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Asymptotic Properties of the Detrended Fluctuation Analysis of Long Range Dependent Processes

Jean-Marc Bardet and Imen Kammoun

Abstract—In the past few years, a certain number of authors have proposed analysis methods of the time series built from a long range dependence noise. One of these methods is the Detrended Fluctuation Analysis (DFA), frequently used in the case of physiological data processing. The aim of this method is to highlight the long-range dependence of a time series with trend. In this study asymptotic properties of the DFA of the fractional Gaussian noise are provided. Those results are also extended to a general class of stationary long-range dependent processes. As a consequence, the convergence of the semi-parametric estimator of the Hurst parameter is established. However, several simple examples also show that this method is not at all robust in the case of trends.

Index Terms—Detrended fluctuation analysis, fractional Gaussian noise, stationary process, self-similar process, Hurst parameter, trend, long-range dependent processes.

I. INTRODUCTION

In the past few years, numerous methods of analysis of a trended long range process have been proposed. One of these methods is the Detrended Fluctuation Analysis (DFA), frequently used in the case of physiological data processing in particular the heartbeat signals recorded on healthy or sick subjects [17], [20], [25]-[27]. Indeed, it can be interesting to find some constants among the fluctuations of physiological data. The Hurst parameter of the original signal, or the self-similarity parameter of the aggregated signal could be a new way of interpreting and explaining a physiological behavior.

The DFA method is a version for time series with trend of the method of aggregated variance used for a long-memory stationary process (see for instance [20]). It consists in 1. aggregating the process by windows with fixed length, 2. detrending the process from a linear regression in each window, 3. computing the standard deviation of the residual errors (the DFA function) for all data, 4. estimating the coefficient of the power law from a log-log regression of the DFA function on the length of the chosen window. After the first stage, the process is supposed to behave like a self-similar process with stationary increments added to a trend. The second stage is supposed to remove the trend. Finally, the third and fourth stages are identical to those of the aggregated method (for zero-mean stationary process).

The processing of experimental data, and in particular physiological data, exhibits a major problem which is the non-stationarity of the signal. Hu et al. [17] have studied different types of non-stationarities associated with examples of trends (linear, sinusoidal and power-law trends) and deduced their effect on an added noise and the kind of competition which exists between this two signals. They have also explained (see Chen et al. [8]) the effects of three other types of non-stationarities, which are often encountered in real data. The DFA method was applied to signals having some segments removed, with random spikes or with different local behavior. The results were compared with the case of stationary correlated signals.

In Taqqu et al. [33], the case of the fractional Gaussian noise (FGN) is studied. A theoretical proof to the power law followed by the expectation of the DFA function of this process is established. This is an important first step in order to prove the convergence of the estimator of the Hurst parameter. The study we propose here constitutes an accomplishment of this work. Indeed the convergence rate of the Hurst parameter estimator is obtained, in a semi-parametric frame.

The paper is organized as follows. In Section II, the DFA method is presented and two general properties are proved. Section III is devoted to providing asymptotic properties (illustrated beforehand by simulations) of the DFA function in the case of the FGN. Section IV contains an extension of these results for a general class of stationary long-range dependent processes. Finally, in Section V, the method is proved not to be robust in different particular cases of trended processes. Indeed the trend is dominant in the case of power law and polynomial trends, where the slope of the DFA log-log regression line for trended processes is always close to 2, or in the case of a piecewise constant trend, where the slope is estimated at 3/2, which dominates the Hurst exponent. The proofs of the different results are in Appendix I.

II. DEFINITIONS AND FIRST PROPERTIES OF THE DFA METHOD

Notation and preliminaries

In the following we shall use the following notations. Let \((Y(1), \ldots, Y(N))\) be a sample of a time series \((Y(n))_{n \in \mathbb{N}}\). Let us denote the "discrete integration" of this sample

\[
X(k) = \sum_{i=1}^{k} Y(i) \quad \text{for } k \in \{1, \ldots, N\}. \tag{1}
\]
For \( j \in \{1, \ldots, [N/n]\} \), let us define \( s_{j,k} = n(j-1)+k \) with \( k = \{1, \ldots, n\} \) and the vectors

\[
X^{(j)} = (X(1 + n(j-1)), \ldots, X(nj))' \\
X^{(j)} = (\tilde{X}(1 + n(j-1)), \ldots, \tilde{X}(nj))'.
\]

Let \( E_j \) be the vector subspace of \( n \) generated by the two vectors of \( n \), \( e_1 = (1, \ldots, 1)' \) and \( e_2 = ((j-1)n+1, (j-1)n+2, \ldots, nj)' \) and \( E_j \) its orthogonal vector subspace. Finally, let us define \( P_A \) the matrix of the orthogonal projection on a vector subspace \( A \) of \( n \).

**The Detrended Fluctuation Analysis (DFA)**

The DFA method was introduced in [26]. The aim of this method is to highlight the self-similarity of a time series with trend.

1) The first step of the DFA method is the calculation of \( (X(1), \ldots, X(N)) \).

2) The second step is a division of \( \{1, \ldots, N\} \) in \([N/n]\) windows of length \( n \) (for \( x \in \), \( [x] \) is the integer part of \( x \)). In each window, the least-square regression line is computed, which represents the linear trend of the process in the window. Then, we denote by \( \tilde{X}_n(k) \) for \( k = 1, \ldots, N \) the process formed by this piecewise linear interpolation. Then the DFA function is the standard deviation of the residuals obtained from the difference between \( X(k) \) and \( \tilde{X}_n(k) \), therefore,

\[
F(n) = \left( \frac{1}{n \cdot [N/n]} \sum_{k=1}^{n[N/n]} \left( X(k) - \tilde{X}_n(k) \right)^2 \right)^{1/2}.
\]

3) The third step consists in a repetition of the second step with different values \( (n_1, \ldots, n_m) \) of the window’s length. Then the graph of \( \log F(n_i) \) by \( \log n_i \) is drawn. The slope of the least-square regression line of this graph provides an estimation of the self-similarity parameter of the \( (X(k))_{k \in \mathbb{N}} \) process or the Hurst parameter of the \( (\tilde{Y}(n))_{n \in \mathbb{N}} \) process (see above for the explanations).

