}^2) \in E^{\oplus} \times E^{\oplus} \\ m_{\hexagon}^1 \leq m_{\hexagon}^2 \end{pmatrix} \Rightarrow V_{m_{\hexagon}^1}^{h}(x) \subseteq V_{m_{\hexagon}^2}^{h}(x) \tag{15}$$

3. equality between iso-valued points:

$$\begin{pmatrix} (x,y) \in D^2 \\ x \in V^h_{m_{\hexagon}}(y) \\ h(x) = h(y) \end{pmatrix} \Rightarrow V^h_{m_{\hexagon}}(x) = V^h_{m_{\hexagon}}(y) \tag{16}$$

4. $\oplus$-translation invariance:

$$c \in E \Rightarrow V^{h \oplus c}_{m_{\hexagon}}(x) = V^h_{m_{\hexagon}}(x) \tag{17}$$

5. $\otimes$-multiplication compatibility:

$$\alpha \in \mathbb{R}^+ \backslash \{0\} \Rightarrow V^{\alpha \otimes h}_{m_{\hexagon}}(x) = V^h_{\frac{1}{\alpha} \otimes m_{\hexagon}}(x) \tag{18}$$

Proof:

1. reflexivity:
   $x \overset{h,m_{\hexagon}}{\longrightarrow} x$, so $x \in V^h_{m_{\hexagon}}(x)$.

2. increasing with respect to $m_{\hexagon}$:

$$\begin{aligned} m^1_{\hexagon} \leq m^2_{\hexagon} &\Rightarrow [h(x) \ominus m^1_{\hexagon}, h(x) \oplus m^1_{\hexagon}] \subseteq [h(x) \ominus m^2_{\hexagon}, h(x) \oplus m^2_{\hexagon}]) \\ &\Rightarrow C_{h^{-1}([h(x) \ominus m^1_{\hexagon}, h(x) \oplus m^1_{\hexagon}])}(x) \subseteq C_{h^{-1}([h(x) \ominus m^2_{\hexagon}, h(x) \oplus m^2_{\hexagon}])}(x) \\ &\Rightarrow V^h_{m^1_{\hexagon}}(x) \subseteq V^h_{m^2_{\hexagon}}(x) \end{aligned}$$

3. equality between iso-valued points:
   Let z be a point in $V^h_{m_{\hexagon}}(x)$. So, there exists a path P^z_x such that:
   $\forall w \in \mathrm{P}^z_x([0,1])\ |h(w) \ominus h(x)|_{\hexagon} \leq m_{\hexagon}$.
   Moreover, x belongs to $V^h_{m_{\hexagon}}(y)$ i.e. there exists a path P^x_y such that:
   $\forall u \in \mathrm{P}^x_y([0,1])\ |h(u) \ominus h(y)|_{\hexagon} \leq m$.
   Thus, there exists a path P^z_y such that $\mathrm{P}^z_y([0,1]) = \mathrm{P}^x_y([0,1]) \cup \mathrm{P}^z_x([0,1])$.
   Consequently, for all t in $\mathrm{P}^z_y([0,1])$, if t belongs to $\mathrm{P}^x_y([0,1])$ then $|h(t) \ominus h(y)|_{\hexagon} \leq m_{\hexagon}$ else t belongs to $\mathrm{P}^z_x([0,1])$ and $|h(t) \ominus h(y)|_{\hexagon} = |h(t) \ominus h(x)|_{\hexagon} \leq m_{\hexagon}$.
   So, for all t in $\mathrm{P}^z_y([0,1])$ $|h(t) \ominus h(y)|_{\hexagon} \leq m_{\hexagon}$ and then $z \in V^h_{m_{\hexagon}}(y)$.
   Conversely, if z belongs to $V^h_{m_{\hexagon}}(y)$ then there exists a path P^z_y such that:
   $\forall w \in \mathrm{P}^z_y([0,1])\ |h(w) \ominus h(y)|_{\hexagon} \leq m$.
   Since x belongs to $V^h_{m_{\hexagon}}(y)$ and $h(y) = h(x)$, then y belongs to $V^h_{m_{\hexagon}}(x)$ (seen with the inverse path $\mathrm{P}^y_x(.) = \widehat{\mathrm{P}^x_y}(.) = \mathrm{P}^x_y(1-.)$).
   So, there exists a path P^z_x such that $\mathrm{P}^z_x([0,1]) = \mathrm{P}^y_x([0,1]) \cup \mathrm{P}^z_y([0,1])$.
   A similar reasoning leads to the expecting result i.e. $z \in V^h_{m_{\hexagon}}(x)$.

4. $\oplus$-translation invariance:

$$\begin{aligned} &(h \oplus c)^{-1}([(h \oplus c)(x) \ominus m_{\hexagon}, (h \oplus c)(x) \oplus m_{\hexagon}]) \\ &= \{y \in D | (h \oplus c)(y) \in [(h \oplus c)(x) \ominus m_{\hexagon}, (h \oplus c)(x) \tilde{\oplus} m_{\hexagon}]\} \\ &= \{y \in D | h(y) \in [h(x) \ominus m_{\hexagon}, h(x) \oplus m_{\hexagon}]\} \\ &= h^{-1}([h(x) \ominus m_{\hexagon}, h(x) \oplus m_{\hexagon}]) \end{aligned}$$

5. $\otimes$-multiplication compatibility:

$$
\begin{aligned}
&(\alpha \otimes h)^{-1}([(\alpha \otimes h)(x) \ominus m_{\hexagon}, (\alpha \otimes h)(x) \oplus m_{\hexagon}]) \\
&= \{y \in D | (\alpha \otimes h)(y) \in [(\alpha \otimes h)(x) \ominus m_{\hexagon}, (\alpha \tilde{\otimes} h)(x) \oplus m_{\hexagon}]\} \\
&= \{y \in D | h(y) \in [h(x) \ominus (\tfrac{1}{\alpha} \otimes m_{\hexagon}), h(x) \oplus (\tfrac{1}{\alpha} \otimes m_{\hexagon})]\} \\
&= h^{-1}([h(x) \ominus (\tfrac{1}{\alpha} \otimes m_{\hexagon}), h(x) \oplus (\tfrac{1}{\alpha} \otimes m_{\hexagon})])
\end{aligned}
$$

