Spin$^c$ Manifolds and Complex Contact Structures

Andrei Moroianu\textsuperscript{1}

\textbf{Abstract} - In this paper we extend our notion of projectable spinors ([9], Ch.1) to the framework of Spin$^c$ manifolds and deduce the basic formulas relating spinors on the base and the total space of Riemannian submersions with totally geodesic one-dimensional fibres. Some geometric applications concerning positive Kähler-Einstein complex contact manifolds (e.g. their characterisation as twistor spaces over positive quaternionic Kähler manifolds) are also given.

1 Introduction

Projectable spinors for Riemannian submersions of spin manifolds arose in a quite natural way ([9], Ch.1) and have led to important geometric applications, as the classification of Kähler manifolds admitting Kählerian Killing spinors ([8]) or results on the spectrum of the Dirac operator for certain classes of Riemannian manifolds ([10]).

In this paper we introduce projectable spinors for Riemannian submersions of Spin$^c$ manifolds, motivated by the following facts: K.-D. Kirchberg and U. Semmelmann discovered that every complex contact manifold of complex dimension $4l + 3$ admitting a Kähler–Einstein metric of positive scalar curvature carries a canonical spin structure with Kählerian Killing spinors [4]. Using this together with the results in [8], we were able to prove the following characterisation of twistor spaces over positive quaternionic Kähler manifolds in half of the possible dimensions:

\textbf{Theorem A.} (cf. [12]) Let $M$ be a compact spin Kähler manifold of positive scalar curvature and complex dimension $4l + 3$. Then the following statements are equivalent:

(i) $M$ is the twistor space of some quaternionic Kähler manifold;
(ii) $M$ is Kähler–Einstein and admits a complex contact structure;
(iii) $M$ admits Kählerian Killing spinors.

By different methods, C. LeBrun independently obtained the following

\textbf{Theorem B.} (cf. [7]) Let $Z$ be a Fano contact manifold. Then $Z$ is a twistor space iff it admits a Kähler–Einstein metric.

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In complex dimensions $4l + 3$, this is a direct corollary of Theorem A. The reasons for which our Theorem A fails to hold in complex dimensions $4l + 1$ are of a topological nature: neither the twistor spaces, nor the complex contact manifolds of complex dimensions $4l + 1$ are spin (with one exception: the complex projective space). On the other hand, each Kähler manifold admits a natural Spin$^c$ structure; it is thus natural to try to extend the above notions to the framework of Spin$^c$ structures, and to generalise the results in [12] to this case.

In order to keep the computations as simple as possible, we do not construct here the whole theory of projectable spinors on Spin$^c$ manifolds, and restrict ourselves to a particular situation which is of special interest to us. Generalisations of the constructions described below can be easily obtained.

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2 Preliminaries

**Definition 2.1** A Spin$^c$ structure on an oriented Riemannian manifold $(M^n, g)$ is given by a $U(1)$ principal bundle $P_{U(1)}M$ and a Spin$^c$ principal bundle $P_{Spin^c}M$ together with a projection $\theta : P_{Spin^c}M \to PSO(n)M \times P_{U(1)}M$ ($PSO(n)M$ means the SO($n$) principal bundle of oriented orthonormal frames on $M$), satisfying

$$\theta(\tilde{u} \tilde{a}) = \theta(\tilde{u}) \xi(\tilde{a}),$$

for every $\tilde{u} \in P_{Spin^c}M$ and $\tilde{a} \in Spin^c_n$, where $\xi$ is the canonical 2-fold covering of Spin$^c_n$ over SO($n$) $\times U(1)$.

Recall that Spin$^c_n = Spin_n \times \mathbb{Z}_2$, $U(1)$, and that $\xi$ is given by $\xi([u, a]) = (\phi(u), a^2)$, where $\phi : Spin_n \to SO(n)$ is the canonical 2-fold covering.

If $M$ has a Spin$^c$ structure, we denote by $\Sigma M$ the associated complex spinor bundle and by $L$ the complex line bundle associated to $P_{U(1)}M$, which is called the auxiliary bundle. On $\Sigma M$ there is a canonical Hermitian product $(.,.)$, with respect to which the Clifford multiplication by vectors is skew-Hermitian:

$$(X \cdot \psi, \varphi) = -(\psi, X \cdot \varphi), \quad \forall X \in TM, \psi, \varphi \in \Sigma M. \tag{1}$$

Every connection form $A$ on $P_{U(1)}M$ defines, together with the Levi-Civita connection of $M$, a covariant derivative on $\Sigma M$ denoted by $\nabla^A$. Correspondingly, we define the Dirac operator as the composition $\gamma \circ \nabla^A$, where $\gamma$ denotes the Clifford contraction. The Dirac operator can be expressed using a local orthonormal frame $\{e_1, \cdots, e_n\}$ as