From the construction of the DFA method, it is interesting to define the restriction of the DFA function in a window. Thus, for \( n \in \{1, \ldots, N\} \), one defines the partial DFA function computed in the \( j \)-th window, i.e.

\[
F^2_j(n) = \frac{1}{n} \sum_{i=n(j-1)+1}^{nj} (X(i) - \tilde{X}_n(i))^2
\]

for \( j \in \{1, \ldots, [N/n]\} \). Then, it is obvious that

\[
F^2(n) = \frac{1}{[N/n]} \sum_{j=1}^{[N/n]} F^2_j(n).
\]

**Remark:** In Hu et al.'s and Kantelhardt et al.'s papers (for details see [17], [19] and [20]), the definition of the time series \( (\tilde{X}(n))_{n \in \mathbb{N}} \) computed from \( (\tilde{Y}(n))_{n \in \mathbb{N}} \) is different from (1), i.e.

\[
\tilde{X}(k) = \sum_{i=1}^{k} (Y(i) - \bar{Y}_N), \text{ for } k \in \{1, \ldots, N\}
\]

with \( \bar{Y}_N = \frac{1}{N} \sum_{j=1}^{N} Y(j) \).

It is obvious that in both definitions, \( (X(k) - \tilde{X}_n(k)) \) is the same and therefore the value of \( F(n) \) is the same.

**Lemma 2.1:** With the previous notations, let \( \tilde{F}(n) \) be the DFA function built from \( (\tilde{X}(k)) \), i.e.

\[
\tilde{F}(n) = \left( \frac{1}{n \cdot [N/n]} \sum_{k=1}^{n[N/n]} \left( \tilde{X}(k) - \tilde{X}_n(k) \right)^2 \right)^{1/2}.\]

Then for \( n \in \{1, \ldots, N\} \), \( F(n) = \tilde{F}(n) \).

**Proof:** In \( j \)-th window, we have \( \tilde{X}^{(j)} = X^{(j)} - e_2 \cdot \bar{Y}_N \),

\[
F^2_j(n) = \frac{1}{n} \left( P_{E_j} \cdot X^{(j)} \right)' \cdot P_{E_j} \cdot X^{(j)}
\]

and \( \tilde{F}^2_j(n) = \frac{1}{n} \left( P_{E_j} \cdot \tilde{X}^{(j)} \right)' \cdot P_{E_j} \cdot \tilde{X}^{(j)} \).

As \( e_2 \cdot \bar{Y}_N \in E_j \), \( P_{E_j} \cdot X^{(j)} = P_{E_j} \cdot \tilde{X}^{(j)} \) and thus \( F^2_j(n) = \tilde{F}^2_j(n) \). This implies that \( F(n) = \tilde{F}(n) \).

In order to simplify the following proofs, we prove that the application of the DFA to a stationary process yields a stationary process again.

**Lemma 2.2:** Let \( \{Y(t), t \geq 0\} \) be a stationary process. Then, the time series \( (F^2_j(n))_{1 \leq j \leq [N/n]} \) is a stationary process.

**Proof:** In each window \( j \),

\[
X^{(j)} = X^{(s_j,1)} : e_1 \equiv X^{(1)} - X^{(1)} \cdot e_1.
\]

Indeed

\[
X^{(j)} - X^{(s_j,1)} \cdot e_1 = (0, Y(s_j,2), \ldots, \sum_{k=2}^{n-1} Y(s_j,k) \sum_{k=2}^{n-1} Y(s_j,k))
\]

and \( X^{(1)} - X^{(1)} \cdot e_1 = (0, Y(2), \ldots, \sum_{k=2}^{n-1} Y(k), \sum_{k=2}^{n-1} Y(k)) \).

As \( \{Y(2), \ldots, Y(n) \} \equiv (Y(s_j,2), \ldots, Y(s_j,n)) \), then with \( g : n \rightarrow n-1 \) a Borelian function defined by \( g(y_2, \ldots, y_n) = (y_2, \ldots, \sum_{k=2}^{n-1} y_k, \sum_{k=2}^{n-1} y_k) \), it is clear that \( g(Y(2), \ldots, Y(n)) \equiv g(Y(s_j,2), \ldots, Y(s_j,n)) \) and therefore (4) is true.

\[
F^2_j(n) = \frac{1}{n} \left( P_{E_j} \cdot X^{(j)} \right)' \cdot P_{E_j} \cdot X^{(j)}
\]

\[
= \frac{1}{n} \left( X^{(j)} - X^{(s_j,1)} \cdot e_1 \right)' \cdot P_{E_j} \cdot \left( X^{(j)} - X^{(s_j,1)} \cdot e_1 \right),
\]

with \( P_{E_j} \cdot e_1 = 0_{\mathbb{R}^n} \). But \( E_1 = E_j \) and thus \( E_j = E_j^{\perp} \).

Therefore, with (4), we obtain \( F^2_j(n) \equiv F^2_1(n) \) for all \( j \in \{1, \ldots, [N/n]\} \).

Moreover, for all \( m \in ^*, (j_1, \ldots, j_m) \in \{1, \ldots, [N/n]\}^m \) and \( t \in ^* \), the same reasoning can be used again
for the case of vectors \((F^2_{j_1}(n), \ldots, F^2_{j_m}(n))\) and \((F^2_{j_1+t}(n), \ldots, F^2_{j_m+t}(n))\). Indeed,

\[
\begin{align*}
(X^{(j_1)})' - X(s_{j_1,1}) \cdot e'_1, \ldots, (X^{(j_m)})' - X(s_{j_m,1}) \cdot e'_1 \end{align*}
\]

\[
\begin{align*}
(X^{(j_1+t)})' - X(s_{j_1+t,1}) \cdot e'_1, \ldots, (X^{(j_m+t)})' - X(s_{j_m+t,1}) \cdot e'_1 \end{align*}
\]

and \(P_{E_{j_1}} = \cdots = P_{E_{j_1+t}} = P_{E_{j_1+2t}} = \cdots = P_{E_{j_m+t}}\). This achieves the proof. 

\[\square\]

III. ASYMPTOTIC PROPERTIES OF THE DFA FUNCTION FOR A FGN

In this section, we study the asymptotic behavior (both the sample size \(N\) and the length of window \(n\) increase to \(\infty\)) of the DFA when \((Y(k))_{k \in \mathbb{Z}}\) is a stationary Gaussian process defined by a fractional Gaussian noise (FGN), i.e., \((X(1), \ldots, X(N))\) is a Gaussian process with stationary increments and called a fractional Brownian motion (FBM). First, let us remind some definitions and properties of both these processes.

**Definition and first properties of the FBM and the FGN**

Let \(\{X^H(t), t \in \text{ }\}\) be a FBM with parameters \(H \in [0,1]\) and \(\sigma^2 > 0\), i.e., a real zero mean Gaussian process satisfying

1. \(X^H(0) = 0\) a.s.
2. \(E[(X^H(t) - X^H(s))^2] = \sigma^2 |t - s|^{2H}\) \(\forall (t, s) \in \mathbb{R}^2\).

