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Abstract. We consider a complete 6–dimensional nearly Kähler manifold together with the first canonical Hermitian connection. We show that if the holonomy of this connection is reducible, then the manifold endowed with a modified metric and almost complex structure is a Kählerian twistor space. This result was conjectured by Reyes–Carrión in [15].

1. Introduction

The class of nearly Kähler manifolds belongs to one of the sixteen classes of almost Hermitian manifolds described by the celebrated Gray–Hervella classification [10]. After Gray had studied them intensively in the seventies, they were somewhat neglected until recently, when Friedrich and Grunewald discovered that a 6–dimensional spin manifold admits Killing spinors if and only if it is nearly Kähler and not Kähler (see [6], [7], [8]). Some years later, this result was also obtained by Bär as a particular case of his geometrical characterization of manifolds with Killing spinors [2]. It is known that a manifold with Killing spinors is locally irreducible [5]. From this point of view, the title of the present note could seem strange to readers who are familiar with nearly Kähler geometry. The explanation is that when speaking about reducible holonomy, we understand here the holonomy of the canonical Hermitian connection, for the reasons presented below.

Twistorial geometry is the second main topic of this paper. Recall that the twistor space Z of a 4–dimensional oriented Riemannian manifold N is the S^2–bundle over N whose fiber over $x \in N$ consists of all almost complex structures on T_xN that are compatible with the metric and the orientation. On Z there is a 1–parameter family of metrics g^t making the bundle projection $Z \to N$ into a Riemannian submersion with totally geodesic fibers (the parameter t is chosen in such a way that the fibers of (Z, g^t) are spheres of Gaussian curvature t). Moreover, Z carries two canonical almost complex structures $\mathcal{J}_\pm$ that are compatible with each of these metrics and are obtained in the following way: the tangent space of Z at some point J_0 splits into vertical and horizontal subspaces via the metric connection. Then $\mathcal{J}_\pm$ at J_0 are equal to the endomorphism induced by J_0 on the horizontal space and to the rotation with angle $\pm\pi/2$ in the vertical direction (see [1], [4] for details).

It is well–known that $\mathcal{J}_-$ is never integrable, and $\mathcal{J}_+$ is integrable if and only if N is a self–dual manifold. In that case, $(Z, \mathcal{J}_+, g^t)$ is Kähler if and only if N is Einstein with positive scalar curvature S and $t = S/3$. On the other hand, if N satisfies the above conditions, the manifold $(Z, \mathcal{J}_-, g^t)$ is quasi–Kähler for every t and it is *nearly Kähler* for exactly one value of t, namely $t = 2S/3$. In [15], Reyes–Carrión shows that the canonical Hermitian connection of $(Z, \mathcal{J}_-, g^t)$ has reduced holonomy (contained in U_2) and conjectures that every complete 6–dimensional nearly Kähler manifold with this property can be obtained from a Kählerian twistor space by changing the metric and the complex structure.

The aim of this paper is to show that this conjecture is true, by proving the following

Theorem 1.1. *Every complete 6–dimensional nearly Kähler, non–Kähler manifold, whose canonical Hermitian connection has reduced holonomy, is homothetic to $\mathbb{CP}^3$ or $F(1,2)$, with their standard nearly Kähler structure coming from the twistor construction.*

The main difficulties in the proof are: 1. to prove that the foliation defined by the holonomy reduction comes from a Riemannian submersion with totally geodesic fibers (here we use Reeb's stability theorem); 2. to check that the conditions needed in order to apply the inverse Penrose construction are satisfied, especially those concerning the antipodal map. We finally remark that some of the ideas used in the last section to complete the proof of Theorem 1.1 are similar to the methods used by Kirchberg for the classification of 6–dimensional manifolds with Kählerian Killing spinors [12].

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2. Preliminaries

In this section we recall some classical facts about nearly Kähler manifolds and Hermitian connections. For most of the proofs we propose [9] and references therein.

Definition 2.1. *An almost complex manifold (M, g, J) is called nearly Kähler if*

$$(\nabla_X J)X = 0 \text{ for all } X \in TM.$$

A nearly Kähler manifold is called strict if we have $\nabla_X J \neq 0$ for all $X \in TM$, and it is called of constant type if for every $x \in M$ and $X, Y \in T_xM$,

$$||(\nabla_X J)Y||^2 = \alpha(||X||^2||Y||^2 - \langle X, Y\rangle^2 - \langle JX, Y\rangle^2),$$

where α is a positive constant.

It is trivial to check that on every nearly Kähler manifold, the tensors $A(X, Y, Z) = \langle(\nabla_X J)Y, Z\rangle$ and $B(X, Y, Z) = \langle(\nabla_X J)Y, JZ\rangle$ are skew–symmetric and have the type $(3, 0) + (0, 3)$ as (real) 3–forms.