$$D = \sum_{i=1}^n e_i \cdot \nabla^A_{e_i}.$$
Suppose now that \((M^{2m}, g, J)\) is a Kähler manifold. We define the twisted Dirac operator \(\tilde{D}\) by
\[
\tilde{D} = \sum_{i=1}^{2m} J(e_i) \cdot \nabla_{e_i} = -\sum_{i=1}^{2m} e_i \cdot \nabla J(e_i),
\]
which satisfies
\[
\tilde{D}^2 = D^2 \quad \text{and} \quad \tilde{D}D + D\tilde{D} = 0. \tag{2}
\]
We also define the complex Dirac operators \(D_{\pm} := \frac{1}{2}(D \mp i\tilde{D})\), and (2) becomes
\[
D_{\pm}^2 = D_{\pm}^2 = 0 \quad \text{and} \quad D_{+}^2 = D_{+} D_{-} + D_{-} D_{+}. \tag{3}
\]
Consider a local orthonormal frame \(\{X_\alpha, Y_\alpha\}\) such that \(Y_\alpha = J(X_\alpha)\). Then \(Z_\alpha = \frac{1}{2}(X_\alpha - iY_\alpha)\) and \(Z_\bar{\alpha} = \frac{1}{2}(X_\alpha + iY_\alpha)\) are local frames of \(T^{1,0}(M)\) and \(T^{0,1}(M)\), and \(D_{\pm}\) can be expressed as
\[
D_{+} = 2 \sum_{\alpha=1}^{m} Z_\alpha \cdot \nabla_{Z_\alpha} \quad \text{and} \quad D_{-} = 2 \sum_{\alpha=1}^{m} Z_\bar{\alpha} \cdot \nabla_{Z_\bar{\alpha}}. \tag{4}
\]
A \(k\)-form \(\omega\) acts on \(\Sigma M\) by
\[
\omega \cdot \Psi := \sum_{i_i < \cdots < i_k} \omega(e_{i_1}, \cdots, e_{i_k}) e_{i_1} \cdot \cdots \cdot e_{i_k} \cdot \Psi,
\]
where \(\{e_i\}\) is a local orthonormal frame on \(M\). With respect to this action, the Kähler form \(\Omega\) (defined by \(\Omega(X, Y) = g(X, JY)\)) satisfies
\[
\Omega = \frac{1}{2} \sum_{i=1}^{2m} J(e_i) \cdot e_i = \frac{1}{2} \sum_{i=1}^{2m} e_i \cdot J(e_i). \tag{5}
\]
For later use let us note that
\[
\sum_{\alpha=1}^{m} Z_\alpha \cdot Z_\bar{\alpha} = -\frac{i}{2} \Omega - \frac{m}{2} \quad \text{and} \quad \sum_{\alpha=1}^{m} Z_\bar{\alpha} \cdot Z_\alpha = \frac{i}{2} \Omega - \frac{m}{2}, \tag{6}
\]
where \(Z_\alpha\) and \(Z_{\bar{\alpha}}\) are local frames of \(T^{1,0}(M)\) and \(T^{0,1}(M)\) as before.

The action of \(\Omega\) on \(\Sigma M\) yields an orthogonal decomposition
\[
\Sigma M = \bigoplus_{r=0}^{m} \Sigma_r M,
\]
where \(\Sigma_r M\) is the eigenbundle associated to the eigenvalue \(i \mu_r = i(m - 2r)\) of \(\Omega\). If we define \(\Sigma_{-1} M = \Sigma_{m+1} M = \{0\}\), then
\[
D_{\pm} \Gamma(\Sigma_r M) \subseteq \Gamma(\Sigma_{r \pm 1} M). \tag{7}
\]
The complex volume element
\[
\omega_C := \bar{\rho}^{n} e_1 \cdot \cdots \cdot e_{2m}
\]
acts on \(\Sigma M\) by Clifford multiplication and its square is the identity. We denote by \(\Sigma^\pm M\) the eigenbundles corresponding to the eigenvalues \(\pm 1\), and it is easy to see that \(\Sigma_r M \subset \Sigma^+ M\) (\(\Sigma_r M \subset \Sigma^- M\)) iff \(r\) is even (respectively odd). If, with respect to the decomposition \(\Sigma M = \Sigma^+ M \oplus \Sigma^- M\), a spinor \(\psi\) is written as \(\psi = \psi_+ + \psi_-\), then we define its conjugate \(\bar{\psi} := \psi_+ - \psi_-\).
3 Projectable spinors to Spin$^c$-Manifolds

As in the case of spin manifolds, projectable spinors may be defined for arbitrary Riemannian submersions of Spin$^c$ manifolds with 1-dimensional totally geodesic fibres, but for the sake of simplicity we treat only a particular case here.

Let $P_{U(1)}M$ be the principal U(1) bundle associated to a Spin$^c$ structure on a Riemannian manifold $(M^n, g)$ of even dimension and suppose that on $P_{U(1)}M$ a connection form $A$ is given. Denote by $\tilde{M} := P_{U(1)}M$ and by $\pi$ the canonical bundle projection. It is well-known that there exists an unique 2-form $\alpha$ on $M$ whose pull-back is just $i$ times the curvature form $dA$ of the connection $A$ (note that $A$ and $dA$ are imaginary-valued forms on $\tilde{M}$). Let $T$ be the (1,1) tensor on $M$ associated to $\alpha$ (defined by $\alpha(X, Y) = g(X, TY)$).

The manifold $\tilde{M}$ carries a canonical 1-parameter family of Riemannian metrics $g_t$ which make the bundle projection $\pi : \tilde{M} \to M$ into a Riemannian submersion with totally geodesic fibres. These metrics are given by

$$g_t(X, Y) = g(\pi_* X, \pi_* Y) - i^2 A(X)A(Y), \quad \forall x \in \tilde{M}, \ X, Y \in T_x \tilde{M},$$

and we denote by $\nabla^A$ the covariant derivative of the Levi-Civita connection of $g_t$ and by $V$ the unit vertical vector field on $(\tilde{M}, \tilde{g})$ satisfying $A(V) = i/t$. This choice of $V$ fixes an orientation for $\tilde{M}$.

Before proceeding, we mention here a simple result relating spin and Spin$^c$ structures, that will be used in a moment.

**Lemma 3.1** A Spin$^c$ structure with trivial auxiliary bundle is canonically identified with a spin structure. Moreover, if the connection $A$ of the auxiliary bundle $L$ is flat, then by this identification $\nabla^A$ corresponds to $\nabla$ on the spinor bundles.

**Proof.** One first remarks that since the U(1) bundle associated to $L$ is trivial, we can exhibit a global section of it, that we will call $\sigma$. Denote by $P_{Spin^c}M$ the inverse image by $\theta$ of $PSO(n) M \times \sigma$. It is straightforward to check that this defines a spin structure on $M$, and that the connection on $P_{Spin^c}M$ restricts to the Levi-Civita connection on $P_{Spin^c}M$ if $\sigma$ can be chosen to be parallel, i.e. if $A$ defines a flat connection.

Q.E.D.

