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ON THE STRUCTURE OF TRIANGULATED CATEGORIES
WITH FINITELY MANY INDECOMPOSABLES

CLAIRE AMIOT

Abstract. We study the problem of classifying triangulated categories with
finite-dimensional morphism spaces and finitely many indecomposables over
an algebraically closed field. We obtain a new proof of the following result
due to Xiao and Zhu: the Auslander-Reiten quiver of such a category is of
the form $\mathbb{Z}\Delta/G$ where $\Delta$ is a disjoint union of simply laced Dynkin diagrams
and $G$ a weakly admissible group of automorphisms of $\mathbb{Z}\Delta$. Then we prove
that for ‘most’ groups $G$, the category $\mathcal{T}$ is standard, i.e. $k$-linearly equivalent
to an orbit category $\mathcal{D}b(\text{mod } \Delta)/\Phi$. This happens in particular when $\mathcal{T}$ is
maximal $d$-Calabi-Yau with $d \geq 2$. Moreover, if $\mathcal{T}$ is standard and algebraic,
we can even construct a triangle equivalence between $\mathcal{T}$ and the corresponding
orbit category. Finally we give a sufficient condition for the category of projectives
of a Frobenius category to be triangulated. This allows us to construct
non standard 1-Calabi-Yau categories using deformed preprojective algebras
of generalized Dynkin type.

Introduction

Let $k$ be an algebraically closed field and $\mathcal{T}$ a small $k$-linear triangulated category
(see [31]) with split idempotents. We assume that

a) $\mathcal{T}$ is Hom-finite, i.e. the space $\text{Hom}_\mathcal{T}(X, Y)$ is finite-dimensional for all
objects $X, Y$ of $\mathcal{T}$.

It follows that indecomposable objects of $\mathcal{T}$ have local endomorphism rings and
that each object of $\mathcal{T}$ decomposes into a finite direct sum of indecomposables [1, 3.3]. We assume moreover that

b) $\mathcal{T}$ is locally finite, i.e. for each indecomposable $X$ of $\mathcal{T}$, there are at most
finitely many isoclasses of indecomposables $Y$ such that $\text{Hom}_\mathcal{T}(X, Y) \neq 0$.

It was shown in [32] that condition b) implies its dual. Condition b) holds in
particular if we have

b') $\mathcal{T}$ is additively finite, i.e. there are only finitely many isomorphism classes
of indecomposables in $\mathcal{T}$.

The study of particular classes of such triangulated categories $\mathcal{T}$ has a long
history. Let us briefly recall some of its highlights:

1) If $A$ is a representation-finite selfinjective algebra, then the stable category
$\mathcal{T}$ of finite-dimensional (right) $A$-modules satisfies our assumptions and is additively
finite. The structure of the underlying $k$-linear category of $\mathcal{T}$ was determined by
C. Riedtmann in [24], [25], [26] and [27].

2) In [11], D. Happel showed that the bounded derived category of the
category of finite-dimensional representations of a representation-finite quiver is
locally finite and described its underlying $k$-linear category.

3) The stable category $\mathcal{CM}(R)$ of Cohen-Macaulay modules over a commuta-
tive complete local Gorenstein isolated singularity $R$ of dimension $d$ is a Hom-finite
triangulated category which is $(d-1)$-Calabi-Yau (cf. for example [17] and [31]).
In [2], M. Auslander and I. Reiten showed that if the dimension of $R$ is 1, then the
category $\mathcal{CM}(R)$ is additively finite and computed the shape of components of its
Auslander-Reiten quiver.

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The cluster category $C_Q$ of a finite quiver $Q$ without oriented cycles was introduced in [7] if $Q$ is an orientation of a Dynkin diagram of type $\mathbb{A}$ and in [6] in the general case. The category $C_Q$ is triangulated [19] and, if $Q$ is representation-finite, satisfies $a)$ and $b')$.

In a recent article [32], J. Xiao and B. Zhu determined the structure of the Auslander-Reiten quiver of a locally finite triangulated category. In this paper, we obtain the same result with a new proof in section 4, namely that each connected component of the Auslander-Reiten quiver of the category $T$ is of the form $\mathbb{Z}\Delta/G$, where $\Delta$ is a simply laced Dynkin diagram and $G$ is trivial or a weakly admissible group of automorphisms. We are interested in the $k$-linear structure of $T$. If the Auslander-Reiten quiver of $T$ is of the form $\mathbb{Z}\Delta$, we show that the category $T$ is standard, i.e. it is equivalent to the mesh category $k(\mathbb{Z}\Delta)$. Then in section 5, we prove that $T$ is standard if the number of vertices of $\Gamma = \mathbb{Z}\Delta/G$ is strictly greater than the number of isoclasses of indecomposables of $\text{mod} \ k \Delta$. In the last section, using [4] we construct examples of non standard triangulated categories such that $\Gamma = \mathbb{Z}\Delta/\tau$.

Finally, in the standard cases, we are interested in the triangulated structure of $T$. For this, we need to make additional assumptions on $T$. If the Auslander-Reiten quiver is of the form $\mathbb{Z}\Delta$, and if $T$ is the base of a tower of triangulated categories [13], we show that there is a triangle equivalence between $T$ and the derived category $D^b(\text{mod} \ k \Delta)$. For the additively finite cases, we have to assume that $T$ is standard and algebraic in the sense of [20]. We then show that $T$ is (algebraically) triangle equivalent to the orbit category of $D^b(\text{mod} \ k \Delta)$ under the action of a weakly admissible group of automorphisms. In particular, for each $d \geq 2$, the algebraic triangulated categories with finitely many indecomposables which are maximal Calabi-Yau of CY-dimension $d$ are parametrized by the simply laced Dynkin diagrams.

Our results apply in particular to many stable categories $\text{mod} \ A$ of representation-finite selfinjective algebras $A$. These algebras were classified up to stable equivalence by C. Riedtmann [25, 27] and H. Asashiba [1]. In [3], J. Białkowski and A. Skowroński give a necessary and sufficient condition on these algebras so that their stable categories $\text{mod} \ A$ are Calabi-Yau. In [13] and [14], T. Holm and P. Jørgensen prove that certain stable categories $\text{mod} \ A$ are in fact $d$-cluster categories. These results can also be proved using our corollary 7.0.6.