□

Figure 4 illustrates the fundamental *geometrical nesting property* of the weak GANs (Eq. 15). These GANs, denoted $V^h_{m_{\hexagon}}(x)$, are called 'Weak' because they do not satisfy the *symmetry*

a. criterion: luminance b. W-GAN sets c. color table

Figure 4. Nesting of weak GAN sets of four seed points (b) using the luminance criterion (a) and different homogeneity tolerances in the CLIP framework: $m = 5, 10, 15, 20$ and 25 encoded by the color table (c). A weak GAN set defined with a certain homogeneity tolerance could be represented by several tinges of the color associated to its seed point.

property, defined in the following sense:

Definition 8 (Symmetric collection of subsets).
A collection $\{A(x)\}_{x\in D}$ of subsets $A(x) \subseteq D$ is called symmetric, if and only if:

$$\forall (x, y) \in D^2 \quad y \in A(x) \Leftrightarrow x \in A(y) \tag{19}$$

Indeed, $\{V^h_{m_{\hexagon}}(x)\}_{x\in D}$ is not a symmetric collection: a one-dimensional counter example is presented in Figure 5, with the CLIP framework selected for the space of criterion mappings.

This notion of symmetry is topologically relevant: it should enable relevant metrics [9] to be built using the GAN paradigm in the field of image analysis (the authors are currently working on topological approaches with respect to the GAN paradigm). Moreover, from a visual point of view, the symmetry property appears closely linked to the human visual perception (as firstly noticed within th gestalt theory,...) [108, 18]. In this way, symmetric GANs are defined in the following.

4.2.2. Strong GANs

In order to get this relevant symmetry property (Eq. 19), a new set of GANs is defined (Def. 9): the *Strong General Adaptive Neighborhoods* (S-GANs). A visual representation of a S-GAN is exposed in Figure 6.

Definition 9 (Strong General Adaptive Neighborhoods).

$$\forall (m_{\hexagon}, h, x) \in E^{\oplus} \times \mathcal{C} \times D \quad N^h_{m_{\hexagon}}(x) = \bigcup_{z\in D} \{V^h_{m_{\hexagon}}(z) | x \in V^h_{m_{\hexagon}}(z)\} \tag{20}$$

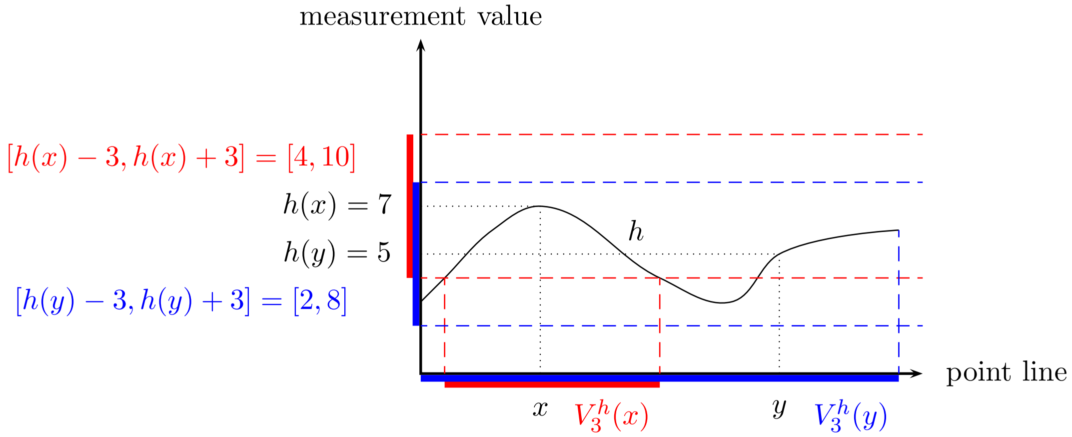


Figure 5. The W-GANs set, $\{V_m^h(z)\}_{z \in D}$, computed within the CLIP framework, is not symmetric (in the sense of Def. 8): $x \in V_3^h(y)$ and $y \notin V_3^h(x)$.

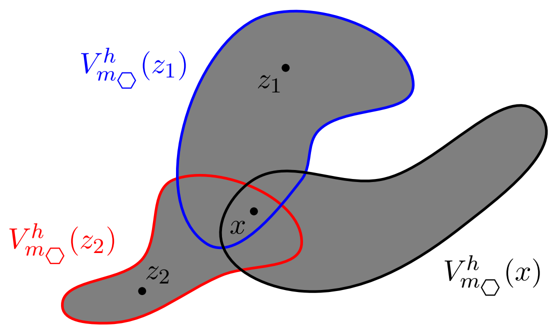


Figure 6. Representation of a strong general adaptive neighborhood $N^h_{m_{\hexagon}}(x)$

These S-GANs satisfy the following properties:

Proposition 10 (Strong General Adaptive Neighborhoods).
Let $(m_{\hexagon}, h, x, y) \in E^{\oplus} \times \mathcal{C} \times D^2$

1. *geometric nesting:*

$$V^h_{m_{\hexagon}}(x) \subseteq N^h_{m_{\hexagon}}(x) \subseteq V^h_{2 \otimes m_{\hexagon}}(x) \tag{21}$$

2. *symmetry:*

$$x \in N^h_{m_{\hexagon}}(y) \Leftrightarrow y \in N^h_{m_{\hexagon}}(x) \tag{22}$$

3. *reflexivity:*

$$x \in N^h_{m_{\hexagon}}(x) \tag{23}$$

4. *increasing with respect to* $m_{\hexagon}$*:*

$$\begin{pmatrix} (m^1_{\hexagon}, m^2_{\hexagon}) \in E^{\oplus} \times E^{\oplus} \\ m^1_{\hexagon} \leq m^2_{\hexagon} \end{pmatrix} \Rightarrow N^h_{m^1_{\hexagon}}(x) \subseteq N^h_{m^2_{\hexagon}}(x) \tag{24}$$

5. *⊕-translation invariance:*

$$c \in E \Rightarrow N^{h \oplus c}_{m_{\bigcirc}}(x) = N^{h}_{m_{\bigcirc}}(x) \tag{25}$$

6. *⊗-multiplication compatibility:*

$$\alpha \in \mathbb{R}^{+}\backslash\{0\} \Rightarrow N^{\alpha \otimes h}_{m_{\bigcirc}}(x) = N^{h}_{\frac{1}{\alpha} \otimes m_{\bigcirc}}(x) \tag{26}$$