**Proposition 3.1** Every Spin$^c$ structure on $M$ induces a canonical spin structure on $\tilde{M}$.

**Proof.** By enlargement of the structure groups, the two-fold covering

$$\theta : P_{Spin^c}M \to PSO(n) M \times P_{U(1)}M,$$
gives a two-fold covering
\[ \theta : P_{\text{Spin}^c_{n+1}} M \rightarrow P_{\text{SO}(n+1)} M \times P_{U(1)} M, \]
which, by pull-back through \( \pi \), gives rise to a \( \text{Spin}^c \) structure on \( \tilde{M} \)
\[ P_{\text{Spin}^c_{n+1}} \tilde{M} \xrightarrow{\pi} P_{\text{Spin}^c_{n+1}} M \]
\[ \pi^* \theta \downarrow \quad \theta \downarrow \]
\[ P_{\text{SO}(n+1)} \tilde{M} \times P_{U(1)} \tilde{M} \xrightarrow{\pi} P_{\text{SO}(n+1)} M \times P_{U(1)} M \]
\[ \downarrow \quad \downarrow \]
\[ \tilde{M} \xrightarrow{\pi} M \]

Using Lemma 3.1 we see that this construction actually yields a \textit{spin} structure on \( \tilde{M} \). Indeed, the pull back of a \( G \) principal bundle \( (P_{U(1)} M, \text{in our situation}) \) with respect to its own projection map is always trivial:
\[ \pi^* P \simeq P \times G \xrightarrow{\pi} P \]
\[ \pi^* \pi \downarrow \quad \pi \downarrow \]
\[ P \xrightarrow{\pi} M \]

Nevertheless, we will continue to call this spin structure the induced \( \text{Spin}^c \) structure on \( \tilde{M} \).

Q.E.D.

The next step is to relate the covariant derivatives of spinors on \( M \) and \( \tilde{M} \). We point out an important detail here: since we are actually interested in \( \tilde{M} \) as \textit{spin} manifold, the connection on \( P_{U(1)} \tilde{M} \) (which defines the covariant derivative of spinors on \( \tilde{M} \)) that we consider, will not be the pull-back connection, but the \textit{flat connection} on the canonically trivial bundle \( P_{U(1)} \tilde{M} \). The following result relates an arbitrary connection on a principal bundle \( \pi : P \rightarrow M \) and the flat connection on \( \pi^* P \rightarrow P \).

\textbf{Lemma 3.2} The connection form \( A_0 \) of the flat connection on \( \pi^* P \) can be related to an arbitrary connection \( A \) on \( P \) by
\[ A_0((\pi^* s)_*(U)) = -A(U), \quad (8) \]
\[ A_0((\pi^* s)_*(X^*)) = A(s, X), \quad (9) \]
where \( U \) is a vertical vector field on \( P \), \( X^* \) is the horizontal lift (with respect to \( A \)) of a vector field \( X \) on \( M \), and \( s \) is a local section of \( P \rightarrow M \).
Proof. The identification \( P \times U(1) \simeq \pi^*P \) is given by \((u,a) \mapsto (u,ua)\), for all \((u,a) \in P \times U(1)\). For some fixed \( u \in P \), take a path \( u_t \) in the fiber over \( x := \pi(u) \) such that \( u_0 = u \) and \( \dot{u}_0 = U \). We define \( a_t \in U(1) \) by \( u_t = s(x)a_t \), so via the above identification we have
\[
  (\pi^*s)(u_t) = (u_t, s(x)) = (u_t, (a_t)^{-1}),
\]
and thus
\[
  A_0((\pi^*s)_*(U)) = -a_0^{-1}\dot{a}_0 = -A(\dot{u}_0) = -A(U).
\]
Similarly, for \( x \in M \) and \( X \in T_xM \), take a path \( x_t \) in \( M \) such that \( x_0 = x \) and \( \dot{x}_0 = X \). Let \( u \in \pi^{-1}(x) \) and \( u_t \) the horizontal lift of \( x_t \) such that \( u_0 = u \). We define \( a_t \in U(1) \) by \( s(x_t) = u_0a_t \), which by derivation gives \( s_*(X) = R_{a_0}\dot{u}_0 + u_0\dot{a}_0 \). Then
\[
  (\pi^*s)(u_t) = (u_t, s(x_t)) = (u_t, a_t),
\]
and thus, using the fact that \( \dot{u}_0 \) is horizontal,
\[
  A_0((\pi^*s)_*(X^*)) = a_0^{-1}\dot{a}_0 = A(s_*(X)).
\]
Q.E.D.

Recall that the complex Clifford representation \( \Sigma_n \) can be made into a \( Cl(n+1) \)-representation by defining
\[
  \mu(e_j) \cdot \psi = \begin{cases} 
  e_j \cdot \psi & \text{for } j \leq n \\
  i\psi & \text{for } j = n+1
\end{cases}
\]
Corresponding to this, we obtain an identification of the pull back \( \pi^*\Sigma M \) with \( \Sigma\bar{M} \), and with respect to this identification, if \( X \) is a vector and \( \psi \) a spinor on \( M \), then
\[
  X^* \cdot \pi^*\psi = \pi^*(X \cdot \psi), \quad (10)
\]
\[
  V \cdot \pi^*\psi = \pi^*(i\psi), \quad (11)
\]
where \( V \) is the unit vertical vector field defined at the beginning of this section.

Definition 3.1 The sections of \( \Sigma\bar{M} \) which can be written as pull-back of sections of \( \Sigma M \) are called projectable spinors.

We now compute the covariant derivative of projectable spinors on \( \bar{M} \) in terms of the covariant derivative of spinors on \( M \).