This paper is organized as follows: In section 1 we prove that $T$ has Auslander-Reiten triangles. Section 2 is dedicated to definitions about stable valued translation quivers and admissible automorphisms groups [12, 13, 8]. We show in section 3 that the Auslander-Reiten quiver of $T$ is a stable valued quiver and in section 4 we reprove the result of J. Xiao and B. Zhu [32]: The Auslander-Reiten quiver is a disjoint union of quivers $\mathbb{Z}\Delta/G$, where $\Delta$ is a Dynkin quiver of type $\mathbb{A}$, $\mathbb{D}$ or $\mathbb{E}$, and $G$ a weakly admissible group of automorphisms. In section 5, we construct a covering functor $D^b(\text{mod} \ k \Delta) \to T$ using Riedtmann’s method [24]. Then, in section 6, we exhibit some combinatorial cases in which $T$ has to be standard, in particular when $T$ is maximal $d$-Calabi-Yau with $d \geq 2$. Section 7 is dedicated to the algebraic case. If $T$ is algebraic and standard, we can construct a triangle equivalence between $T$ and an orbit category. If $P$ is a $k$-category such that $\text{mod} \ P$ is a Frobenius category satisfying certain conditions, we will prove in section 8 that $P$ has naturally a triangulated structure. This allows us to deduce in section 9 that the category $\text{proj}P^f(\Delta)$ of the projective modules over a deformed preprojective algebra of generalized Dynkin type $\mathbb{A}$ is naturally triangulated and to reduce the classification of the additively finite triangulated categories which are 1-Calabi-Yau to that of the deformed preprojective algebras in the sense of [4].
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Notation and terminology

We work over an algebraically closed field $k$. By a triangulated category, we mean a $k$-linear triangulated category $T$. We write $S$ for the suspension functor of $T$ and $U \xrightarrow{u} V \xrightarrow{v} W \xrightarrow{w} SU$ for a distinguished triangle. We say that $T$ is Hom-finite if for each pair $X, Y$ of objects in $T$, the space $\text{Hom}_T(X,Y)$ is finite-dimensional over $k$. The category $T$ will be called a Krull-Schmidt category if each object is isomorphic to a finite direct sum of indecomposable objects and the endomorphism ring of an indecomposable object is a local ring. This implies that idempotents of $T$ split [11, 3.2]. The category $T$ will be called locally finite if for each indecomposable $X$ of $T$, there are only finitely many isoclasses of indecomposables $Y$ such that $\text{Hom}_T(X,Y) \neq 0$. This property is selfdual by [32, prop 1.1].

The Serre functor will be denoted by $\nu$ (see definition in section 1). The Auslander-Reiten translation will always be denoted by $\tau$ (section 1).

Let $\mathcal{T}$ and $\mathcal{T}'$ be two triangulated categories. An $\mathcal{S}$-functor $(F, \phi)$ is given by a $k$-linear functor $F : \mathcal{T} \to \mathcal{T}'$ and a functor isomorphism $\phi$ between the functors $F \circ S$ and $S' \circ F$, where $S$ is the suspension of $\mathcal{T}$ and $S'$ the suspension of $\mathcal{T}'$. The notion of $\nu$-functor, or $\tau$-functor is then clear. A triangle functor is an $\mathcal{S}$-functor $(F, \phi)$ such that for each triangle $U \xrightarrow{u} V \xrightarrow{v} W \xrightarrow{w} SU$ of $\mathcal{T}$, the sequence $FU \xrightarrow{Fu} FV \xrightarrow{Fv} FW \xrightarrow{\phi_v \circ Fu} S'FU$ is a triangle of $\mathcal{T}'$.

The category $\mathcal{T}$ is Calabi-Yau if there exists an integer $d > 0$ such that we have a triangle functor isomorphism between $S^d$ and $\nu$. We say that $\mathcal{T}$ is maximal Calabi-Yau if $\mathcal{T}$ is $d$-Calabi-Yau and if for each covering functor $\mathcal{T}' \to \mathcal{T}$ with $\mathcal{T}'$ $d$-Calabi-Yau, we have a $k$-linear equivalence between $\mathcal{T}$ and $\mathcal{T}'$.

For an additive $k$-category $\mathcal{E}$, we write $\text{mod} \mathcal{E}$ for the category of contravariant finitely presented functors from $\mathcal{E}$ to $\text{mod} k$ (section 1), and if the projectives of $\text{mod} \mathcal{E}$ coincide with the injectives, $\text{mod} \mathcal{E}$ will be the stable category.

1. Serre duality and Auslander-Reiten triangles

1.1. Serre duality. Recall from [23] that a Serre functor for $\mathcal{T}$ is an autoequivalence $\nu : \mathcal{T} \to \mathcal{T}$ together with an isomorphism $D\text{Hom}_\mathcal{T}(X,?) \simeq \text{Hom}_\mathcal{T}(?,\nu X)$ for each $X \in \mathcal{T}$, where $D$ is the duality $\text{Hom}_k(?,k)$.

Theorem 1.1.1. Let $\mathcal{T}$ be a Krull-Schmidt, locally finite triangulated category. Then $\mathcal{T}$ has a Serre functor $\nu$.

Proof. Let $X$ be an object of $\mathcal{T}$. We write $X^\wedge$ for the functor $\text{Hom}_\mathcal{T}(?,X)$ and $F$ for the functor $D\text{Hom}_\mathcal{T}(X,?)$. Using the lemma [23, I.1.6] we just have to show that $F$ is representable. Indeed, the category $\mathcal{T}^{\text{op}}$ is locally finite as well. The proof is in two steps.

Step 1: The functor $F$ is finitely presented.

Let $Y_1, \ldots, Y_r$ be representatives of the isoclasses of indecomposable objects of $\mathcal{T}$ such that $FY_i$ is not zero. The space $\text{Hom}(Y_i^\wedge, F)$ is finite-dimensional over $k$. Indeed it is isomorphic to $FY_i$ by the Yoneda lemma. Therefore, the functor $\text{Hom}(Y_i^\wedge, F) \otimes_k Y_i^\wedge$ is representable. We get an epimorphism from a representable
functor to $F$:

$$\bigoplus_{i=1}^{r} \text{Hom}(Y_i^\wedge, F) \otimes_k Y_i^\wedge \rightarrow F.$$  

By applying the same argument to its kernel we get a projective presentation of $F$ of the form $U^\wedge \rightarrow V^\wedge \rightarrow F \rightarrow 0$, with $U$ and $V$ in $T$.

Step 2: A cohomological functor $H : T^{\text{op}} \rightarrow \text{mod } k$ is representable if and only if it is finitely presented.