Proof:

1. geometric nesting:
Since x belongs to $V^h_{m_{\bigcirc}}(x)$, $V^h_{m_{\bigcirc}}(x)$ is included in $N^h_{m_{\bigcirc}}(x)$.
Let y be a point in $N^h_{m_{\bigcirc}}(x)$. So, there exists z in D such that y belongs to $V^h_{m_{\bigcirc}}(z)$ (with the path P^y_z and x belongs to $V^h_{m_{\bigcirc}}(z)$ (with the path P^x_z).
Thus, the path P^y_x such that $\mathrm{P}^y_x([0,1]) = \check{\mathrm{P}}^x_z([0,1]) \cup \mathrm{P}^y_z([0,1])$ is well-defined.
Let w in $\mathrm{P}^y_x([0,1])$.
If w belongs to $\mathrm{P}^y_z([0,1])$ then $|h(w) \ominus h(y)|_{\bigcirc} \leq m_{\bigcirc} \leq 2 \otimes m_{\bigcirc}$, else
w belongs to $\check{\mathrm{P}}^x_z([0,1]) = \mathrm{P}^z_x([0,1])$ and so
$|h(w) \ominus h(y)|_{\bigcirc} \leq |h(w) \ominus h(z)|_{\bigcirc} \oplus |h(z) \ominus h(y)|_{\bigcirc} \leq m_{\bigcirc} \oplus m_{\bigcirc} = 2 \tilde{\otimes} m.$
Consequently, $y \in V^h_{2 \otimes m_{\bigcirc}}(x)$.

2. symmetry:
If y belongs to $N^h_{m_{\bigcirc}}(x)$, there exists z in D such that y and x both belong to $V^h_{m_{\bigcirc}}(z)$.
So, $N^h_{m_{\bigcirc}}(y)$ holds x by definition.

3-6. these properties are inferred from the correspondent properties of the W-GAN sets (Prop. 7).

□

These S-GANs respect the GAN paradigm (Subsection 4.1) through the geometric nesting property.
In the next subsection, theses S-GANs are used for the definition of Adaptive Structuring Elements required for the so-called *General Adaptive Neighborhood Mathematical Morphology* (GANMM).

4.3. GAN Mathematical Morphology

Using abstract linear algebra (Subsection 2.2) and ordered sets theory (Subsection 2.3), it is possible to examine and propose entirely new operations and structures for image processing. Nevertheless, it is not enough satisfactory, since the available notions do not enable to handle with a sufficiently powerful image representation and to achieve performing image processing techniques. In addition, it is also necessary to resort to other mathematical fields, such as topology, functional analysis, . . .

In the following of this paper, the GANIP approach is then particularly studied in the context of Mathematical Morphology (MM) whose analysis is based on set theory, integral geometry, and lattice algebra [96]. The origin of MM stems from the study of the geometry of porous media by Matheron [59] who proposed the first morphological transformations for investigating the geometry of the objects of a binary image. MM can be defined as a theoretical framework for the analysis of spatial structures [89] characterized by a cross-fertilization between applications, methodologies, theory, and algorithms. It leads to several processing tools in the aim of image

filtering, image segmentation and classification, image measurement, pattern recognition, or texture analysis and synthesis [96].
Mathematical Morphology (MM) needs a *complete lattice* structure [90] to be mathematically well-defined.

Definition 11 (Complete lattice).
The set L is a complete lattice, if and only if:

1. *L is provided with a partial order relation,*
2. *for each collection $\{X_i\}_{i\in I}$ (finite or not) of elements belonging to L, there exists in L, a greatest lower bound (or supremum) $\bigvee_i X_i$, and a least upper bound (or infimum) $\bigwedge_i X_i$.*

Thus, searching to apply the GANIP approach in the context of MM, the GLIP framework of image mappings (Subsection 4.2), $\mathcal{I}$, has to be structured as a complete lattice. However, the ordered vector space $\mathcal{I} = (\tilde{E}^D, \oplus, \otimes, \geq)$ is naturally a complete lattice:

1. $\geq$ is a partial order relation,
2. the supremum and infimum derive directly from those of the real number interval E:
   for each collection $\{f_i\}_{i\in I}$ (finite or not) of image mappings belonging to $\mathcal{I}$,

$$\forall x \in D \quad \left(\bigvee_{i\in I} f_i\right)(x) = \bigvee_{i\in I} f_i(x) \tag{27}$$

$$\forall x \in D \quad \left(\bigwedge_{i\in I} f_i\right)(x) = \bigwedge_{i\in I} f_i(x) \tag{28}$$

Consequently, the GAN paradigm could be applied to Mathematical Morphology, in the so-called General Adaptive Neighborhood Mathematical Morphology (GANMM). First notions and results have been reported in [22, 23, 24].
In this paper, only the flat MM (ie, with structuring elements as subsets of $D \subseteq \mathbb{R}^2$) is considered, but the approach is not restricted and can also address the case of functional MM (ie, with functional structuring elements as functions from a subset of D into $\tilde{E}$) [19].

4.3.1. Adaptive Structuring Elements

The two fundamental operators of Mathematical Morphology are mappings that commute with the infimun and supremum operations, called respectively *erosion* and *dilation* (Def. 12). To each morphological dilation there corresponds a unique morphological erosion, through a *duality relation*, and vice versa.
Two operators ψ and ϕ defines an *adjunction* or a *morphological duality* [90] if and only if:

$$\forall (f,g) \in \mathcal{I} \quad \psi(f) \leq g \Leftrightarrow f \leq \phi(g)$$

Definition 12 (Dilation/Erosion).
The dilation and erosion of an image $f \in \mathcal{I}$ by a SE, denoted B, are respectively defined as:

$$\mathrm{D}_B(f) \; : \; \begin{cases} D & \rightarrow \quad \tilde{E} \\ x & \mapsto \quad \bigvee_{w\in \check{B}(x)} f(w) \end{cases} \tag{29}$$

$$\mathrm{E}_B(f) \; : \; \begin{cases} D & \rightarrow \quad \tilde{E} \\ x & \mapsto \quad \bigwedge_{w\in B(x)} f(w) \end{cases} \tag{30}$$

where $B(x)$ denotes the structuring element B located at point x, and $\check{B}(x)$ its reflected subset.