Proposition 3.2 The covariant derivative \( \nabla \) on \( \Sigma\bar{M} \) induced by the Levi-Civita connection on \((\bar{M},g_\ell)\) and the flat connection on \( \pi^*P_{U(1)}M \) satisfies
\[
  \nabla_X \pi^*\psi = \pi^*(\nabla_X^A\psi - i\frac{t}{4}T(X) \cdot \bar{\psi}) \quad \forall X \in TM, \quad (12)
\]
\[
  \nabla_V \pi^*\psi = -\pi^*(\frac{L}{4}\alpha \cdot \psi + \frac{i}{2t} \psi) \quad \forall V \in T\bar{M}, \quad (13)
\]
PROOF. Recall that the curvature form $dA$ of the principal $U(1)$ bundle $\tilde{M} \to M$ satisfies

$$dA = -i\pi^* \alpha.$$  \hspace{1cm} (14)

The metric $g_t$ is given by

$$g_t(X, Y) = g(\pi_* X, \pi_* Y) - t^2 A(X)A(Y), \ \forall X, Y \in T\tilde{M}. \hspace{1cm} (15)$$

If $V$ denotes as before the unit vertical vector field, then $A(V) = i/t$, and we obtain

$$g_t(\nabla^t_\cdot, V, Y^*) = \frac{1}{2} g_t(V, [X^*, Y^*]) = \frac{t^2}{2} A(V)A([X^*, Y^*])$$

$$= \frac{it}{2} A([X^*, Y^*]) = \frac{-it}{2} dA(X^*, Y^*)$$

$$= -\frac{t}{2} \alpha(X, Y) = \frac{t}{2} g_t(T(X)^*, Y^*),$$

so finally

$$\nabla^t_\cdot, V = \frac{t}{2} T(X)^*. \hspace{1cm} (16)$$

Consider the pull-back $\pi^* \psi$ of a spinor field $\psi = [\alpha, \xi]$, where $\xi : U \subset M \to \Sigma_n$ is a vector valued function, and $\sigma$ is a local section of $P_{\text{Spin}}(\tilde{M})$ whose projection onto $P_{SO(n)}\tilde{M}$ is a local orthonormal frame $(X_1, ..., X_n)$ and whose projection onto $P_{U(1)}\tilde{M}$ is a local section $s$. Then $\pi^* \psi$ can be expressed as $\pi^* \psi = [\pi^* \sigma, \pi^* \xi]$, and it is easy to see that the projection of $\pi^* \sigma$ onto $P_{SO(n+1)}\tilde{M}$ is the local orthonormal frame $(X_1^*, ..., X_n^*, V)$ and its projection onto $P_{U(1)}\tilde{M}$ is just $\pi^* s$.

Using Lemma 3.2 and (16) we obtain

$$\nabla^t_\cdot, \pi^* \psi = [\pi^* \sigma, X^*(\pi^* \xi)] + \frac{1}{2} \sum_{j < k} g_t(\nabla^t_\cdot, X_j^*, X_k^*, X_j^* \cdot X_k^* \cdot \pi^* \psi$$

$$+ \frac{1}{2} \sum_{j} g_t(\nabla^t_\cdot, X_j^*, V) X_j^* \cdot V \cdot \pi^* \psi + \frac{1}{2} A_0((\pi^* s)_\cdot X^*) \pi^* \psi$$

$$= [\pi^* \sigma, \pi^*(X(\xi))] + \frac{1}{2} \sum_{j < k} g(\nabla X_j, X_k) \pi^*(X_j \cdot X_k \cdot \psi)$$

$$- \frac{1}{2} \frac{it}{2} \sum_{j} g(T(X), X_j) \pi^*(X_j \cdot \tilde{\psi}) + \frac{1}{2} A(s \cdot X) \pi^* \psi$$

$$= \pi^* ([\sigma, (X(\xi))] + \frac{1}{2} \sum_{j < k} g(\nabla X_j, X_k) X_j \cdot X_k \cdot \psi$$

$$- \frac{it}{4} T(X) \cdot \tilde{\psi} + \frac{1}{2} A(s \cdot X) \psi)$$

$$= \pi^*(\nabla X^\psi - \frac{it}{4} T(X) \cdot \tilde{\psi}).$$
and similarly,
\[
\nabla^*_V \pi^* \psi = [\pi^* \sigma, V(\pi^* \xi)] + \frac{1}{2} \sum_{j<k} g_l(\nabla^*_V X^*_j, X^*_k) X^*_j \cdot X^*_k \cdot \pi^* \psi \\
+ \frac{1}{2} \sum_j g_l(\nabla^*_V X^*_j, V) X^*_j \cdot V \cdot \pi^* \psi + \frac{1}{2} A_0(\pi^* s, V) \pi^* \psi \\
= \frac{t}{2} \frac{1}{2} \sum_{j<k} g(T(X_j), X_k) \pi^* (X_j \cdot X_k \cdot \psi) - \frac{1}{2} A(V) \pi^* \psi \\
= -\pi^* \left( \frac{t}{4} \alpha \cdot \psi + \frac{i}{2t} \psi \right).
\]

Q.E.D.

We now particularise the above results to the case where $M$ is a Kähler-Einstein manifold $(M^n, g, J)$ of positive scalar curvature, and the auxiliary bundle $L$ of the Spin$^c$ structure on $M$ is a root of the canonical bundle $K$, i.e. $L^{\otimes r} = K$ for some $r \in \mathbb{N}^*$. The canonical connection on $K$, whose curvature form is just $-i \rho$ ($\rho$ is the Ricci form), induces then a connection $A$ on $L$, whose curvature form $\omega$ satisfies $\omega = -i \rho / r$. As before, we denote by $\tilde{M}$ the U(1) principal bundle associated to $L$. By rescaling the metric on $M$ if necessary, we can suppose that the scalar curvature of $M$ is equal to $n(n+2)$, and thus $\rho = (n+2) \Omega$. The 2-form $\alpha$ on $M$ defined at the beginning of this section is given in this case by
\[
\alpha = \frac{n+2}{r} \Omega, \tag{17}
\]
so the above proposition becomes