Every nearly Kähler manifold of real dimension 2 or 4 is Kähler. In dimension 6, a nearly Kähler manifold is either Kähler, or strict and of constant type (see [9]).

Later on, we will need the following classical relation between the covariant derivative of the almost complex structure J and its Nijenhuis tensor N.

Lemma 2.2. *For every nearly Kähler manifold (M, g, J) we have*

$$N(X,Y) = J(\nabla_X J)Y, \ \forall X, Y \in TM. \tag{1}$$

Proof. Straightforward computation using the definition of N

$$4N(X,Y) = [X,Y] + J[JX,Y] + J[X,JY] - [JX,JY],$$

and the antysymmetry of the tensors A and B defined above. □

The space of Hermitian connections on an almost Hermitian manifold M (*i.e.* the ones which preserve the metric and the complex structure) is an affine space modelled on $\Lambda^1 M \otimes \Lambda^{(1,1)} M$. If ∇ denotes the Levi–Civita connection, then every Hermitian connection $\bar{\nabla}$ satisfies

$$\bar{\nabla} = \nabla - \frac{1}{2} J \nabla J + H, \quad H \in \Lambda^1 M \otimes \Lambda^{(1,1)} M.$$

In other words, $\bar{\nabla} - \nabla$ has two components, one in $\Lambda^1 M \otimes \Lambda^{(0,2)+(2,0)} M$, which is fixed, and the other in $\Lambda^1 M \otimes \Lambda^{(1,1)} M$, which is free.

Definition 2.3. *The canonical Hermitian connection of an almost Hermitian manifold is the unique Hermitian connection $\bar{\nabla}$ such that the $\Lambda^1 M \otimes \Lambda^{(1,1)} M$–part of $\bar{\nabla} - \nabla$ vanishes.*

Lemma 2.4. *If $\bar{\nabla}$ denotes the canonical Hermitian connection of a nearly Kähler manifold, then $\bar{\nabla}(\nabla J) = 0$.*

Proof. We use the formula (2.9) of [9]

$$2\langle \nabla^2_{W,X}(J)Y, Z\rangle = -\sigma_{X,Y,Z}\langle (\nabla_W J)X, (\nabla_Y J)JZ\rangle,$$

and compute

$$\begin{aligned}
\bar{\nabla}(\nabla J)(W,X,Y,Z) &= W\langle (\nabla_X J)Y, Z\rangle - \langle (\nabla_{\bar{\nabla}_W X} J)Y, Z\rangle - \langle (\nabla_X J)\bar{\nabla}_W Y, Z\rangle \\
&\quad -\langle (\nabla_X J)Y, \bar{\nabla}_W Z\rangle \\
&= \langle \nabla^2_{W,X}(J)Y, Z\rangle + \frac{1}{2}\Big(\langle (\nabla_{J(\nabla_W J)X} J)Y, Z\rangle \\
&\quad + \langle (\nabla_X J)J(\nabla_W J)Y, Z\rangle + \langle (\nabla_X J)Y, J(\nabla_W J)Z\rangle \Big) \\
&= \langle \nabla^2_{W,X}(J)Y, Z\rangle + \frac{1}{2}\Big(- \langle (\nabla_{JZ} J)Y, (\nabla_W J)X\rangle \\
&\quad - \langle (\nabla_W J)Y, (\nabla_X J)JZ\rangle + \langle (\nabla_X J)JY, (\nabla_W J)Z\rangle \Big) \\
&= \langle \nabla^2_{W,X}(J)Y, Z\rangle + \frac{1}{2}\sigma_{X,Y,Z}\langle (\nabla_W J)X, (\nabla_Y J)JZ\rangle = 0.
\end{aligned}$$

□

Consequently, in dimension 6, the holonomy group of $\bar{\nabla}$ is contained in SU_3 (as $\nabla J + i * \nabla J$ is a complex volume form). This remark was made by Reyes–Carrión, who proved the previous result for the particular case of 6–dimensional nearly Kähler manifolds (see [15], Claim 4.15).