The definition of those operators entails the notion of *reflected* SEs [90], in order to get this morphological duality, necessary to the building of morphological filters.

Definition 13 (Reflected subset).
The reflected subset of $A(x) \subseteq D$, element of a collection $\{A(z)\}_{z\in D}$, is defined as:

$$\check{A}(x) = \{z; x \in A(z)\} \tag{31}$$

The notion of *autoreflectedness* is then defined as following:

Definition 14 (Autoreflected subset). *The subset $A(x) \subseteq D$, element of a collection $\{A(z)\}_{z\in D}$ is autoreflected if and only if:*

$$\check{A}(x) = A(x) \tag{32}$$

Remark 15. *The term autoreflectedness is introduced in place of symmetry that is generally used in literature [89], so as to avoid the confusion with the geometrical symmetry. The autoreflected subset $A(x) \subseteq D$ of a collection $\{A(z)\}_{z\in D}$ is generally not symmetric with respect to the point x. Nevertheless, autoreflectedness is linked to symmetry, in the sense of Def. 8:*

$$\begin{aligned}(\forall x \in D \quad A(x) \text{ is autoreflected }) &\overset{Def.14}{\Longleftrightarrow} (\forall x \in D \quad A(x) = \check{A}(x)) \\ &\overset{Def.13}{\Longleftrightarrow} (\forall (x,y) \in D^2 \quad y \in A(x) \Leftrightarrow x \in A(y)) \\ &\overset{Def.8}{\Longleftrightarrow} \{A(x)\}_{x\in D} \text{ is a symmetric collection}\end{aligned} \tag{33}$$

The basic idea in the General Adaptive Neighborhood Mathematical Morphology is to substitute the usual Structuring Elements (SEs) by General Adaptive Neighborhoods (GANs).

Although autoreflectedness is not necessary in the general framework of spatially-variant mathematical morphology, as formally proposed by Charif-Chefchaouni and Schonfeld [10] and practically used by Cuisenaire [17]; Lerallut et al. [51], it is however relevant for the three main following reasons [24]:

1. it is more adapted to image analysis for topological and visual reasons,
2. both dualities by adjunction and by opposite for dilation and erosion are satisfied,
3. it allows to simplify mathematical expressions of morphological operators, without increasing computational complexity of algorithms.

From this point, autoreflected adaptive structuring elements are considered in this paper. The GANs employed as ASEs will be the S-GANs (Paragraph 4.2.2), denoted $N^h_{m_{\bigcirc}}$, which satisfy the autoreflectedness condition (or, in an equivalent manner, the symmetry condition in the sense of Def. 8).

Definition 16 (Adaptive Structuring Elements).
The Adaptive Structuring Elements required for the GANMM are the S-GANs, whose definition is called back below:

$$\forall (m_{\bigcirc}, h, x) \in E^{\oplus} \times \mathcal{C} \times D \quad N^h_{m_{\bigcirc}}(x) = \bigcup_{z\in D} \{V^h_{m_{\bigcirc}}(z) | x \in V^h_{m_{\bigcirc}}(z)\} \tag{34}$$

In this way, reflected ASEs will not be necessary to the definition of the dual operators of adaptive dilation and erosion (Def. 18 below).
These adaptive SEs satisfy the properties stated in Proposition 10 above, and then respect the

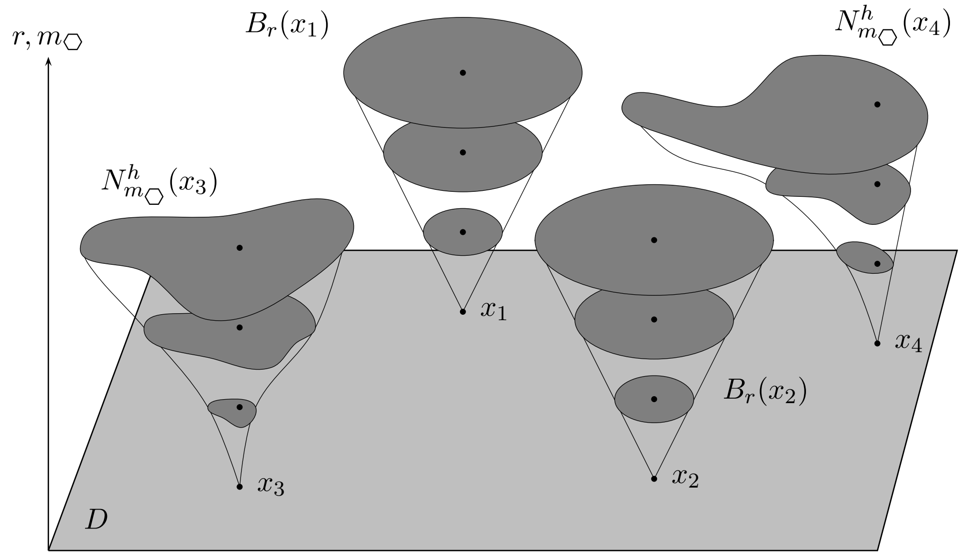


Figure 7. Example of adaptive $N^h_{m_{\hexagon}}$ and non-adaptive B_r structuring elements with three values both for the homogeneity tolerance parameter $m_{\hexagon}$, and for the disks radius r. The shape of $B_r(x_1)$ and $B_r(x_2)$ are identical and $\{B_r(x)\}_r$ is a family of homothetic sets for each point $x \in D$. On the contrary, the shape of $N^h_{m_{\hexagon}}(x_3)$ and $N^h_{m_{\hexagon}}(x_4)$ are dissimilar and $\{N_{m_{\hexagon}}(x)\}_{m_{\hexagon}}$ is not a family of homothetic sets.

AN paradigm through the geometrical nesting property (Prop. 10.1).

Figure 7 compares the shape of usual SEs $B_r(x)$ as disks of radius $r \in \mathbb{R}^+$ to the one of adaptive SEs $N^h_{m_{\hexagon}}(x)$ as sets self-defined with regard to the criterion mapping h and the homogeneity tolerance $m_{\hexagon} \in E^{\oplus}$.
The next step is to define basic adaptive operators of MM in order to build (adaptive) morphological filters.