Let $U^\wedge \xrightarrow{u^\wedge} V^\wedge \xrightarrow{\phi} H \xrightarrow{\nu} 0$ be a presentation of $H$. We form a triangle $U \xleftarrow{u} V \xrightarrow{v} W \xrightarrow{w} SU$. We get an exact sequence

$$U^\wedge \xrightarrow{u^\wedge} V^\wedge \xrightarrow{v^\wedge} W^\wedge \xrightarrow{w^\wedge} (SU)^\wedge.$$  

Since the composition of $\phi$ with $u^\wedge$ is zero and $H$ is cohomological, the morphism $\phi$ factors through $v^\wedge$. But $H$ is the cokernel of $u^\wedge$, so $v^\wedge$ factors through $\phi$. We obtain a commutative diagram:

$$\begin{array}{ccc}
U^\wedge & \xrightarrow{u^\wedge} & V^\wedge \\
\downarrow & & \downarrow \\
H & \xrightarrow{i} & (SU)^\wedge \\
\uparrow & & \phi' \\
W^\wedge & \xrightarrow{w^\wedge} & (SU)^\wedge \\
\end{array}$$

The equality $\phi' \circ i \circ \phi = \phi' \circ v^\wedge = \phi$ implies that $\phi' \circ i$ is the identity of $H$ because $\phi$ is an epimorphism. We deduce that $H$ is a direct factor of $W^\wedge$. The composition $i \circ \phi' = e^\wedge$ is an idempotent. Then $e \in \text{End}(W)$ splits and we get $H = W'^\wedge$ for a direct factor $W'$ of $W$.

\[ \square \]

1.2. Auslander-Reiten triangles.

**Definition 1.2.1.** [10] A triangle $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} SX$ of $T$ is called an Auslander-Reiten triangle or AR-triangle if the following conditions are satisfied:

- (AR1) $X$ and $Z$ are indecomposable objects;
- (AR2) $w \neq 0$;
- (AR3) if $f : W \rightarrow Z$ is not a retraction, there exists $f' : W \rightarrow Y$ such that $vf' = f$;
- (AR3') if $g : X \rightarrow V$ is not a section, there exists $g' : Y \rightarrow V$ such that $g'u = g$.

Let us recall that, if (AR1) and (AR2) hold, the conditions (AR3) and (AR3') are equivalent. We say that a triangulated category $T$ has Auslander-Reiten triangles if, for any indecomposable object $Z$ of $T$, there exists an AR-triangle ending at $Z$: $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} SX$. In this case, the AR-triangle is unique up to triangle isomorphism inducing the identity of $Z$.

The following proposition is proved in [23, Proposition I.2.3]

**Proposition 1.2.1.** The category $T$ has Auslander-Reiten triangles.

The composition $\tau = S^{-1}\nu$ is called the Auslander-Reiten translation. An AR-triangle of $T$ ending at $Z$ has the form:

$$\tau Z \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \nu Z.$$
2. Valued translation quivers and automorphism groups

2.1. Translation quivers. In this section, we recall some definitions and notations concerning quivers [8]. A quiver $Q = (Q_0, Q_1, s, t)$ is given by the set $Q_0$ of its vertices, the set $Q_1$ of its arrows, a source map $s$ and a tail map $t$. If $x \in Q_0$ is a vertex, we denote by $x^+$ the set of direct successors of $x$, and by $x^-$ the set of its direct predecessors. We say that $Q$ is locally finite if for each vertex $x \in Q_0$, there are finitely many arrows ending at $x$ and starting at $x$ (in this case, $x^+$ and $x^-$ are finite sets). The quiver $Q$ is said to be without double arrows, if two different arrows cannot have the same tail and source.

Definition 2.1.1. A stable translation quiver $(Q, \tau)$ is a locally finite quiver without double arrows with a bijection $\tau : Q_0 \rightarrow Q_0$ such that $(\tau x)^+ = x^-$ for each vertex $x$. For each arrow $\alpha : x \rightarrow y$, let $\sigma_{\alpha}$ the unique arrow $\tau y \rightarrow x$.

Note that a stable translation quiver can have loops.

Definition 2.1.2. A valued translation quiver $(Q, \tau, a)$ is a stable translation quiver $(Q, \tau)$ with a map $a : Q_1 \rightarrow \mathbb{N}$ such that $a(\alpha) = a(\sigma_{\alpha})$ for each arrow $\alpha$. If $\alpha$ is an arrow from $x$ to $y$, we write $a_{xy}$ instead of $a(\alpha)$.

Definition 2.1.3. Let $\Delta$ be an oriented tree. The repetition of $\Delta$ is the quiver $Z\Delta$ defined as follows:

- $(Z\Delta)_0 = Z \times \Delta_0$
- $(Z\Delta)_1 = Z \times \Delta_1 \cup \sigma(Z \times \Delta_1)$ with arrows $(n, \alpha) : (n, x) \rightarrow (n, y)$ and $\sigma(n, \alpha) : (n-1, y) \rightarrow (n, x)$ for each arrow $\alpha : x \rightarrow y$ of $\Delta$.

The quiver $Z\Delta$ with the translation $\tau(n, x) = (n-1, x)$ is clearly a stable translation quiver which does not depend (up to isomorphism) on the orientation of $\Delta$ (see [24]).

2.2. Groups of weakly admissible automorphisms.

Definition 2.2.1. An automorphism group $G$ of a quiver is said to be admissible [24] if no orbit of $G$ intersects a set of the form $\{x\} \cup x^+$ or $\{x\} \cup x^-$ in more than one point. It said to be weakly admissible [8] if, for each $g \in G - \{1\}$ and for each $x \in Q_0$, we have $x^+ \cap (gx)^+ = \emptyset$.