3. Reducible nearly Kähler manifolds

Definition 3.1. *A nearly Kähler manifold is called reducible if the holonomy representation of the canonical Hermitian connection is reducible.*

Let M be a 6–dimensional nearly Kähler manifold such that the holonomy group of the canonical Hermitian connection $\bar{\nabla}$ is strictly contained in SU_3. As every maximal subgroup of SU_3 is conjugate to U_2, this condition is equivalent to the reducibility of M. We fix such a U_2 containing the holonomy group, which defines a $\bar{\nabla}$–parallel complex line sub–bundle of TM, henceforth denoted by $\mathcal{V}$ (for "vertical"). Since $\nabla_X Y = \bar{\nabla}_X Y$ as soon as X and Y are complex linearly dependent, we deduce that $\mathcal{V}$ is totally geodesic for ∇ (so in particular integrable). The orthogonal complement of $\mathcal{V}$ in TM will be denoted by $\mathcal{H}$ (for "horizontal"), and the restriction of $\mathcal{H}$ to every integral manifold S of $\mathcal{V}$ will be identified with the normal bundle of S. We write $X^{\mathcal{V}}$ and $X^{\mathcal{H}}$ for the "vertical" and "horizontal" part of any vector X on M with respect to this decomposition. For any fixed leaf S of the vertical distribution, we define a covariant derivative $\mathcal{L}^{\mathcal{H}}$ on the normal bundle of S by

$$\mathcal{L}^{\mathcal{H}}_V X = (\mathcal{L}_V X)^{\mathcal{H}},$$

where V and X are vector fields on M tangent to $\mathcal{V}$ and $\mathcal{H}$, respectively. It is straightforward to check that this is a covariant derivative and that its curvature vanishes.

Definition 3.2. *The parallel transport on $\mathcal{H}|_S$, with respect to the covariant derivative $\mathcal{L}^{\mathcal{H}}$ is called the Lie transport.*

Lemma 3.3. *The covariant derivatives $\mathcal{L}^{\mathcal{H}}$, ∇ and $\bar{\nabla}$ are related by*

$$\mathcal{L}^{\mathcal{H}}_V X = \nabla_V X + \frac{1}{2} J(\nabla_V J)X = \bar{\nabla}_V X + J(\nabla_V J)X,$$

for every vertical vector V and horizontal vector field X.

Proof. We extend V to a vertical vector field and remark that $\nabla_V X = \bar{\nabla}_V X + \frac{1}{2} J(\nabla_V J)X$ is horizontal, so

$$\begin{aligned} \mathcal{L}^{\mathcal{H}}_V X &= (\nabla_V X - \nabla_X V)^{\mathcal{H}} = \nabla_V X - (\bar{\nabla}_X V)^{\mathcal{H}} - \frac{1}{2} J(\nabla_X J)V \\ &= \nabla_V X - \frac{1}{2} J(\nabla_X J)V. \end{aligned}$$

□

Let (M, g, J) be a reducible strictly nearly Kähler 6–dimensional manifold and let $TM = \mathcal{V} \oplus \mathcal{H}$ be the corresponding decomposition of the tangent bundle. For the rest of this paper, we will normalize the metric such that the scalar curvature of M is equal to 30 (*i.e.* such that M has constant type 1).

Lemma 3.4. *$\mathcal{V}$ is a totally geodesic distribution of TM and the integral manifolds of $\mathcal{V}$ are round spheres of Gaussian curvature 8.*

Proof. The first assertion has already been proven above. Let X and V be vector fields of $\mathcal{H}$ and $\mathcal{V}$, respectively. We compute

$$0 = \langle \bar\nabla_V V, \bar\nabla_X X\rangle = \langle \nabla_V V, \nabla_X X\rangle = \langle \nabla_X \nabla_V V, X\rangle.$$

Consequently,

$$\begin{aligned} R(X,V,V,X) &= -\langle \nabla_V\nabla_X V + \nabla_{[X,V]}V, X\rangle = \langle \nabla_X V, \nabla_V X\rangle + \frac{1}{2}\langle J(\nabla_{[X,V]}J)V, X\rangle \\ &= \langle \nabla_X V, \nabla_V X\rangle + \frac{1}{2}\langle J(\nabla_X J)V, \nabla_X V - \nabla_V X\rangle \\ &= \langle \bar\nabla_X V, \bar\nabla_V X\rangle - \frac{1}{4}\langle J(\nabla_X J)V, J(\nabla_V J)X\rangle \\ &= \frac{1}{4}|X|^2|V|^2. \end{aligned}$$

This, together with the fact that $\mathrm{Ric} = 5g$, shows that $R(JV,V,V,JV) = 4|V|^4$. The latter statement follows easily, as the round 2–sphere is the unique space form that is also complex (thus oriented). ☐

Proposition 3.5. *There exists a Riemannian manifold N and a Riemannian submersion with totally geodesic fibers $\pi : M \to N$ such that the tangent space of the fiber through any point $x \in M$ is just $\mathcal{V}_x$.*

Proof. The previous lemma shows that every maximal leaf of V is compact and simply connected. By the theorem of stability of Reeb [14], any such leaf has a basis of *saturated* tubular neighborhoods, *i.e.*, equal to a union of leaves. It follows then that the space of leaves, denoted by N, is a compact Hausdorff topological space.

