Garside and locally Garside categories

François Digne, Jean Michel

GARSIDE AND LOCALLY GARSIDE CATEGORIES

F. DIGNE AND J. MICHEL

1. Introduction

This text is our version of (locally) Garside categories. Our motivation is the example of section 6, which we needed in September 2004 to understand some Deligne-Lusztig varieties. Since this example works naturally in the setting of arbitrary Coxeter groups, at that time we only considered the general case of categories which are locally Garside. Krammer has independently introduced the notion of (full) Garside categories [Krammer1], [Krammer2].

The things we added since 2004 are that we noticed that it makes sense to consider categories which are only left or right locally Garside, and that a sufficient condition to make things work is a Noetherianness property (before that, we imposed the homogeneity which comes from an additive length). We also added a discussion of the relation between our definitions and the notion of Garside categories, for which we use the definition introduced by Bessis [Be1]. We define what we call left Garside categories in this context; part of this reflects inspiring discussions we had with Bessis and with Krammer in april 2006.

The notion of Garside category has recently been used by Bessis [Be2] to obtain deep theorems about braid groups of complex reflection groups. We have a joint project with David Bessis and Daan Krammer to write a general survey about the subject. This text should be taken as our initial contribution to this project.

2. Locally Garside and Garside categories

We adopt conventions for categories which are consistent with those for monoids and with those in algebraic topology: we write xy for the composed of the morphisms $A \xrightarrow{x} B$ and $B \xrightarrow{y} C$. We consider only small categories. The morphisms have a natural preorder given by left divisibility: if $x = yz$, we say that y is a left divisor or a left factor of x, which we denote $y \preccurlyeq x$. We write $y \prec x$ if in addition $y \neq x$. And we say that x is a right multiple of y (we have also evidently the corresponding notions when exchanging left and right). We will write $x \in C$ to say that x is a *morphism* in C. We will write $A \in \mathrm{Obj}(C)$ to say that A is an object of C.

Definition 2.1. *We say that a preordered set is* Noetherian *if there does not exist any bounded infinite strictly increasing sequence.*

Notice that such a set is then a poset. We say that a category is left Noetherian if left divisibility induces a Noetherian poset on the morphisms. This notion also makes sense (and we will use it) for a subset of a category.

We call right lcm (resp. left gcd) a least upper bound (resp. largest lower bound) for left divisibility. An lcm is unique for a poset.

Date: December 26th, 2006.

Definition 2.2. *A (small) category C is left locally Garside if*

(i) *It is left Noetherian.*
(ii) *It has the left cancellation property, i.e. $xy = xz$ implies $y = z$ (in other words, every morphism is an epimorphism).*
(iii) *Two morphisms which have some common right multiple have a right lcm.*

A left locally Garside monoid is the monoid of morphisms of a left locally Garside category with only one object.

In a locally Garside category, complements are well defined, that is, when $x \preccurlyeq y$ there is a unique z, the *complement* to y of x, such that $x = yz$.

We say that a subcategory C_1 is stable by complement if $x, y \in C_1$ and $x \preccurlyeq y$ in C implies that $x \preccurlyeq y$ in C_1.

Given a subcategory stable by complement, we say it is stable by lcm if when $x, y \in C_1$ have an lcm z in C then z is in C_1. From the stability by complement it follows that z is an lcm in C_1. The following lemma is a transposition to categories of a result of [Godelle].

Lemma 2.3. *Let C be a left locally Garside category; if C_1 is a subcategory stable by complement and by lcm, then C_1 is left locally Garside.*

Proof. Axioms 2.2(i) and 2.2(ii) for C_1 are inherited from the same axioms for C. Let us check axiom 2.2(iii). If two morphisms have a common right multiple in C_1 they have an lcm in C which is an lcm in C_1 by assumption. □

Before going further, let us look at consequences and equivalent ways of formulating the third axiom in the setting of posets.

Lemma 2.4. *In a Noetherian poset where any two elements which have an an upper bound have a least upper bound, the same holds for any family of elements.*

Proof. Let $\mathcal{F}$ be a bounded above subset of the poset. By Noetherianity there exist maximal elements in $\mathcal{F}$. As two maximal elements of $\mathcal{F}$ have a least upper bound by assumption, they have to be equal, so there is a unique maximal element, which is a least upper bound for $\mathcal{F}$. □

Lemma 2.5. *In a Noetherian poset which has a minimal element and where any two elements which have an an upper bound have a least upper bound, any family of elements has a largest lower bound.*

Proof. Let $\mathcal{F}$ be the family in the statement. The elements of $\mathcal{F}$ have at least one lower bound, the minimal element of the poset. Now we can apply the previous lemma to the family of the lower bounds of $\mathcal{F}$, which have thus a least upper bound. This least upper bound is smaller than all the elements of $\mathcal{F}$, so is a largest lower bound for $\mathcal{F}$. □

In the context of left divisibility, we get that axiom 2.2(iii) implies

Corollary 2.6. *In a left locally Garside category, a family of morphisms with the same source have a left gcd.*

There are contexts where the above corollary is equivalent to the third axiom.

Definition 2.7. *We say that a poset is* Artinian *or if there does not exist any bounded infinite strictly decreasing sequence.*

We note that for left divisibility any family of morphisms with same source has always a lower bound, the identity morphism. It follows from the previous lemmas, using the reverse order, that

Corollary 2.8. *A category which satisfies axioms 2.2(i) and (ii), which is left Artinian and such that any two morphisms with same source have a left gcd, is left locally Garside (i.e. satisfies 2.2(iii)).*

The notion of right locally Garside category is obtained by exchanging left and right in the definition of a left locally Garside category.

A locally Garside category is a category which is both locally left Garside and locally right Garside.

Lemma 2.9. *A subset of a category stable by left divisibility, left Noetherian and which verifies a weak form of the left cancellation property, that is $xy = x$ implies $y = 1$, is also* right Artinian.

Proof. If a_n is a strictly decreasing sequence for the order $\succcurlyeq$ and if $a_n = b_n a_{n+1}$ then $c_n = b_1 b_2 \dots b_n$ is an increasing sequence for $\preccurlyeq$ all terms of which left divide a_1. It is strictly increasing since $b_1 \dots b_{n-1} = b_1 \dots b_n \Rightarrow b_n = 1 \Rightarrow a_n = a_{n+1}$, the first implication by the weak form of left cancellation. □

It follows from 2.8 and 2.9 that for a locally Garside category we can replace axiom 2.2(iii) by the existence of a left gcd for all pairs of morphisms with same source and similarly on the right.

The three following definitions are adaptations to one-sided Garside categories of the definitions of [Be1]. Definition 2.12 is almost equivalent to [Be1, 2.5].

Definition 2.10. *A left Garside category C is a left locally Garside category such that there exists an endofunctor Φ of C and a natural transformation Δ from the identity functor to Φ such that the set of left divisors of Δ generate C.*

We denote by $A \xrightarrow{\Delta_A} \Phi(A)$ the natural transformation applied to the object A; in the above the left divisors of Δ mean the divisors of the various Δ_A as A runs over the objects of C.

For right Garside, we change also the direction of the natural transformation.

Definition 2.11. *A right Garside category C is a right locally Garside category such that there exists an endofunctor Φ of C and a natural transformation Δ from Φ to the identity functor such that the set of right divisors of Δ generate C.*

Finally, we define Garside:

Definition 2.12. *A Garside category C is a right and left Garside category such that the functor Φ for the right Garside structure is the inverse of Φ for the left Garside structure, and such that the left and right Δ coincide.*

By saying that the right and left Δ coincide, we mean that Δ_A for the left Garside structure is the same as $\Delta_{\Phi(A)}$ for the right Garside structure.

We will show (*cf.* 5.4) that a left Garside category which is right Noetherian and such that Φ has an inverse is Garside.

In the case of a Garside monoid identified with the endomorphisms of a one-object Garside category, the functor Φ is the conjugation by the element Δ of the monoid.

3. Germs for locally Garside categories

We introduce a convenient technique for constructing locally Garside categories by introducing the notion of a *germ*, which is some kind of generating set for categories, and giving conditions on a germ for the generated category to be locally Garside. This section is an adaptation in the context of categories of section 2 of [Bessis-Digne-Michel]; the main technical difference being that here we assume neither atomicity nor the existence of a length function: they are replaced by the Noethianness property.

Definition 3.1. *A germ $(P, \mathcal{O})$ is a pair consisting of a set $\mathcal{O}$ of objects, and a set P of morphisms (which have a source and a target, which are objects), with a partially defined "composition" map $m : P \times P \to P$. For $a, b \in P$ we will write "$ab \in P$" to mean that $m(a, b)$ is defined; and in this situation we denote ab for $m(a, b)$; we abbreviate $ab \in P$ and $c = ab$ to $c = ab \in P$. If we denote by $P(A, B)$ the set of morphisms in $\mathcal{O}$ of source A and target B, we require the following axioms:*

(i) *For all $A \in \mathcal{O}$, there exists $1_A \in P(A, A)$ such that for any $a \in P(B, A)$ (resp. any $a' \in P(A, B)$) $a = a.1_A \in P$ (resp. $a' = 1_A.a' \in P$).*
(ii) *For $a, b, c \in P$, we have $ab, (ab)c \in P$ if and only if $bc, a(bc) \in P$ and in this case $a(bc) = (ab)c$.*

We will write 1 instead of 1_A when the context makes clear that the source of this morphism is A. A *path* in P is a sequence of morphisms $(p_1, \ldots, p_n)$ such that the target of p_i is the source of p_{i+1}. If $(x_1, \ldots, x_n)$ is a path such that for some bracketing of this sequence the product $x_1 \ldots x_n$ is defined in P, then by axiom 3.1 (ii) the product is also defined, and has the same value, for any bracketing of the sequence. We will denote by $x_1 \ldots x_n \in P$ this situation (and $x_1 \ldots x_n$ the product when this situation occurs).

Definition 3.2. *The category generated by the germ $(P, \mathcal{O})$ is the category with objects $\mathcal{O}$ defined by generators and relations as follows: the generators are P, and the relations are $ab = c$ whenever $c = ab \in P$.*

We write $C(P, \mathcal{O})$, or $C(P)$ when there is no ambiguity, for the category generated by the germ $(P, \mathcal{O})$. We can give an explicit model for the morphisms of $C(P)$ in terms of equivalences classes of paths in P. The equivalence relations between paths is generated by the elementary equivalences:

$$(p_1, \ldots, p_{i_1}, p_i, p_{i+1}, \ldots, p_n) \sim (p_1, \ldots, p_{i_1}, p'p'', p_{i+1}, \ldots, p_n)$$

when $p_i = p'p'' \in P$, and $(1) \sim ()$.

The composition of morphisms in $C(P)$ is defined by the concatenation of paths. The next lemma shows that this extends the partial product in P.

Lemma 3.3. *Let $(x_1, \ldots, x_n)$ be a path equivalent to the the single-term path (y). Then $x_1 \ldots x_n \in P$ and $x_1 \ldots x_n = y$.*

Proof. The assumption implies that there exists a sequence of elementary equivalences

$$l_0 = (x_1, \ldots, x_n) \sim l_1 \sim \cdots \sim l_k = (y)$$

where each equivalence $l_{j-1} \to l_j$ is either

- a contraction

$$(p_1, \ldots, p_{i-1}, p', p'', p_{i+1}, \ldots, p_m) \to (p_1, \ldots, p_{i-1}, p'p'', p_{i+1}, \ldots, p_m)$$

- an expansion
$$(p_1,\dots,p_{i-1},p'p'',p_{i+1},\dots,p_m)\to(p_1,\dots,p_{i-1},p',p'',p_{i+1},\dots,p_m)$$

where $p'p''\in P$. We may assume that k is minimal. If there is any expansion in the sequence, let $(p_1,\dots,p_{i-1},p'p'',p_{i+1},\dots,p_m)\to(p_1,\dots,p_{i-1},p',p'',p_{i+1},\dots,p_m)$ be the last one. Since all subsequent steps are contractions we have m subsequent steps and $y=p_1\dots p_{i-1}p'p''p_{i+1}\dots p_m\in P$. Since any bracketing of the sequence $(p_1,\dots,p_{i-1},p',p'',p_{i+1},\dots,p_m)$ has the same value, we see that we could start with the bracketing $\dots(p'p'')\dots$, and thus get from $(p_1,\dots,p_{i-1},p'p'',p_{i+1},\dots,p_m)$ to (y) in $m-1$ steps instead of $m+1$ whence a contradiction unless there are only contractions in a minimal sequence of equivalences leading to (y), whence the result. □

We have the following

Corollary 3.4. *P identifies with a subset of $C(P)$ stable by taking left or right factors.*

Proof. Indeed, if two morphisms of P are equal in $C(P)$ the above lemma (using the particular case $n=1$) shows that they are equal in P. And a left or right factor in $C(P)$ of y in the above lemma is a product $x_1\dots x_i$ or $x_i\dots x_n$ and is thus in P. □

Just as for a category, we say that a germ $(P,\mathcal{O})$ is left Noetherian if left divisibility induces a Noetherian poset on P.

Let us remark that a germ with a superadditive length, that is, a function $P\xrightarrow{l}\mathbb{Z}_{\geq 0}$ such that $l(ab)\geq l(a)+l(b)$ and $l(a)=0\Leftrightarrow a=1$ is left and right Noetherian.

Lemma 3.5. *Let C be a category and P be a set of morphisms which generates C. Let X be a set of morphisms of C with same source satisfying*

(i) *X is stable by taking left factors,*
(ii) *X is a bounded Noetherian poset for left divisibility,*
(iii) *If $x\in X$, $y,z\in P$ and $xy,xz\in X$ then y and z have a common right multiple m such that $xm\in X$.*

Then X is the set of left divisors of some morphism of C.