4.3.2. Fundamental Adaptive Morphological Operators and Filters

The adaptive flat MM is then considered with the ASEs as subsets in D.
The fundamental morphological dual operators of *adaptive dilation* and *adaptive erosion* are respectively defined as:

Definition 17 (Adaptive Dilation/Erosion).
$\forall (m_{\hexagon}, h) \in E^{\oplus} \times \mathcal{C}$

$$\mathrm{D}^h_{m_{\hexagon}} : \begin{cases} \mathcal{I} \rightarrow \mathcal{I} \\ f \mapsto \mathrm{D}^h_{m_{\hexagon}}(f) \end{cases} \tag{35}$$

$$\text{where} \quad \mathrm{D}^h_{m_{\hexagon}}(f) : \begin{cases} D \rightarrow \tilde{E} \\ x \mapsto \bigvee_{w \in N^h_{m_{\hexagon}}(x)} f(w) \end{cases} \tag{36}$$

$$\mathrm{E}^h_{m_{\hexagon}} : \begin{cases} \mathcal{I} \rightarrow \mathcal{I} \\ f \mapsto \mathrm{E}^h_{m_{\hexagon}}(f) \end{cases} \tag{37}$$

$$\text{where} \quad \mathrm{E}^h_{m_{\hexagon}}(f) : \begin{cases} D \rightarrow \tilde{E} \\ x \mapsto \bigwedge_{w \in N^h_{m_{\hexagon}}(x)} f(w) \end{cases} \tag{38}$$

The following example (Fig. 8) illustrates the application of the usual and adaptive morphological operators of dilation and erosion on the 'Lena' image. The adaptive operators do not damaged

a. original image f b. $\mathrm{D}_{B_2}(f)$ c. $\mathrm{D}_{B_2}(f)$

d. $\mathrm{D}_{20}^{f}(f)$ e. $\mathrm{E}_{20}^{f}(f)$

Figure 8. Original 'Lena' image (a). Usual dilation (b) and erosion (c) of the original image using a disk of radius 2 as SE. Adaptive dilation (d) and erosion (e) of the original image using ASEs computed in the CLIP framework on the luminance criterion.

the spatial structures contrary to the usual ones.
Next, the lattice theory of increasing mappings [90] from $\mathcal{I}$ into itself allows to create in many ways more complex morphological operators. They can solve a broad variety of problems in image analysis and nonlinear filtering. More precisely, the two transformations defined by elementary composition of the adaptive dilation and the adaptive erosion, called *adaptive opening* and *adaptive closing*, are *morphological filters* (increasing and idempotent operators) [91]. They are respectively defined as:

Definition 18 (Adaptive Opening/Closing).
$\forall (m_{\hexagon}, h) \in E^{\oplus} \times \mathcal{C}$

$$\mathrm{O}_{m_{\hexagon}}^{h} : \begin{cases} \mathcal{I} & \rightarrow \mathcal{I} \\ f & \mapsto \mathrm{D}_{m_{\hexagon}}^{h}(\mathrm{E}_{m_{\hexagon}}^{h}(f)) \end{cases} \tag{39}$$

$$\mathrm{C}_{m_{\hexagon}}^{h} : \begin{cases} \mathcal{I} & \rightarrow \mathcal{I} \\ f & \mapsto \mathrm{E}_{m_{\hexagon}}^{h}(\mathrm{D}_{m_{\hexagon}}^{h}(f)) \end{cases} \tag{40}$$

The adaptive operators of dilation, erosion, closing and opening satisfy the following properties:

Proposition 19 (Adaptive Dilation/Erosion/Closing/Opening).
Let $(m_{\hexagon}, h, f, f_1, f_2) \in E^{\oplus} \times \mathcal{C} \times \mathcal{I}^3$.

1. *increasing:*

$$f_1 \le f_2 \Rightarrow \begin{cases} \mathrm{D}^h_{m_{\hexagon}}(f_1) \le \mathrm{D}^h_{m_{\hexagon}}(f_2) \\ \mathrm{E}^h_{m_{\hexagon}}(f_1) \le \mathrm{E}^h_{m_{\hexagon}}(f_2) \\ \mathrm{C}^h_{m_{\hexagon}}(f_1) \le \mathrm{C}^h_{m_{\hexagon}}(f_2) \\ \mathrm{O}^h_{m_{\hexagon}}(f_1) \le \mathrm{O}^h_{m_{\hexagon}}(f_2) \end{cases} \tag{41}$$

2. *adjunction (morphological duality):*

$$\mathrm{D}^h_{m_{\hexagon}}(f_1) \le f_2 \Leftrightarrow f_1 \le \mathrm{E}^h_{m_{\hexagon}}(f_2) \tag{42}$$

3. *extensiveness, anti-extensiveness:*

$$\mathrm{O}^h_{m_{\hexagon}}(f) \le f \le \mathrm{C}^h_{m_{\hexagon}}(f) \tag{43}$$

4. *distributivity with* $\bigvee$, $\bigwedge$*:*

$$\forall (f_i) \in T^I \quad \begin{cases} \bigvee_{i \in I} [\mathrm{D}^h_{m_{\hexagon}}(f_i)] = \mathrm{D}^h_{m_{\hexagon}}(\bigvee_{i \in I} [f_i]) \\ \bigwedge_{i \in I} [\mathrm{E}^h_{m_{\hexagon}}(f_i)] = \mathrm{E}^h_{m_{\hexagon}}(\bigwedge_{i \in I} [f_i]) \end{cases} \tag{44}$$

where I *is an index set (finite or not).*

5. *duality with respect to opposite* $\tilde{\ominus}$*:*

$$\begin{cases} \tilde{\ominus}\, \mathrm{D}^h_{m_{\hexagon}}(f) = \mathrm{E}^h_{m_{\hexagon}}(\tilde{\ominus}\, f) \\ \tilde{\ominus}\, \mathrm{C}^h_{m_{\hexagon}}(f) = \mathrm{O}^h_{m_{\hexagon}}(\tilde{\ominus}\, f) \end{cases} \tag{45}$$

6. *idempotence:*

$$\begin{cases} \mathrm{C}^h_{m_{\hexagon}}(\mathrm{C}^h_{m_{\hexagon}}(f)) = \mathrm{C}^h_{m_{\hexagon}}(f) \\ \mathrm{O}^h_{m_{\hexagon}}(\mathrm{O}^h_{m_{\hexagon}}(f)) = \mathrm{O}^h_{m_{\hexagon}}(f) \end{cases} \tag{46}$$