**Proposition 3.3** If $M$ is a Spin$^c$ Kähler-Einstein manifold of positive scalar curvature and the auxiliary bundle $L$ of the Spin$^c$ structure on $M$ is a $r$-root of the canonical bundle $K$, then the covariant derivative $\nabla^l$ on $\Sigma \tilde{M}$ induced by the Levi-Civita connection on $(\tilde{M}, g_l)$ and the flat connection on $\pi^* P_{U(1)} M$ satisfies
\[
\nabla^l_{X^*} (\pi^* \psi) = \pi^* (\nabla^l_X \psi - i \frac{t(n+2)}{4r} J(X) \cdot \tilde{\psi}) \quad \forall X \in TM, \tag{18}
\]
\[
\nabla^l_{V^*} \pi^* \psi = -\pi^* \left( \frac{t(n+2)}{4r} \Omega \cdot \psi + \frac{i}{2t} \psi \right). \tag{19}
\]

The formula (16) allows us to compute the Ricci tensor $\text{Ric}^l$ of the Riemannian manifold $(\tilde{M}, g_l)$. If we denote by $a := \frac{(n+2)}{2r}$, then
\[
\text{Ric}^l(V, V) = na^2 \quad , \quad \text{Ric}^l(X^*, V) = 0, \tag{20}
\]

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\[ \text{Ric}'(X^*, Y^*) = (n + 2 - 2a^2)g(X, Y). \]  

Let us take \( t_0 = \frac{2r}{n+2} \) and denote \( \tilde{g} := g_{o0}, \tilde{\nabla} := \nabla^0. \) The vertical vector field \( V \) defines then an Einstein–Sasakian structure on the manifold \( (\bar{M}, \tilde{g}) \) (cf. [2]). We can synthesise the above results in the following

**Theorem 3.1** Let \( (M^n, g, J) \) be a Kähler-Einstein manifold with scalar curvature \( R = n(n + 2), L := K^2 \) a root of the canonical bundle \( K \) and \( \bar{M} \) the associated \( U(1) \) principal bundle with connection form \( A \), induced by the Levi-Civita connection on \( K \). Then the following hold:

(i) There is a canonical metric \( \bar{g} \) on \( \bar{M} \) making the bundle projection \( \pi : \bar{M} \to M \) into a Riemannian submersion with totally geodesic fibres, and satisfying \( \tilde{\nabla}_X V = J(X)^* \).

(ii) With respect to the metric \( \bar{g}, V \) defines a regular Einstein–Sasakian structure on \( \bar{M} \). The length of the fibres of the corresponding \( S^1 \)-action is constant and equal to \( \frac{2\pi r}{n+2} \).

(iii) The \( \text{Spin}^c \) structure on \( M \) defined by \( (L, A) \) induces a canonical spin structure on \( \bar{M} \) and every spinor field on \( M \) induces a projectable spinor field \( \pi^* \psi \) on \( \bar{M} \), satisfying

\[ \tilde{\nabla}_X (\pi^* \psi) = \pi^*(\nabla^A_X \psi - \frac{i}{2} J(X) \cdot \bar{\psi}) \quad \forall X \in TM, \]

\[ \nabla_V \pi^* \psi = -\frac{1}{2} \pi^*(\Omega \cdot \psi + \frac{i(n + 2)}{2r} \psi). \]

4 Complex contact structures

**Definition 4.1** (cf. [5]) Let \( M^{2m} \) be a complex manifold of complex dimension \( m = 2k + 1 \). A complex contact structure is a family \( \mathcal{C} = \{(U_i, \omega_i)\} \) satisfying the following conditions:

(i) \( \{U_i\} \) is an open covering of \( M \).

(ii) \( \omega_i \) is a holomorphic 1-form on \( U_i \).

(iii) \( \omega_i \wedge (\partial \omega_i)^k \in \Gamma(\Lambda^{m,0} M) \) is different from zero at every point of \( U_i \).

(iv) \( \omega_i = f_{ij} \omega_j \) in \( U_i \cap U_j \), where \( f_{ij} \) is a holomorphic function on \( U_i \cap U_j \).

Let \( \mathcal{C} = \{(U_i, \omega_i)\} \) be a complex contact structure. Then there exists an associated holomorphic line sub–bundle \( L_{\mathcal{C}} \subset \Lambda^{1,0}(M) \) with transition functions \( \{f_{ij}^{-1}\} \) and local sections \( \omega_i \). It is easy to see that

\[ \mathcal{D} := \{ Z \in T^{1,0}M \mid \omega(Z) = 0, \forall \omega \in L_{\mathcal{C}} \} \]
is a codimension 1 maximally non–integrable holomorphic sub–bundle of $T^3,0M$, and conversely, every such bundle defines a complex contact structure. From condition (iii) immediately follows the isomorphism $L_{c}^{k+1} \cong K$, where $K = \Lambda^{m,0}(M)$ denotes the canonical bundle of $M$.

From now on, $M$ will denote a Kähler–Einstein manifold of odd complex dimension $m = 4l + 1$ with positive scalar curvature, admitting a complex contact structure $C$. The manifold $M$ is compact, by Myers’ Theorem. By rescaling the metric on $M$ if necessary, we can suppose that the scalar curvature of $M$ is equal to $2m(2m + 2)$, and thus the Ricci form $\rho$ and the Kähler form $\Omega$ are related by $\rho = (2m + 2)\Omega$. The main objective of this section is to construct the analogues of Kählerian Killing spinors ([3], [4], [8]) for a certain Spin$^c$ structure on $M$, determined by $C$. This is done just as in [4].

The collection $(U_i, \omega_i \wedge (\partial \omega_i)^I)$ defines a holomorphic line bundle $L_i \subset \Lambda^{2l+1,0}M$, and from the definition of $C$ we easily obtain

$$L_i \cong L_{c}^{l+1}.$$  \hfill (24)

We now fix some $(U, \omega) \in C$ and define a local section $\psi_C$ of $\Lambda^{0,2l+1}M \otimes L_{c}^{l+1}$ by

$$\psi_C|_U := |\xi_\tau|^{-2} \otimes \xi_\tau,$$  \hfill (25)

where $\tau := \omega \wedge (\partial \omega)^I$ and $\xi_\tau$ is the element corresponding to $\tau$ through the isomorphism (24). The fact that $\psi_C$ does not depend of the element $(U, \omega) \in C$ shows that it actually defines a global section $\psi_C$ of $\Lambda^{0,2l+1}M \otimes L_{c}^{l+1}$.