Proof. Since X is a bounded Noetherian poset for $\preccurlyeq$, there exists a maximal element $g\in X$ for $\preccurlyeq$. Let us prove by contradiction that X is the set of left divisors of g. First we notice that otherwise $E=\{x\prec g\mid \exists u\in P, xu\in X, xu\not\preccurlyeq g\}$ is not empty: indeed let $y\in X$ be such that $y\not\preccurlyeq g$ and let x be a maximal common factor of y and g; then x in E, since if we write $y=xu_1\dots u_k$ with $u_i\in P$ and k minimal, then $k\neq 0$ and $xu_1\neq x$, thus $xu_1\not\preccurlyeq g$ (by maximality of x). Let now $x\in E$ be maximal for $\preccurlyeq$ and let u be as in the definition of E. As $x\prec g$, there is $v\in P$ such that $x\prec xv\preccurlyeq g$. As xu and xv are both in X, the assumption on X implies that u and v have a common multiple m such that $xm\in X$. As xm is a right multiple of xu we have $xm\not\preccurlyeq g$. Thus if v' is a maximal such that $v\preccurlyeq v'\preccurlyeq m$ and $xv'\preccurlyeq g$, we have $x\prec xv'\prec xm$ and $xv'\in E$ (taking for the u in the definition of E any element of P such that $v'u\preccurlyeq m$), which contradicts the maximality of x. □

Definition 3.6. *A germ $(P,\mathcal{O})$ is left locally Garside if*

(G1) *It is left Noetherian.*

(G2) *If two morphisms in P have a common right multiple in P, they have a right lcm in P.*
(G3) *If two morphisms $u, v \in P$ have a right lcm $\Delta_{u,v} \in P$ and if $x \in P$ is such that $xu, xv \in P$ then $x\Delta_{u,v} \in P$.*
(G4) *For $z \in C(P)$ and $x, y \in P$, the equality $zx = zy$ implies $x = y$.*

Remark 3.7. We note that 3.6(G4) is the only axiom which does not involve only a check on elements of P. However, in practical applications, it will be easy to check since it is automatically verified if there is an injective map compatible with multiplication from P into a category with the left cancellation property.

We have a weak form of right cancellation

Lemma 3.8. *If P is a germ satisfying (G1) and (G4) then the equality $xy = y \in P$ implies $x = 1$.*

Proof. From $xy = y$ we deduce that for all n we have $x^n y = y$, so x^n is an increasing sequence for $\preccurlyeq$ which is bounded by y so has to be constant for n large enough by (G1). But $x^n = x^{n+1}$ implies $x = 1$ by (G4). □

We will show that the category generated by a locally Garside germ is locally Garside by directly constructing normal forms for elements of $C(P)$. We fix now a locally Garside germ $(P, \mathcal{O})$.

Proposition 3.9. *Any family of morphisms in P with same source have a left gcd.*

Proof. This is a consequence of lemma 2.5, whose assumption is true by 3.6(G2). □

Proposition 3.10. *If $x, y \in P$ are such that the target of x is the source of y, then there is a unique maximal z such that $z \preccurlyeq y$ and $xz \in P$.*

Proof. This time we apply lemma 3.5 to the set X of u such that $u \preccurlyeq y$ and $xu \in P$. X inherits left Noetherianity from P thus it is enough to check that if $u, v, w \in P$ are such that $uv, uw \in X$ then they have a right lcm $\Delta_{v,w}$ and $xu\Delta_{v,w} \in P$ (which will imply $u\Delta_{v,w} \in X$). As uv and uw are left factors of $y \in P$, by axiom 3.6 (G4) they have a common multiple, in P by corollary 3.4, thus by 3.6 (G2) they have a lcm $\Delta_{v,w}$ which by axiom 3.6 (G3) satisfies $xu\Delta_{v,w} \in P$. □

Definition 3.11. *Under the assumptions of proposition 3.10 we set $\alpha_2(x, y) = xz$ and we write $\omega_2(x, y)$ for the morphism $t \in P$ (unique by axiom 3.6 (G4)) such that $y = zt$. Thus $xy = \alpha_2(x, y)\omega_2(x, y)$.*

Proposition 3.12. *For $x, y, z, xy \in P$ we have*
(i) $\alpha_2(xy, z) = \alpha_2(x, \alpha_2(y, z))$.
(ii) $\omega_2(xy, z) = \omega_2(x, \alpha_2(y, z))\omega_2(y, z)$.

Proof. Let us show (i). Define $u, v \in P$ by $\alpha_2(xy, z) = xyu$ and $\alpha_2(y, z) = yv$. As $yu \in P$, $u \preccurlyeq z$, we have by definition $yu \preccurlyeq yv$. Similarly $xyu \preccurlyeq \alpha_2(x, yv) = \alpha_2(x, \alpha_2(y, z))$; let thus $u' \in P$ be such that $\alpha_2(x, \alpha_2(y, z)) = xyuu'$. Since $xyuu' \preccurlyeq xyv$ by axiom 3.6 (G4) have $uu' \preccurlyeq v$; as $v \preccurlyeq z$ we have $uu' \preccurlyeq z$, and as $xyuu' \in P$, we have $u' = 1$ by maximality of u in the definition of $\alpha_2(xy, z)$, which gives (i).

Let us show (ii). Using $xyz = \alpha_2(xy, z)\omega_2(xy, z)$ and

$$\alpha_2(xy, z)\omega_2(x, \alpha_2(y, z))\omega_2(y, z) = \alpha_2(x, \alpha_2(y, z))\omega_2(x, \alpha_2(y, z))\omega_2(y, z) = x\alpha_2(y, z)\omega_2(y, z) = xyz,$$

which comes from (i) and definition 3.11, we will get (ii) if we can simplify $\alpha_2(xy, z)$ between these two expressions for xyz. We apply axiom 3.6 (G4) if we show that both sides of (ii) lie in P. It is the case for $\omega_2(xy, z) \in P$ by definition, thus we have to show that $\omega_2(x\alpha_2(y, z))\omega_2(y, z) \in P$. Define $u \in P$ by $\alpha_2(y, z) = yu$, so that $u\omega_2(y, z) = z$, and $u_1 \in P$ by $\alpha_2(x, \alpha_2(y, z)) = xyu_1$, so that $xyu_1\omega_2(x, \alpha_2(y, z)) = x\alpha_2(y, z) = xyu$. Then $u_1\omega_2(x, \alpha_2(y, z)) \in P$ as it is a right factor of $\alpha_2(y, z) \in P$, thus $u_1\omega_2(x, \alpha_2(y, z)) = u$ (by axiom 3.6(G4)) thus $u_1\omega_2(x, \alpha_2(y, z))\omega_2(y, z) = u\omega_2(y, z) = z$ which implies that $\omega_2(x\alpha_2(y, z))\omega_2(y, z) \in P$ as it is a right factor of an element of P. □

Proposition 3.13. *There is a unique map $\alpha : C(P) \to P$ which is the identity on P, such that for $x, y \in P$ we have $\alpha(xy) = \alpha_2(x, y)$, and such that for any $u, v \in C(P)$ we have $\alpha(uv) = \alpha(u\alpha(v))$. In addition $\alpha(u)$ is the unique maximal left factor in P of u.*

Proof. We will define α on the paths in P, and then check that our definition is compatible with elementary equivalence. As α is the identity on P, we need that $\alpha(()) = 1$ and that $\alpha((y)) = y$ for $y \in P$. The conditions we want impose that

$$\alpha(p_1, \dots, p_k) = \alpha_2(p_1, \alpha(p_2, \dots, p_k)). \tag{3.14}$$

By induction on k, this already shows that α is unique. We will now show by induction on k that α is compatible with the elementary equivalence $(p_1, \dots, p_k) \sim (p_1, \dots, p_ip_{i+1}, \dots, p_k)$ when $p_ip_{i+1} \in P$. If this equivalence is applied at a position $i > 1$, formula 3.14 shows that compatibility for paths of length $k-1$ implies compatibility for paths of length k. If $i = 1$ we have to compare $\alpha_2(p_1, \alpha(p_2, \dots, p_k))$ and $\alpha_2(p_1p_2, \alpha(p_3, \dots, p_k))$. But $\alpha_2(p_1p_2, \alpha(p_3, \dots, p_k)) = \alpha_2(p_1, \alpha_2(p_2, \alpha(p_3, \dots, p_k)))$, by 3.12 (i) and $\alpha_2(p_2, \alpha(p_3, \dots, p_k)) = \alpha(p_2, p_3, \dots, p_k)$ by 3.14, whence the result that α is well defined by 3.14 on $C(P)$.

Similarly, if $u = (u_1, \dots, u_m)$ and $v = (v_1, \dots, v_n)$, we show that $\alpha(uv) = \alpha(u\alpha(v))$ by induction on $m + n$. Indeed

$$\alpha(u_1, \dots, u_m, v_1, \dots, v_n) = \alpha_2(u_1, \alpha(u_2, \dots, u_m, v_1, \dots, v_n)) = \\ \alpha_2(u_1, \alpha(u_2, \dots, u_m, \alpha(v))) = \alpha(u_1, u_2, \dots, u_m, \alpha(v)),$$

by respectively 3.14, the induction hypothesis, and 3.14 again.

Finally we show that $\alpha(u)$ is the maximal left factor in P of u. It is by definition an element of P which is a left factor of u. If we have another expression $u = pv$ with $p \in P$ then $\alpha(u) = \alpha(pv) = \alpha(p\alpha(v)) = \alpha_2(p, \alpha(v))$ so $\alpha(u)$ is a right multiple of p. □

Proposition 3.15. *There is a unique map $\omega : C(P) \to C(P)$ such that for $x, y \in P$ we have $\omega(xy) = \omega_2(x, y)$, and such that for any $u, v \in C(P)$ we have $\omega(uv) = \omega(u\alpha(v))\omega(v)$.*

Proof. As in the previous proposition we define ω on paths by induction. We must have $\omega(x) = 1$ for $x \in P$ and for a path of length $k \geq 2$ we must have

$$\omega(p_1, \dots, p_k) = \omega_2(p_1, \alpha(p_2 \dots p_k))\omega(p_2, \dots, p_k) \tag{3.16}$$

This proves the unicity of ω, and again we show by induction on k that this is compatible with elementary equivalence. Again, we come to the case of an elementary equivalence occurring in the first term, *i.e.*, to compare $\omega(p_1, p_2, \dots, p_k)$ and

$\omega(p_1p_2, p_3, \ldots, p_k)$ when $p_1p_2 \in P$. We have

$$\omega(p_1, \ldots, p_k) = \omega_2(p_1, \alpha(p_2 \ldots p_k))\omega(p_2, \ldots, p_k) =$$
$$\omega_2(p_1, \alpha(p_2 \ldots p_k))\omega_2(p_2, \alpha(p_3 \ldots p_k))\omega(p_3, \ldots, p_k) =$$
$$\omega_2(p_1p_2, \alpha(p_3 \ldots p_k))\omega(p_3, \ldots, p_k) = \omega(p_1p_2, p_3, \ldots, p_k)$$

by respectively 3.16, 3.16, 3.12 (ii) and 3.16 whence the result.

We show similarly for $u = (u_l, u_2, \ldots, u_m)$ and $v = (v_1, \ldots, v_n)$ that $\omega(uv) = \omega(u\alpha(v))\omega(v)$ by induction on $m + n$. We have

$$\begin{aligned}
\omega(uv) &= \omega(u_1, \ldots, u_m, v_1, \ldots, v_m)\\
&= \omega_2(u_1, \alpha(u_2 \ldots u_m v_1 \ldots v_n))\omega(u_2, \ldots, u_m, v_1, \ldots, v_n)\\
&= \omega_2(u_1, \alpha(u_2 \ldots u_m v_1 \ldots v_n))\omega(u_2, \ldots, u_m, \alpha(v_1 \ldots v_n))\omega(v_1, \ldots, v_n)\\
&= \omega_2(u_1, \alpha(u_2 \ldots u_m \alpha(v_1 \ldots v_n)))\omega(u_2, \ldots, u_m, \alpha(v_1 \ldots v_n))\omega(v_1, \ldots, v_n)\\
&= \omega(u_1, u_2, \ldots, u_m, \alpha(v_1 \ldots v_n))\omega(v_1, \ldots, v_n),
\end{aligned}$$

by respectively 3.16, the induction hypothesis, 3.13 and 3.16, whence the result. □

We are now ready to define normal forms for morphisms in $C(P)$.

Definition 3.17. *We call* normal form *of a morphism $x \in C(P), x \neq 1$ a decomposition $x = x_1 \ldots x_k$ such that $x_i \in P, x_k \neq 1$ and such that for all i we have $x_i = \alpha(x_i \ldots x_k)$.*

We notice that we have $x_i \neq 1$ for all i since an element $\neq 1$ has a non-trivial α. We declare that the normal form of 1 is the trivial decomposition ($k = 0$).

We will show the existence of normal forms in 3.20 and their unicity in 3.23. We first show another characterization.

Proposition 3.18. *A decomposition $x_1 \ldots x_k$ where $x_i \in P, x_k \neq 1$ is a normal form if and only if for all $i \leq k-1$ the decomposition $x_i x_{i+1}$ is a normal form.*

Proof. If $x_1, \ldots, x_k$ is a normal form then $x_i = \alpha(x_i \ldots x_k) = \alpha(x_i\alpha(x_{i+1} \ldots x_k)) = \alpha(x_i x_{i+1})$ thus $x_i x_{i+1}$ is a normal form. Conversely, assuming by decreasing induction on i, that $\alpha(x_{i+1} \ldots x_k) = x_{i+1}$, we have $\alpha(x_i \ldots x_k) = \alpha(x_i\alpha(x_{i+1} \ldots x_k)) = \alpha(x_i x_{i+1}) = x_i$. □

The above statement implies that any product of consecutive terms in a normal forms is itself a normal form.

Proposition 3.19. *If $x_1 \ldots x_k$ is a normal form and $y \in P$, there exist decompositions $x_i = x'_i x''_i$ such that either $(yx'_1)(x''_1 x'_2) \ldots (x''_{k-1} x'_k)x''_k$, if $x''_k \neq 1$, or $(yx'_1)(x''_1 x'_2) \ldots (x''_{k-1} x'_k)$ otherwise, is a normal form of $yx_1 \ldots x_k$.*

Proof. This is obtained by recursively applying $\alpha(ab) = \alpha(a\alpha(b))$: we first write $\alpha(yx) = yx'_1$ where $x_1 = x'_1 x''_1$, then $\alpha(x''_1 x_2 \ldots x_k) = \alpha(x''_1\alpha(x_2 \ldots x_k)) = \alpha(x''_1 x_2) = x''_1 x'_2$ etc... □

Corollary 3.20. *Normal forms exist.*

Proof. We proceed by induction on k for an $x \in C(P)$ of the form $x = p_1 \ldots p_k$ with $p_i \in P$. By induction we may as well assume that $p_2 \ldots p_k$ is a normal form. The previous proposition shows then how to construct a normal form for x. □

We now show that $C(P)$ has the let cancellation property. We will deduce it from the following property of ω.