Consider a leaf $S \in N$ and a point $x \in S$. We will construct a C^∞ map on N around S as follows: consider U a sufficiently small saturated tubular neighbourhood such that

(1) There is a metric defined on U such that the orthogonal projection $p_U : U \longrightarrow S$ is a submersion isomorphic to the normal bundle of S in M;
(2) any leaf $S' \subset U$ is transverse to the fibers of p.

Then $p|_{S'}$ is a covering, thus a diffeomorphism. A chart on any fiber of p thus provides a chart on the space of leaves contained in U (by identifying a leaf S' with its intersection with the considered fiber).

So the projection $M \xrightarrow{\pi} N$, whose fibers are the leaves tangent to $\mathcal{V}$, is a submersion, and actually a locally trivial fibration. We need to prove that the metric g on M induces a "quotient" metric g^N on N. It is clear that, for each $x \in S$,

$$\mathcal{H}_x \xrightarrow{\pi_*} T_S N$$

is an isomorphism from the fiber in x of the normal bundle of S and the tangent space in S at N, which induces a *quotient* metric g^x on T_SN. All we need to prove is that this metric is independent of $x \in S$.

For this, we first remark that an $\mathcal{L}^{\mathcal{H}}$–parallel section of the normal bundle of S projects, via π, on a unique vector in T_SN. Indeed, let Y be such an $\mathcal{L}^{\mathcal{H}}$–parallel vector field along S, let x be a point of S and let X be the projection on N of Y_x. The horizontal lift $\tilde{X}$ of X is then $\mathcal{L}^{\mathcal{H}}$–parallel because for every vertical vector field V, the bracket $[V, \tilde{X}]$ is projectable over 0, *i.e.* it is vertical. As $\tilde{X}$ and Y are both $\mathcal{L}^{\mathcal{H}}$–parallel and coincide in x, they are equal, thus proving our assertion.

The independence of the quotient metric g^x of x is then equivalent to the $\mathcal{L}^{\mathcal{H}}$–invariance of $g|_{\mathcal{H}}$ and is given by the following

Lemma 3.6. *The Lie transport consists of isometries of $\mathcal{H}|_S$:*

$$\mathcal{L}^{\mathcal{H}}_V g|_{\mathcal{H}} = 0, \ \forall V \in T_xS.$$

Proof. From Lemma 3.3 we have

$$\mathcal{L}^{\mathcal{H}}_V g|_{\mathcal{H}}(X, Y) = \bar{\nabla}_V g|_{\mathcal{H}}(X, Y) + \langle J(\nabla_V J)X, Y\rangle + \langle J(\nabla_V J)Y, X\rangle = 0,$$

where the first term of the right–hand side obviously vanishes, and the sum of the following two is zero as the tensor B (defined in Sect. 2) is skew–symmetric. □

Thus, the quotient metric g^N on N is well–defined, and the projection is a Riemannian submersion whose fibers are, by definition, totally geodesic. □

4. The Kähler structure of a reducible nearly Kähler manifold

Consider a new metric $\bar{g}$ and an almost complex structure $\bar{J}$ on M defined in the following way.

$$\bar{g}(X, Y) = g(X, Y) \text{ if } X \text{ or } Y \text{ belong to } \mathcal{H}, \quad \bar{g}(X, Y) = 2g(X, Y) \text{ if } X, Y \in \mathcal{V}.$$

$$\bar{J}(X) = J(X) \text{ if } X \in \mathcal{H}\ , \quad \bar{J}(X) = -J(X) \text{ if } X \in \mathcal{V}.$$

Proposition 4.1. *The manifold $(M, \bar{g}, \bar{J})$ is Kähler.*

Proof. Let N and $\bar{N}$ be the Nijenhuis tensors of J and $\bar{J}$, respectively. Since $\mathcal{V}$ is integrable, $\bar{J}$–invariant and 2–dimensional, $\bar{N}(X, Y) = 0$ whenever X and Y belong to $\mathcal{V}$.