Here are some properties of a FBM \(\{X^H(t), t \in \text{ }\}\) (see more details in Samorodnitsky and Taqqu [30])

- The process \(\{X^H(t), t \in \text{ }\}\) has stationary increments. As a consequence, if we denote \(\{Y^H(t), t \in \text{ }\}\) the process defined by \(Y^H(t) = X^H(t + 1) - X^H(t)\) for \(t \in \mathbb{R}\), then \(\{Y^H(t), t \in \text{ }\}\) is a zero-mean stationary Gaussian process called a FGN.
- \(\{X^H(t), t \in \text{ }\}\) is a self-similar process satisfying \(\forall c > 0, X^H(ct) \equiv c^H X^H(t)\) and \(H\) is also called the exponent (or parameter) of self-similarity.
- The covariance function of a FBM \(\{X^H(t), t \in \text{ }\}\), for all \((s, t) \in \mathbb{R}^2\) is

\[
\text{Cov}(X^H(t), X^H(s)) = \frac{\sigma^2}{2} (|s|^{2H} + |t|^{2H} - |t - s|^{2H}).
\]

(5)

- The covariance function of a FGN \(\{Y^H(t), t \in \text{ }\}\), for all \((s, t) \in \mathbb{R}^2\) is

\[
\text{Cov}(Y^H(t), Y^H(s)) = \frac{\sigma^2}{2} (|t - s + 1|^{2H} + |t - s - 1|^{2H} - 2|t - s|^{2H}).
\]

Therefore, \(\text{Cov}(Y^H(t), Y^H(s)) \sim H(2H - 1)|t - s|^{2H - 2}\) when \(|t - s| \rightarrow \infty\): when \(1/2 < H < 1\), \(Y^H\) is a long memory process (see also (11) below) and \(H\) is the Hurst (or long range dependent) parameter of \(Y^H\).

**Some numerical results of the DFA applied to the FGN**

The following Figures 1 and 2 show an example of the DFA method applied to a FGN with different values of \(H\) (\(H = 0.6\) in the first figure and \(H = 0.2, 0.4, 0.5, 0.7, 0.8\) in the second one, with \(N = 10000\) in both cases). Such a sample path is generated with a circulant matrix algorithm (see for instance Bardet et al., [7]).

On the right side of Figure 2 appear the different estimations of \(H\) computed from the DFA method. Those values have to be compared with theoretical ones. The results seem to be quite good and it seems that, under certain conditions, the asymptotic behavior of the DFA function \(F(n)\) can be written as

\[
F(n) \simeq c(\sigma, H) \cdot n^H
\]

(7)

where \(c\) is a positive function depending only on \(\sigma\) and \(H\) (see its expression above).

The approximation (7) explains that the slope of the least-square regression line of \((\log F(n_i))\) by \((\log(n_i))\) for different (6) values of \(n_i\) provides an estimation of \(H\). We now provide a mathematical proof of this result.

Let \(\{X^H(t), t \geq 0\}\) be a FBM, built as a cumulated sum of a FGN \(\{Y^H(t), t \geq 0\}\). We first give some asymptotic properties of \(F^2_1(n)\).
Property 3.1: Let \( \{X^H(t), t \geq 0\} \) be a FBM with parameters \( 0 < H < 1 \) and \( \sigma^2 > 0 \). Then, for \( n \) and \( j \) large enough,

1. \( (F_2^2(n)) = \sigma^2 f(H) \cdot n^{2H} \left(1 + O\left(\frac{1}{n}\right)\right) \),
2. \( \text{Var}(F_2^2(n)) = \sigma^4 g(H) \cdot n^{4H} \left(1 + O\left(\frac{1}{n}\right)\right) \),
3. \( \text{Cov}(F_2^2(n), F_j^2(n)) = \sigma^4 h(H) \cdot n^{4H} \cdot j^{2H-3} \cdot \left(1 + O\left(\frac{1}{j}\right)\right) \),

with \( f(H) = \frac{(1-H)(2H+1)(H+1)(H+2)}{(2H+1)(H+2)} \), \( g(H) = \frac{1}{H^2(1-H)(2H-1)^2} \), \( h(H) = \frac{1}{48(H+1)(2H+1)(2H+3)} \).

The proofs of these results (and of the others) are provided in Appendix I.

In order to obtain a central limit theorem for the logarithm of the DFA function, we consider normalized DFA functions

\[
\tilde{S}_j(n) = \frac{F_2^2(n)}{n^{2H} \sigma^2 f(H)} \quad \text{and} \quad \tilde{S}(n) = \frac{F_2^2(n)}{n^{2H} \sigma^2 f(H)}
\]

for \( n \in \{1, \ldots, N\} \) and \( j \in \{1, \ldots, N/n\} \).

As a consequence, for \( n \in \{1, \ldots, N\} \), the stationary time series \( (\tilde{S}_j(n))_{1 \leq j \leq [N/n]} \) satisfy

\[
\begin{cases}
(\tilde{S}_j(n)) = 1 + O\left(\frac{1}{n}\right) \\
\text{Var}(\tilde{S}_j(n)) = \frac{g(H)}{f(H)^2} \left(1 + O\left(\frac{1}{n}\right)\right) \\
\text{Cov}(\tilde{S}_1(n), \tilde{S}_j(n)) = \frac{h(H)}{f(H)^2} \left(1 + O\left(\frac{1}{n}\right)\right) \left(1 + O\left(\frac{1}{j}\right)\right)
\end{cases}
\]

Under certain conditions on the asymptotic length \( n \) of the windows, one proves a central limit theorem satisfied by the logarithm of the empirical mean \( \tilde{S}(n) \) of the random variables \( (\tilde{S}_j(n))_{1 \leq j \leq [N/n]} \).

Property 3.2: Under the previous assumptions and notations, let \( n \in \{1, \ldots, N\} \) be such that \( N/n \rightarrow \infty \) and \( N/n^3 \rightarrow 0 \) when \( N \rightarrow \infty \). Then

\[
\sqrt{\frac{N}{n}} \cdot \log(\tilde{S}(n)) \xrightarrow{n \to \infty} N(0, \gamma^2(H)),
\]

where \( \gamma^2(H) > 0 \) depends only on \( H \).