We now recall ([6], Appendix D) that $\Lambda^{0,1}M$ is just the spinor bundle associated to the canonical Spin$^c$ structure on $M$, whose auxiliary line bundle is $K^{-1}$, so that $\Lambda^{0,1}M \otimes L_{c}^{l+1}$ is actually the spinor bundle associated to the Spin$^c$ structure on $M$ with auxiliary bundle $L = K^{-1} \otimes L_{c}^{l+1} \cong L_{c}^{-2l+1} \otimes L_{c}^{2l+1} \cong L_c$. The section $\psi_C$ is thus a spinor lying in $\Lambda^{0,2l+1}M \otimes L_{c}^{l+1} \cong \Sigma_{2l+1}M$, so

$$\Omega \cdot \psi_C = -i \psi_C.$$  \hfill (26)

**Proposition 4.1** The spinor field $\psi_C$ satisfies $\nabla_Z \psi_C = 0, \forall Z \in T^{1,0}M$ (in particular $D_\pm \psi_C = 0$), and

$$D^2 \psi_C = D_- D_+ \psi_C = \frac{l+1}{2l+1} (\frac{1}{2} R \psi_C - i\rho \cdot \psi_C),$$  \hfill (27)

where $R$ is the scalar curvature of $M$.

**Proof.** This is actually a variant of Proposition 5 from [4], the only difference being that $\xi_\tau$ (\(\Psi_\omega\) in their notations) is not any more a section of $K^{1/2}$, but of $K^{(l+1)/(2l+1)}$, so the coefficients $1/2$ in formulas (8) and (9) of [4] have to be replaced by $\frac{l+1}{2l+1}$. 

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Q.E.D.

Using (26), (27) and the fact that \( \rho = \frac{1}{8t^2} R \Omega = (8l + 4) \Omega \), we obtain

**Corollary 4.1** The spinor field \( \psi_C \) is an eigenspinor of \( D^2 \) with respect to the eigenvalue \( 16l(l + 1) \).

Let us introduce some notations

\[
\psi_- := \psi_C \in \Gamma(\Sigma_{2l+1} M), \quad \psi_+ := \frac{1}{4l + 4} D\psi_C \in \Gamma(\Sigma_{2l+2} M).
\] (28)

By integration over \( M \) we immediately obtain from the above Corollary

\[
|\psi_-|_{L^2}^2 = \frac{l + 1}{l} |\psi_+|_{L^2}^2.
\] (29)

**Proposition 4.2** The following relations hold

\[
\nabla_Z \psi_- = 0, \ \forall Z \in T^{1,0} M,
\] (30)

\[
\nabla_Z \psi_- + \bar{Z} \cdot \psi_+ = 0, \ \forall Z \in T^{0,1} M,
\] (31)

\[
\nabla_Z \psi_+ = 0, \ \forall Z \in T^{0,1} M,
\] (32)

\[
\nabla_Z \psi_+ + Z \cdot \psi_- = 0, \ \forall Z \in T^{1,0} M.
\] (33)

**Proof.** The first relation is part of Proposition 4.1. In order to prove (31), let us consider the local frames of \( T^{1,0}(M) \) and \( T^{0,1}(M) \) introduced in Section 2: \( Z_\alpha = \frac{1}{2} (X_\alpha - \imath Y_\alpha) \) and \( \bar{Z}_\alpha = \frac{1}{2} (X_\alpha + \imath Y_\alpha) \), where \( Y_\alpha = J(X_\alpha) \), and \( \{ X_\alpha, Y_\alpha \} \) is a local orthonormal frame of \( TM \). From (30) we find \( \nabla_{Z_\alpha} \psi_- = \nabla_{X_\alpha} \psi_- = \imath \nabla_{Y_\alpha} \psi_- \), so using (6) and (28) gives

\[
0 \leq \sum_{\alpha=1}^m |\nabla_{Z_\alpha} \psi_- + \bar{Z}_\alpha \cdot \psi_+|^2 \\
\quad = \sum_{\alpha=1}^m |\nabla_{X_\alpha} \psi_-|^2 - 2 \text{Re} \sum_{\alpha=1}^m (\psi_+, Z_\alpha \cdot \nabla_{Z_\alpha} \psi_-) - \sum_{\alpha=1}^m (\psi_+, Z_\alpha \cdot Z_\alpha \cdot \psi_+) \\
\quad = \frac{1}{2} |\nabla \psi_-|^2 - \text{Re}(\psi_+, D_+ \psi_-) - \frac{1}{2} (\psi_+ , (\imath \Omega - m) \psi_+) \\
\quad = \frac{1}{2} |\nabla \psi_-|^2 - (4l + 4) |\psi_+|^2 + \frac{1}{2} (4l + 4) |\psi_+|^2.
\]
D), Corollary 4.1 and (29), we obtain

\[ |F|^2_{L^2} = \frac{1}{2} (D^* D \psi_+ \psi_+, \psi_-)_{L^2} - (4l + 4) |\psi_+|^2_{L^2} + \frac{1}{2} (4l + 4) |\psi_+|^2_{L^2}, \]

\[ = \frac{1}{2} (D^2 \psi_- - \frac{1}{4} R \psi_- + \frac{i}{2} \frac{1}{2l + 1} \rho \cdot \psi_- \psi_-, \psi_-)_{L^2} - (2l + 2) |\psi_+|^2_{L^2}, \]

\[ = |\psi_-|^2_{L^2} \left( 8l(l + 1) - \frac{(8l + 2)(8l + 4)}{8} + \frac{i}{2} \left( -i(8l + 4) - 2l \right) \right) = 0, \]

thus proving that \( F = 0 \) and consequently (31). In order to check the last two equations one has to make use of the operator \( \tilde{D} \). From \( D_- \psi_- = 0 \) we find

\[ 0 = \frac{1}{4l + 4} D^2 \psi_- = D_+ \psi_+, \tag{34} \]

so

\[ \tilde{D} \psi_+ = -iD \psi_+. \tag{35} \]