Note that an admissible automorphism group is a weakly admissible automorphism group. Let us fix a numbering and an orientation of the simply-laced Dynkin trees.

\[ \mathbb{A}_n : \begin{array}{c} 1 \rightarrow 2 \rightarrow \cdots \rightarrow n-1 \rightarrow n \\ n-1 \rightarrow n \end{array} \]

\[ \mathbb{D}_n : \begin{array}{c} 1 \rightarrow 2 \cdots \rightarrow n-2 \rightarrow n \\ n \rightarrow n \end{array} \]

\[ \mathbb{E}_n : \begin{array}{c} 1 \rightarrow 2 \rightarrow 3 \rightarrow 4 \rightarrow \cdots \rightarrow n \\ 4 \rightarrow 3 \rightarrow 2 \rightarrow n \end{array} \]

Let $\Delta$ be a Dynkin tree. We define an automorphism $S$ of $Z\Delta$ as follows:

- if $\Delta = \mathbb{A}_n$, then $S(p, q) = (p + q, n + 1 - q)$;
• if $\Delta = \mathbb{D}_n$ with $n$ even, then $S = \tau^{-n+1}$;
• if $\Delta = \mathbb{D}_n$ with $n$ odd, then $S = \tau^{-n+1}\phi$ where $\phi$ is the automorphism of $\mathbb{D}_n$, which exchanges $n$ and $n-1$;
• if $\Delta = \mathbb{E}_6$, then $S = \phi\tau^{-6}$ where $\phi$ is the automorphism of $\mathbb{E}_6$ which exchanges 2 and 5, and 1 and 6;
• if $\Delta = \mathbb{E}_7$, then $S = \tau^{-9}$;
• if $\Delta = \mathbb{E}_8$, then $S = \tau^{-15}$.

In [24, Anhang 2], Riedtmann describes all admissible automorphism groups of Dynkin diagrams. Here is a more precise result:

**Theorem 2.2.1.** Let $\Delta$ be a Dynkin tree and $G$ a non trivial group of weakly admissible automorphisms of $\mathbb{Z}\Delta$. Then $G$ is isomorphic to $\mathbb{Z}$, and here is a list of its possible generators:

- if $\Delta = \mathbb{A}_n$ with $n$ odd, possible generators are $\tau^r$ and $\phi\tau^r$ with $r \geq 1$, where $\phi = \tau^{n+2}S$ is an automorphism of $\mathbb{Z}\Delta$ of order 2;
- if $\Delta = \mathbb{A}_n$ with $n$ even, then possible generators are $\rho^r$, where $r \geq 1$ and where $\rho = \tau^{\frac{n}{2}}$. (Since $\rho^2 = \tau^{-1}$, $\tau^r$ is a possible generator.)
- if $\Delta = \mathbb{D}_n$ with $n \geq 5$, then possible generators are $\tau^r$ and $\tau^r\phi$, where $r \geq 1$ and where $\phi = (n-1,n)$ is the automorphism of $\mathbb{D}_n$, exchanging $n$ and $n-1$.
- if $\Delta = \mathbb{D}_4$, then possible generators are $\phi\tau^r$, where $r \geq 1$ and where $\phi$ belongs to $\mathfrak{S}_3$, the permutation group on 3 elements seen as subgroup of automorphisms of $\mathbb{D}_4$.
- if $\Delta = \mathbb{E}_6$, then possible generators are $\tau^r$ and $\phi\tau^r$, where $r \geq 1$ and where $\phi$ is the automorphism of $\mathbb{E}_6$, exchanging 2 and 5, and 1 and 6.
- if $\Delta = \mathbb{E}_7$ with $n = 7,8$, possible generators are $\tau^r$, where $r \geq 1$.

The unique weakly admissible automorphism group which is not admissible exists for $\mathbb{A}_n$, $n$ even, and is generated by $\rho$.

3. **Property of the Auslander-Reiten translation**

We define the Auslander-Reiten quiver $\Gamma_T$ of the category $\mathcal{T}$ as a valued quiver $(\Gamma, a)$. The vertices are the isoclasses of indecomposable objects. Given two indecomposable objects $X$ and $Y$ of $\mathcal{T}$, we draw one arrow from $x = [X]$ to $y = [Y]$ if the vector space $R(X,Y)/R^2(X,Y)$ is not zero, where $R(?,?)$ is the radical of the bifunctor $\text{Hom}_\mathcal{T}(?,?)$. A morphism of $R(X,Y)$ which does not vanish in the quotient $R(X,Y)/R^2(X,Y)$ will be called irreducible. Then we put

$$a_{xy} = \dim_k R(X,Y)/R^2(X,Y).$$

Remark that the fact that $\mathcal{T}$ is locally finite implies that its AR-quiver is locally finite. The aim of this section is to show that $\Gamma_T$ with the translation $\tau$ defined in the first part is a valued translation quiver. In other words, we want to show the proposition:

**Proposition 3.0.2.** If $X$ and $Y$ are indecomposable objects of $\mathcal{T}$, we have the equality

$$\dim_k R(X,Y)/R^2(X,Y) = \dim_k R(\tau Y, X)/R^2(\tau Y, X).$$

Let us recall some definitions [11].

**Definition 3.0.2.** A morphism $g : Y \to Z$ is called sink morphism if the following hold:

1. $g$ is not a retraction;
2. if $h : M \to Z$ is not a retraction, then $h$ factors through $g$;
3. if $u$ is an endomorphism of $Y$ which satisfies $gu = u$, then $u$ is an automorphism.
Dually, a morphism \( f : X \to Y \) is called source morphism if the following hold:

1. \( f \) is not a section;
2. if \( h : X \to M \) is not a section, then \( h \) factors through \( f \);
3. if \( u \) is an endomorphism of \( Y \) which satisfies \( uf = f \), then \( u \) is an automorphism.

These conditions imply that \( X \) and \( Z \) are indecomposable. Obviously, if \( X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} SX \) is an AR-triangle, then \( u \) is a source morphism and \( v \) is a sink morphism. Conversely, if \( v \in \text{Hom}_T(X,Y) \) is a sink morphism (or if \( u \in \text{Hom}_T(X,Y) \) is a source morphism), then there exists an AR-triangle \( X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} SX \) (see \([11, I 4.5]\)).

The following lemma (and the dual statement) is proved in \([28, 2.2.5]\).

**Lemma 3.0.3.** Let \( g \) be a morphism from \( Y \to Z \), where \( Z \) is indecomposable and \( Y = \bigoplus_{i=1}^r Y_i^{n_i} \) is the decomposition of \( Y \) into indecomposables. Then the morphism \( g \) is a sink morphism if and only if the following hold:

1. For each \( i = 1, \ldots, r \) and \( j = 1, \ldots, n_i \), the morphism \( g_{i,j} \) belongs to the radical \( \text{Rad}(Y_i, Z) \).
2. For each \( i = 1, \ldots, r \), the family \( (g_{i,j})_{j=1,\ldots,n_i} \) forms a \( k \)-basis of the space \( \text{Rad}(Y_i, Z)/\text{Rad}^2(Y_i, Z) \).
3. If \( h \in \text{Hom}_T(Y', Z) \) is irreducible and \( Y' \) indecomposable, then \( h \) factors through \( g \) and \( Y' \) is isomorphic to \( Y_i \) for some \( i \).