7. *increasing, decreasing with respect to* m*:*

$$\begin{pmatrix} (m^1_{\hexagon}, m^2_{\hexagon}) \in E^{\oplus} \times E^{\oplus} \\ m^1_{\hexagon} \le m^2_{\hexagon} \end{pmatrix} \Rightarrow \begin{cases} \mathrm{D}^h_{m^1_{\hexagon}}(f) \le \mathrm{D}^h_{m^2_{\hexagon}}(f) \\ \mathrm{E}^h_{m^1_{\hexagon}}(f) \ge \mathrm{E}^h_{m^2_{\hexagon}}(f) \end{cases} \tag{47}$$

8. $\oplus$*-translation invariance:*

$$c \in \tilde{E} \Rightarrow \begin{cases} \mathrm{D}^h_{m_{\hexagon} \oplus c}(f) = \mathrm{D}^h_{m_{\hexagon}}(f) \\ \mathrm{E}^h_{m_{\hexagon} \oplus c}(f) = \mathrm{E}^h_{m_{\hexagon}}(f) \\ \mathrm{C}^h_{m_{\hexagon} \oplus c}(f) = \mathrm{C}^h_{m_{\hexagon}}(f) \\ \mathrm{O}^h_{m_{\hexagon} \oplus c}(f) = \mathrm{O}^h_{m_{\hexagon}}(f) \end{cases} \tag{48}$$

9. $\otimes$*-multiplication compatibility:*

$$\alpha \in \mathbb{R}^+ \backslash \{0\} \Rightarrow \begin{cases} \mathrm{D}_{m_{\bigcirc}}^{\alpha \otimes h}(f) = \mathrm{D}_{\frac{1}{\alpha} \otimes m_{\bigcirc}}^{h}(f) \\ \mathrm{E}_{m_{\bigcirc}}^{\alpha \otimes h}(f) = \mathrm{E}_{\frac{1}{\alpha} \otimes m_{\bigcirc}}^{h}(f) \\ \mathrm{C}_{m_{\bigcirc}}^{\alpha \otimes h}(f) = \mathrm{C}_{\frac{1}{\alpha} \otimes m_{\bigcirc}}^{h}(f) \\ \mathrm{O}_{m_{\bigcirc}}^{\alpha \otimes h}(f) = \mathrm{O}_{\frac{1}{\alpha} \otimes m_{\bigcirc}}^{h}(f) \end{cases} \tag{49}$$

10. $\tilde{\oplus}$*-translation commutativity:*

$$c \in E \Rightarrow \begin{cases} \mathrm{D}_{m_{\bigcirc}}^{h}(f \,\tilde{\oplus}\, c) = \mathrm{D}_{m_{\bigcirc}}^{h}(f) \,\tilde{\oplus}\, c \\ \mathrm{E}_{m_{\bigcirc}}^{h}(f \,\tilde{\oplus}\, c) = \mathrm{E}_{m_{\bigcirc}}^{h}(f) \,\tilde{\oplus}\, c \\ \mathrm{C}_{m_{\bigcirc}}^{h}(f \,\tilde{\oplus}\, c) = \mathrm{C}_{m_{\bigcirc}}^{h}(f) \,\tilde{\oplus}\, c \\ \mathrm{O}_{m_{\bigcirc}}^{h}(f \,\tilde{\oplus}\, c) = \mathrm{O}_{m_{\bigcirc}}^{h}(f) \,\tilde{\oplus}\, c \end{cases} \tag{50}$$

11. $\tilde{\otimes}$*-multiplication commutativity:*

$$\alpha \in \mathbb{R} \Rightarrow \begin{cases} \mathrm{D}_{m_{\bigcirc}}^{h}(\alpha \,\tilde{\otimes}\, f) = \alpha \,\tilde{\otimes}\, \mathrm{D}_{m_{\bigcirc}}^{h}(f) \\ \mathrm{E}_{m_{\bigcirc}}^{h}(\alpha \,\tilde{\otimes}\, f) = \alpha \,\tilde{\otimes}\, \mathrm{E}_{m_{\bigcirc}}^{h}(f) \\ \mathrm{C}_{m_{\bigcirc}}^{h}(\alpha \,\tilde{\otimes}\, f) = \alpha \,\tilde{\otimes}\, \mathrm{C}_{m_{\bigcirc}}^{h}(f) \\ \mathrm{O}_{m_{\bigcirc}}^{h}(\alpha \,\tilde{\otimes}\, f) = \alpha \,\tilde{\otimes}\, \mathrm{O}_{m_{\bigcirc}}^{h}(f) \end{cases} \tag{51}$$

12. connectivity:

$$\begin{pmatrix} \mathcal{I} = \mathcal{C} \\ f \in \mathcal{I} \end{pmatrix} \Rightarrow \begin{cases} f \mapsto \mathrm{D}_{m_{\bigcirc}}^{f}(f) \\ f \mapsto \mathrm{E}_{m_{\bigcirc}}^{f}(f) \\ f \mapsto \mathrm{C}_{m_{\bigcirc}}^{f}(f) \\ f \mapsto \mathrm{O}_{m_{\bigcirc}}^{f}(f) \end{cases} \textit{are connected operators.} \tag{52}$$

Proof:

1-6. These properties are inferred from the lattice theory of increasing mappings [90, 91].

7-9. It is directly inferred from the properties 4-6 of the S-GANs (Prop. 10) representing the ASEs.

10-11. The proofs are straightforward.

12. connected operators :
Let g be in $\mathcal{I} = \mathcal{C}$.
For all (x, y) neighboring points (with the usual Euclidean topology on $D \subseteq \mathbb{R}^2$),
if $g(x) = g(y)$ then $N_{m_{\bigcirc}}^{g}(x) = N_{m_{\bigcirc}}^{g}(y)$. So, $D_{m_{\bigcirc}}^{g}(g)(x) = D_{m_{\bigcirc}}^{g}(g)(y)$ and $E_{m_{\bigcirc}}^{g}(g)(x) = E_{m_{\bigcirc}}^{g}(g)(y)$.
Thereafter, the closing and the opening are connected operators by composition of connected operators [93].