If X and Y are both horizontal vector fields, (1) implies that $\bar{N}(X, Y)^{\mathcal{H}} = N(X, Y)^{\mathcal{H}} = 0$. Moreover, as $\mathcal{H}$ is $\bar{\nabla}$–invariant, we have

$$[X, Y]^{\mathcal{V}} = (\nabla_X Y - \nabla_Y X)^{\mathcal{V}} = \frac{1}{2}(J(\nabla_X J)Y - J(\nabla_Y J)X)^{\mathcal{V}} = J(\nabla_X J)Y,$$

so

$$\bar{N}(X, Y)^{\mathcal{V}} = J(\nabla_X J)Y - J(\nabla_{JX} J)JY - J(J(\nabla_{JX} J)Y + J(\nabla_X J)JY) = 0.$$

Finally, consider vector fields X and V belonging to $\mathcal{H}$ and $\mathcal{V}$, respectively. Since $\nabla_V X = \bar\nabla_V X + \frac{1}{2}J(\nabla_V J)X$ is horizontal, we deduce that

$$\begin{aligned}\bar N(V,X)^{\mathcal{V}} &= ([V,X]+[JV,JX]-\bar J[V,JX]+\bar J[JV,X])^{\mathcal{V}}\\ &= (-\nabla_X V-\nabla_{JX}JV+J\nabla_{JX}V-J\nabla_X JV)^{\mathcal{V}}\\ &= (-(\nabla_{JX}J)V-J(\nabla_X J)V)^{\mathcal{V}}=0.\end{aligned}$$

To compute the horizontal part of $\bar N(V,X)$ we may suppose (from the tensoriality of this expression) that X is the lift of a vector field from N, so that $[V,X]^{\mathcal{H}}=[JV,X]^{\mathcal{H}}=0$. Then we have

$$\begin{aligned}\bar N(V,X)^{\mathcal{H}} &= ([JV,JX]+J[V,JX])^{\mathcal{H}}\\ &= (\nabla_{JV}J)X+J(\nabla_{JV}X)^{\mathcal{H}}-\nabla_{JX}(JV)^{\mathcal{H}}\\ &\quad +J((\nabla_V J)X+J(\nabla_V X)^{\mathcal{H}}-\nabla_{JX}V^{\mathcal{H}})\\ &= 2J(\nabla_V J)X+J(\nabla_X JV)^{\mathcal{H}}+\frac{1}{2}J(\nabla_{JX}J)JV-(\nabla_X V)^{\mathcal{H}}-\frac{1}{2}(\nabla_{JX}J)V\\ &= 2J(\nabla_V J)X-\frac{1}{2}(\nabla_X J)JV+\frac{1}{2}J(\nabla_{JX}J)JV+\frac{1}{2}J(\nabla_X J)V-\frac{1}{2}(\nabla_{JX}J)V\\ &= 0.\end{aligned}$$

This proves that $\bar J$ is integrable. Consider now a local unit vector field V of $\mathcal{V}$ and write locally $\Omega=\Omega^{\mathcal{V}}+\Omega^{\mathcal{H}}$, where $\Omega^{\mathcal{V}}=V\wedge JV$. Then the Kähler form of $(M,\bar g,\bar J)$ is just $\Omega^{\mathcal{H}}-2\Omega^{\mathcal{V}}=\Omega-3\Omega^{\mathcal{V}}$. Obviously, $d\Omega(X,Y,Z)=3g((\nabla_X J)Y,Z)$, so in order to prove the Lemma we have to check that

$$d\Omega^{\mathcal{V}}(X,Y,Z)=g((\nabla_X J)Y,Z) \tag{2}$$

for every vector fields X,Y,Z, such that each of them belongs either to $\mathcal{V}$ or $\mathcal{H}$. As $d\Omega^{\mathcal{V}}=dV\wedge JV-V\wedge dJV$, both sides of (2) vanish if neither of X,Y,Z belong to $\mathcal{V}$. A straightforward computation shows that the same holds if more than one field of X,Y,Z belongs to $\mathcal{V}$. Suppose now that X,Y belong to $\mathcal{H}$ and Z belongs to $\mathcal{V}$. Without loss of generality we may suppose that $Z=V$, and we compute

$$\begin{aligned}d\Omega^{\mathcal{V}}(X,Y,V) &= -V\wedge dJV(X,Y,V)=-dJV(X,Y)=g(\nabla_Y JV,X)-g(\nabla_X JV,Y)\\ &= 2g((\nabla_Y J)V,X)-g(\nabla_Y V,JX)+g(\nabla_X V,JY)\\ &= 2g((\nabla_X J)Y,V)-g(\frac{1}{2}J(\nabla_Y J)V,JX)+g(\frac{1}{2}J(\nabla_X J)V,JY)\\ &= g((\nabla_X J)Y,V).\end{aligned}$$

□

5. The twistor structure

The following two lemmas hold in the general context of a Riemannian submersion with totally geodesic fibers [11].