Using this lemma, it is easy to see that proposition \( 3.0.2 \) holds. Thus, the Auslander-Reiten quiver \( \Gamma_T = (\Gamma, \tau, \mathcal{A}) \) of the category \( T \) is a valued translation quiver.

### 4. Structure of the Auslander-Reiten quiver

This section is dedicated to an other proof of a theorem due to J. Xiao and B. Zhu \([13]\):

**Theorem 4.0.4.** \([13]\) Let \( T \) be a Krull-Schmidt, locally finite triangulated category. Let \( \Gamma \) be a connected component of the AR-quiver of \( T \). Then there exists a Dynkin tree \( \Delta \) of type \( A, D, \text{ or } E \), a weakly admissible automorphism group \( G \) of \( \mathbb{Z}\Delta \) and an isomorphism of valued translation quiver

\[ \theta : \Gamma \sim \mathbb{Z}\Delta/G. \]

The underlying graph of the tree \( \Delta \) is unique up to isomorphism (it is called the type of \( \Gamma \)), and the group \( G \) is unique up to conjugacy in \( \text{Aut}(\mathbb{Z}\Delta) \).

In particular, if \( T \) has an infinite number of isoclasses of indecomposable objects, then \( G \) is trivial, and \( \Gamma \) is the repetition quiver \( \mathbb{Z}\Delta \).

#### 4.1. Auslander-Reiten quivers with a loop

In this section, we suppose that the Auslander-Reiten quiver of \( T \) contains a loop, i.e. there exists an arrow with same tail and source. Thus, we suppose that there exists an indecomposable \( X \) of \( T \) such that

\[ \dim_k \text{Rad}(X, X)/\text{Rad}^2(X, X) \geq 1. \]

**Proposition 4.1.1.** Let \( X \) be an indecomposable object of \( T \). Suppose that we have \( \dim_k \text{Rad}(X, X)/\text{Rad}^2(X, X) \geq 1. \) Then \( \tau X \) is isomorphic to \( X \).

To prove this, we need a lemma.

**Lemma 4.1.2.** Let \( X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} \cdots \xrightarrow{f_n} X_{n+1} \) be a sequence of irreducible morphisms between indecomposable objects with \( n \geq 2 \). If the composition \( f_n \circ f_{n-1} \cdots f_1 \) is zero, then there exists an \( i \) such that \( \tau^{-1}X_i \) is isomorphic to \( X_{i+2} \).
Proof. The proof proceeds by induction on $n$. Let us show the assertion for $n = 2$. Suppose $X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} X_3$ is a sequence such that $f_2 \circ f_1 = 0$. We can then construct an AR-triangle:

$$
\xymatrix{ X_1 \ar[r]^{(f_1,f_2)^T} \ar[d]_{(f_1,0)} & X_2 \oplus X_1^{(g_1,g_2)} \ar[r]_{\tau^{-1}X_1} & SX_1 \ar[d]_{\beta} \\
X_3}
$$

The composition $f_2 \circ f_1$ is zero, thus the morphism $f_2$ factors through $g_1$. As the morphisms $g_1$ and $f_2$ are irreducible, we conclude that $\beta$ is a retraction, and $X_3$ a direct summand of $\tau^{-1}X_1$. But $X_1$ is indecomposable, so $\beta$ is an isomorphism between $X_3$ and $\tau^{-1}X_1$.

Now suppose that the property holds for an integer $n - 1$ and that we have $f_n \ldots f_1 = 0$. If the composition $f_{n-1} \ldots f_1$ is zero, the proposition holds by induction. So we can suppose that for $i \leq n - 2$, the objects $\tau^{-1}X_i$ and $X_{i+2}$ are not isomorphic. We show now by induction on $i$ that for each $i \leq n - 1$, there exists a map $\beta_i : \tau^{-1}X_i \to X_{i+1}$ such that $f_n \ldots f_{i+1} = \beta_i g_i$ where $g_i : X_{i+1} \to \tau^{-1}X_i$ is an irreducible morphism. For $i = 1$, we construct an AR-triangle:

$$
\xymatrix{ X_1 \ar[r]^{(f_1,f_2)^T} \ar[d]_{(f_1,0)} & X_2 \oplus X_1^{(g_1,g_2)} \ar[r]_{\tau^{-1}X_1} & SX_1 \ar[d]_{\beta} \\
X_{n+1}}
$$

As the composition $f_n \ldots f_1$ is zero, we have the factorization $f_n \ldots f_2 = \beta_1 g_1$.

Now for $i$, as $\tau^{-1}X_{i-1}$ is not isomorphic to $X_{i+1}$, there exists an AR-triangle of the form:

$$
\xymatrix{ X_i^{(g_{i-1},f_i,f_i)^T} \ar[r]_{\tau^{-1}X_{i-1}} \ar[d]_{(-\beta_{i-1},f_{n}f_{i+1},0)} & X_{i+1} \oplus X_i^{(g_i,g_i,g_i)} \ar[r]_{\tau^{-1}X_i} & SX_i \ar[d]_{\beta_i} \\
X_{n+1}}
$$

By induction, $-\beta_{i-1} g_{i-1} + f_n \ldots f_{i+1} f_i$ is zero, thus $f_n \ldots f_{i+1}$ factors through $g_i$. This property is true for $i = n - 1$, so we have a map $\beta_{n-1} : \tau^{-1}X_{n-1} \to X_{n+1}$ such that $\beta_{n-1} g_{n-1} = f_n$. As $g_{n-1}$ and $f_n$ are irreducible, we conclude that $\beta_{n-1}$ is an isomorphism between $X_{n+1}$ and $\tau^{-1}X_{n-1}$.

Now we are able to prove proposition 4.1.1. There exists an irreducible map $f : X \to X$. Suppose that $X$ and $\tau X$ are not isomorphic. Then from the previous lemma, the endomorphism $f^n$ is non zero for each $n$. But since $T$ is a Krull-Schmidt, locally finite category, a power of the radical $R(X,X)$ vanishes. This is a contradiction.

4.2. Proof of theorem 4.0.4. Let $\bar{\Gamma} = (\bar{\Gamma}_0, \bar{\Gamma}_1, \bar{a})$ be the valued translation quiver obtained from $\Gamma$ by removing the loops, i.e. we have $\bar{\Gamma}_0 = \Gamma_0$, $\bar{\Gamma}_1 = \{ \alpha \in \Gamma_1$ such that $s(\alpha) \neq t(\alpha) \}$, and $\bar{a} = a_{\Gamma_1}$.