Proposition 3.21. *If $x \in C(P)$, then $\omega(x)$ is the unique $y \in C(P)$ such that $x = \alpha(x)y$.*

Proof. We show that $x = \alpha(x)y$ implies $y = \omega(x)$ by induction on the number of terms in a normal form of y. If $y = 1$ then $x \in P$ and the result holds. The assumptions at step k are now that $x = \alpha(x)y$ and that some decomposition $y = y_1 \dots y_k$ is a normal form. By the induction hypothesis and the equality $y = y_1(y_2 \dots y_k)$ we get $\omega(y) = y_2 \dots y_k$, thus $y = \alpha(y)\omega(y)$. Thus $\omega(x) = \omega(\alpha(x)y) = \omega(\alpha(x)\alpha(y))\omega(y)$. On the other hand $\alpha(x) = \alpha(\alpha(x)y) = \alpha(\alpha(x)\alpha(y)) = \alpha_2(\alpha(x), \alpha(y))$. Thus $\omega(\alpha(x)\alpha(y)) = \omega_2(\alpha(x), \alpha(y)) = \alpha(y)$ where the last equality is by definition of ω_2. Putting things together we get $\omega(x) = \alpha(y)\omega(y) = y$. □

Corollary 3.22. *$C(P)$ has the left cancellation property.*

Proof. We want to show that for any $x, y, z \in C(p)$ the equality $xy = xz$ implies $y = z$. By induction on the number of terms in a decomposition of x into a product of elements of P, we may assume that $x \in P$. Define b by $\alpha(xy) = xb$; then b is unique since P has the left cancellation property. Let y' be an element such that $by' = y$, which is possible since $b \preccurlyeq \alpha(y) \preccurlyeq y$. By proposition 3.21 we have $y' = \omega(xy)$. Thus if similarly we define z' as an element such that $bz' = z$ we have $z' = \omega(xy) = y'$ thus $z = bz' = by' = y$. □

Corollary 3.23. *Normal forms are unique.*

Proof. If $x = x_1 \dots x_k$ is a normal form then $x_1 = \alpha(x)$ is uniquely defined and by proposition 3.21 we have $x_2 \dots x_k = \omega(x)$; we conclude by induction on k. □

For $x \in C(P)$, we denote by $\nu(x)$ the minimum number of terms in a decomposition of x into a product of elements of P.

Lemma 3.24. *The normal form of x has $\nu(x)$ terms.*

Proof. The proof is by induction on $\nu(x)$. We assume the result for $\nu(x) = k - 1$ and will prove it for $\nu(x) = k$. Let then $x = x_1 \dots x_k$ be a minimal decomposition of x. By induction, the normal form of $x_2 \dots x_k$ has $k - 1$ terms, so we may as well assume that $x_2 \dots x_k$ is normal. By lemma 3.19 the normal form of x has $k - 1$ or k terms. Thus it has k terms, whence the result. □

Lemma 3.25. *If x is a right factor of y then $\nu(x) \leq \nu(y)$.*

Proof. Since an element can be obtained from a right factor by repeatedly multiplying on the left by elements of P, proposition 3.19 shows that a right factor has less terms in its normal form. □

If $x \in C(P)$ has normal form $x = x_1 \dots x_n$ we have $\omega^k(x) = x_{k+1} \dots x_n$ for $k \geq 0$ (it is 1 if $k \geq n$), and the k-th term of the normal form of x is $\alpha(\omega^{k-1}(x))$.

Lemma 3.26. *For $a \in P$ and $x \in C(P)$, if $\omega^k(x) = \omega^k(ax)$ for some k then $\alpha(\omega^{k-1}(ax)) \succcurlyeq \alpha(\omega^{k-1}(x))$.*

Proof. In this proof (only) we will still call normal form a product with a certain number of trailing 1's. Let $x = x_1 \dots x_n$ be a normal form of x. Then by proposition 3.19 which is still valid with our present definition of normal forms, we can write $x_i = x_i' x_i''$ for all i, so that $(ax_1')(x_1''x_2') \dots (x_{n-1}''x_n')x_n''$ is a normal form of ax. By assumption $(x_k''x_{k+1}') \dots (x_{n-1}''x_n')x_n'' = \omega^k(ax) = \omega^k(x) = x_{k+1} \dots x_n$, so identifying these two normal forms we get $x_n'' = 1$ and $x_i''x_{i+1}' = x_{i+1} = x_{i+1}'x_{i+1}''$ for $i \geq k$, whence by decreasing induction on i and lemma 3.8 we get $x_i'' = 1$ so that $x_i = x_i'$ for $i \geq k$. Whence $\alpha(\omega^{k-1}(ax)) = x_{k-1}''x_k' = x_{k-1}''x_k = x_{k-1}''\alpha(\omega^{k-1}(x))$. □

Proposition 3.27. *$C(P)$ is Noetherian for left divisibility.*

Proof. We have to show that no infinite sequence $x_1 \prec x_2 \prec x_3 \cdots \prec x_n \prec \cdots \preccurlyeq x$ exists; we proceed by induction on $\nu(x)$. If $\nu(x) = 1$ the sequence consists of elements of P which contradicts the Noetherianity of P. The sequence $\alpha(x_i)$ is non-decreasing and bounded by $\alpha(x)$ so is constant at some stage by the Noetherianity of P. Truncating the previous terms and simplifying by the common value a_1 of $\alpha(x_i)$, we get a an infinite sequence bounded by $a_1^{-1}x$. If $\nu(a_1^{-1}x) < \nu(x)$ then we are done by induction. Otherwise we can repeat the same argument for another step, introducing the common value a_2 of $\alpha(a_1^{-1}x_i)$, etc...; after k such steps we will still have $\nu((a_1 \dots a_k)^{-1}x) = \nu(x)$. But this implies by lemma 3.26 that $\omega^{\nu(x)-1}((a_1 \dots a_h)^{-1}x)$ is a decreasing sequence of elements of P, so it has to be constant at some stage. Truncating at this stage we may assume that the last term of the normal form of $\nu((a_1 \dots a_h)^{-1}x)$ is equal to the last term of the normal form of x. Lemma 3.26 gives then that $\alpha(\omega^{\nu(x)-2}((a_1 \dots a_h)^{-1}x))$ is decreasing for right divisibility so has to be constant at some stage. Truncating again we can assume that in the whole process the last two terms of the normal form of $(a_1 \dots a_h)^{-1}x$ are constant. Going on we come to a point where $(a_1 \dots a_h)^{-1}x$ itself is constant which means, again by 3.26, that $a_h = 1$ for h large enough. □

Proposition 3.28. *If two elements of P have a common right multiple in $C(P)$ then they have a right lcm in P (which is also their lcm in $C(P)$.*

Proof. We first observe that if u, v have a common multiple $x \in C(P)$ they have a common multiple in P, which is $\alpha(x)$. We may then apply 3.6(G2) to conclude. □

Proposition 3.29. *Any family of elements of $C(P)$ who have a common right multiple has a right lcm. If the family is a subset of P then the lcm is in P.*

Proof. Assume $(x_i)_{i \in I}$ have a common right multiple. We apply lemma 3.5 to the set X of elements which divide all the common multiples of the x_i. It inherits Noetherianity from $C(P)$, and if $u, v \in P$ and $x \in C(P)$ are such that $xu, xv \in X$ then any right multiple of xu and xv is of the form xz where u and v divide z (by the cancellation property in $C(P)$); thus the right lcm $\Delta_{u,v}$ of u and v (which exists by 3.28) divides z. Thus $x\Delta_{u,v} \in X$. We may thus apply 3.5 and the elements of X are the divisors of an element which must be the lcm of the x_i. The second statement comes from the fact that if the x_i are in P and divide x then they divide $\alpha(x)$ and so does their lcm. □

Proposition 3.30. *Any family of morphisms in $C(P)$ has a left gcd.*

Proof. This is a consequence of lemma 2.5, whose assumption is true by 3.29. □

At this stage we have proved the following

Theorem 3.31. *If the germ P is left locally Garside, so is $C(P)$.*

This has a converse:

Theorem 3.32. *Let C be a left locally Garside category C. Then $(P, \mathcal{O})$, where $\mathcal{O}$ is the set of objects of C and where P is a set of morphisms of C which generate C, stable by taking left factors and right factors, and stable by taking right lcm when they exist, is a left locally Garside germ; for this germ, $C = C(P)$.*

Proof. We first check that a P such as above is a germ. Axiom (i) of a germ is clear. Axiom (ii) ("associativity") holds for a set of morphisms as soon as they are stable by taking left and right factors. The axioms for a locally Garside germ 3.6 are immediate except perhaps axiom 3.6 (G3) (for 3.6(G4) see remark 3.7). If $u, v \in P$ have a right lcm $\Delta_{u,v}$ and if $x \in P$ is such that $xu, xv \in P$ then $x\Delta_{u,v}$ is the right lcm in C of xu and xv thus is in P.

All the relations of $C(P)$ hold in C, thus we have a functor $C(P) \xrightarrow{i} C$ which is clearly surjective since P generates C. We have to see that i is injective.

Let us define a function $\alpha : C \to P$ defined for $x \in C$ by taking the largest (for left divisibility) factor of x in P; this exists, since P is stable by right lcm. The formula $\alpha(xy) = \alpha(x\alpha(y))$ holds when $x \in P$ since $\alpha(xy)$ is by definition of the form $xa \in P$ where $xa \preccurlyeq xy$ thus $a \preccurlyeq y$ by the cancellation property thus $a \preccurlyeq \alpha(y)$.

To see that i is injective it is enough to show that $i \circ \alpha = \alpha \circ i$; indeed, by induction on $\nu(x)$ for $x \in C(P)$ we have $i(x) = i(y) \Rightarrow \alpha(i(x)) = \alpha(i(y)) \Rightarrow i(\alpha(x)) = i(\alpha(y)) \Rightarrow \alpha(x) = \alpha(y)$, the last equality since i is injective on P. By the left cancellation property in C this implies $i(\omega(x)) = i(\omega(y))$ and we conclude by induction.

Let us show that for any $x \in C(P)$ we have $i \circ \alpha(x) = \alpha \circ i(x)$ by induction on $\nu(x)$. Let $x_1 \dots x_n$ be the normal form of x. Then $\alpha(i(x_1 \dots x_n)) = \alpha(i(x_1)i(x_2 \dots x_n)) = \alpha(i(x_1)\alpha(i(x_2 \dots x_n))) = \alpha(i(x_1)i(x_2))$ where the last equality is by the induction hypothesis. We are thus reduced to the case $n = 2$, *i.e.* to show that if $x, y \in P$ and $\alpha_2(x, y) = x$, then $\alpha(xy) = x$ in C. But this is clear by the definitions of α in C and α_2 in P. □

Subgerms, and fixed points.

Definition 3.33. *If $(P, \mathcal{O})$ is a germ, we call* subgerm *of P a pair $(P_1, \mathcal{O}_1)$ obtained by taking a part $\mathcal{O}_1$ of the objects $\mathcal{O}$ and a part P_1 of the morphisms between objects in $\mathcal{O}_1$ which is stable by the partial multiplication in P, and contains the morphisms 1_A for $A \in \mathcal{O}_1$.*

It is straightforward to check that a subgerm is a germ. Note, however, that left divisibility might be quite different in P_1: it is possible that $a, ab \in P_1$ but $b \in P - P_1$ in which case we do not have $a \preccurlyeq ab$ in P_1. If the divisibility in P_1 is the restriction of the divisibility in P, we say that P_1 is stable by complement. As in the case of categories we say that a subgerm P_1 stable by complement is stable by lcm if for any two morphisms in P_1 which have a common multiple in P_1, their lcm in P is in P_1 (so is an lcm in P_1).

Lemma 3.34. *If P is a left locally Garside germ, and P_1 a subgerm stable by complement and lcm, then P_1 is left locally Garside.*

Proof. Axiom 3.6(G1) is clearly inherited from P to P_1. Axiom 3.6(G4) is also inherited from P, using the natural functor $C(P_1) \to C(P)$ which is injective on P_1

(since its restriction to P_1 restricts to the injection $P_1 \to P$, because $P \to C(P)$ is injective).

Let us check 3.6(G2). If two elements have a common multiple in P_1, they have an lcm in P since P is locally Garside. That lcm is in P_1 and is an lcm in P_1 by assumption.

Let us check 3.6(G3). If two elements u, v have a lcm $\Delta_{u,v}$ in P_1, then by assumption their lcm in P is in P_1, and must thus be equal to $\Delta_{u,v}$. Thus if $xu, xv \in P_1$ then $x\Delta_{u,v} \in P$ thus $x\Delta_{u,v} \in P_1$ since P_1 is stable by partial multiplication. □

Lemma 3.35. *If, under the assumptions of 3.34, in addition P_1 is stable by α_2 (that is when $x, y \in P_1$ then $\alpha_2(x, y) \in P_1$), then $C(P_1)$ injects in $C(P)$.*

Proof. We have to show that the natural functor $C(P_1) \xrightarrow{i} C(P)$ which sends a path to the corresponding path is injective. Since i is injective on P_1 it is enough to show that i preserves normal forms; by the local characterization of normal forms it is enough to show that the image of a 2-term normal form is a normal form. But that is a consequence of the fact that P_1 is stable by α_2: indeed, if (x, y) is a 2-term normal form in P_1 and $\alpha_2(x, y) = xz$ in P then by assumption we have $xz \in P_1$, whence $z \in P_1$ and z divides y in P_1, as P_1 is stable by complement, so that $z = 1$ since (x, y) is normal in P_1. □

Proposition 3.36. *Let P be a left locally Garside germ and let σ be an automorphism of $C(P)$ stabilizing P; let P^σ (resp. $C(P)^\sigma$) be the subgerm (resp. the subcategory) of the σ-fixed morphisms and objects of P (resp. $C(P)$); then P^σ is a left locally Garside germ and $C(P)^\sigma = C(P^\sigma)$.*

Proof. The unicity of complement, lcm and α_2 shows that P^σ is stable by complement, lcm and α_2. Thus P^σ is a left locally Garside germ and the natural functor $C(P^\sigma) \xrightarrow{i} C(P)^\sigma$ is injective. As, given a σ-fixed morphism of $C(P)$, all the terms of its normal form are in P^σ by the unicity of normal forms, we get that i is surjective. □

A counterexample. We give an example to show that the endomorphisms of an object in a locally Garside category are not necessarily a locally Garside monoid.