□

Remark 20. *The connectivity property (Eq. 52) allows to define several connected operators, which are of great morphological importance. Consequently, the building by composition or combination with the supremum and the infimum of these ones define connected operators too [93].*

Hereafter, the operators :

$$\mathrm{OC}^h_{m_\bigcirc} = \mathrm{O}^h_{m_\bigcirc} \circ \mathrm{C}^h_{m_\bigcirc} \quad (53)$$
$$\mathrm{CO}^h_{m_\bigcirc} = \mathrm{C}^h_{m_\bigcirc} \circ \mathrm{O}^h_{m_\bigcirc} \quad (54)$$
$$\mathrm{OCO}^h_{m_\bigcirc} = \mathrm{O}^h_{m_\bigcirc} \circ \mathrm{C}^h_{m_\bigcirc} \circ \mathrm{O}^h_{m_\bigcirc} \quad (55)$$
$$\mathrm{COC}^h_{m_\bigcirc} = \mathrm{C}^h_{m_\bigcirc} \circ \mathrm{O}^h_{m_\bigcirc} \circ \mathrm{C}^h_{m_\bigcirc} \quad (56)$$

called respectively *adaptive opening-closing*, *adaptive closing-opening*, *adaptive opening-closing-opening* and *adaptive closing-opening-closing* are (adaptive) morphological filters [60], and in addition connected operators when $\mathcal{I} = \mathcal{C}$ (i.e. with the luminance criterion).

4.3.3. ***Adaptive Sequential Morphological Operators***

The collections of adaptive morphological filters $\{\mathrm{O}^h_{m_\bigcirc}\}_{m_\bigcirc \geq 0_\bigcirc}$ and $\{\mathrm{C}^h_{m_\bigcirc}\}_{m_\bigcirc \geq 0_\bigcirc}$ are generally not a *size distribution* and *anti-size distribution* respectively [91], since the notion of semi-group is generally not satisfied [90]. A counter-example is given in [19]. However, those ordered collections of filters (size and anti-size distributions) are particularly useful in multiscale image processing. Therefore, such GAN-based families are built by naturally reiterating adaptive dilation or erosion, in order to define new fundamental morphological operators, and thereafter, advanced operators. Explicitly, *adaptive sequential dilation, erosion, closing and opening* are respectively defined as:

Definition 21 (Adaptive Sequential Morphological Operators).
$\forall (m_\bigcirc, p, h) \in E^{\oplus} \times \mathbb{N} \times \mathcal{C}$

$$\mathrm{D}^h_{m_\bigcirc,p} : \begin{cases} \mathcal{I} \rightarrow \mathcal{I} \\ f \mapsto \underbrace{\mathrm{D}^h_{m_\bigcirc} \circ \cdots \circ \mathrm{D}^h_{m_\bigcirc}}_{p\ times}(f) \end{cases} \quad (57)$$

$$\mathrm{E}^h_{m_\bigcirc,p} : \begin{cases} \mathcal{I} \rightarrow \mathcal{I} \\ f \mapsto \underbrace{\mathrm{E}^h_{m_\bigcirc} \circ \cdots \circ \mathrm{E}^h_{m_\bigcirc}}_{p\ times}(f) \end{cases} \quad (58)$$

$$\mathrm{C}^h_{m_\bigcirc,p} : \begin{cases} \mathcal{I} \rightarrow \mathcal{I} \\ f \mapsto \mathrm{E}^h_{m_\bigcirc,p} \circ \mathrm{D}^h_{m_\bigcirc,p}(f) \end{cases} \quad (59)$$

$$\mathrm{O}^h_{m_\bigcirc,p} : \begin{cases} \mathcal{I} \rightarrow \mathcal{I} \\ f \mapsto \mathrm{D}^h_{m_\bigcirc,p} \circ \mathrm{E}^h_{m_\bigcirc,p}(f) \end{cases} \quad (60)$$

The morphological duality (Prop. 20 below) between adaptive sequential dilation (Eq. 57) and adaptive sequential erosion (Eq. 58) allows the operators of adaptive sequential closing (Eq. 59) and adaptive sequential opening (Eq. 60) to be idempotent. Consequently, it allows both the adaptive sequential closing and the adaptive sequential opening to be morphological filters, since in addition, these operators are increasing by composition of increasing operators.

Proposition 22 (Adjunction Sequential Dilation - Sequential Erosion).
$\forall (m_\bigcirc, p, h) \in E^{\oplus} \times \mathbb{N} \times \mathcal{C}$ $\mathrm{D}^h_{m_\bigcirc,p}$ *and* $\mathrm{E}^h_{m_\bigcirc,p}$ *define a morphological duality.*
Proof:
Let (f, g) be in $\mathcal{I}^2$.
If $\mathrm{D}^h_{m_\bigcirc,p}(f) \leq g$ then $\mathrm{E}^h_{m_\bigcirc,p} \circ \mathrm{D}^h_{m_\bigcirc,p}(f) \leq \mathrm{E}^h_{m_\bigcirc,p}(g)$.

Beyond,

$$\begin{aligned}
\mathrm{E}^h_{m_{\bigcirc},p} \circ \mathrm{D}^h_{m_{\bigcirc},p}(f) &= \underbrace{\mathrm{E}^h_{m_{\bigcirc}} \circ \cdots \circ \mathrm{E}^h_{m_{\bigcirc}}}_{p-1 \text{ times}} \circ \mathrm{E}^h_{m_{\bigcirc}} \circ \mathrm{D}^h_{m_{\bigcirc}} \circ \underbrace{\mathrm{D}^h_{m_{\bigcirc}} \circ \cdots \circ \mathrm{D}^h_{m_{\bigcirc}}}_{p-1 \text{ times}}(f) \\
&= \underbrace{\mathrm{E}^h_{m_{\bigcirc}} \circ \cdots \circ \mathrm{E}^h_{m_{\bigcirc}}}_{p-1 \text{ times}} \circ \mathrm{C}^h_{m_{\bigcirc}} \circ \underbrace{\mathrm{D}^h_{m_{\bigcirc}} \circ \cdots \circ \mathrm{D}^h_{m_{\bigcirc}}}_{p-1 \text{ times}}(f) \\
&\geq \underbrace{\mathrm{E}^h_{m_{\bigcirc}} \circ \cdots \circ \mathrm{E}^h_{m_{\bigcirc}}}_{p-1 \text{ times}} \circ \underbrace{\mathrm{D}^h_{m_{\bigcirc}} \circ \cdots \circ \mathrm{D}^h_{m_{\bigcirc}}}_{p-1 \text{ times}}(f) \\
&\geq \cdots \geq \mathrm{E}^h_{m_{\bigcirc}} \circ \mathrm{D}^h_{m_{\bigcirc}}(f) \geq f
\end{aligned}$$

Thus, $f \leq \mathrm{E}^h_{m_{\bigcirc},p} \circ \mathrm{D}^h_{m_{\bigcirc},p}(f) \leq \mathrm{E}^h_{m_{\bigcirc},p}(g)$ □

Moreover, the morphological filters $\mathrm{C}^h_{m,p}$ and $\mathrm{O}^h_{m,p}$ generate size and anti-size distributions, i.e.: for all f in $\mathcal{I}$, $\{\mathrm{C}^h_{m,p}(f)\}_{p\geq 0}$ and $\{\mathrm{O}^h_{m,p}(f)\}_{p\geq 0}$ are ordered collections of image mappings with regard to the order relation of the GLIP framework $\mathcal{I}$.