Lemma 5.1. *Let γ be a geodesic on N, $\gamma(0) = y \in N$, and let $x \in \pi^{-1}(y)$. Then the horizontal lift $\tilde{\gamma}$ of γ through x (i.e., $\tilde{\gamma}(0) = x$) is a geodesic for any of the metrics g or $\bar{g}$.*

Lemma 5.2. *For every curve γ in N, the flow of the vector field $\tilde{\dot{\gamma}}$ — which is the horizontal lift of $\dot{\gamma}$, defined over $\pi^{-1}(\gamma)$ — consists of isometries between the fibers.*

Let S be a fiber over $x \in N$ of the above submersion and let $c(t)$ be a closed geodesic on S such that $|\dot{c}(0)| = 1/2$ (in particular $c(0) = c(2\pi)$ and c maps the interval $[0, 2\pi)$ bijectively on S). Denote by $V := 2\dot{c}(t)$ the unit vector field tangent to c.

Lemma 5.3. *Let X be a vector in T_xN and let $\tilde{X}$ be its horizontal lift to M along S. Then the vector field along c given by*

$$Y_{c(t)} = J\tilde{X}\cos t - (\nabla_V J)\tilde{X}\sin t \tag{3}$$

is projectable over a vector on N.

Proof. On the first hand, it is clear that $V(\sin t) = 2\cos t$ and $V(\cos t) = -2\sin t$. Next, using Lemma 3.3 we obtain

$$\nabla_V \tilde{X} = -\frac{1}{2} J(\nabla_V J)\tilde{X},$$

so

$$\begin{aligned}
\mathcal{L}^{\mathcal{H}}_V(J\tilde{X}) &= \nabla_V(J\tilde{X}) + \frac{1}{2}J(\nabla_V J)J\tilde{X} \\
&= (\nabla_V J)\tilde{X} + \frac{1}{2}(\nabla_V J)\tilde{X} + \frac{1}{2}J(\nabla_V J)J\tilde{X} \\
&= 2(\nabla_V J)\tilde{X}.
\end{aligned}$$

Finally, again by Lemma 3.3, together with Lemma 2.4,

$$\begin{aligned}
\mathcal{L}^{\mathcal{H}}_V((\nabla_V J)\tilde{X}) &= \bar{\nabla}_V((\nabla_V J)\tilde{X}) + J(\nabla_V J)(\nabla_V J)\tilde{X} \\
&= \bar{\nabla}_V(\nabla J)(V, \tilde{X}) + (\nabla_{\bar{\nabla}_V V} J)\tilde{X} + (\nabla_V J)\bar{\nabla}_V \tilde{X} - J\tilde{X} \\
&= (\nabla_V J)(-J(\nabla_V J)\tilde{X}) - J\tilde{X} = -2J\tilde{X}.
\end{aligned}$$

These relations show that $\mathcal{L}^{\mathcal{H}}_V Y = 0$, so Y is indeed projectable.

□

We define *the antipodal map* $\tau : M \longrightarrow M$ as follows: Let $x \in S$, a fiber of $\pi : M \longrightarrow N$. Then, by definition, $\tau(x)$ is the point $\bar{x} \in S$ antipodal to x (S is isometric to a sphere), thus, with the notations in the above Lemma, if $x = c(0)$, then $\bar{x} = c(\pi)$.

Proposition 5.4. *The antipodal map τ is an anti–holomorphic isometry of $(M, \bar{g}, \bar{J})$.*

Proof. Fix a point $x \in M$ projecting on $y \in N$. It is obvious that τ, restricted to a fiber, is an anti–holomorphic isometry. We will prove that

$$\tau_*|_{\mathcal{H}} : \mathcal{H}_x \longrightarrow \mathcal{H}_{\bar{x}}$$

is an isometry that anti–commutes with the complex structure for such fixed x.

Take the exponential chart on N around y. By Lemma 5.1, the horizontal lifts at x resp. $\bar{x}$ of the geodesics $\gamma_X(t) := \exp(tX),\ X \in T_yN$ through y are geodesics $\gamma^x_{\tilde{X}}(t)$ resp. $\gamma^{\bar{x}}_{\tilde{X}}(t)$ of M through x, resp. $\bar{x}$, both of them tangent to the horizontal lift $\tilde{X}$ of X. Note that these lifts do not depend on which of the metrics g or $\bar{g}$ is used. By Lemma 5.2, the distance between $\gamma^x_{\tilde{X}}(t)$ and $\gamma^{\bar{x}}_{\tilde{X}}(t)$ is constant, so Lemma 3.4 shows that $\gamma^x_{\tilde{X}}(t)$ and $\gamma^{\bar{x}}_{\tilde{X}}(t)$ are antipodal for each t. Therefore

$$\tau_*(\tilde{X}_x) = \tilde{X}_{\bar{x}}, \tag{4}$$

showing that τ_* sends $\mathcal{H}_x$ isometrically onto $\mathcal{H}_{\bar{x}}$. Thus τ is an isometry.