Let $(P, \mathcal{O})$ be the germ where $\mathcal{O} = \{X, Y\}$ and where there are seven morphisms: $s, t \in \mathrm{End}(Y)$, two elements $a, b \in \mathrm{Hom}(X, Y)$, two elements $u, v \in \mathrm{Hom}(Y, X)$, plus one additional morphism resulting from the only composition defined in P, given by $as = bt$. The axioms of a germ as well as 3.6(G2) and 3.6(G3) are easy (associativity and 3.6 (G3) are empty, and the only morphisms having a common right multiple are a and b, and this multiple is unique; the same holds on the left for s and t). Let us prove 3.6 (G4) and its right analogue. There is an additive length on $C(P)$ defined by $l(a) = l(b) = l(s) = l(t) = 1$. If $zx = zy$ for $x, y \in P$ and $z \in C(P)$, then x and y have the same length. If this length is 2 then $x = y$ since there is only one element of length 2 in P. Assume then the length is 1. If x is neither s nor t, no relation in $C(P)$ for the word zx can involve x, thus $x = y$. It remains to consider the case $x = s$ and $y = t$, *i.e.* an equality $zs = zt$. If z has no decomposition ending by a again no relation can involve the terminal s. If $z = z'a$, no relation in $C(P)$ can change the terminal a into another morphism, in particular into b. Thus the word $z'at$ cannot be changed into $z'as$ so we are finished. A similar reasoning applies on the right, which finishes the proof that $C(P)$ is locally Garside.

On the other hand, in $\mathrm{End}(X)$ the morphisms a and b have two minimal-length common right multiples $asu = btu$ and $asv = btv$, thus no lcm.

4. Atoms

We call *atom* a morphism in a category (resp. a germ) which does not admit any proper right or left factor.

Note that if the category is left Noetherian and has the left cancellation property, then by 3.8 having no proper left factor is equivalent to having no proper right factor.

We say that a germ (resp. a category) is *atomic* if any morphism in the germ (resp. category) is a product of atoms.

In an atomic category C, a set P of morphisms generates C if and only if it contains the atoms of C.

Proposition 4.1. *A category C which has the left cancellation property and is right and left Noetherian (e.g. a locally Garside category) is atomic.*

Proof. Let us show that any morphism $f \in C$ is a product of atoms. By the analogue of 2.9 on the right, which is applicable thanks to 3.8, we know that C is left Artinian, thus f has a left factor which has no proper left factor, and which is thus an atom by the remark above.

Let thus a_1 be an atom which is a left factor of f and let f_1 be such that $f = a_1 f_1$. We may similarly write $f_1 = a_2 f_2$ where a_2 is an atom which is a left factor of f_1, etc... and if f is not equal to a finite product $a_1 \dots a_n$ we would get an infinite increasing sequence of factors of f which would contradict left Noetherianity. □

Remark 4.2. The cancellation property is necessary in the the above proposition. An example of a left and right Noetherian monoid without atoms is given by the set $\{x_i\}_{i\in\mathbb{Z}_{\geq 0}} \cup \{x_\infty\}$ where x_i is the infinite sequence beginning by i times 1 followed by all 0s and x_∞ is the infinite sequence with all terms equal to 1; the product is by term-wise multiplication. We have $x_i x_j = x_{\inf(i,j)}$.

Proposition 4.3. *Under the assumptions of 3.36, if in addition $C(P)$ is atomic, then $C(P^\sigma)$ is also atomic with atoms the right lcm of orbits of atoms of P (for the orbits which have a common multiple) which are not right multiples of another such lcm.*

Proof. Let us first see that any element $x \in C(P^\sigma)$ is divisible on the left by such an lcm. Let $s \in P$ be an atom such that $s \preccurlyeq x$. Then any element of the orbit of s also divides x, thus their lcm $\overline{s}$, which is in P^σ, also does divide x. By the left cancellation property, if we write $x = \overline{s}x_1$, then we also have $x_1 \in C(P)^\sigma$; we can apply the same process to x_1 to get an x_2, etc.... By 2.9 the sequence x_i is finite thus x is a finite product of such lcm. Finally, such an lcm which is not divisible by another is clearly an atom in $C(P)^\sigma$. □

We will now give conditions in terms of atoms which imply the properties for locally Garside.

Proposition 4.4. *A Noetherian atomic germ P satisfying 3.6 (G4) satisfies (G2) and (G3) if and only if*

(G2′) *If two atoms have a right common multiple in P then they have a right lcm in P.*

(G3′) *If two atoms s and t have a right lcm $\Delta_{s,t} \in P$ and $xs, xt \in P$ then $x\Delta_{s,t} \in P$.*

Proof. These conditions are necessary, as a special case of 3.6(G2) and 3.6(G3).

Let us show that (G2′) implies (G2). Assume $x, y \in P$ have a common right multiple in P. We apply lemma 3.5 to the set X of elements of P which are left factors of all common right multiples of x and y, taking for the P of 3.5 the atoms in P. We may do so since X inherits Noetherianity from P, and the assumption of 3.5 comes from (G4) and (G2′): if $z \in X$ and s and t are atoms such that $zs, zt \in X$, by (G4) s and t have a common right multiple, thus by (G2′) they have a right lcm $\Delta_{s,t}$ and $z\Delta_{s,t} = \operatorname{lcm}(zs, zt)$ is in X. The common multiple of elements of X given by 3.5 is the desired lcm.

We now show (G3). Let $u, v, x \in P$ be such that u and v have a right lcm $\Delta_{u,v} \in P$ and such that $xu, xv \in P$. This time we apply 3.5 to $X = \{y \in P \mid xy \in P \text{ and } y \preccurlyeq \Delta_{u,v}\}$, taking again for the P of 3.5 the atoms. Again X inherits Noetherianity from P; we have $u, v \in X$ by assumption. Assume now that $y \in X$ and the atoms s and t are such that $ys, yt \in X$, *i.e.* $xys, xyt \in P$ and $ys \preccurlyeq \Delta_{u,v}, yt \preccurlyeq \Delta_{u,v}$. By (G4) s and t have a common right multiple P, thus by (G2′) they have a lcm $\Delta_{s,t} \in P$ and by (G3′) $xy\Delta_{s,t} \in P$. Thus $y\Delta_{s,t} \in X$ and the assumption of 3.5 holds. The common multiple of the elements of X given by 3.5 is necessarily $\Delta_{u,v}$ since it is a multiple of both u and v. Thus $\Delta_{u,v} \in X$ which implies $x\Delta_{u,v} \in P$. □

5. Garside categories

In a left Garside category C the smallest set containing the left divisors of Δ and stable by taking left and right factors forms a left locally Garside germ P such that $C = C(P)$ by 3.32. The elements of P are called the *simples* of the category.

Remark 5.1. For a left locally Garside category C, we could call *simples of C* the set of morphisms of a chosen germ. Note that if C is in addition right Noetherian, there exists always a minimal such set, which is the minimal set P of morphisms of C stable by taking left and right factors and right lcm's and generating C; indeed this set exists and is unique, since C itself has these properties and an intersection of sets with these properties is also a set with these properties by 4.1 (the only non-trivial property to check for an intersection is that it still generates C; but 4.1 shows that a subset generates C if and only if it contains the atoms, which is a condition stable by intersection). Then P is a locally Garside germ and $C = C(P)$ by 3.32.

The set of simples in a left Garside category is not necessarily minimal in the sense above.

A simple f has a *complement to* Δ denoted $\tilde{f}$, and defined by $f\tilde{f} = \Delta$ (it is unique by the left cancellation property). If $f \in C$, as Δ is a natural transformation from the identity to Φ we have $f\Delta = \Delta\Phi(f)$ whence, using left cancellation by f, we get $\Delta = \tilde{f}\Phi(f)$, which can also be written $\tilde{\tilde{f}} = \Phi(f)$.

If Φ is an automorphism this shows that the set of left factors of Δ is the same as the set of right factors of Δ.

Remark 5.2. In a left Garside category, a set P of morphisms stable by taking left factors and complements to Δ is stable by taking right factors. Indeed, if ab and a are in P then $ab\widetilde{(ab)} = a\tilde{a}$ thus $b\widetilde{(ab)} = \tilde{a}$ thus $b \in P$ as a left factor of $\tilde{a}$.

Proposition 5.3. *In a left Garside category, left divisibility makes the set of morphisms with same source into a lattice.*

Proof. It is enough to show that any two morphisms with the same source have a right lcm. If they are simple, they divide Δ so we are done. Otherwise, given $x \in C$, we show by induction on n that $x \preccurlyeq \Delta^n$ where n is the number of terms of the normal form of x with respect to P. Indeed, if $x = x_1 \ldots x_n$ is the normal form, by induction $x_2 \ldots x_n \preccurlyeq \Delta^{n-1}$ thus $x_1 \ldots x_n \preccurlyeq x_1\Delta^{n-1}$. But by definition of Φ we have $x_1\Delta^{n-1} = \Delta^{n-1}\Phi^{n-1}(x_1) \preccurlyeq \Delta^n$. $\square$

Proposition 5.4. *A left Garside category which is right Noetherian and such that Φ is an automorphism is Garside.*

Proof. We first show that such a category C has the right cancellation property. Indeed, if $xa = ya$ we have seen in the proof of 5.3 that $a \preccurlyeq \Delta^n$ for some n whence $x\Delta^n = y\Delta^n \Leftrightarrow \Delta^n\Phi^n(x) = \Delta^n\Phi^n(y)$ which implies by left cancellation that $x = y$.

We then observe that for two simples f, g we have $g \preccurlyeq f \Leftrightarrow \tilde{g} \succcurlyeq \tilde{f}$ (the implication from left to right uses the left cancellation property and from right to left the right cancellation property). This implies that a left gcd of f and g transports by $\tilde{}$ to a left lcm of $\tilde{f}$ and $\tilde{g}$, and conversely a right lcm transports to a right gcd.

We can argue similarly for arbitrary morphisms by considering the complement to a suitable Δ^n instead of the complement to Δ.

We thus get that C is right locally Garside. Since as remarked above, the fact that Φ is an autoequivalence implies that the right divisors of Δ are the same as the left divisors, the category is right Garside for the same Δ and Φ^{-1}, so we are done. $\square$

The following proposition points to a possible alternative definition of left Garside categories.

Proposition 5.5. *A left locally Garside category which has a germ P as in 3.32 such that the morphisms in P with a given source have a right lcm is left Garside.*

Proof. Given an object A, we define Δ_A to be the right lcm of the morphisms of source A. The elements of P are the left divisors of the Δ_A so these divisors generate the category. On the morphisms of P we define an operation $f \mapsto \tilde{f}$ by the equality $f\tilde{f} = \Delta$, using cancellation. We then define a functor Φ which maps A to the target of Δ_A and a map f to $\tilde{\tilde{f}}$. To show that Φ is a functor, since any morphism is a composition of elements of P, it is enough to check that it is compatible with partial composition.

For $f, g, fg \in P$ we have $fg(fg)\tilde{} = \Delta = f\tilde{f}$ so that $g(fg)\tilde{} = \tilde{f}$, whence $g(fg)\tilde{}\,\tilde{\tilde{f}} = \Delta = g\tilde{g}$. By left cancellation, this gives $(fg)\tilde{}\,\tilde{\tilde{f}} = \tilde{g}$, whence $(fg)\tilde{}\,\tilde{\tilde{f}}\tilde{\tilde{g}} = \Delta = (fg)\tilde{}(fg)\tilde{\tilde{}}$ and by cancellation $\tilde{\tilde{f}}\tilde{\tilde{g}} = (fg)\tilde{\tilde{}}$ which is what we wanted.

Finally we note that the equality $f\Delta = \Delta\Phi(f)$ for $f \in P$ extends to the same equality for arbitrary maps in the category, which shows that Δ is indeed a natural transformation from the identity functor to Φ. $\square$

6. The conjugacy category

Conjugation in a monoid or a category is defined as: w is conjugate to w' if there exists x such that $xw' = wx$. In a category, this condition implies that w and w' are endomorphisms of some object.

Definition 6.1. *Given a category C, the* conjugacy category *of C is the category whose objects are the endomorphisms of C and where* $\operatorname{Hom}(w, w') = \{x \in C \mid wx = xw'\}$.

We can extend this definition to simultaneous conjugation of a family of elements, to get the simultaneous conjugacy category. If $wx = xw'$ as in the above definition we will write $w' = w^x$ and $w = {}^x w'$.

Proposition 6.2. *If C is a left (resp. right) locally Garside category, its (simultaneous) conjugacy category is also. Further, one can take as simples for the conjugacy category the morphisms which are induced by simples of C.*

Proof. Let us denote by $\mathcal{C}$ the conjugacy category of C. Since $\mathcal{C}$ clearly inherits Noetherianity and cancellability from C, we have just to show the existence of lcm for morphisms which have a common multiple. We will actually show that lcm and gcd in C of morphisms of the conjugacy category are lcm and gcd in the conjugacy category.

We can rephrase the condition $x \in \operatorname{Hom}_{\mathcal{C}}(w, ?)$ as $x \preccurlyeq wx$. If we look at simultaneous conjugation of a family $\mathcal{F}$, it will be the simultaneous condition $x \preccurlyeq wx$ for all $w \in \mathcal{F}$. Suppose that $x, y, w \in C$ are such that $x \preccurlyeq wx$ and that $xy \preccurlyeq wxy$; define w' by $xw' = wx$; then by cancellation $y \preccurlyeq w'y$, so that x and xy in $\mathcal{C}$ imply $y \in \mathcal{C}$. Suppose now that $x \preccurlyeq wx$ and $y \preccurlyeq wy$, and that x and y have a right lcm z in C. Then using the left cancellation property we see that wz is the right lcm of wx and wy thus $x \preccurlyeq wz$ and $y \preccurlyeq wz$ from which it follows that $z \preccurlyeq wz$, *i.e.* $z \in \operatorname{Hom}_{\mathcal{C}}(w, ?)$, so is the right lcm of x and y in $\mathcal{C}$ by the first part of the proof.

Similarly the condition $x \in \operatorname{Hom}_{\mathcal{C}}(?, w)$ can be written $xw \succcurlyeq x$, and if $x, y \in \operatorname{Hom}_{\mathcal{C}}(?, w)$ have a left lcm z we get that $z \in \operatorname{Hom}_{\mathcal{C}}(?, w)$ and is the left lcm of x and y in $\mathcal{C}$.

The second assertion of the proposition, follows from the fact that if $x \preccurlyeq wx$ then $\alpha(x) \preccurlyeq \alpha(wx) = \alpha(w\alpha(x)) \preccurlyeq w\alpha(w)$ which shows that $\alpha(x) \in \operatorname{Hom}_{\mathcal{C}}(w, ?)$ (and similarly on the right). ☐

The following is a straightforward consequence of the proposition:

Corollary 6.3. *If P is a germ for C and if we take the germ for the conjugacy category as in the above proposition, then the normal form of a morphism in the conjugacy category of C is identical to its normal form in C.*

The locally Garside category $B^+(\mathcal{I})$. The locally Garside category that we will consider in this subsection is related to the study of the normalizer of the submonoid generated by a part of the atoms in an Artin monoid, which has been done by Paris and Godelle.