This result can be obtained for different lengths of windows satisfying the conditions \( N/n \rightarrow \infty \) and \( N/n^3 \rightarrow 0 \). Let \( (n_1, \ldots, n_m) \) be such different window lengths. Then, one can write for \( N \) and \( n_i \) large enough

\[
\log(\tilde{S}(n_i)) \simeq \frac{1}{\sqrt{[N/n_i]}} \cdot \varepsilon_i \quad \implies \quad \log(F(n_i)) \simeq H \cdot \log(n_i) + \frac{1}{2} \cdot \log(\sigma^2 f(H)) + \frac{1}{\sqrt{[N/n_i]}} \cdot \varepsilon_i,
\]

with \( \varepsilon_i \sim N(0, \gamma^2(H)) \). As a consequence, a linear regression of \( \log(F(n_i)) \) on \( \log(n_i) \) provides an estimation of \( H \). More precisely,

Proposition 3.3: Under the previous assumptions and notations, let \( n \in \{1, \ldots, N\} \), \( m \in \mathbb{N} \setminus \{1\} \), \( r_i \in \{1, \ldots, [N/n]\} \) for each \( i \) with \( r_1 < \cdots < r_m \) and \( n_i = r_i n \) be such that \( N/n \rightarrow \infty \) and \( N/n^3 \rightarrow 0 \) when \( N \rightarrow \infty \). Let \( \hat{H} \) be the estimator of \( H \) from the linear regression of \( \log(F(r_i \cdot n)) \) on \( \log(r_i \cdot n) \), i.e.,

\[
\hat{H} = \frac{\sum_{i=1}^m (\log(F(r_i \cdot n)) - \log(F(n))) \cdot (\log(r_i \cdot n) - \log(n))}{\sum_{i=1}^m (\log(r_i \cdot n) - \log(n))^2}.
\]

Then \( \hat{H} \) is a consistent estimator of \( H \) such that

\[
[(\hat{H} - H)^2] \leq C(H, m, r_1, \ldots, r_m) \frac{1}{[N/n]} \quad (10)
\]

with \( C(H, m) > 0 \).

IV. EXTENSION OF THE RESULTS FOR A GENERAL CLASS OF A LONG-RANGE DEPENDENT PROCESS

Let \( \{Y^k(k), k \in \mathbb{Z}\} \) be a stationary zero mean long-range dependent process with a Hurst parameter \( H \in [1/2, 1] \). More precisely, let \( r_Y(k) \) be the autocorrelation function of this process and let us assume that there exists a slowly varying function \( L(k) \) such that

\[
r_Y(k) \sim k^{2H-2} L(k) \quad \text{as} \quad k \to \infty.
\]

Under different additional assumptions on \( Y \), Davydov [9], Taqqu [32], Dobrushin and Major [11], Giraitis and Surgailis [15] and other authors have studied the asymptotic behavior of the Donsker line and obtained the following convergence,

\[
(L(n)^{-1/2} n^{-H} \sum_{i=1}^{[nt]} Y(i))_{t>0} \xrightarrow{n \to \infty} \mathbb{D} \quad \sigma \cdot B_H(t)_{t>0}
\]

with \( \sigma > 0 \) and \( B_H \) a fractional Brownian motion. Remind that \( Z = \{Z(k), k \in \mathbb{Z}\} \) is a linear process when

\[
Z(k) = \sum_{i=-\infty}^{\infty} a_i \xi_{k-i} \quad \text{for} \quad k \in \mathbb{Z},
\]

with \( (a_k) \) a sequence of real numbers and \( (\xi_n) \) a sequence of zero mean i.i.d.r.v. Then,

Theorem 4.1: (Davydov, Taqqu) Let \( Y = \{Y(k), k \in \mathbb{Z}\} \) be a stationary zero mean long-range dependent process satisfying assumption (11). Then, if:

- \( Y \) is a linear process,
- or \( Y \) is a function of a Gaussian process with Hermite rank \( r = 1 \),

then (12) holds, and the convergence takes place in the Skorohod space.

Limit theorems are also obtained by Dobrushin and Major [11], Giraitis and Surgailis [15] and Ho and Hsing [16] for sums of polynomials of linear (or moving average) process with slowly decreasing coefficients \( a_i \). It is obtained under the hypothesis that \( (\xi_n) \) are i.i.d standard normal random variable and that the Polynomial Hermit rank satisfies \( 2r < (1-H)^{-1} \).
So, in this case of general class of LRD process, the aggregated process \(X(k)\) has roughly the same behavior as a fractional Brownian motion and the previous asymptotic results of the DFA method can be applied. But Property 3.1 and Proposition 3.3 cannot be proved under so general assumptions. Indeed, the proofs of such results use a very precise expression of the covariance and a stronger version of assumption (11) is necessary. Hence, the covariance \(r_Y\) of the stationary process \(Y\) is now supposed to satisfy \(r_Y \in \mathcal{H}(H, \beta, C)\) with
\[
\mathcal{H}(H, \beta, C) = \left\{ r, r(k) = C \cdot k^{2H-2}(1 + O(1/k^3)) \right\}
\]
when \(k \to \infty\), (13)
with \(1/2 < H < 1\), \(C > 0\) and \(\beta > 0\). In such a semi-parametric frame, the previous proofs are still valid and:

**Theorem 4.2:** Let \(Y = \{Y(k), k \in \mathbb{N}\}\) be a Gaussian stationary zero mean long-range dependent process with covariance \(r_Y \in \mathcal{H}(H, \beta, C)\). Then, Property 3.1 holds with the addition of \(O(1/n^2)\) in each expansion. Moreover, if \(N = o(n^{\max(2H+1,3)})\), Property 3.2 and Proposition 3.3 hold.

As a consequence of this theorem, if \(0 < \beta \leq 1\), the DFA method provides a semi-parametric estimator of \(H\) with the well-known minimax rate of convergence for the estimation of Hurst parameter in this semi-parametric setting (see for instance Giraitis et al. [13]). i.e.,
\[
\limsup_{N \to \infty} \sup_{r_Y \in \mathcal{H}(H, \beta, C)} \frac{\hat{H} - H}{\sqrt{N}} = 0.
\]

However, if \(\beta > 1\), this result is replaced with
\[
\limsup_{N \to \infty} \sup_{r_Y \in \mathcal{H}(H, \beta, C)} \frac{(\hat{H} - H)^2}{N^{2\beta/(1+2\beta)}} < +\infty.
\]

Case of power law and polynomial trends

First, let us assume that there exists \(\lambda > 0\) and \(a \in \mathbb{R}\) such that
\[
f(t) = at^{\lambda+1} - (t-1)^{\lambda+1}, \quad \text{for } t \geq 1.
\]

Then, the associated integrated function is
\[
g(k) = \sum_{i=1}^{k} f(i) = ak^{\lambda+1}.
\]

For this kind of trend,

**Property 5.1:** For \(f(t) = at^{\lambda+1} - (t-1)^{\lambda+1}\), with \(\gamma(a, N, \lambda)\) a real number depending only on \(a\), \(N\) and \(\lambda\),
\[
\log F_f(n) \approx 2 \log n + \gamma(a, N, \lambda) \quad \text{for } n \to \infty.
\]

Thus, it appears that a linear regression of \(\log F_f(n)\) and \(\log(n)\) for different values of \(n\) will provide a slope 2 for any \(\lambda > 0\). This result is confirmed by several simulations made for various values of \(\lambda > 0\), \(a\) and \((n_1, \ldots, n_m)\).