On the other hand, the *Lie transport* of a horizontal vector field $\tilde{X}$ along a vertical curve always projects onto $\pi_*(\tilde{X})$. Then, according to the previous lemma, with the notations therein, we obtain that

$$\tau_*(Y_{c(0)}) = Y_{c(\pi)}, \tag{5}$$

where we assume that the vertical geodesic $c : \mathbb{R} \longrightarrow M$ satisfies $c(0) = x$ (and, therefore, $c(\pi) = \bar{x}$). Combining (3), (4) and (5) yields

$$\tau_*(J\tilde{X}_x) = -J\tilde{X}_{\bar{x}} = -J\tau_*(\tilde{X}_x),$$

for every vector $X \in T_yN$, so τ is anti–holomorphic. $\square$

Proposition 5.5. *The normal bundle $N(S)$ of any fiber S is isomorphic to $\mathcal{O}(1)\oplus\mathcal{O}(1)$ as a complex bundle (and trivial as a real bundle).*

Proof. The holomorphic structure of $N(S)$ is determined by the $\bar{\partial}$ operator, which is equal to the $(0,1)$–part (for the complex structure $\bar{J}$) of the Levi–Civita connection $\tilde{\nabla}$ of $\bar{g}$, the Kähler metric on M. As S is totally geodesic for $\tilde{\nabla}$, we can write

$$\bar{\partial}_V Y := \frac{1}{2}(\tilde{\nabla}_V Y + \bar{J}\tilde{\nabla}_{\bar{J}V}Y) = \frac{1}{2}(\tilde{\nabla}_V Y - J\tilde{\nabla}_{JV}Y),\ V \in TS, Y \in N(S) \subset TM.$$

The first step is to show that all horizontal lifts of vectors on N are holomorphic. Let Y and Z be two such lifts along a fixed fiber S and let V be an arbitrary vector field tangent to S.

Of course, Y and Z are $\mathcal{L}^{\mathcal{H}}$–parallel, so using Lemma 3.3, the Koszul formula and the well–known O'Neill formulas ([13]) we get

$$\begin{aligned}
2\bar{g}(\bar{\partial}_V Y, Z) &= \bar{g}(\tilde{\nabla}_V Y, Z) + \bar{g}(\tilde{\nabla}_{JV} Y, JZ)\\
&= g(\nabla_V Y, Z) - \frac{1}{2}g([Y,Z],V) + g(\nabla_{JV}Y, JZ) - \frac{1}{2}g([Y,JZ],JV)\\
&= 2(\nabla_V Y, Z) + 2g(\nabla_{JV}Y, JZ) = -g(J(\nabla_V J)Y, Z) - g(J(\nabla_{JV}J)Y, JZ)\\
&= g(J(\nabla_Y J)V, Z) + g((\nabla_Y J)JV, Z) = 0.
\end{aligned}$$

Thus a basis in T_xN (where $x := \pi(S)$) gives us a *real*–linearly independent system of nowhere–vanishing holomorphic sections of $N(S)$.

Now, it is known [9] that the first Chern class of a nearly Kähler manifold vanishes. We have then:

$$0 = c_1(TM, J) = c_1(TS, J) + c_1(N(S), J),$$

thus the first Chern class of the holomorphic bundle $N(S)$, for the *opposite* orientation (induced by $\bar{J}$) on S, is equal to the Chern class of the tangent bundle of $\mathbb{CP}^1$, *i.e.*, its Chern number is equal to 2.

By a theorem of Grothendieck, every holomorphic bundle over $\mathbb{CP}^1$ is isomorphic to a direct sum of powers of the tautological line bundle. Thus, as a holomorphic bundle, $N(S)$ is isomorphic to $\mathcal{O}(a) \oplus \mathcal{O}(b)$, with $a, b \in \mathbb{Z}$, $a + b = 2$. We will show now that, on this bundle, the existence of four holomorphic sections that are real–linearly independent at each point implies $a = b = 1$. First, it is easy to see that a and b are both non–negative. Suppose, for instance, that $b < 0$. As $\mathcal{O}(b)$ has no holomorphic sections, any holomorphic section of $\mathcal{O}(a) \oplus \mathcal{O}(b)$ belongs actually to $\mathcal{O}(a)$, and, hence, vanishes somewhere. The other case to be excluded is $a = 2$, $b = 0$. In this case, the $\mathcal{O}(b)$–component of each holomorphic section is a complex number, so we can pick, in our real 4–dimensional space of holomorphic sections, one section whose $\mathcal{O}(b)$–component vanishes. Then as before, this section will vanish somewhere on S, a contradiction. Thus a and b have to be equal to 1. (Of course, in this case, the above system of sections turns out to be a *basis* of holomorphic sections of $N(S)$.)