Let (W, S) be a Coxeter system, and let $(B^+, \mathbf{S})$ be the corresponding Artin monoid. Recall that B^+ is a locally Garside monoid, with germ the canonical lift $\mathbf{W}$ of W in B^+ consisting of the elements whose length with respect to $\mathbf{S}$ is equal to the length of their image in W with respect to S (see *e.g.*, [Michel]). Let $\mathbf{I}_0 \subset \mathbf{S}$ and let $\mathcal{I}$ be the set of conjugates of $\mathbf{I}_0$. Since conjugacy preserves the length (measured with respect to the generating set $\mathbf{S}$), we see that any element of $\mathcal{I}$ is also a subset of $\mathbf{S}$. Let $\mathcal{C}_{\mathcal{I}}$ be the connected component of the (simultaneous) conjugacy category whose objects are $\mathcal{I}$. As the monoid B^+ is locally Garside the category $\mathcal{C}(\mathcal{I})$ is left locally Garside.

We denote by $B^+_{\mathbf{I}}$ the submonoid of B^+ generated by a set $\mathbf{I}\subset\mathbf{S}$.

We recall some definitions and results from [DMR].

Proposition 6.4. (i) *Any* $\mathbf{b}\in B^+$ *has a maximal left divisor in* $B^+_{\mathbf{I}}$*, denoted* $\alpha_{\mathbf{I}}(\mathbf{b})$*. We denote by* $\omega_{\mathbf{I}}(\mathbf{b})$ *the unique element such that* $\mathbf{b}=\alpha_{\mathbf{I}}(\mathbf{b})\omega_{\mathbf{I}}(\mathbf{b})$*.*

(ii) *Let* $\mathbf{I},\mathbf{J}\subset\mathcal{I}$ *and* $\mathbf{b}\in B^+$*; then* ${}^{\mathbf{b}}B^+_{\mathbf{J}}\subset B^+_{\mathbf{I}}$ *if and only if* ${}^{\omega_{\mathbf{I}}(\mathbf{b})}\mathbf{J}=\mathbf{I}$*.*

(iii) *Let* $\mathbf{I},\mathbf{J}\subset\mathcal{I}$ *and let* $\mathbf{b}_1,\mathbf{b}_2\in B^+$ *be such that* ${}^{\mathbf{b}_1}\mathbf{J}=\mathbf{I}$ *and* $\alpha_{\mathbf{I}}(\mathbf{b}_1)=1$*; then* $\alpha_{\mathbf{J}}(\mathbf{b}_2)=1$ *if and only if* $\alpha_{\mathbf{I}}(\mathbf{b}_1\mathbf{b}_2)=1$*.*

(iv) *If* $\mathbf{b}_1,\mathbf{b}_2\in B^+$ *satisfy* $\mathbf{I}^{\mathbf{b}_1}\subset\mathbf{S}$*,* $\mathbf{I}^{\mathbf{b}_2}\subset\mathbf{S}$ *and* $\alpha_{\mathbf{I}}(\mathbf{b}_1)=\alpha_{\mathbf{I}}(\mathbf{b}_2)=1$*, then their right lcm* $\mathbf{c}$ *also satisfies* $\alpha_I(\mathbf{c})=1$

Proof. (i) is [DMR, 2.1.5](ii) and (ii) results from [DMR, 2.3.10]. Let us prove (iii). For $\mathbf{s}\in\mathbf{I}$ there exists $\mathbf{s}'\in\mathbf{J}$ such that $\mathbf{s}\mathbf{b}_1=\mathbf{b}_1\mathbf{s}'$. This element is then a common multiple of $\mathbf{s}$ and $\mathbf{b}_1$ and has to be their lcm since $\mathbf{s}'$ is an atom of B^+. So $\mathbf{s}\preccurlyeq\mathbf{b}_1\mathbf{b}_2$ if and only if $\mathbf{b}_1\mathbf{s}'\preccurlyeq\mathbf{b}_1\mathbf{b}_2$, *i.e*, $\mathbf{s}'\preccurlyeq\mathbf{b}_2$ whence the equivalence of $\alpha_{\mathbf{J}}(\mathbf{b}_2)=1$ and $\alpha_{\mathbf{I}}(\mathbf{b}_1\mathbf{b}_2)=1$.

To prove (iv) we will actually show the stronger statement that if in B^+ we have $\mathbf{b}\preccurlyeq\mathbf{c}$, $\mathbf{I}^{\mathbf{b}}\subset\mathbf{S}$ and $\alpha_{\mathbf{I}}(\mathbf{b})=1$ then $\mathbf{b}\preccurlyeq\omega_{\mathbf{I}}(\mathbf{c})$. We proceed by induction on the length of $\alpha_{\mathbf{I}}(\mathbf{c})$. If $\alpha_{\mathbf{I}}(\mathbf{c})=1$ the result is trivial. Otherwise there exists $s\in\mathbf{I}$, $s\preccurlyeq\alpha_{\mathbf{I}}(\mathbf{c})$. By the assumption $\mathbf{I}^{\mathbf{b}}\subset\mathbf{S}$ there exists $\mathbf{t}\in\mathbf{S}$ such that $\mathbf{s}\mathbf{b}=\mathbf{b}\mathbf{t}$ or equivalently $\mathbf{s}^{-1}\mathbf{b}=\mathbf{b}\mathbf{t}^{-1}$. If we write $\mathbf{c}=\mathbf{b}\mathbf{a}$ with $\mathbf{a}\in B^+$ then by assumption $\mathbf{s}^{-1}\mathbf{c}=\mathbf{b}\mathbf{t}^{-1}\mathbf{a}$ is positive *i.e*, we have $\mathbf{b}\mathbf{t}^{-1}=\mathbf{x}\mathbf{a}^{-1}$ for some $\mathbf{x}\in B^+$. As $\mathbf{b}\mathbf{t}^{-1}=\mathbf{s}^{-1}\mathbf{b}\notin B^+$, we have $\mathbf{b}\not\succcurlyeq\mathbf{t}$, whence by unicity of irreducible fractions (see [Michel, 3.2]) $\mathbf{x}=\mathbf{b}\mathbf{y}$ and $\mathbf{a}=\mathbf{t}\mathbf{y}$ for some $\mathbf{y}\in B^+$; thus $\mathbf{t}^{-1}\mathbf{a}\in B^+$, *i.e.* $\mathbf{b}\preccurlyeq\mathbf{s}^{-1}\mathbf{c}$. We then conclude by induction on the length of $\alpha_{\mathbf{I}}(\mathbf{c})$ that $\mathbf{b}\preccurlyeq\omega_{\mathbf{I}}(\mathbf{s}^{-1}\mathbf{c})=\omega_{\mathbf{I}}(\mathbf{c})$. ☐

Statement (ii) in the above proposition is a motivation for restricting the next definition to elements such that $\alpha_{\mathbf{I}}(\mathbf{b})=1$ (we "lose nothing" by doing so).

The following definition makes sense by 6.4(iii)

Definition 6.5. *We define* $B^+(\mathcal{I})$ *as the category whose set of objects is* $\mathcal{I}$ *and such that the morphisms from* $\mathbf{I}$ *to* $\mathbf{J}$ *are the elements* $\mathbf{b}\in B^+$ *such that* $\mathbf{I}^{\mathbf{b}}=\mathbf{J}$ *and* $\alpha_{\mathbf{I}}(\mathbf{b})=1$ *(such a morphism will be denoted* $(\mathbf{I},\mathbf{b},\mathbf{J})$*).*

By 6.4(iii) and 6.4(iv) the subcategory $B^+(\mathcal{I})$ of $\mathcal{C}(\mathcal{I})$ satisfies the assumptions of lemma 2.3 and similarly on the right, so it is locally Garside.

We now get a germ for $B^+(\mathcal{I})$ from the germ $\mathbf{W}$ of the locally Garside monoid B^+. By 6.3 we have a germ P for $\mathcal{C}(\mathcal{I})$ consisting of the elements of $\mathbf{W}$ which are in $\mathcal{C}(\mathcal{I})$.

Proposition 6.6. *Let* $\mathbf{b}$ *be a morphism of* $B^+(\mathcal{I})$*; then all the terms of the normal form in* $\mathcal{C}(\mathcal{I})$ *of* $\mathbf{b}$ *are in* $B^+(\mathcal{I})$*.*

Proof. Let $\mathbf{b}=\mathbf{w}_1\ldots\mathbf{w}_k$ be the normal form of $\mathbf{b}\in\mathrm{Hom}_{B^+(\mathcal{I})}(\mathbf{I},\mathbf{J})$ in $\mathcal{C}(\mathcal{I})$ (*i.e.* in B^+). As $\mathbf{w}_i\in\mathcal{C}(\mathcal{I})$, we have $\mathbf{I}_i={}^{\mathbf{w}_{i+1}\ldots\mathbf{w}_k}\mathbf{J}\subset\mathbf{S}$ for all i. Now, as ${}^{\mathbf{w}_1\ldots\mathbf{w}_{i-1}}\alpha_{\mathbf{I}_i}(\mathbf{w}_i\ldots\mathbf{w}_k)\subset B^+_{\mathbf{J}}$, so divides $\alpha_{\mathbf{I}}(\mathbf{b})$, this element has to be 1, whence the result. ☐

Corollary 6.7. *The set of* $(\mathbf{I},\mathbf{w},\mathbf{J})$ *in* $\mathcal{C}(\mathcal{I})$ *such that* $\mathbf{w}\in\mathbf{W}$ *and* $\alpha_{\mathbf{I}}(\mathbf{w})=1$ *is a germ for* $B^+(\mathcal{I})$*.*

We now identify the germ of the above corollary with a germ constructed in W. It will be convenient to work with roots instead of subsets of the generators. We use the standard geometric realization of W as a reflection group in an $\mathbb{R}$-vector space V endowed with a basis Π in bijection with S. The set of roots, denoted by Φ is the set $W\Pi$. We denote by Φ^+ (resp. Φ^-) the elements of Φ which are linear combinations with positive (resp. negative) coefficients of Π; a basic property is that $\Phi = \Phi^+ \coprod \Phi^-$. For $\alpha \in \Pi$ let s_α be the corresponding element of S (a reflection with root α). For $I \subset \Pi$ we denote by W_I the subgroup of W generated by $\{s_\alpha \mid \alpha \in I\}$; we say that I is spherical if W_I is finite and we then denote by w_I its longest element. A subset $I \subset \Pi$ corresponds to a subset $\mathbf{I} \subset \mathbf{S}$. We denote by the same letter the conjugacy class $\mathcal{I}$ and the corresponding orbit of subsets of Π. We say that $w \in W$ is I-reduced if $w^{-1}I \in \Phi^+$. Being I-reduced corresponds to the lift $\mathbf{w} \in \mathbf{W}$ having $\alpha_{\mathbf{I}}(\mathbf{w}) = 1$. So the germ P identifies with the set of (I, w, J) such that $I, J \in \mathcal{I}$ and $wJ = I$. The product $(I, w, J)(J, w', K)$ is defined in P if and only if $l(ww') = l(w) + l(w')$ and is then equal to (I, ww', K).

We now describe the atoms of $B^+(\mathcal{I})$ using the results of [Brink-Howlett]. If I is a subset of Π and $\alpha \in \Pi$ is such that $\Phi_{I\cup\{\alpha\}} - \Phi_I$ is finite, then by [Brink-Howlett] there exists a unique $v(\alpha, I) \in W_{I\cup\{\alpha\}}$ such that $v(\alpha, I)(\Phi^+_{I\cup\{\alpha\}} - \Phi^+_I) \subseteq \Phi^-_{I\cup\{\alpha\}}$ and $J = v(\alpha, I)I \subseteq I \cup \{\alpha\}$; when $I \cup \{\alpha\}$ is spherical then $v(\alpha, I) = w_{I\cup\{\alpha\}}w_I$. We have $(J, v(\alpha, I), I) \in P$ when $I \in \mathcal{I}$

Proposition 6.8. *The atoms of P are the elements $(J, v(\alpha, I), I)$ for $I \in \mathcal{I}$ and $\alpha \in \Pi - I$ such that $\Phi_{\{\alpha\}\cup I} - \Phi_I$ is finite.*

Proof. By [Brink-Howlett, 3.2] the elements $(J, v(\alpha, I), I)$ as in the proposition generate the monoid. They are atoms because by [Brink-Howlett, 4.1] the lcm of two such elements, when it exists, has length strictly larger than either of them. $\square$

The spherical case. We show now that $B^+(\mathcal{I})$ is Garside when $\mathbf{W}$ is finite. We recall that in that case B^+ is a Garside monoid, with $\mathbf{w_S}$ as Δ. We denote by $\mathbf{s} \mapsto \bar{\mathbf{s}}$ the involution on $\mathbf{S}$ given by $\mathbf{s} \mapsto {}^{\mathbf{w_S}}\mathbf{s}$. This extends naturally to involutions on $\mathcal{I}$ and on B^+ that we denote in the same way. We define the functor Φ by $\Phi(\mathbf{I}) = \bar{\mathbf{I}}$ and $\Phi((\mathbf{J}, \mathbf{w}, \mathbf{I})) = (\bar{\mathbf{J}}, \bar{\mathbf{w}}, \bar{\mathbf{I}})$. The natural transformation Δ is given by the collection of morphisms $(\mathbf{J}, \mathbf{w_J}^{-1}\mathbf{w_S}, \bar{\mathbf{J}})$. The properties which must be satisfied by Δ and Φ are easily checked.

7. A result à la Deligne for locally Garside categories

In this section we prove a simply connectedness property for the decompositions into simples for a map in a locally Garside category. This result is similar (but weaker, see the remark after 7.1) to Deligne's result in [Deligne], but the proof is much simpler and the result is sufficient for the applications that we have in mind. The present proof follows a suggestion by Serge Bouc to use a version of [Bouc, lemma 6].

Let P a left locally Garside germ and fix $g \in C(P)$ with $g \neq 1$. We denote by $E(g)$ the set of decompositions of g into a product of elements of P different from 1.

Then $E(g)$ is a poset, the order being defined by

$$(g_1, \ldots, g_{i-1}, g_i, g_{i+1}, \ldots, g_n) > (g_1, \ldots, g_{i-1}, a, b, g_{i+1}, \ldots, g_n)$$

if $ab = g_i \in P$.