We take a local orthonormal frame \( e_i \) and write (using (1), (5), (28) and (35))

\[ 0 \leq \sum_{j=1}^{n} |\nabla e_j \psi_+ + \frac{1}{2} (e_j - iJ(e_j)) \psi_-|^2 \]

\[ = |\nabla \psi_+|^2 - \Re((D + i\tilde{D}) \psi_+ \psi_-) \]

\[ = \frac{1}{4} \sum_{j=1}^{n} ((\nabla e_j \psi_+ \psi_-) (e_j - iJ(e_j)) \psi_-) \]

\[ = |\nabla \psi_+|^2 - 2\Re(D \psi_+ \psi_-) + ((m - i\Omega) \cdot \psi_-, \psi_-) \]

\[ = |\nabla \psi_+|^2 - 8l |\psi_-|^2 + 4l |\psi_-|^2 := |G|^2 \]

Just as before, we compute the integral over \( M \) of the positive function \( |G|^2 \), namely

\[ |G|^2_{L^2} = |\nabla \psi_+|^2_{L^2} - 4l |\psi_-|^2_{L^2} \]

\[ = \left( D^2 \psi_+ - \frac{1}{4} \right) R \psi_+ + \frac{i}{2} \frac{1}{2l + 1} \rho \cdot \psi_- \psi_+ \right)_{L^2} - 4l |\psi_-|^2_{L^2} \]

\[ = \left( |\psi_+|^2_{L^2} \left( 16l(l + 1) - \frac{(8l + 2)(8l + 4)}{4} + \frac{i}{2} \left( -3i(8l + 4) - 2l + 1 \right) \right) \right) = 0, \]

thus proving \( G = 0 \). Consequently \( \nabla X \psi_+ + \frac{1}{2} (X - iJ(X)) \cdot \psi_- = 0, \forall X \in TM \), which is equivalent to (32) and (33).

Q.E.D.

The above proposition motivates the following
**Definition 4.2** A section $\psi$ of the spinor bundle of a given $\text{Spin}^c$ structure on a Kähler manifold $(M^{4l+2}, g, J)$ satisfying
\[
\nabla^A_X \psi = \frac{1}{2} X \cdot \psi + \frac{i}{2} J X \cdot \bar{\psi}, \quad \forall X \in TM
\]
is called a Kählerian Killing spinor.

Defining $\psi := \psi_+ - \psi_-$ we immediately obtain the

**Corollary 4.2** Let $C$ be a complex contact structure on a Kähler–Einstein manifold $(M^{4l+2}, g, J)$. Then the $\text{Spin}^c$ structure on $M$ with auxiliary bundle $L_C$ carries a Kählerian Killing spinor $\psi \in \Gamma(\Sigma_{2l+1} \oplus \Sigma_{2l+2} M)$.

## 5 Geometric consequences

We can now state the main application of the above results:

**Theorem 5.1** Let $M$ be a compact Kähler manifold of positive scalar curvature and complex dimension $4l + 1$. Then the following statements are equivalent:

(i) $M$ is the twistor space of some quaternionic Kähler manifold;

(ii) $M$ is Kähler-Einstein and admits a complex contact structure;

(iii) There exist a $\text{Spin}^c$ structure on $M$ with auxiliary bundle $\Sigma M$ such that $L_{\mathbb{C}^{2l+1}} \cong \Lambda^{4l+1,0} M$ and $\Sigma M$ carries a Kählerian Killing spinor $\psi \in \Gamma(\Sigma_{2l+1} \oplus \Sigma_{2l+2} M)$.

**Proof.** The implications (i)$\implies$ (ii) and (ii)$\implies$ (iii) follow directly from [13] and Corollary 4.2 respectively.

Suppose now that (iii) holds. The proof of (iii)$\implies$ (i) parallels that of [8]. We first show that $M$ is Kähler–Einstein. Let $\psi \in \Gamma(\Sigma_{2l+1} \oplus \Sigma_{2l+2} M)$ be a spinor field on $M$ which satisfies (36). Taking the covariant derivative with respect to an arbitrary vector field $Y$ we obtain
\[
\nabla^A_Y \nabla^A_X \psi = \frac{1}{4} (X \cdot Y + JX \cdot JY) \cdot \psi + \frac{i}{4} (X \cdot JY - JX \cdot Y) \cdot \bar{\psi} + \nabla^A_{\nabla^A_Y X} \psi, \quad (37)
\]

which easily implies
\[
\mathcal{R}^A_{Y,X} \psi = \frac{1}{2} (X \cdot Y + JX \cdot JY + 2g(X, Y)) \cdot \psi - ig(X, JY) \bar{\psi}. \quad (38)
\]
A local computation shows that the curvature operator $\mathcal{R}^A$ on the spinor bundle is given by the formula

$$\mathcal{R}^A = \mathcal{R} + \frac{i}{2} \omega,$$  \hspace{2cm} (39)

where $i \omega := -\frac{i}{2l+1} \rho$ is the curvature form of the auxiliary bundle $L$, and

$$\mathcal{R}_{X,Y} = \frac{1}{2} \sum_{j<k} R(X, Y, e_j, e_k) e_j \cdot e_k.$$  \hspace{2cm} (40)

in a local orthonormal frame \{\{e_1, ..., e_n\}. Using the first Bianchi identity for the curvature tensor one obtains ([2], p.16)

$$\sum_i e_i \cdot \mathcal{R}_{e_i,X} = \frac{1}{2} \text{Ric}(X),$$  \hspace{2cm} (41)

so, by (39) and (41),

$$\sum_j e_j \cdot \mathcal{R}^A_{e_j,X} \psi = \sum_j e_j \cdot (\mathcal{R}_{e_j,X} \psi + \frac{i}{2} \omega(e_j, X) \psi) = \frac{1}{2} \text{Ric}(X) \cdot \psi - \frac{i}{2} X \int \omega \cdot \psi.$$  \hspace{2cm} (42)