Lemma 4.2.1. The quiver $\bar{\Gamma} = (\bar{\Gamma}_0, \bar{\Gamma}_1, \bar{a})$ with the translation $\tau$ is a valued translation quiver without loop.

Proof. We have to check that the map $\sigma$ is well-defined. But from proposition 4.1.1 if $\alpha$ is a loop on a vertex $x$, $\sigma(\alpha)$ is the unique arrow from $\tau x = x$ to $x$, i.e.
σ(α) = α. Thus ˜Γ is obtained from Γ by removing some σ-orbits and it keeps the structure of stable valued translation quiver.

Now, we can apply Riedtmann’s Struktursatz [2] and the result of Happel-Preiser-Ringel [13]. There exist a tree Δ and an admissible automorphism group G (which may be trivial) of ZΔ such that ˜Γ is isomorphic to ZΔ/G as a valued translation quiver. The underlying graph of the tree Δ is then unique up to isomorphism and the group G is unique up to conjugacy in Aut(ZΔ). Let x be a vertex of Δ. We write ˜x for the image of x by the map:

\[ \Delta \longrightarrow \mathbb{Z} \Delta \overset{\tau}{\longrightarrow} \mathbb{Z} \Delta / G \cong \hat{\Gamma} \longrightarrow \Gamma. \]

Let C : Δ0 × Δ0 → Z be the matrix defined as follows:

- C(x, y) = -a_{xy} (resp. -a_{yx}) if there exists an arrow from x to y (resp. from y to x) in Δ,
- C(x, x) = 2 - a_{xy},
- C(x, y) = 0 otherwise.

The matrix C is symmetric; it is a ‘generalized Cartan matrix’ in the sense of [12]. If we remove the loops from the ‘underlying graph of C’ (in the sense of [12]), we get the underlying graph of Δ.

In order to apply the result of Happel-Preiser-Ringel [12, section 2], we have to show:

**Lemma 4.2.2.** The set Δ0 of vertices of Δ is finite.

**Proof.** Riedtmann’s construction of Δ is the following. We fix a vertex x0 in ˜Γ0. Then the vertices of Δ are the paths of ˜Γ beginning on x0 and which do not contain subpaths of the form ασ(α), where α is in ˜Γ1. Now suppose that Δ0 is an infinite set. Then for each n, there exists a sequence:

\[ x_0 \xrightarrow{\alpha_1} x_1 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_{n-1}} x_{n-1} \xrightarrow{\alpha_n} x_n \]

such that τx_{i+2} ≠ x_i. Then there exist some indecomposables X_0, ..., X_n such that the vector space \( R(X_{i-1}, X_i)/R^2(X_{i-1}, X_i) \) is not zero. Thus from the lemma [4.1.3] there exists irreducible morphisms \( f_i : X_{i-1} \rightarrow X_i \) such that the composition \( f_n f_{n-1} \cdots f_1 \) does not vanish. But the functor \( \text{Hom}_T(X_0, ?) \) has finite support. Thus there is an indecomposable Y which appears an infinite number of times in the sequence \( (X_i) \). But since \( R^N(Y, Y) \) vanishes for an N, we have a contradiction. □

Let \( S \) a system of representatives of isoclasses of indecomposables of \( T \). For an indecomposable \( Y \) of \( T \), we put

\[ l(Y) = \sum_{M \in S} \dim_k \text{Hom}_T(M, Y). \]

This sum is finite since \( T \) is locally finite.

**Lemma 4.2.3.** For \( x \in \Delta_0 \), we write \( d_x = l(\bar{x}) \). Then for each \( x \in \Delta_0 \), we have:

\[ \sum_{y \in \Delta_0} d_y C_{xy} = 2. \]

**Proof.** Let \( X \) and \( U \) be indecomposables of \( T \). Let

\[ X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} SX \]

be an AR-triangle. We write \( (U, ?) \) for the cohomological functor \( \text{Hom}_T(U, ?) \). Thus, we have a long exact sequence:

\[ (U, S^{-1}Z) \xrightarrow{S^{-1}w} (U, X) \xrightarrow{u^*} (U, Y) \xrightarrow{v^*} (U, Z) \xrightarrow{w^*} (U, SX). \]
Let $S_Z(U)$ be the image of the map $w_*$. We have the exact sequence:

$$0 \to S_{S^*}(U) \to (U,X) \xrightarrow{w_*} (U,Y) \xrightarrow{v_*} (U,Z) \xrightarrow{u_*} S_Z(U) \to 0.$$ 

Thus we have the following equality:

$$\dim_k S_Z(U) + \dim_k S_{S^*}(U) + \dim_k (U,Y) = \dim_k (U,X) + \dim_k (U,Z).$$

If $U$ is not isomorphic to $Z$, each map from $U$ to $Z$ is radical, thus $S_Z(U)$ is zero. If $U$ is isomorphic to $Z$, the map $w_*$ factors through the radical of $\text{End}(Z)$, so $S_Z(Z)$ is isomorphic to $k$. Then summing the previous equality when $U$ runs over $S$, we get:

$$l(X) + l(Z) = l(Y) + 2.$$ 

Clearly $l$ is $\tau$-invariant, thus $l(Z)$ equals $l(X)$. If the decomposition of $Y$ is of the form $\bigoplus_{i=1}^r Y_i^{n_i}$, we get:

$$l(Y) = \sum_i n_i l(Y_i) = \sum_{i,X \to Y_i \in \Gamma} a_{XY_i}l(Y_i) + \sum_{i,X \to Y_i \in \Gamma} a_{XX^i}l(X).$$

We deduce the formula:

$$2 = (2 - a_{XX})l(X) - \sum_{i,X \to Y_i \in \Gamma} a_{XY_i}l(Y_i).$$

Let $x$ be a vertex of the tree $\Delta$ and $\varpi$ its image in $\hat{\Gamma}$. Then an arrow $\varpi \to Y$ in $\hat{\Gamma}$ comes from an arrow $(x,0) \to (y,0)$ in $\mathbb{Z}\Delta$ or from an arrow $(x,0) \to (y,-1)$ in $\mathbb{Z}\Delta$, i.e. from an arrow $(y,0) \to (x,0)$. Indeed the projection $\mathbb{Z}\Delta \to \mathbb{Z}\Delta/G$ is a covering. From this we deduce the following equality:

$$2 = (2 - a_{\varpi\varpi})d_x - \sum_{y,x \to y \in \Delta} a_{\varpi\varpi}d_y - \sum_{y,y \to x \in \Delta} a_{\varpi\varpi}d_y = \sum_{y \in \Delta} d_y C_{xy}. \quad \Box$$

Now we can prove theorem 4.0.4. The matrix $C$ is a ‘generalized Cartan matrix’. The previous lemma gives us a subadditive function which is not additive. Thus by [12], the underlying graph of $C$ is of ‘generalized Dynkin type’. As $C$ is symmetric, the graph is necessarily of type $A$, $D$, $E$, or $L$. But this graph is the graph $\Delta$ with the valuation $a$. We are done in the cases $A$, $D$, or $E$.