We recall the definition of the notion of homotopy in a poset (which is nothing but a translation of the notion of homotopy in a simplicial complex isomorphic to

E as a poset). A path from x_1 to x_k in E is a sequence $x_1 \dots x_k$ where each x_i is comparable to x_{i+1}. The composition of paths is defined by concatenation. We denote homotopy by $\sim$. It is the finest equivalence relation on paths compatible with concatenation and generated by the two following elementary relations: $xyz \sim xz$ if $x \leq y \leq z$ and $xyx \sim x$ (resp. $yxy \sim y$) when $x \leq y$. Homotopy classes form a groupoid, as the composition of a paths with source x and of the inverse path is the constant path at x. For $x \in E$ we denote by $\Pi_1(E, x)$ the fundamental group of E with base point x, which is the group of homotopy classes of loops starting from x.

A poset E is said to be *simply connected* if it is connected (there is a path linking any two elements of E) and if the fundamental group with some (or any) base point is trivial.

Note that a poset with a smallest or largest element x is simply connected since any path $(x, y, z, t, \dots, x)$ is homotopic to $(x, y, x, z, x, t, x, \dots, x)$ which is homotopic to the trivial loop.

Theorem 7.1. *(Deligne) The set $E(g)$ is simply connected.*

In fact Deligne, in his more specific setting, proves the stronger result that $E(g)$ is contractible.

Proof. First we prove a version of a lemma from [Bouc] on order preserving maps between posets. For a poset E we put $E_{\geq x} = \{x' \in E \mid x' \geq x\}$, which is a simply connected subposet of E since it has a smallest element. If $f : X \to Y$ is an order preserving map it is compatible with homotopy (it corresponds to a continuous map between simplicial complexes), so it induces a homomorphism $f^* : \Pi_1(X, x) \to \Pi_1(Y, f(x))$.

Lemma 7.2. *(Bouc) Let $f : X \to Y$ an order preserving map between two posets. We assume that Y is connected and that for any $y \in Y$ the poset $f^{-1}(Y_{\geq y})$ is connected and non empty. Then f^* is surjective. If moreover $f^{-1}(Y_{\geq y})$ is simply connected for all y then f^* is an isomorphism.*

Proof. Let us first show that X is connected. Let $x, x' \in X$; we choose a path $y_0 \dots y_n$ in Y from $y_0 = f(x)$ to $y_n = f(x')$. For $i = 0, \dots, n$, we choose $x_i \in f^{-1}(Y_{\geq y_i})$ with $x_0 = x$ and $x_n = x'$. Then if $y_i \geq y_{i+1}$ we have $f^{-1}(Y_{\geq y_i}) \subset f^{-1}(Y_{\geq y_{i+1}})$ so that there exists a path in $f^{-1}(Y_{\geq y_{i+1}})$ from x_i to x_{i+1}; otherwise $y_i < y_{i+1}$, which implies $f^{-1}(Y_{\geq y_i}) \supset f^{-1}(Y_{\geq y_{i+1}})$ and there exists a path in $f^{-1}(Y_{\geq y_i})$ from x_i to x_{i+1}. Concatenating these paths gives a path connecting x and x'.

We fix now $x_0 \in X$. Let $y_0 = f(x_0)$. We prove that $f^* : \Pi_1(X, x_0) \to \Pi_1(Y, y_0)$ is surjective. Let $y_0 y_1 \dots y_n$ with $y_n = y_0$ be a loop in Y. We lift arbitrarily this loop into a loop $x_0 — \cdots — x_n$ in X as above, (where $x_i — x_{i+1}$ stands for a path from x_i to x_{i+1} which is either in $f^{-1}(Y_{\geq y_i})$ or in $f^{-1}(Y_{\geq y_{i+1}})$. Then the path $f(x_0 — x_1 — \cdots — x_n)$ is homotopic to $y_0 \dots y_n$; this can be seen by induction: let us assume that $f(x_0 — x_1 \cdots — x_i)$ is homotopic to $y_0 \dots y_i f(x_i)$; then the same property holds for $i + 1$: indeed $y_i y_{i+1} \sim y_i f(x_i) y_{i+1}$ as they are two paths in a simply connected set which is either $Y_{\geq y_i}$ or $Y_{\geq y_{i+1}}$; similarly we have

$f(x_i)y_{i+1}f(x_{i+1}) \sim f(x_i—x_{i+1})$. Putting things together gives

$$\begin{aligned} y_0\dots y_i y_{i+1} f(x_{i+1}) &\sim y_0 y_1 \dots y_i f(x_i) y_{i+1} f(x_{i+1}) \\ &\sim f(x_0—\cdots—x_i) y_{i+1} f(x_{i+1}) \\ &\sim f(x_0—\cdots—x_i—x_{i+1}). \end{aligned}$$

We now prove injectivity of f^* when all $f^{-1}(Y_{\geq y})$ are simply connected.

We first prove that if $x_0—\cdots—x_n$ and $x'_0—\cdots—x'_n$ are two loops lifting the same loop $y_0\dots y_n$, then they are homotopic. Indeed, we get by induction on i that $x_0—\cdots—x_i—x'_i$ and $x'_0—\cdots—x'_i$ are homotopic paths, using the fact that x_{i-1}, x_i, x'_{i-1} and x'_i are all in the same simply connected sub-poset, namely either $f^{-1}(Y_{\geq y_{i-1}})$ or $f^{-1}(Y_{\geq y_i})$.

It remains to prove that we can lift homotopies, which amounts to show that if if we lift as above two loops which differ by an elementary homotopy, the liftings are homotopic. If $yy'y \sim y$ is an elementary homotopy with $y < y'$ (resp. $y > y'$), then $f^{-1}(Y_{\geq y'}) \subset f^{-1}(Y_{\geq y})$ (resp. $f^{-1}(Y_{\geq y}) \subset f^{-1}(Y_{\geq y'})$) and the lifting of $yy'y$ constructed as above is in $f^{-1}(Y_{\geq y})$ (resp. $f^{-1}(Y_{\geq y'})$) so is homotopic to the trivial path. If $y < y' < y''$, a lifting of $yy'y''$ constructed as above is in $f^{-1}(Y_{\geq y})$ so is homotopic to any path in $f^{-1}(Y_{\geq y})$ with the same endpoints. $\square$

We now prove 7.1. By 2.9 $C(P)$ is right Artinian. Thus if 7.1 is not true there exists $g \in C(P)$ which is minimal for right divisibility such that $E(g)$ is not simply connected. Let T be the set of elements of P which are left divisors of g. By 3.29, for any $I \subset T$ the elements of I have an lcm Δ_I. We put $E_I(g) = \{(g_1,\dots,g_n) \in E(g) \mid \forall s \in I,\ s \preccurlyeq g_1\}$. The set $E_I(g)$ is the set of decompositions of g whose first term is left divisible by Δ_I.

We claim that $E_I(g)$ is simply connected for $I \neq \emptyset$. In the following, if $a \preccurlyeq b$, we denote by $a^{-1}b$ the element c such that $b = ac$. We apply 7.2 to the map $f : E_I(g) \to E(\Delta_I^{-1}g)$ defined by

$$(g_1,\dots,g_n) \mapsto \begin{cases} (g_2,\dots,g_n) & \text{if } g_1 = \Delta_I \\ (\Delta_I^{-1}g_1, g_2, \dots g_n) & \text{otherwise} \end{cases}.$$

This map preserves the order and any set $f^{-1}(Y_{\geq (g_1,\dots,g_n)})$ has a least element, namely $(\Delta_I, g_1, \dots, g_n)$, so is simply connected. As by minimality of g $E(\Delta_I^{-1}g)$ is simply connected 7.2 implies that $E_I(g)$ is simply connected as claimed.

We now apply 7.2 to the map $f : E(g) \to Y = \mathcal{P}(T) - \{\emptyset\}$ defined by $(g_1,\dots,g_n) \mapsto \{s \in T \mid s \preccurlyeq g_1\}$, where $\mathcal{P}(T)$ is ordered by inclusion. This map is order preserving since $(g_1,\dots,g_n) < (g'_1,\dots,g'_n)$ implies $g_1 \preccurlyeq g'_1$. We have $f^{-1}(Y_{\geq I}) = E_I(g)$, so this set is simply connected Since $\mathcal{P}(T) - \{\emptyset\}$, having a greatest element, is simply connected 7.2 gives that $E(g)$ is simply connected, whence the theorem. $\square$

8. The categories associated to P^n

Let P be a left locally Garside germ. For any positive integer n we define a germ P_n whose objects are the paths of length n in P and such that a morphism $\mathbf{a} \xrightarrow{f} \mathbf{b}$ where $\mathbf{a} = (a_1,\dots,a_n)$ and $\mathbf{b} = (b_1,\dots,b_n)$, is given by a sequence f_i for $1 \leq i \leq n+1$, where f_i is a morphism from the source of a_i to the source of b_i for $i \leq n$ and f_{n+1} is a morphism from the target of a_n to the target of b_n, with the additional condition that $f_i \preccurlyeq a_i$ for $i \leq n$ and that, if we define f'_i by $a_i = f_i f'_i$, using left cancellability, we then have $b_i = f'_i f_{i+1}$ for $1 \leq i \leq n$.

The composition of two morphisms $(a_1,\dots,a_n)\xrightarrow{f}(b_1,\dots,b_n)\xrightarrow{g}(c_1,\dots,c_n)$ is defined in P_n when $f_ig_i\preccurlyeq a_i$. We then set $(fg)_i=f_ig_i$, which satisfies the conditions for being in P_n; indeed, if we define $(fg)'_i$ by $a_i=(fg)_i(fg)'_i$ the equality to prove $c_i=(fg)'_i(fg)_{i+1}$ is equivalent by left cancellation to $(fg)_ic_i=a_i(fg)_{i+1}$, which is true since $f_ig_ic_i=f_ig_ig'_ig_{i+1}=f_ib_ig_{i+1}=f_if'_if_{i+1}g_{i+1}=a_if_{i+1}g_{i+1}$.

Divisibility in P_n is then given by the following result:

Lemma 8.1. *The morphism* $\mathbf{a}\xrightarrow{f}\mathbf{b}$ *left divides in* P_n *the morphism* $\mathbf{a}\xrightarrow{g}\mathbf{c}$ *if and only if* $f_i\preccurlyeq g_i$. *Then there is a unique morphism* $\mathbf{b}\xrightarrow{h}\mathbf{c}$ *such that* $fh=g$, *where* h_i *is given by* $f_ih_i=g_i$, *using the left cancellation property.*

Proof. By the description of the product it is clear that if h is such that $fh=g$ then $f_ih_i=g_i$. Let us see that conversely this implies that h is a morphism from $\mathbf{b}$ to $\mathbf{c}$. Indeed $g_ic_i=a_ig_{i+1}\Leftrightarrow f_ih_ic_i=f_if'_if_{i+1}h_{i+1}$ which implies $h_ic_i=b_ih_{i+1}$, so if we define h'_i by $b_i=h_ih'_i$ we get $c_i=h'_ih_{i+1}$ as wanted. □

Lemma 8.2. P_n *is a germ.*

Proof. Axiom 3.1 (i) is clear, the identity morphism being given by the sequence $f_i=1$.

Let us check axiom 3.1 (ii). Let us consider three morphisms $\mathbf{a}\xrightarrow{f}\mathbf{b}\xrightarrow{g}\mathbf{c}\xrightarrow{h}\mathbf{d}$. From the definition of the product in P_n, since when they are defined, we have $(fg)_i=f_ig_i$ and $(gh)_i=g_ih_i$, the condition for $(fg)h\in P_n$ and for $f(gh)\in P_n$ is the same, namely that $f_ig_ih_i\preccurlyeq a_i$, and both products are defined by the sequence $f_ig_ih_i$ so are equal. □

We will also consider the two subgerms of P_n defined by one of the two additional conditions:

Definition 8.3.
(i) *The subgerm* $P_n(\mathrm{Id})$ *has the same objects as* P_n *and its morphisms verify the additional condition* $f_1=f_{n+1}=1$.
(ii) *Let* F *be a functor from* $C(P)$ *to itself. The objects of the subgerm* $P_n(F)$ *are the paths* $(a_1,\dots,a_n)$ *such that the target of* a_n *is the image by* F *of the source of* a_1, *and the morphisms of* $P_n(F)$ *are the morphisms of* P_n *verifying the condition* $f_{n+1}=F(f_1)$.

Note that the condition on the objects of $P_n(F)$ is such that they have an identity morphism. A connected component of the category $C(P_n(\mathrm{Id}))$ is a "category of decompositions" of a given morphism in $C(P)$, while a connected component of $C(P_n(F))$ corresponds to a connected component of the "category of F-twisted conjugacy" for $C(P)$. Note that $P_n(\mathrm{Id})$ is a subgerm of P_n stable by taking left and right factors, while $P_n(F)$ is not.

The germ $P_n(\mathrm{Id})$ was inspired by a conversation with Daan Krammer. The germ $P_n(F)$ mimics the "divided categories" of David Bessis.

Since $(fg)_i=f_ig_i$ we can extend the map $f\mapsto f_i:P_n\to P$ to a map $f\mapsto f_i:C(P_n)\to C(P)$; this corresponds to the "product of the i-th column", equal to

$f_i = f_{1i}f_{2i}\dots f_{ni}$, in the following picture of a map in $C(P_n)$.

$$
\begin{tikzcd}
{} \arrow[r, "s_1"] \arrow[d, "f_{11}"'] & {} \arrow[r, "s_2"] \arrow[d, "f_{12}"'] & \cdots & {} \arrow[r, "s_n"] \arrow[d] & {} \arrow[d, "f_{1(n+1)}"] \\
{} \arrow[r] \arrow[d, "f_{21}"'] & {} \arrow[r] \arrow[d, "f_{22}"'] & \cdots & {} \arrow[r] \arrow[d] & {} \arrow[d, "f_{2(n+1)}"] \\
{} \arrow[r] \arrow[d, dotted, no head] & {} \arrow[r] \arrow[d, dotted, no head] & \cdots & {} \arrow[r] \arrow[d, dotted, no head] & {} \arrow[d, dotted, no head] \\
{} \arrow[r, "t_1"] & {} \arrow[r, "t_2"] & \cdots & {} \arrow[r, "t_n"] & {}
\end{tikzcd}
$$

Theorem 8.4. *P_n is left locally Garside.*

Proof. Let us check Noetherianity (3.6 (G1)). Let us consider an increasing sequence (g_k) of morphisms all dividing a morphism f from $(a_1, \dots, a_n)$ to $(b_1, \dots, b_n)$. By lemma 8.1 this increasing sequence corresponds to an increasing sequence of left factors of f_i for each i . By the Noetherianity of P each of these sequence becomes constant at some stage so g_k itself becomes constant and we are done.