On the other hand, a straightforward computation using (38) and the fact that $\psi \in \Gamma(\Sigma_{2l+1} M \oplus \Sigma_{2l+2} M)$ yields

$$\sum_j e_j \cdot \mathcal{R}^A_{e_j,X} \psi = (4l + 2) X \cdot \psi + i JX \cdot \bar{\psi} + JX \cdot \Omega \cdot \psi$$

$$= (4l + 2) X \cdot \psi - 2i JX \cdot \psi,$$

which, together with (42), gives

$$\left(\frac{1}{2} \text{Ric}(X) - (4l + 2) X \right) \cdot \psi = \frac{i}{2l+1} J \left(\frac{1}{2} \text{Ric}(X) - (4l + 2) X \right) \cdot \psi.$$  \hspace{2cm} (43)

As $\psi$ never vanishes, if the equality $A \cdot \psi = iB \cdot \psi$ holds for some real vectors $A, B$, then $|A| = |B|$. The above formula thus shows that $\text{Ric}(X) = (8l + 4) X$, $\forall X \in TM$, so $M$ is Kähler–Einstein with scalar curvature $R = (8l + 2)(8l + 4)$.

From Theorem 3.1 we deduce that the principal $\text{U}(1)$ bundle $\widetilde{M}$ associated to $L$ admits a canonical metric $\bar{g}$ and a canonical spin structure such that the spinor $\pi^* \psi$ induced by $\psi$ satisfies

$$\bar{\nabla}_X (\pi^* \psi) = \pi^* (\nabla^A_X \psi - \frac{i}{2} J(X) \cdot \bar{\psi}) = \pi^* \left(\frac{1}{2} X \cdot \psi\right), \quad \forall X \in TM,$$  \hspace{2cm} (44)

$$\bar{\nabla}_V \pi^* \psi = -\frac{1}{2} \pi^* (\Omega \cdot \psi + \frac{i(8l + 4)}{2(2l+1)} \psi) = \pi^* \left(\frac{i}{2} \bar{\psi}\right),$$  \hspace{2cm} (45)

and (10), (11) show that $\pi^* \psi$ is a Killing spinor on $\widetilde{M}$. 

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The spinor field $\pi^*\psi$ induces then a parallel spinor $\Psi$ on the cone $C\tilde{M}$ over $\tilde{M}$, which is a Kähler manifold (cf. [1], [8], [11]). Moreover, using (45) we can compute the action of the Kähler form of $C\tilde{M}$ on $\Psi$, and obtain that $\Psi \in \Sigma_{2l+3} C\tilde{M}$. From C. Bär’s classification [1] we know that the restricted holonomy group of $C\tilde{M}$ is one of the following: $\text{SU}(4l+2)$, $\text{Sp}(2l+1)$ or 0. The fixed points of the spin representation of $\text{SU}(4l+2)$ lie in $\Sigma_0$ and $\Sigma_{2l+2}$, so as $\Psi$ is a parallel spinor in $\Sigma_{2l+3} C\tilde{M}$, the restricted holonomy group of $C\tilde{M}$ cannot be equal to $\text{SU}(4l+2)$. This implies that the universal covering of $C\tilde{M}$ is hyperkähler, and thus that the universal covering of $\tilde{M}$ is 3–Sasakian (see [1]).

Let us denote by $\tilde{M}'$ the $U(1)$ bundle associated to some maximal root of $L$. Using the Gysin exact sequence we deduce that $\tilde{M}'$ is simply connected (see [2], p.85). Moreover, there exists a canonical covering projection $\tilde{M}' \to \tilde{M}$, thus proving that $\tilde{M}'$ is the universal covering of $\tilde{M}$. Consequently, $(\tilde{M}', \tilde{g}')$ is a 3–Sasakian manifold, where $\tilde{g}'$ is the metric induced from $\tilde{g}$ via the covering projection. On the other hand, the unit vertical vector field $V'$ on $\tilde{M}'$ defines a Sasakian structure, since this is true for its projection $V$ on $\tilde{M}$. It is well known that any Sasakian structure on a 3–Sasakian manifold $\mathbb{P}^{4k-1}$ of non-constant sectional curvature belongs to the 2–sphere of Sasakian structures. Indeed, the cone $CP$ over $P$ has restricted holonomy $\text{Sp}(k)$, and since the centraliser of $\text{Sp}(k)$ in $U(2k)$ is just $\text{Sp}(1)$, every Kähler structure on $CP$ must belong to the 2–sphere of Kähler structures of $CP$, which is equivalent to our statement.

Now, $\tilde{M}'$ is regular in the direction of $V'$, so an old result of Tanno implies that it is actually a regular 3–Sasakian manifold (cf. [14]). It is then well known that the quotient of $\tilde{M}'$ by the corresponding $\text{SO}(3)$ action is a quaternionic Kähler manifold of positive scalar curvature, say $N$, and that the twistor space over $N$ is biholomorphic to the quotient of $\tilde{M}'$ by each of the $S^1$ actions given by the Sasakian vector fields, so in particular to $M$, which is the quotient of $\tilde{M}'$ by the $S^1$ action generated by $V'$.

Q.E.D.

From Theorem A and Theorem 5.1 we immediately obtain the result of LeBrun mentioned in Section 1:

**Corollary 5.1** Let $Z$ be a Fano contact manifold. Then $Z$ is a twistor space iff it admits a Kähler–Einstein metric.

**References**


Institut für Reine Mathematik
Humboldt-Universität zu Berlin
Ziegelstr. 13a
D-10099 Berlin

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