This result can also be used to deduce similar results for polynomial trends.

**Property 5.2:** Let us assume that there exists \(p \in \mathbb{N}\) and a family \((a_j)_{0 \leq j \leq p}\) with \(a_p \neq 0\) such that
\[
f(k) = a_p k^p + \cdots + a_0.
\]

Then,
\[
\Rightarrow \log F_{a_p k^p + \cdots + a_0}(n) \approx 2 \log n + \gamma(a_p, N, p) \quad \text{for } n \to \infty.
\]
Using relations (14) and (15), the previous results for trends can be used for deducing the behavior of the DFA function of trended long range dependent processes. Hence, in both previous cases of trends, there exists $C > 0$ such that

$$F_{Y+t}^2(n) = C \cdot n^4 N^{2\lambda-2} \left(1 + O \left( \frac{1}{n} \right) + O \left( \frac{1}{\log n} \right) \right)$$

$$+ \sigma^2 f(H) \cdot n^{2H} \left(1 + O \left( \frac{1}{\min(1,\beta)} \right) \right)$$

$$\approx C \cdot n^4 N^{2\lambda-2}.$$

where the $n \times 1$ vector $(g(k) - \hat{g}_n(k))_{1 \leq k \leq n} = P_{E^+} \cdot G^{(1)}$ with:

$$G^{(1)} = (a_0 \cdot 1, \ldots, a_0 \cdot t_1, (a_0 - a_1)t_1 + a_1 \cdot (t_1 + 1), \ldots, (a_0 - a_1)t_1 + a_1 \cdot n)'.$$

Then,

$$\sum_{k=1}^{\tau_n} (g(k) - \hat{g}_n(k))^2 = \left( J_{\tau_n} \cdot P_{E^+} \cdot G^{(1)} \right)' \cdot \left( J_{\tau_n} \cdot P_{E^+} \cdot G^{(1)} \right)$$

where $J_{\tau_n}$ is a square matrix of order $n$ with ones in the $\tau_n$ first terms of the diagonal and zeros elsewhere. When we approximate sums by integrals, this expression can be written as follows:

$$\sum_{k=1}^{\tau_n} (g(k) - \hat{g}_n(k))^2 = n^3 \left( \int_0^1 a_0y - (a_0x \cdot x \leq \frac{1}{\lambda} + (a_1x + (a_0 - a_1) \frac{t_1}{n}) x > \frac{1}{\lambda} \right) (4 - 6(x + y) + 12xy) dx dy \cdot \left(1 + O \left( \frac{1}{n} \right) \right).$$

For $\tau \in ]0, \frac{1}{2}[$, the second term can be developed in the same way by replacing $J_{\tau_n}$ by $J_{n-\tau_n}$ which is $I_d - J_{\tau_n}$. Then, this term can be approximated by:

$$\sum_{k=1}^{n} (g(k) - \hat{g}_n(k))^2 = n^3 \left( \int_0^1 (a_0 - a_1) \frac{t_1}{n} + a_1y - (a_0x \cdot x \leq \frac{1}{\lambda} + (a_1x + (a_0 - a_1) \frac{t_1}{n}) x > \frac{1}{\lambda} \right) (4 - 6(x + y) + 12xy) dx dy \cdot \left(1 + O \left( \frac{1}{n} \right) \right).$$

Then after developing the two terms, we deduce that there exists a positive number $c(a_0, \ldots, a_i, t_{i_p}, \tau)$ such that the partial DFA function in the $j_p$-th window, where $t_{i_p} \in [(j_p - 1)n + \tau_n, j_p n - \tau_n]$ for $p \in \{1, \ldots, r\}$ and $n$ large enough, can be written as:

$$F_{f,j_p}^2(n) \geq c(a_0, \ldots, a_i, t_{i_p}, \tau)n^2.$$

Then if we suppose that there exists only one change point or a definite number of windows $j_1, \ldots, j_r$, there exists $c'(a_0, \ldots, a_i, t_{i_1}, \ldots, t_{i_r}, \tau) > 0$ such that the DFA function relating to $f$ is:

$$F_{f,j}^2(n) = \frac{1}{[N/n]} \sum_{j=j_1}^{j_r} F_{f,j}^2(n) \geq c'(a_0, \ldots, a_i, t_{i_1}, \ldots, t_{i_r}, \tau)n^3N^{-1} \left(1 + O \left( \frac{1}{n} \right) \right).$$

If we consider the form signal generated by the superposition between the trend and a long range dependent process, we point out that:

$$F_{f,j}^2(n) = \sigma^2 f(H) \cdot n^{2H} \left(1 + O \left( \frac{1}{\min(1,\beta)} \right) \right) \cdot \left[ \left( \frac{1}{n} \right) \right].$$

We can deduce, from the previous conditions on $n$ and $N (N/n \to \infty$ and $N = o(n^{\min(3, 2\beta + 1)})$, that the trend is dominant for large $n$.

Then, for different values $(n_1, \ldots, n_m)$, the graph tracing the relation between log $F_{f,Y}(n_i)$ and log $(n_i)$ (Figure 4), shows a slope estimated at $\frac{3}{2}$. 