The case $L_n$ occurs when the AR-quiver contains at least one loop. We can see $L_n$ as $\alpha_n$ with valuations on the vertices with a loop. Then, it is obvious that the automorphism groups of $\mathbb{Z}L_n$ are generated by $\tau^r$ for an $r \geq 1$. But proposition 4.1.1 tell us that a vertex $x$ with a loop satisfies $\tau x = x$. Thus $G$ is generated by $\tau$ and the AR-quiver has the following form:

$$1 \to 2 \to 3 \to \cdots \to n$$

This quiver is isomorphic to the quiver $\mathbb{Z}A_{2n}/G$ where $G$ is the group generated by the automorphism $\tau^n S = \rho$.

The suspension functor $S$ sends the indecomposables on indecomposables, thus it can be seen as an automorphism of the AR-quiver. It is exactly the automorphism $S$ defined in section 2.2.

As shown in [18], it follows from the results of [19] that for each Dynkin tree $\Delta$ and for each weakly admissible group of automorphisms $G$ of $\mathbb{Z}\Delta$, there exists a locally finite triangulated category $T$ such that $\Gamma_T \simeq \mathbb{Z}\Delta/G$. This category is of the form $T = D^b(\text{mod} \, k\Delta)/\varphi$ where $\varphi$ is an auto-equivalence of $D^b(\text{mod} \, k\Delta)$. 

5. Construction of a covering functor

From now, we suppose that the AR-quiver $\Gamma$ of $\mathcal{T}$ is connected. We know its structure. It is natural to ask: Is the category $\mathcal{T}$ standard, i.e. equivalent as a $k$-linear category to the mesh category $k(\Gamma)$? First, in this part we construct a covering functor $F : k(\mathcal{Z}\Delta) \to \mathcal{T}$.

5.1. Construction. We write $\pi : \mathcal{Z}\Delta \to \Gamma$ for the canonical projection. As $G$ is a weakly admissible group, this projection verifies the following property: if $x$ is a vertex of $\mathcal{Z}\Delta$, the number of arrows of $\mathcal{Z}\Delta$ with source $x$ is equal to the number of arrows of $\mathcal{Z}\Delta/G$ with source $\pi x$. Let $S$ be a system of representatives of the isoclasses of indecomposables of $\mathcal{T}$. We write $\text{ind}\mathcal{T}$ for the full subcategory of $\mathcal{T}$ whose set of objects is $S$. For a tree $\Delta$, we write $k(\mathcal{Z}\Delta)$ for the mesh category (see [24]). Using the same proof as Riedtmann [24], one shows the following theorem:

Theorem 5.1.1. There exists a $k$-linear functor $F : k(\mathcal{Z}\Delta) \to \text{ind}\mathcal{T}$ which is surjective and induces bijections:

$$\bigoplus_{Fz = Fy} \text{Hom}_{k(\mathcal{Z}\Delta)}(x, z) \to \text{Hom}_{\mathcal{T}}(Fx, Fy),$$

for all vertices $x$ and $y$ of $\mathcal{Z}\Delta$.

5.2. Infinite case. If the category $\mathcal{T}$ is locally finite not finite, the constructed functor $F$ is immediately fully faithful. Thus we get the corollary.

Corollary 5.2.1. If $\text{ind}\mathcal{T}$ is not finite, then we have a $k$-linear equivalence between $\mathcal{T}$ and the mesh category $k(\mathcal{Z}\Delta)$.

5.3. Uniqueness criterion. The covering functor $F$ can be see as a $k$-linear functor from the derived category $\mathcal{D}^b(\text{mod } k\mathcal{Z}\Delta)$ to the category $\mathcal{T}$. By construction, it satisfies the following property called the AR-property:

For each AR-triangle $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} S X$ of $\mathcal{D}^b(\text{mod } k\mathcal{Z}\Delta)$, there exists a triangle of $\mathcal{T}$ of the form $FX \xrightarrow{Ff} FY \xrightarrow{Fg} FZ \xrightarrow{\epsilon} SFX$.

In fact, thanks to this property, $F$ is determined by its restriction to the subcategory $\text{proj} k\mathcal{Z}\Delta = k(\Delta)$, i.e. we have the following lemma:

Lemma 5.3.1. Let $F$ and $G$ be $k$-linear functors from $\mathcal{D}^b(\text{mod } k\mathcal{Z}\Delta)$ to $\mathcal{T}$. Suppose that $F$ and $G$ satisfy the AR-property and that the restrictions $F|_{k(\mathcal{Z}\Delta)}$ and $G|_{k(\mathcal{Z}\Delta)}$ are isomorphic. Then the functors $F$ and $G$ are isomorphic as $k$-linear functors.

Proof. It is easy to construct this isomorphism by induction using the (TR3) axiom of the triangulated categories (see [22]).

6. Particular cases of $k$-linear equivalence

From now we suppose that the category $\mathcal{T}$ is finite, i.e. $\mathcal{T}$ has finitely many isoclasses of indecomposable objects.

6.1. Equivalence criterion. Let $\Gamma$ be the AR-quiver of $\mathcal{T}$ and suppose that it is isomorphic to $\mathcal{Z}\Delta/G$. Let $\varphi$ be a generator of $G$. It induces an automorphism in the mesh category $k(\mathcal{Z}\Delta)$ that we still denote by $\varphi$. Then we have the following equivalence criterion:

Proposition 6.1.1. The categories $k(\Gamma)$ and $\text{ind}\mathcal{T}$ are equivalent as $k$-categories if and only if there exists a covering functor $F : k(\mathcal{Z}\Delta) \to \text{ind}\mathcal{T}$ and an isomorphism of functors $\Phi : F \circ \varphi \to F$. 
The proof consists in constructing a \(k\)-linear equivalence between \(\text{ind}T\) and the orbit category \(k(\mathbb{Z}\Delta)/\mathbb{Z}\Delta\) using the universal property of the orbit category (see [13]), and then constructing an equivalence between \(k(\mathbb{Z}\Delta)/\mathbb{Z}\Delta\) and \(k(\Gamma)\).

