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Multiple Qubits as Symplectic Polar Spaces of Order Two

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Abstract

It is surmised that the algebra of the Pauli operators on the Hilbert space of N -qubits is embodied in the geometry of the symplectic polar space of rank N and order two, $W_{2N-1}(2)$. The operators (discarding the identity) answer to the points of $W_{2N-1}(2)$, their partitionings into maximally commuting subsets correspond to spreads of the space, a maximally commuting subset has its representative in a maximal totally isotropic subspace of $W_{2N-1}(2)$ and, finally, “commuting” translates into “collinear” (or “perpendicular”).

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It is well known that a complete basis of operators in the Hilbert space of N -qubits, $N \geq 2$, can be given in terms of the Pauli operators — tensor products of classical 2×2 Pauli matrices. Although the Hilbert space in question is 2^N -dimensional, the operators' space is of dimension 4^N . Excluding the identity matrix, the set of $4^N - 1$ Pauli operators can be partitioned into $2^N + 1$ subsets, each comprising $2^N - 1$ mutually commuting elements [1]. The purpose of this note is to put together several important facts supporting the view that this operators' space can be identified with $W_{2N-1}(q = 2)$, the symplectic polar space of rank N and order two.

A (finite-dimensional) classical polar space (see [2–6] for more details) describes the geometry of a d -dimensional vector space over the Galois field $GF(q)$, $V(d, q)$, carrying a non-degenerate reflexive sesquilinear form σ . The polar space is called symplectic, and usually denoted as $W_{d-1}(q)$, if this form is bilinear and alternating, i.e., if $\sigma(x, x) = 0$ for all $x \in V(d, q)$; such a space exists only if $d = 2N$, where N is called its rank. A subspace of $V(d, q)$ is called totally isotropic if σ vanishes identically on it. $W_{2N-1}(q)$ can then be regarded as the space of totally isotropic subspaces of $PG(2N - 1, q)$, the ordinary $(2N - 1)$ -dimensional projective space over $GF(q)$, with respect to a symplectic form (also known as a null polarity), with its maximal totally isotropic subspaces, also called generators G , having dimension $N - 1$. For $q = 2$ this polar space contains

$$|W_{2N-1}(2)| = |PG(2N - 1, 2)| = 2^{2N} - 1 = 4^N - 1 \quad (1)$$

points and

$$|\Sigma(W_{2N-1}(2))| = (2 + 1)(2^2 + 1) \dots (2^N + 1) \quad (2)$$

generators [2–4]. An important object associated with any polar space is its *spread*, i.e., a set of generators partitioning its points. A spread S of $W_{2N-1}(q)$ is an $(N - 1)$ -spread of its ambient projective space $PG(2N - 1, q)$ [4, 5, 7], i.e., a set of $(N - 1)$ -dimensional subspaces of $PG(2N - 1, q)$ partitioning its points. The cardinalities of a spread and a generator of $W_{2N-1}(2)$ thus read

$$|S| = 2^N + 1 \quad (3)$$

and

$$|G| = 2^N - 1, \tag{4}$$

respectively [2, 3]. Finally, it needs to be mentioned that two distinct points of $W_{2N-1}(q)$ are called perpendicular if they are “isotropically” collinear, i. e., joined by a totally isotropic line of $W_{2N-1}(q)$; for $q = 2$ there are

$$\#\Delta = 2^{2N-1} \tag{5}$$

points that are *not* perpendicular to a given point of $W_{2N-1}(2)$ [2, 3].

Now, in light of Eq. (1), we can identify the Pauli operators with the points of $W_{2N-1}(2)$. If, further, we identify the operational concept “commuting” with the geometrical one “perpendicular,” from Eqs. (3) and (4) we readily see that the points lying on generators of $W_{2N-1}(2)$ correspond to maximally commuting subsets (MCSs) of operators and a spread of $W_{2N-1}(2)$ is nothing but a partitioning of the whole set of operators into MCSs. From Eq. (2) we then infer that the operators’ space possesses $(2+1)(2^2+1)\dots(2^N+1)$ MCSs and, finally, Eq. (5) tells us that there are 2^{2N-1} operators that do *not* commute with a given operator; the last two statements are, for $N > 2$, still conjectures to be rigorously proven. However, the case of two-qubits ($N = 2$) is recovered in full generality [1, 8, 9], with the geometry behind being that of the *generalized quadrangle of order two* [9] — the simplest nontrivial symplectic polar space.¹

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References

- [1] Lawrence, J., Brukner, Č., and Zeilinger, A., “Mutually unbiased binary observable sets on N qubits,” *Physical Review A* 65, 032320 (2002).
- [2] Ball, S., “The geometry of finite fields,” *Quaderni Elettronici del Seminario di Geometria Combinatoria 2E* (2001); available on-line from <http://www.mat.uniroma1.it/~combinat/quaderni/>.
- [3] Cameron, P. J., “Projective and polar spaces,” available on-line from <http://www.maths.qmw.ac.uk/~pjc/pps/>.
- [4] De Clerck, F., and Van Maldeghem, H., “Ovoids and spreads of polar spaces and generalized polygons,” a lecture given at an intensive course on “Galois Geometry and Generalized Polygons,” University of Ghent, April 14–25, 1998; available from <http://cage.rug.ac.be/~fdc/intensivecourse/ovspr.ps>.
- [5] Hirschfeld, J. W. P., and Thas, J. A., “General Galois Geometries” (Oxford University Press, Oxford, 1991).
- [6] Payne, S. E., and Thas, J. A., “Finite Generalized Quadrangles” (Pitman, London, 1984).
- [7] Thas, J. A., “Ovoids and spreads in classical polar spaces,” *Geom. Dedicata* 10, 135–144 (1981).
- [8] Wootters, W. K., “Picturing qubits in phase space,” *IBM J. Res. Dev.* 48, 99–110 (2004).
- [9] Saniga, M., Planat, M., and Pracna, P., “Projective ring line encompassing two-qubits,” *Theoretical and Mathematical Physics*, submitted; quant-ph/0611063-v4.

¹This object can also be recognized as the projective line over the Jordan system of the full 2×2 matrix ring with coefficients in $GF(2)$ [9].