![Graph showing relation between log $F_{f,Y}(n_i)$ and log $(n_i)$](image-url)
Based estimator can be computed by Mallat’s fast cascade and with its notations, we obtain
\[
F_1^2(n) = \frac{1}{n} \left( X^{(1)} - P_{E_1} \cdot X^{(1)} \right) \cdot \left( X^{(1)} - P_{E_1} \cdot X^{(1)} \right).
\]
As a consequence,
\[
\langle F_1^2(n) \rangle = \frac{1}{n} \left( \text{trace}(\Sigma_n) - \text{trace}(P_{E_1} \cdot \Sigma_n) \right),
\]
where \(\Sigma_n\) is the covariance matrix of \(X^{(1)}\) and is such that
\[
\Sigma_n = \text{Cov}(X(i), X(j))_{1 \leq i,j \leq n} = \frac{\sigma^2}{2} \left( |i|^{2H} + |j|^{2H} - |i-j|^{2H} \right)_{1 \leq i,j \leq n}.
\]
But, \(\text{trace}(\Sigma_n) = \sigma^2 \sum_{i=1}^{n} |i|^{2H} = \sigma^2 n^{2H+1} \left( \frac{1}{n} \sum_{i=1}^{n} \frac{i^{2H}}{n} \right)\),
\[
= \sigma^2 n^{2H+1} \left( \int_0^1 x^{2H} dx + O\left(\frac{1}{n}\right) \right).
\]
Therefore, on the one hand,
\[
\text{trace}(\Sigma_n) = \frac{\sigma^2}{2H+1} n^{2H+1} \cdot \left( 1 + O\left(\frac{1}{n}\right) \right) \tag{17}
\]
and on the other hand, it is well known that \(P_{E_1}\) is a \((n \times n)\) square matrix such that
\[
P_{E_1} = \frac{2}{n(n-1)} \left( (2n+1) - 3(i+j) + 6 \frac{i \cdot j}{1+n} \right)_{1 \leq i,j \leq n}.
\]
Then, after some straightforward computations, we obtain the formula
\[
\text{trace}(P_{E_1} \cdot \Sigma_n) = \frac{\sigma^2 n^{2H+1} n^2}{n(n-1)} \sum_{p=1}^{n} \sum_{q=1}^{n} \left[ \frac{1}{n^2} \left( (2+\frac{1}{n}) - 3 \right) \frac{p+q}{n} + \frac{6 \cdot p \cdot q}{n(1+n)} \right] \left[ \frac{2H}{n} |p|^{2H} - \frac{2H}{n} |q|^{2H} \right].
\]
In order to clarify the formula, we approximate these sums by integrals
\[
\text{trace}(P_{E_1} \cdot \Sigma_n) = \sigma^2 n^{2H+1} \cdot \left( 1 + O\left(\frac{1}{n}\right) \right) \int_0^1 \int_0^1 \left[ (2 - 3(x+y) + 6xy) \left( x^{2H} + y^{2H} - |x-y|^{2H} \right) \right] dxdy.
\]
After the calculation of this integral and a simplification with formula (17), we get the result
\[
\text{trace}(\Sigma_n) - \text{trace}(P_{E_1} \cdot \Sigma_n) = \sigma^2 f(H) \cdot n^{2H+1} \cdot \left( 1 + O\left(\frac{1}{n}\right) \right)
\]
and therefore the formula of \(\langle F_1^2(n) \rangle\).

2. From the previous notations and the property of the trace of a product of matrices,
\[
\text{Var}(F_1^2(n)) = \frac{1}{n^2} \left[ \text{trace}(\Sigma_n^2 - \Sigma_n - \Sigma_n^2) \right] - \left( \sum_{E_1} \left( X^{(1)} \cdot P_{E_1} \cdot X^{(1)} \right) \right)^2,
\]
\[
= \frac{1}{n^2} \left[ \text{trace}(\Sigma_n^2 - \Sigma_n - \Sigma_n^2) \right] \tag{18}
\]
APPENDIX I

VI. CONCLUSION

In the semi-parametric frame of a long-memory stationary process, we have shown, using the DFA method, that the estimator of the long range dependence parameter is convergent with a reasonable (but not always optimal) convergence rate. However, in numerous cases of trended long-range dependent processes (with perhaps the only exception of a constant trend), this estimator does not converge. Indeed the effect of long-memory property is dominated by the effect of the trend in the case of power law or polynomial trends. The slope for the trended processes of the DFA log-log regression is always close to and in the case of a piecewise constant trend, the slope is estimated at \(\frac{1}{2}\). The DFA method is therefore not robust at all and should not be applied for trended processes.

The wavelet-based method provides a more efficient and robust estimator of the Hurst parameter especially when a polynomial trended LRD (or self-similar) process is considered. Indeed, Abry et al. [1] remarked that all polynomial trend of degree \(M\) is without effects on the estimator of the Hurst parameter as soon as the mother wavelet has its \(M\) first vanishing moments. Therefore, the larger \(M\), the more robust the estimator is. Moreover, in the semi-parametric frame of general class of stationary Gaussian LRD processes, it was established by Moulines et al. [24] that the estimator of the Hurst parameter converges with an optimal convergence rate (following the minimax criteria) when an optimal length of windows is known. Bardet et al. [6] proposed an adaptive estimator and obtained an optimal convergence rate up to logarithmic factor. Finally, the wavelet based estimator can be computed by Mallat’s fast cascade algorithm which is a very fast algorithm (the equivalent for wavelet transform of FFT for Fourier transform) for computing wavelet coefficients. Thus, computing time of wavelet based estimator is smaller than DFA estimator one.
The development of the first term provides the following asymptotic expansion

\[
\text{trace}(\Sigma_n \cdot \Sigma_n) = \frac{\sigma^4}{4} \sum_{i=1}^{n} \sum_{p=1}^{n} (|x|^2 + |y|^{2H} - |x - y|^{2H})^2 = \frac{\sigma^4}{4} n^{4H+2} \left(1 + O\left(\frac{1}{n}\right)\right) \int_0^1 \int_0^1 |x|^2 + |y|^{2H} - |x - y|^{2H}^2 \, dx \, dy.
\]

The calculation of this integral provides the following simplified expression

\[
\text{trace}(\Sigma_n \cdot \Sigma_n) = \frac{\sigma^4}{4} n^{4H+2} \left(1 + O\left(\frac{1}{n}\right)\right).
\]

The same development can be made for the second term

\[
\text{trace}(P_{E_1} \cdot \Sigma_n \cdot \Sigma_n) = \frac{\sigma^4}{2} n^{4H+2} \left(1 + O\left(\frac{1}{n}\right)\right).
\]

After the computation of this last integral, and using relations (18) and (19)

\[
\text{trace}(\Sigma_n \cdot \Sigma_n) - \text{trace}(P_{E_1} \cdot \Sigma_n \cdot \Sigma_n) = \sigma^4 \cdot g(H) n^{4H+2} \left(1 + O\left(\frac{1}{n}\right)\right),
\]

with

\[
\sigma_{k,k'}^{(1,j)} = \frac{\sigma^2}{2} \left( |k + n|^{2H} + |k'|^{2H} - |k - k' + n|^{2H} \right)_{1 \leq k,k' \leq n}
\]

and with \(P_{E_1} = (p_{i,j})_{1 \leq i,j \leq n} \) such that

\[
p_{i,j} = \frac{2}{n(n-1)} \left( (2n+1) - 3(i+j) + 6 \frac{i \cdot j}{n} \right).
\]