We now check left cancellability (3.6 (G4)). Assume that we have an equality $fg = fh$ where $f \in C(P_n)$ and $g, h \in P_n$. Then $f_i g_i = (fg)_i = (fh)_i = f_i h_i$ for all i, and by left cancellability in $C(P)$ we deduce $g_i = h_i$ for all i q.e.d.

We now check axiom 3.6 (G2). If f and g have a common right multiple h in P_n, then by lemma 8.1 for all i the morphism h_i is a right multiple of f_i and g_i, so f_i and g_i have a right lcm k_i in P. From $f_i \preccurlyeq a_i$ and $g_i \preccurlyeq a_i$ we get $k_i \preccurlyeq a_i$, so k_i defines a morphism k in P_n which is clearly an lcm for f and g.

The axiom 3.6 (G3) can be similarly deduced from the corresponding axiom in P. $\square$

Theorem 8.5. (i) *If F preserves right lcms, the category $C(P_n(F))$ is left Garside.*
(ii) *If $C(P)$ is left Garside, then $C(P_n)$ also.*
(iii) *$C(P_n(\mathrm{Id}))$ is left Garside.*

Proof. We first check that the above categories are left locally Garside. We have seen this for $C(P_n)$ in 8.4. For $C(P_n(\mathrm{Id}))$ and $C(P_n(F))$, since $P_n(\mathrm{Id})$ and $P_n(F)$ are subgerms of the left locally Garside germ P_n, by lemma 3.34 we have just to check that they are stable by right complement and lcm. The stability by complement is clear from the formula $(fg)_i = f_i g_i$ for a product, since then if $(fg)_i$ and f_i satisfy the condition for $P_n(\mathrm{Id})$ (resp. $P_n(F)$) then g_i will also satisfy it. Similarly, since by the proof of 8.4 the lcm of f and g is obtained by taking the lcm of f_i and g_i, it will obviously satisfy the condition for $P_n(\mathrm{Id})$, and also for $P_n(F)$ using the assumption that F preserves lcms.

Thus, by 5.5, we just have to check that in each of these categories the morphisms in the germ with a given source have a right lcm.

For $C(P_n)$, let Δ_A be the natural transformation starting from the object A corresponding to the left Garside structure on $C(P)$, and let $\mathbf{a} = (a_1, \dots, a_n)$ be an object of P_n. Then $f_i = a_i$ for $i \le n$ and $f_{n+1} = \Delta_A$, where A is the target of a_n, defines a morphism f from $\mathbf{a}$ in P_n. This morphism is clearly multiple of any other morphism from $\mathbf{a}$.

For the category $C(P_n(F))$, we take the morphism given by $f_i = a_i$ for $i \le n$ and $f_{n+1} = F(a_1)$. It is clear that it is a multiple in P_n of any morphism from

$(a_1,\dots,a_n)$ which is in in $P_n(F)$; it is also clear that the quotient is in $P_n(F)$ since $g_{n+1}h_{n+1} = F(g_1h_1)$ and $g_{n+1} = F(g_1)$ imply $h_{n+1} = F(h_1)$ by cancellation.

Finally for $C(P_n(\mathrm{Id}))$, we define by induction for $i \geq 2$ morphisms f'_{i-1} and f_i by the rules $a_{i-1} = f_{i-1}f'_{i-1}$ and $f'_{i-1}f_i = \alpha(f'_{i-1}a_i)$. If we have another morphism g from $\mathbf{a}$, we see by the same induction that $g_i \preccurlyeq f_i$ and $g'_i \succcurlyeq f'_i$. $\square$

Let us spell out the value of Φ in the first two categories.

For $C(P_n)$, we have $\Phi((a_1,\dots,a_n)) = (a_2,\dots,a_n,\Delta_A)$ where A is the target of a_n. If $\mathbf{a} \xrightarrow{f} \mathbf{b}$ is given by f_i, we have $\Phi(f)_i = f_{i+1}$ for $i \leq n$ and $\Phi(f)_{n+1} = \Psi(f_{n+1})$ where Ψ is the endofunctor corresponding to the assumed left Garside structure on $C(P)$.

In $C(P_n(F))$, if $\mathbf{a} = (a_1,\dots,a_n)$ then $\Phi(\mathbf{a}) = (a_2,\dots,a_n,F(a_1))$; and $\mathbf{a} \xrightarrow{f} \mathbf{b}$ is given by f_i, we have $\Phi(f)_i = f_{i+1}$ for $i \leq n$ and $\Phi(f)_{n+1} = F(f_2)$.

The case of a right locally Garside P. In the case where P is right and left locally Garside, we can compute the normal form of a morphism and will deduce that a morphism f is determined by the f_i.

Lemma 8.6. *Let f and g be morphisms in P_n such that the target of f is the source of g; then $\alpha(fg)$ is the morphism whose i-th component $\alpha(fg)_i$ is the left gcd of f_ig_i and s_i for $i = 1,\dots,n$ and $\alpha(fg)_{n+1} = \alpha(f_{n+1}g_{n+1})$.*

Proof. We have the following commutative diagram

$$\begin{tikzcd}
{} \arrow[r,"s_1"] \arrow[d,"f_1"'] & {} \arrow[r,"s_2"] \arrow[d,"f_2"] & \cdots & {} \arrow[r,"s_n"] \arrow[d] & {} \arrow[d,"f_{n+1}"] \\
{} \arrow[r] \arrow[d,"g_1"'] & {} \arrow[r] \arrow[d,"g_2"] & \cdots & {} \arrow[r] \arrow[d] & {} \arrow[d,"g_{n+1}"] \\
{} \arrow[r,"t_1"'] \arrow[ur,"x_1"'] & {} \arrow[r,"t_2"'] & \cdots & {} \arrow[r,"t_n"'] \arrow[ur,"x_n"'] & {}
\end{tikzcd}$$

For $i = 1,\dots,n$, let f_ia_i be the left gcd of f_ig_i and s_i and let $f_{n+1}a_{n+1} = \alpha(f_{n+1}g_{n+1})$; we put $g_i = a_ib_i$ for $i = 1\dots,n+1$ and $s_i = f_ia_iv_i$ for $i = 1,\dots,n$. Let w_i be the morphism x_ia_i. The diagram is the following

$$\begin{tikzcd}
{} \arrow[r,"s_1"] \arrow[d,"f_1"'] & {} \arrow[r,"s_2"] \arrow[d,"f_2"] & \cdots & {} \arrow[r,"s_n"] \arrow[d] & {} \arrow[d,"f_{n+1}"] \\
{} \arrow[d,"a_1"'] & {} \arrow[d,"a_2"] & & {} \arrow[d] & {} \arrow[d,"a_{n+1}"] \\
{} \arrow[r,"v_1f_2a_2"] \arrow[uur,"v_1"'] \arrow[d,"b_1"'] & {} \arrow[r,"v_2f_3a_3"] \arrow[d,"b_2"] & \cdots & {} \arrow[r] \arrow[uur,"v_n"'] \arrow[d] & {} \arrow[d,"b_{n+1}"] \\
{} \arrow[r,"t_1"'] \arrow[ur,"w_1"'] & {} \arrow[r,"t_2"'] & \cdots & {} \arrow[r,"t_n"'] \arrow[ur,"w_n"'] & {}
\end{tikzcd}$$

We prove that the there exists a morphism $h \in P_n$ such that $h_i = f_ia_i$ which is equivalent to proving that $v_if_{i+1}a_{i+1}$ is in P. We claim that $v_if_{i+1}a_{i+1}$ is the left lcm of $f_{i+1}a_{i+1}$ and w_i for $i = 1,\dots,n$: indeed it is a common left multiple and if the left lcm was smaller then b_i and v_i would have a non trivial common left divisor y_i which would give a common left divisor $f_ia_iy_i$ of s_i and f_ig_i greater than their gcd f_ia_i. We conclude as the lcm of two morphisms in P is in P.

So we have a morphism $h \in P_n$ which divides fg. There cannot be a greater simple divisor k of fg as k_i has to divide s_i and $f_i g_i$ for $i = 1 \dots n$ and k_{n+1} has to be a simple divisor of $f_{n+1}g_{n+1}$. $\square$

Note that in the above proof we have used a left lcm. It is the only place where we use the fact that P is right locally Garside.

Proposition 8.7. *Let f be a morphism in $C(P_n)$ with source $(s_1, \dots, s_n)$; then $\alpha(f)$ is the simple morphism with same source such that $\alpha(f)_i = \gcd(f_i, s_i)$ for $i = 1, \dots, n$ and $\alpha(f)_{n+1} = \alpha(f_{n+1})$.*

Proof. We write $f = f^1 \dots f^k$ with $f^i \in P_n$. The proof is by induction on k. The above lemma proves the result for $k = 2$. We have $\alpha(f) = \alpha(f^1\alpha(f^2 \dots f^k))$. By the induction hypothesis applied to $f^2 \dots f^k$, the following diagram represents $f^1\alpha(f^2 \dots f^k)$:

$$\begin{tikzcd}
\cdot \arrow[r, "s_1"] \arrow[d, "f_1^1"'] & \cdot \arrow[r, "s_2"] \arrow[d, "f_2^1"] & \cdots & \cdot \arrow[r, "s_n"] \arrow[d] & \cdot \arrow[d, "f_{n+1}^1"] \\
\cdot \arrow[r, "t_1"] \arrow[ur] \arrow[d, "g_1"'] & \cdot \arrow[r, "t_2"] \arrow[d, "g_2"] & \cdots & \cdot \arrow[r, "t_n"] \arrow[ur] \arrow[d] & \cdot \arrow[d, "g_{n+1}"] \\
\cdot \arrow[r] \arrow[ur] & \cdot \arrow[r] & \cdots & \cdot \arrow[r] \arrow[ur] & \cdot
\end{tikzcd}$$

where $g_i = \gcd(t_i, (f^2 \dots f^k)_i)$ for $i \le n$ and $g_{n+1} = \alpha((f^2 \dots f^k)_{n+1})$. We apply now the previous lemma to the two term product $f^1\alpha(f^2 \dots f^k)$. We will be done if $\gcd(f_i^1 g_i, s_i) = \gcd(f_i, s_i)$ *i.e*, $\gcd(f_i^1 \gcd(t_i, (f^2 \dots f^k)_i), s_i) = \gcd(f_i, s_i)$ for $i \le n$ and $\alpha(f_{n+1}^1 \alpha((f^2 \dots f^k)_{n+1}) = \alpha(f_{n+1})$. The latter is true by the properties of α. The former is true as the right hand side is a multiple of f_i^1 so has to be the product of f_i^1 by a common divisor of $(f^2 \dots f^k)_i$ and t_i. $\square$

Corollary 8.8. *Let f and g be two morphisms in $C(P_n)$ with same source; then $f \preccurlyeq g$ if and only if $f_i \preccurlyeq g_i$ for all i*

Proof. Assume that $f_i \preccurlyeq g_i$ for all i. If f and g are in P_n then we are done by lemma 8.1. In general we prove the result by induction on the length of the normal form of f. We first show that $\alpha(f) \preccurlyeq \alpha(g)$: let $(s_1, \dots, s_n)$ be the common source of f and g; we have $\alpha(f)_i = \gcd(f_i, s_i) \preccurlyeq \gcd(g_i, s_i) = \alpha(g)_i$ for $i \le n$ and $\alpha(f)_{n+1} = \alpha(f_{n+1}) \preccurlyeq \alpha(g_{n+1}) = \alpha(g)_{n+1}$, whence the result as $\alpha(f)$ and $\alpha(g)$ are two elements of P_n. After simplifying by $\alpha(f)$ we can apply the induction hypothesis which gives that $\omega(f) \preccurlyeq \alpha(f)^{-1}g$, whence $f \preccurlyeq g$. The converse is clear. $\square$

Corollary 8.9. *A morphism $f \in C(P_n)$ is determined uniquely by its source and the morphisms $f_i \in C(P)$*

Proof. If two morphisms f and g have same source and $f_i = g_i$ for all i, then by the previous corollary they divide each other, so are equal. $\square$

Note that corollaries 8.8 and 8.9 are true for any subcategory of $C(P_n)$. Proposition 8.7 is true in $C(P_n(\mathrm{Id}))$. In $C(P_n(F))$ it has to be modified as follows:

Corollary 8.10. *Let P be as above and F be as in 8.3(ii); if f is a morphism in $C(P_n(F))$ with source $(s_1, \dots, s_n)$ then $\alpha(f)$ is the simple morphism with same source such that $\alpha(f)_i = \gcd(f_i, s_i)$ for $i = 1, \dots, n$.*

Proof. It is clear that the morphism defined by these conditions is the greatest divisor of the $\alpha(f)$ computed in P_n which is in $P_n(F)$. $\square$

More on $C(P_n(\mathrm{Id}))$. We now return to the case of an only left Garside P and look at $C(P_n(\mathrm{Id}))$.

Lemma 8.11. *In $P_n(\mathrm{Id})$, there is at most one morphism between two objects.*

Proof. Indeed, if $(a_1,\ldots,a_n) \xrightarrow{f} (b_1,\ldots,b_n)$ in $P_n(\mathrm{Id})$ then we have $f_1 = 1$ thus f is determined by increasing induction on i, using the cancellation property, by the equations $a_i = f_i f_i'$ and $b_i = f_i' f_{i+1}$. □

Let $\mathbf{a} = (a_1,\ldots,a_n)$ be an object of P_n and let $(b_1,\ldots,b_n)$ be the normal form of $a_1a_2\ldots a_n$ in $C(P)$, completed if needed by ones; *i.e.*, the sequence b_i is defined by $b_i = \alpha(\omega^{i-1}(a_1\ldots a_n))$ for all i (we then have $b_i = 1 \Rightarrow b_k = 1 \forall k \geq i$). In this situation we set $\mathrm{NF}(\mathbf{a}) = (b_1,\ldots,b_n)$. Using *e.g.* inductively 3.19 we can always construct at least one morphism in $C(P_n(\mathrm{Id}))$ from $\mathbf{a}$ to $\mathrm{NF}(\mathbf{a})$. In particular two objects are in the same connected component if and only if the product of their terms is the same.