□

Thus M is a complex three–manifold, fibered over N such that the fibers are embedded rational curves whose normal bundle is isomorphic to $\mathcal{O}(1) \oplus \mathcal{O}(1)$, and endowed with an anti–holomorphic involution τ with no fixed points. By the *reverse Penrose construction* (see [1] or [3], Chapter 13), M is diffeomorphic to the twistor space of N (with its natural complex structure associated to a canonical anti–self–dual conformal structure on N). All we need to prove now is that the metric g^N on N belongs to this conformal class.

The canonical conformal structure of N is defined, via the reverse Penrose construction, by its isotropic vectors (lying in $T_xN \otimes \mathbb{C}$, for $x \in N$); these correspond to somewhere vanishing holomorphic sections of $N(S)$, where $S = \pi^{-1}(x)$. Such a section is equal to

$$\tilde{X} + J\tilde{Y}, \text{ for some vectors } X, Y \in T_xN,$$

and it vanishes somewhere only if $\|X\| = \|Y\|$ and $X \perp Y$. (Actually, these conditions suffice for the considered section to vanish somewhere: using Lemma 5.3, one can prove that, for $\|X\| = 1$, $\{J\tilde{X}_p\}_{p \in S}$ is the two–sphere of *all* unitary vectors in T_xN, orthogonal to X).

Thus the metric g^N on N is anti–self–dual, its twistor space (endowed with its *integrable* complex structure) is isomorphic to $(M, \bar{J})$, and the metric on it comes, in the standard way, from the metric on N. As this is a Kähler manifold, N needs to be either (S^4, can) or $(\mathbb{CP}^2, g_{FS})$ (the Fubini–Study metric on $\mathbb{CP}^2$) – with the reverse orientation on $\mathbb{CP}^2$ –, hence (M, J) is biholomorphically $\mathbb{CP}^3$ or $F(1,2)$, the manifold of flags in $\mathbb{C}^3$. This proves Theorem 1.1.

References

[1] M.F. ATIYAH, N.J. HITCHIN, I.M. SINGER, *Self–duality in four–dimensional Riemannian geometry,* Proc. R. Soc. Lond. A. **362** (1978), 425–461.

[2] C. BÄR, *Real Killing spinors and holonomy,* Commun. Math. Phys. **154** (1993), 509-521.

[3] A. L. BESSE, *Einstein Manifolds,* Springer–Verlag, New York 1987.

[4] J. EELLS, S. SALAMON, *Twistorial construction of harmonic maps of surfaces into four–manifolds,* Ann. Scuola Norm. Sup. Pisa Cl. Sci. **12** (1985), 589–640.

[5] Th. FRIEDRICH, *A remark on the first eigenvalue of the Dirac operator on 4–dimensional manifolds,* Math. Nachr. **102** (1981), 53–56.

[6] Th. FRIEDRICH, R. GRUNEWALD, *On Einstein Metrics on the Twistor Space of a Four–Dimensional Riemannian Manifold,* Math. Nachr. **123** (1985), 55–60.

[7] Th. FRIEDRICH, R. GRUNEWALD, *On the first eigenvalue of the Dirac operator on 6–dimensional manifolds,* Ann. Global Anal. Geom. **3** (1985), 265–273.

[8] R. GRUNEWALD, *Six–Dimensional Riemannian Manifolds with real Killing Spinors,* Ann. Global Anal. Geom. **8** (1990), 43–59.

[9] A. GRAY, *The Structure of Nearly Kähler manifolds,* Math. Ann. **223** (1976), 233–248.

[10] A. GRAY, L. M. HERVELLA, *The sixteen classes of almost Hermitian manifolds and their linear invariants,* Ann. Mat. Pura Appl. **123** (1980), 35–58.

[11] R. HERMANN, *A sufficient condition that a mapping of Riemannian manifolds be a fibre bundle,* Proc. Amer. Math. Soc. **11** (1960), 236–242.

[12] K.-D. KIRCHBERG, *Compact Six–Dimensional Kähler Spin Manifolds of Positive Scalar Curvature with the Smallest Possible First Eigenvalue of the Dirac Operator,* Math. Ann. **282** (1988), 157–176.

[13] B. O'NEILL, *The fundamental equations of a submersion,* Michigan Math J. **13** (1966), 459–469.

[14] G. REEB, *Stabilité des feuilles compactes à groupe de Poincaré fini,* C. R. Acad. Sci. Paris **228** (1949), 47–48.

[15] R. REYES–CARRIÓN, *Some special geometries defined by Lie groups,* PhD Thesis, Oxford, 1993.

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