Now, we consider the asymptotic expansion of this formula when \(n\) is large enough

\[
\text{Cov}(F_1^2(n), F_2^2(n)) = \frac{\sigma^4}{4} n^{4H} \left(1 + O\left(\frac{1}{n}\right)\right) \int_0^1 \int_0^1 |x| + |y|^{2H} + |x - y|^{2H}^2 \, dx \, dy.
\]

In order to obtain an asymptotic expansion of this formula when \(j\) is large enough (i.e. both windows are taken away from one another), a Taylor expansion in \(j\) up to order 3 is necessary. After calculating and simplifying the integrals, we get the result. \(\square\)

**Proof of Property 3.2:** We divide the proof into 3 steps:

- **Step 1:** one proves that \([N/n] \cdot \text{Var}(\tilde{S}(n)) \to \gamma^2(H),\) where \(\gamma^2(H)\) depends only on \(H\), when \([N/n] \to \infty.\) Indeed,

\[
\text{Var}(\tilde{S}(n)) = \frac{1}{[N/n]^2} \sum_{j=1}^{[N/n]} \sum_{j'=1}^{[N/n]} \text{Cov}(\tilde{S}_j(n), \tilde{S}_{j'}(n)) = \frac{1}{[N/n]} \cdot \text{Var}(\tilde{S}_j(n)) + \frac{2}{[N/n]^2} \sum_{j=1}^{[N/n]} \left( [\frac{N}{n}] - j \right) \text{Cov}(\tilde{S}_j(n), \tilde{S}_{j'}(n))\]

due to the stationarity.

However, with properties (9), one deduces that when \([N/n] \to \infty,\) \(\sum_{j=1}^{[N/n]} \text{Cov}(\tilde{S}_j(n), \tilde{S}_{j'}(n))\) and \(\sum_{j=1}^{[N/n]} \text{Cov}(\tilde{S}_j(n), \tilde{S}_{j'}(n))\) converge, because there exist \(C \geq 0\) such that \(|C| \leq C \cdot j^{2H-3}\) and \(0 < H < 1.\)

Therefore, there exists \(\gamma^2(H)\) depending only on \(H\) such that

\[
\lim_{[N/n] \to \infty} [N/n] \cdot \text{Var}(\tilde{S}(n)) = \gamma^2(H).
\]

- **Step 2:** the proof of a central limit theorem for \(\tilde{S}(n)\) when \([N/n] \to \infty\) can be obtained from the same method as in the proof of Proposition 2.1 in Bardet [4] (Theorem 3 in Soulier [31] leads to the same result).

Indeed, \(\tilde{S}(n) = \frac{1}{n^{2H+1} \sigma^2 f(H) \cdot [N/n]} \sum_{i=1}^{n/[N/n]} Z_i^2,\) where the zero-mean Gaussian vector \(Z = (Z_1, \ldots, Z_{n/[N/n]})\) has the covariance matrix \(P \cdot \Sigma \cdot P,\) where \(P\) is a diagonal block matrix with each block consisting of \((n,n)\) matrix \(P_{E_1}^j \) and \(\Sigma\) is the covariance matrix of an FBM time series (each \((n,n)\)}
block is $\Sigma^{(i,j)}$ with the previous notations). Using a Lindeberg condition, $\hat{S}(n)$ satisfies the following central limit theorem:

$$\sqrt{[N/n]} \cdot (\hat{S}(n) - (S(n))) \xrightarrow{\mathcal{L}} N(0, \gamma^2(H)), \quad (22)$$

if $\lambda = \|P \cdot \Sigma \cdot P\|$, the supremum of the eigenvalues of the symmetrical matrix $P \cdot \Sigma \cdot P$, is such that

$$\lambda = o\left(\frac{1}{\sqrt{[N/n]}}\right). \quad (23)$$

But, using Lemma 4.1 and following the proof of Proposition 2.1 in Bardet [4],

$$\lambda \leq \frac{1}{n^{2H+1} \sigma^2 f(H) \cdot [N/n]} \cdot \max_{i \in \{1, \ldots, [N/n]\}} \left(\sum_{j=1}^{n-\lfloor [N/n] \rfloor} |\text{Cov}(Z_i, Z_j)|\right) \leq \frac{1}{\sqrt{2[N/n]} \cdot \max_{i \in \{1, \ldots, [N/n]\}} \left(\sum_{j=1}^{N/n} \sqrt{\text{Cov}(\hat{S}_i(n), \hat{S}_j(n))}\right)} \leq \frac{1}{\sqrt{2[N/n]} \cdot \left(\sum_{j=1}^{[N/n]} \sqrt{\text{Cov}(\hat{S}_1(n), \hat{S}_j(n))}\right)} \leq \frac{\sqrt{2}}{[N/n]} \cdot \left(\sum_{j=1}^{[N/n]} \sqrt{\text{Cov}(\hat{S}_1(n), \hat{S}_j(n))}\right).$$

So, there exists $C(H) > 0$ depending only on $H$ such that

$$\lambda \leq C(H) \cdot \frac{1}{[N/n]} \sum_{j=1}^{[N/n]} \left(\sqrt{2^{2H-3} + \frac{C}{n}}\right) \text{ third line of (9)} \leq C'(H) \cdot \left(\frac{[N/n]}{n^{H-3/2}} + \frac{1}{n}\right). \quad (24)$$

Therefore if $\frac{1}{n} = o\left(\frac{1}{\sqrt{[N/n]}}\right)$ (i.e. $N = o(n^3)$), (23) and (22) are proved.

Step 3: Now, $\hat{S}(n) = 1 + O\left(\frac{1}{n}\right)$ for $n$ large enough. Then, if $\sqrt{[N/n]} \cdot \frac{1}{n} \to 0$, that is $N/n^3 \to 0$, $\sqrt{[N/n]} \cdot (\hat{S}(n) - 1) \xrightarrow{\mathcal{L}} N(0, \gamma^2(H)).$

The classical Delta method allows the passage between a central limit theorem for $\hat{S}(n)$ and a central limit theorem for $\log(\hat{S}(n))$ (thanks to the regularity properties of the function logarithm).