Lemma 8.12. *In $C(P_n(\mathrm{Id}))$, there is a unique morphism $\mathbf{a} \to \mathrm{NF}(\mathbf{a})$.*

Proof. We first show that any morphism from $\mathbf{a}$ to $\mathrm{NF}(\mathbf{a})$ has $\Delta_{\mathbf{a}}$ as a left factor. Let $f = f_1f_2\ldots f_m$ be such a morphism, where $f_i \in P_n(\mathrm{Id})$. By definition of $\Delta_{\mathbf{a}}$ we have $f_1 \preccurlyeq \Delta_{\mathbf{a}}$. Using left cancellability we define g_1 by $\Delta_{\mathbf{a}} = f_1 g_1$. Since f_2 and g_1 both divide $\Delta_{\mathbf{a}_1}$, where $\mathbf{a}_1$ is the target of f_1 they have a right lcm of the form $f_2g_2 = g_1h_2$. By induction on i we can extend this process to get morphisms $g_i, h_i \in P_n(\mathrm{Id})$ such that $f_ig_i = g_{i-1}h_i$. As $\mathrm{NF}(\mathbf{a})$ is a final object in $C(P_n(\mathrm{Id}))$ we have $g_{m+1} = \mathrm{Id}$ whence $f_1f_2\ldots f_m = \Delta_{\mathbf{a}}h_2\ldots h_m$.

By induction, and using Noetherianity of $C(P_n(\mathrm{Id}))$, we can express any map from $\mathbf{a}$ to $\mathrm{NF}(\mathbf{a})$ as a (necessarily unique) finite product of Δ's, whence the lemma. □

Lemma 8.13. *In $C(P_n(\mathrm{Id}))$, there is at most one morphism between two objects.*

Proof. Suppose there exists two distincts morphisms f, g from $\mathbf{a}$ to $\mathbf{b}$. The fact that one morphism exists implies that $\mathrm{NF}(\mathbf{a}) = \mathrm{NF}(\mathbf{b})$. By composing f and g with the canonical morphism from $\mathbf{b}$ to $\mathrm{NF}(\mathbf{b}) = \mathrm{NF}(\mathbf{a})$, we get two morphisms from $\mathbf{a}$ to $\mathrm{NF}(\mathbf{a})$, which are distinct by the left cancellation property of $C(P_n(\mathrm{Id}))$. This contradicts 8.12. □

The category $C(P_\bullet(\mathrm{Id}))$. We will now consider a category whose objects can be identified to all possible decompositions of a morphism of $C(P)$ into elements of P. We first define a germ $P_\bullet(\mathrm{Id})$ whose set of objects is the union of the set of objects of all P_n for $n \geq 1$; this germ is thus graded. For morphisms, we start by taking all the morphisms of $\bigcup_n P_n(\mathrm{Id})$ as morphisms of degree 0. We will also add some morphisms of positive degree. If $\mathbf{a} = (a_1,\ldots,a_m)$ is an object of degree m we denote by $\mathbf{a}^{[k]}$ the object $(a_1,\ldots,a_m,\underbrace{1,\ldots,1}_{k})$ of degree $m+k$. Then we add a morphism $i_{\mathbf{a},k}$ from $\mathbf{a}$ to $\mathbf{a}^{[k]}$ which we declare to be of degree k. We add to the germ the products of a morphism $i_{\mathbf{a},k}$ with a morphism of degree 0. Finally we add the relations (*i.e.* define the following products) $i_{\mathbf{a},k}i_{\mathbf{a}^{[k]},l} = i_{\mathbf{a},k+l}$ and, for each morphism $\mathbf{a} \xrightarrow{f} \mathbf{b}$ of degree 0 between objects of degree m, the relations $fi_{\mathbf{b},k} = i_{\mathbf{a},k}f^{[k]}$ where $f^{[k]}$ is defined by $f_i^{[k]} = f_i$ for $i \leq m$ and $f_{m+1} = \ldots = f_{m+k+1} = 1$.

It follows from these relations that any product of $i_{\mathbf{a},k}$'s and of *one* morphism of degree 0 is in $P_\bullet(\mathrm{Id})$, and that a morphism in $P_\bullet(\mathrm{Id})$ is unique given its source and target (using 8.11).

The category $\mathcal{C}(P_\bullet(\mathrm{Id}))$ generated by $P_\bullet(\mathrm{Id})$ inherits a grading.

Proposition 8.14. *The category $\mathcal{C}(P_\bullet(\mathrm{Id}))$ is left locally Garside.*

Proof. The axioms for a germ are clear. The locally Garside germ axiom 3.6(G1) is also clear (in a bounded increasing sequence the degree becomes constant and we are reduced to the case of $P_n(\mathrm{Id})$ and 3.6(G4) is clear, using the unicity of morphisms between two objects.

We prove now 3.6(G3). Consider two maps $i_{\mathbf{a},k}f$ and $i_{\mathbf{a},l}g$. We may assume that $k \leq l$. Then $i_{\mathbf{a},l}f^{[l-k]}$ is a multiple of $i_{\mathbf{a},k}f$ and $i_{\mathbf{a},l}$ times the lcm of $f^{[l-k]}$ and g is the lcm of $i_{\mathbf{a},k}f$ and $i_{\mathbf{a},l}g$. Indeed any multiple of $i_{\mathbf{a},k}f$ of degree l is of the form $i_{\mathbf{a},l}h$ where by cancellation we must have $f \preccurlyeq i_{\mathbf{a}^{[k]},l-k}h$; since any morphism of degree $l-k$ extending f must start by $fi_{\mathbf{b},l-k} = i_{\mathbf{a}^{[k]},l-k}f^{[l-k]}$ (where $\mathbf{b}$ is the target of f) we have $f^{[l-k]} \preccurlyeq h$.

The proof of 3.6 (G3) is similar. □

Remark 8.15. Note that $C(P_n(\mathrm{Id}))$ is the full subcategory of $\mathcal{C}(P_\bullet(\mathrm{Id}))$ obtained by restricting the objects to those of P^n.

We can extend 8.12 to $\mathcal{C}(P_\bullet(\mathrm{Id}))$:

Lemma 8.16. *For any k there is a unique morphism $\mathbf{a} \to \mathrm{NF}(\mathbf{a})^{[k]}$.*

Proof. Using the relations in $C(P_\bullet(\mathrm{Id}))$, any morphism from $\mathbf{a}$ to $\mathrm{NF}(\mathbf{a})^{[k]}$ is of the form $i_{\mathbf{a},k}f$ where f is a morphism of degree 0 from $\mathbf{a}^{[k]}$ to $\mathrm{NF}(\mathbf{a})^{[k]}$. Since $\mathrm{NF}(\mathbf{a})^{[k]}$ is clearly the normal form of $\mathbf{a}^{[k]}$ we get the result by 8.12, using remark 8.15. □

Corollary 8.17. *Two objects of $\mathcal{C}(P_\bullet(\mathrm{Id}))$ are in the same connected component if and only if the product of their terms is the same.*

Proposition 8.18. *Let $\mathcal{O}$ be a functor from $C(P_\bullet(\mathrm{Id}))$ to a groupoid. Let us call* elementary isomorphism *a map of the form $\mathcal{O}(f)\mathcal{O}(i_{\mathbf{a}',1})^{-1}$ or $\mathcal{O}(i_{\mathbf{a}',1})\mathcal{O}(f)^{-1}$ where $\mathbf{a} \xrightarrow{f} \mathbf{a}'^{[1]} \in P_\bullet$ is of the form $\mathbf{a} = (a_1,\ldots,a_i,a_{i+1},\ldots,a_n) \xrightarrow{f} \mathbf{a}'^{[1]} = (a_1,\ldots,a_{i-1},a_ia_{i+1},\ldots,a_n,1)$. Then all the compositions of elementary isomorphisms between two objects in the image of $\mathcal{O}$ are equal.*

Proof. Given $\mathbf{a} = (a_1,\ldots,a_n)$ an object of P_n, and given $k \geq n$, let $g_{\mathbf{a},k}$ be the image by $\mathcal{O}$ of the unique map in $C(P_\bullet(\mathrm{Id}))$ between $\mathbf{a}$ and $NF(\mathbf{a})^{[k-\deg(\mathbf{a})]}$ (*cf.* 8.16). Then for any $k \geq n$ we have $g_{\mathbf{a},k} = \mathcal{O}(f)g_{\mathbf{a}'^{[1]},k}$ and $\mathcal{O}(i_{\mathbf{a}',1})g_{\mathbf{a}'^{[1]},k} = g_{\mathbf{a}',k}$ thus the elementary morphism $\mathcal{O}(f)\mathcal{O}(i_{\mathbf{a}',1})^{-1}$ between $\mathcal{O}(\mathbf{a})$ and $\mathcal{O}(\mathbf{a}')$ is equal to $g_{\mathbf{a},k}g^{-1}_{\mathbf{a}'^{[1]},k}$. It follows that, for k larger than the degree of all the objects involved, we find by composing the above formula along a path of elementary isomorphisms, that a composition of elementary isomorphisms between $\mathbf{a}$ and $\mathbf{b}$ is equal to $g_{\mathbf{a},k}g^{-1}_{\mathbf{b},k}$. Thus all such compositions are equal. □

An application to Deligne-Lusztig varieties. We give an application of the last proposition to the existence of generalized Schubert cells associated to the elements of the braid monoid. Let $\mathbf{G}$ be a reductive group over an algebraically closed field. Let W be the Weyl group, identified to the set of orbits of $\mathbf{G}$ on $\mathcal{B}\times\mathcal{B}$, where $\mathcal{B}$ is the variety of Borel subgroups. Let $B^+(W)$ the corresponding Artin-Tits monoid, and let $\mathbf{W}$ be the germ of simple elements of $B^+(W)$ (naturally in bijection with W), so $B^+(W) = C(\mathbf{W})$. To an object $(\mathbf{w}_1, \ldots, \mathbf{w}_n)$ of $C(\mathbf{W}_\bullet(\mathrm{Id}))$ we attach the variety

$$\mathcal{O}(\mathbf{w}_1, \ldots, \mathbf{w}_n) = \{(\mathbf{B}_1, \ldots, \mathbf{B}_{n+1}) \in \mathcal{B}^{n+1} \mid (\mathbf{B}_i, \mathbf{B}_{i+1}\}) \in \mathcal{O}(w_i)\},$$

where w_i is the image of $\mathbf{w}_i$ in W and $\mathcal{O}(w_i)$ is the orbit of $\mathbf{G}$ in $\mathcal{B}\times\mathcal{B}$ corresponding to w_i. To a morphism $\mathbf{w} \xrightarrow{f} \mathbf{v}$ of $C(\mathbf{W}_\bullet(\mathrm{Id}))$, given by $\mathbf{w}_i = f_i f'_i$, and $\mathbf{v}_i = f'_i f_{i+1}$ we associate the isomorphism $\mathcal{O}(f) : \mathcal{O}(\mathbf{w}) \to \mathcal{O}(\mathbf{v})$ which sends $\mathbf{B}_k$ to the unique Borel subgroup $\mathbf{B}'_k$ such that $(\mathbf{B}_k, \mathbf{B}'_k, \mathbf{B}_{k+1}) \in \mathcal{O}(f_k, f'_k)$ and $\mathbf{B}'_{n+1} = \mathbf{B}_{m+1}$. To the morphism $i_{\mathbf{w},k}$ we associate the isomorphism which maps $(\mathbf{B}_1, \ldots, \mathbf{B}_{n+1})$ to $(\mathbf{B}_1, \ldots, \mathbf{B}_n, \underbrace{\mathbf{B}_{n+1}, \ldots, \mathbf{B}_{n+1}}_{k+1})$.

Proposition 8.19. *$\mathcal{O}$ is a functor from $C(\mathbf{W}_\bullet(\mathrm{Id}))$ to the category of quasi-projective varieties with isomorphisms.*

Proof. We need to check that if $\mathbf{w} \xrightarrow{f} \mathbf{v}$ and $\mathbf{v} \xrightarrow{g} \mathbf{u}$ are such that $f, g, fg \in \mathbf{W}_\bullet(\mathrm{Id})$, then $\mathcal{O}(f)\mathcal{O}(g) = \mathcal{O}(fg)$. This results from the fact that if $(\mathbf{B}_k, \mathbf{B}'_k, \mathbf{B}_{k+1}) \in \mathcal{O}(f_k, f'_k)$, $(\mathbf{B}_{k+1}, \mathbf{B}'_{k+1}, \mathbf{B}_{k+2}) \in \mathcal{O}(f_{k+1}, f'_{k+1})$ and $(\mathbf{B}'_k, \mathbf{B}''_k, \mathbf{B}'_{k+1}) \in \mathcal{O}(g_k, g'_k)$ then $(\mathbf{B}_k, \mathbf{B}''_k, \mathbf{B}_{k+1}) \in \mathcal{O}(f_k g_k, (fg)'_k)$ since $g'_k = (fg)'_k f_{k+1}$. ☐

If $\mathbf{b} = \mathbf{w}_1 \ldots \mathbf{w}_n$, since we can pass from any decomposition of $\mathbf{b}$ into a product of elements of $\mathbf{W}$ to another by elementary isomorphisms, it follows that varieties associated to decompositions of the same element $\mathbf{b}$ of $B^+(W)$ are canonically isomorphic. Passing to the projective limit of these isomorphisms, we can define a variety $\mathcal{O}(\mathbf{b})$ associated to an element of $B^+(W)$. Note that we could also have applied theorem 7.1 to this situation, as Deligne did in [Deligne].

References

[Be1] D. Bessis, Garside categories, periodic loops and cyclic sets `math.GR/0610778`.

[Be2] D. Bessis, Finite complex reflection arrangements are $K(\pi, 1)$ `math.GT/0610777`.

[Bessis-Digne-Michel] D. Bessis, F. Digne and J. Michel, Springer theory in braid groups and the Birman-Ko-Lee monoid", *Pacific J. of Math* **205** (2002) 287–309.

[Brink-Howlett] B. Brink and R. B. Howlett, Normalizers of parabolic subgroups in Coxeter groups, *Inventiones Math.* **136**, (1999), 323–351.

[Bouc] S. Bouc, Homologie de certains ensembles de 2-sous-groupes des groupes symétriques, *Journal of Algebra* **150**, 158–186 (1992).

[Deligne] P. Deligne, Action du groupe des tresses sur une catégorie, *Inventiones math.* **128**, 159–175 (1997).

[DMR] F. Digne, J. Michel, R. Rouquier, Cohomologie des variétés de Deligne-Lusztig, *Advances in Math.* **209** 749–822 (2007).

[Godelle] Parabolic subgroups of Garside groups, *preprint.*

[Krammer1] D. Krammer, A class of Garside groupoid structures on the pure braid group `math.GR/0509165`.

[Krammer2] D. Krammer, The braid group of $\mathbb{Z}^n$ `math.GR/0603399`.

[Michel] J. Michel, A note on words in braid monoids, *J. of Algebra,* **215** (1999) 366–377.