Theorem 23 (Size and Anti-Size Distributions).
$\forall (m_{\bigcirc}, h) \in E^{\oplus} \times \mathcal{C}$

$$\{\mathrm{O}^h_{m_{\bigcirc},p}\}_{p\geq 0} \quad \textit{is a size distribution} \tag{61}$$

$$\{\mathrm{C}^h_{m_{\bigcirc},p}\}_{p\geq 0} \quad \textit{is an anti-size distribution} \tag{62}$$

Proof:
Let $f \in \mathcal{I}$ and $(p,q) \in \mathbb{N}^2$ such that $p \geq q$.

$$\begin{aligned}
\mathrm{O}^h_{m_{\bigcirc},p} &\leq \mathrm{D}^h_{m_{\bigcirc},q} \circ \mathrm{D}^h_{m_{\bigcirc},p-q} \circ \mathrm{E}^h_{m_{\bigcirc},p-q} \circ \mathrm{E}^h_{m_{\bigcirc},q} \\
&\leq \mathrm{D}^h_{m_{\bigcirc},q} \circ \mathrm{O}^h_{m_{\bigcirc},p-q} \circ \mathrm{E}^h_{m_{\bigcirc},q} \\
&\leq \mathrm{D}^h_{m_{\bigcirc},q} \circ \mathrm{E}^h_{m_{\bigcirc},q} \\
&\leq \mathrm{O}^h_{m_{\bigcirc},q}.
\end{aligned}$$

$$\begin{aligned}
\mathrm{C}^h_{m_{\bigcirc},p} &\geq \mathrm{E}^h_{m_{\bigcirc},q} \circ \mathrm{E}^h_{m_{\bigcirc},p-q} \circ \mathrm{D}^h_{m_{\bigcirc},p-q} \circ \mathrm{D}^h_{m_{\bigcirc},q} \\
&\geq \mathrm{E}^h_{m_{\bigcirc},q} \circ \mathrm{C}^h_{m_{\bigcirc},p-q} \circ \mathrm{D}^h_{m_{\bigcirc},q} \\
&\geq \mathrm{E}^h_{m_{\bigcirc},q} \circ \mathrm{D}^h_{m_{\bigcirc},q} \\
&\geq \mathrm{C}^h_{m_{\bigcirc},q}
\end{aligned}$$

□

Thus, Theorem 23 allows to define the GAN-based extension of the well-known Alternating Sequential Filters (ASF) [92]. They are based on compositions of increasingly more severe openings and closings. So, the *adaptive alternating sequential filters* are defined as:

Definition 24 (Adaptive Alternating Sequential Filters).
$\forall (m,n,h) \in E^{\oplus} \times \mathbb{N}\backslash\{0\} \times \mathcal{C} \quad \forall (p_i) \in \mathbb{N}^{[\![1,n]\!]}$ *increasing sequence*

$$\mathrm{ASFOC}^h_{m_{\bigcirc},n} : \begin{cases} \mathcal{I} \rightarrow \mathcal{I} \\ f \mapsto \mathrm{OC}^h_{m_{\bigcirc},p_n} \circ \cdots \circ \mathrm{OC}^h_{m_{\bigcirc},p_1}(f) \end{cases} \tag{63}$$

$$\mathrm{ASFCO}^h_{m_{\bigcirc},n} : \begin{cases} \mathcal{I} \rightarrow \mathcal{I} \\ f \mapsto \mathrm{CO}^h_{m_{\bigcirc},p_n} \circ \cdots \circ \mathrm{CO}^h_{m_{\bigcirc},p_1}(f) \end{cases} \tag{64}$$

On the whole, the practical results and interests of such GAN-based morphological operators, in relation to the usual ones, are exposed in Part II [21] of the present paper. GANIP-based applications are achieved in the field of image filtering, image segmentation and image enhancement.

5. Conclusion and Prospects

In this part I, the General Adaptive Neighborhood Image Processing (GANIP) approach has been exposed from a theoretical point of view. GAN-based operators depend on the image context with intrinsically and locally defined operational windows. It allows to get a connection with the physical and/or physiological image settings, with general linear image processing frameworks, using concepts and structures from abstract linear algebra. Moreover, a significant spatially-adaptive analysis is achieved with the help of an analyzing criterion which is added to the definition of the usual Adaptive Neighborhoods. Thereafter, the GANIP approach has been more particularly studied in the context of Mathematical Morphology. In this way, the connectivity property of the new adaptive morphological operators, satisfied in several and relevant cases, theoretically highlights the morphological and topological relevance of the proposed approach. Indeed, only advanced operators of Mathematical Morphology [16], based on reconstruction processes using geodesic [35] concepts, satisfy this connectivity property of powerful topological importance.
Several application examples exposed in Part II [21] emphasize this theoretical advantage. Moreover, the settings of general linear image processing frameworks enables to choose the most appropriate framework compatible with the application to be processed. More precisely, the Part II [21] practically shows that the Logarithmic Image Processing framework is needed in presence of locally small lightening changes in scene illumination.
Furthermore, the General Adaptive Neighborhood Image Processing approach promise large prospects, more particularly in other fields than mathematical morphology, such as convolution analysis, order filtering, differential and integral calculus ...

Finally, the authors [19] are currently studying the size distributions induced by families of adaptive morphological operators, the transforms defined with selected criterion other than 'luminance' and 'contrast', multiscale metrics based on GANs sets, and the use of concepts and notions of generalized topologies [9] within the GANs framework.

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