**Proof of Proposition 3.3:** It is possible to write $H = (1, 0) \cdot (Z' \cdot Z)^{-1} \cdot Z' \cdot F$, where $Z$ is the $(m, 2)$ matrix such that $Z = \begin{pmatrix} \log(r_1 \cdot n) & 1 \\ \vdots & \vdots \\ \log(r_m \cdot n) & 1 \end{pmatrix}$ and $F = \left(\frac{\log(F(r_1 \cdot n))}{\log(F(r_m \cdot n))}\right)$. Then

$$\frac{\hat{S}(n) - 1}{\sqrt{\text{Var}(H)}} = (1, 0) \cdot (Z' \cdot Z)^{-1} \cdot Z' \cdot F \cdot Z' \cdot Z^{-1} \cdot (1, 0)' \leq \|Z' \cdot Z^{-1} \cdot Z'\| \cdot \|\text{Cov}(F)\| \leq \|Z'^{-2} \cdot 2m \cdot \prod_{i=1}^{m} r_i \cdot \gamma^2(H) \cdot \frac{1}{[N/n]}.$$

Since $\|Z' \cdot Z^{-1} \cdot Z'\|$ only depends on $r_1, \ldots, r_m$, the proof of Proposition 3.3 is completed. □

**Proof of Theorem 4.2:** From the assumptions on $Y$ and $r_Y$, if $i \geq j \geq 1$,

$$\text{Cov}(X(i), X(j)) = \sum_{k=1}^{i} \sum_{\ell=1}^{j} \text{Cov}(Y(k), Y(\ell)) = \sum_{k=1}^{i} (i-k)r_Y(k) + \sum_{k=1}^{j} (j-k)r_Y(k) - \sum_{k=1}^{i-j} (i-j-k)r_Y(k).$$

As a consequence, for all $(i, j) \in \{1, \ldots, n\}^2$,

$$\text{Cov}(X(i), X(j)) = C \cdot \left(\int_{0}^{1} (1-u)u^{2H-2}du\right) \cdot \left(j^{2H} \left(1 + O\left(\frac{1}{\min(3,1)}\right)\right) - \frac{1}{(1+|i-j|\min(3,1))}\right).$$

Now, this covariance can be used for the proofs replacing the previous ones. This implies

1. $(F_2\gamma(n)) = \sigma^2 f(H) \cdot n^{2H} \left(1 + O\left(\frac{1}{n^{\min(3,1)}}\right)\right),$
2. $\text{Var}(F_2\gamma(n)) = \sigma^4 g(H) \cdot n^{4H} \left(1 + O\left(\frac{1}{n^{\min(3,1)}}\right)\right),$
3. $\text{Cov}(F_2\gamma(n), F_2\gamma(n)) = \sigma^4 h(H) \cdot n^{4H} \cdot j^{2H-3} \cdot \left(1 + O\left(\frac{1}{n^{\min(3,1)}}\right)\right).$

with $\sigma^2 = 2C \cdot \left(\int_{0}^{1} (1-u)u^{2H-2}du\right)$. The proofs of property 3.2 are the same as in the case of the FGN except that in (24),

$$\lambda \leq C(H) \cdot \frac{1}{\sqrt{[N/n]} \cdot \sum_{j=1}^{[N/n]} \left(\sqrt{2^{2H-3} + \frac{C}{n}}\right)} \leq C'(H) \cdot \left(\frac{[N/n]}{n^{H-3/2}} + \frac{1}{n} + \frac{1}{n^3}\right).$$

So, if $\frac{1}{n} + \frac{1}{n^2} = o\left(\frac{1}{\sqrt{[N/n]}}\right)$ therefore $N = o(n^{\max(2/3,1,3)})$, the central limit theorem as well as Proposition 3.3 are proved following the same proof as in the case of the FGN. □

**Proof of Property 5.1:** In the $j$-th window, with $\frac{1}{n} \in \{1, \ldots, [N/n]\}$, let us consider $E_j$ the vector
subspace defined above and define the vector 
\( G^{(j)} = \alpha((1 + n(j - 1))^\beta + 1, \ldots, (nj)^{\beta+1}) \). We have
\[
F_{2,j}^2(n) = \frac{1}{n} \left( G^{(j)}' \cdot G^{(j)} - G^{(j)}' \cdot P_{E_j} \cdot G^{(j)} \right)
\]

An explicit asymptotic expansion (in \( n \) and \( N \)) of this partial DFA function can be obtained by approximating sums by integrals. Then,
\[
F_{2,j}^2(n) = a_n^{2\lambda+2} \left( 1 + O\left( \frac{1}{n} \right) \right) \left[ \int_0^1 \int_0^1 (x + j - 1)^{2\lambda-2} - (4 - 6(x + y) + 12xy)(x + j - 1)^{\lambda+1}(y + j - 1)^{\lambda+1} \right] \, dx \, dy
\]
Moreover, using Taylor expansion in \( j \) up to order 3, one obtains
\[
F_{2,j}^2(n) = \alpha(a, \lambda) \cdot n^{2\lambda+2} j^{2\lambda-2} \left( 1 + O\left( \frac{1}{n} \right) + O\left( \frac{1}{j} \right) \right), \tag{25}
\]
and it implies that the DFA function relating to \( f \) can be written as
\[
F_{2,j}^2(n) = \frac{1}{[N/n]} \sum_{j=1}^{[N/n]} F_{2,j}^2(n) = \beta(a, \lambda) \cdot n^4 N^{2\lambda-2} \left( 1 + O\left( \frac{1}{n} \right) + O\left( \frac{1}{N} \right) \right),
\]
with \( \alpha(a, \lambda), \beta(a, \lambda) \) two positive numbers depending only on \( a \) and \( \lambda \). □

Proof of Property 5.2: Let \( f(k) = a_p k^p + \cdots + a_0 \implies g(k) = \sum_{i=1}^{K} f(i) = b_{p+1} k^{p+1} + \cdots + b_0 \), with \( b_{p+1} \neq 0 \), i.e. the associated integrated function is also a polynomial function. From the expression of the partial DFA function and with the asymptotic expansion (25) depending on the degree \( \lambda \), for \( n \) and \( N \) large enough,

\[
F_{a_p k^p + \cdots + a_0}^2(n) = F_{a_p k^p + \cdots + a_0}^2(n) \left( 1 + O\left( \frac{1}{n} \right) + O\left( \frac{1}{N} \right) \right)
\]

the power of \( n \) in the partial DFA function relating to \( a_p k^p \) is greater than the ones in the partial DFA function relating to the other monomines. This approximation leads to the following expression of the DFA function of a polynomial function,
\[
F_{a_p k^p + \cdots + a_0}^2(n) = \beta(b_{p+1}) \cdot n^4 N^{2\lambda-2} \left( 1 + O\left( \frac{1}{n} \right) + O\left( \frac{1}{N} \right) \right).
\]

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REFERENCES