### 6.2. Cylindric case for \(A_n\)

**Theorem 6.2.1.** If \(\Delta = A_n\) and \(\varphi = \tau^r\) for some \(r \geq 1\), then there exists a functor isomorphism \(\Phi : F \circ \varphi \to F\), i.e. for each object \(x\) of \(k(\mathbb{Z}\Delta)\) there exists an automorphism \(\Phi_x\) of \(Fx\) such that for each arrow \(\alpha : x \to y\) of \(\mathbb{Z}\Delta\), the following diagram commutes:

\[
\begin{array}{ccc}
Fx & \xrightarrow{\Phi_x} & Fx \\
F\alpha & \downarrow & F\varphi \alpha \\
Fy & \xrightarrow{\Phi_y} & Fy.
\end{array}
\]

To prove this, we need the following lemma:

**Lemma 6.2.2.** Let \(\alpha : x \to y\) be an arrow of \(ZA_n\) and let \(c\) be a path from \(x\) to \(\tau^r y\), \(r \in \mathbb{Z}\), which is not zero in the mesh category \(k(ZA_n)\). Then \(c\) can be written \(c'\alpha\) where \(c'\) is a path from \(y\) to \(\tau^r y\) (up to sign).

**Proof.** There is a path from \(x\) to \(\tau^r y\), thus, we have \(\text{Hom}_{k(\mathbb{Z}\Delta)}(x, \tau^r y) \simeq k\), and \(x\) and \(\tau^r y\) are opposite vertices of a ‘rectangle’ in \(ZA_n\). This implies that there exists a path from \(x\) to \(\tau^r y\) beginning by \(\alpha\). \(\square\)

**Proof. (of theorem 6.2.1)** Combining proposition 6.1.1 and lemma 5.3.1, we have just to construct an isomorphism between the restriction of \(F\) and \(F \circ \varphi\) to a subquiver \(A_n\).

Let us fix a full subquiver of \(ZA_n\) of the following form:

\[
\begin{array}{cccccccc}
& x_1 & \overset{\alpha_1}{\longrightarrow} & x_2 & \overset{\alpha_2}{\longrightarrow} & \cdots & \overset{\alpha_{n-1}}{\longrightarrow} & x_n \\
\end{array}
\]

such that \(x_1, \ldots, x_n\) are representatives of the \(r\)-orbits in \(ZA_n\). We define the \((\Phi_z)_i\) by induction. We fix \(\Phi_{x_1} = Id_{Fx_1}\). Now suppose we have constructed some automorphisms \(\Phi_{x_1}, \ldots, \Phi_{x_{i-1}}\) such that for each \(j \leq i\) the following diagram is commutative:

\[
\begin{array}{ccc}
F_{x_{j-1}} & \xrightarrow{\Phi_{x_{j-1}}} & F_{x_{j-1}} \\
F\alpha_{j-1} & \downarrow & F\varphi \alpha_{j-1} \\
F_{x_j} & \xrightarrow{\Phi_{x_j}} & F_{x_j}.
\end{array}
\]

The composition \((F \varphi \alpha_i) \circ \Phi_{x_i}\) is in the morphism space \(\text{Hom}_T(Fx_i, Fx_{i+1})\), which is isomorphic, by theorem 5.1.1, to the space \(\bigoplus_{Fz=Fx_{i+1}} \text{Hom}_{k(\mathbb{Z}\Delta)}(x_i, z)\).

Thus we can write

\[
(F \varphi \alpha_i) \Phi_{x_i} = \lambda F\alpha_i + \sum_{z \neq x_{i+1}} F\beta_z
\]

where \(\beta_z\) belongs to \(\text{Hom}_{k(\mathbb{Z}\Delta)}(x_i, z)\) and \(Fz = Fx_{i+1}\). But \(Fz\) is equal to \(Fx_{i+1}\) if and only if \(z\) is of the form \(\tau^r x_{i+1}\) for an \(l\) in \(\mathbb{Z}\). By the lemma, we can write \(\beta_z = \beta_z' \alpha_i\). Thus we have the equality:

\[
(F \varphi \alpha_i) \Phi_{x_i} = F(\lambda Id_{x_{i+1}} + \sum_z \beta_z') F\alpha_i.
\]
The scalar $\lambda$ is not zero. Indeed, $\Phi_{x_i}$ is an automorphism, thus the image of $(F\varphi\alpha_i)\Phi_{x_i}$ is not zero in the quotient
$$R(Fx_i, Fx_{i+1})/R^2(Fx_i, Fx_{i+1}).$$
Thus $\Phi_{x_{i+1}} = F(\lambda Id_{x_{i+1}} + \sum \beta_\ell)$ is an automorphism of $Fx_{i+1}$ which verifies the commutation relation
$$(F\varphi\alpha_i) \circ \Phi_{x_i} = \Phi_{x_{i+1}} \circ F\alpha_i.$$

6.3. Other standard cases. In the mesh category $k(\mathbb{Z}\Delta)$, where $\Delta$ is a Dynkin tree, the length of the non zero paths is bounded. Thus there exist automorphisms $\varphi$ such that, for an arrow $\alpha : x \to y$ of $\Delta$, the paths from $x$ to $\varphi^r y$ vanish in the mesh category for all $r \neq 0$. In other words, for each arrow $\alpha : x \to y$ of $\mathbb{Z}\Delta$, we have:

$$\text{Hom}_{k(\mathbb{Z}\Delta)/\varphi}(x, y) = \bigoplus_{r \in \mathbb{Z}} \text{Hom}_{k(\mathbb{Z}\Delta)}(x, \varphi^r y) = \text{Hom}_{k(\mathbb{Z}\Delta)}(x, y) \simeq k,$$

where $k(\mathbb{Z}\Delta)/\varphi^2$ is the orbit category (see section 5.1).

**Lemma 6.3.1.** Let $T$ be a finite triangulated category with AR-quiver $\Gamma = \mathbb{Z}\Delta/G$. Let $\varphi$ be a generator of $G$ and suppose that $\varphi$ verifies for each arrow $x \to y$ of $\mathbb{Z}\Delta$

$$\bigoplus_{r \in \mathbb{Z}} \text{Hom}_{k(\mathbb{Z}\Delta)}(x, \varphi^r y) = \text{Hom}_{k(\mathbb{Z}\Delta)}(x, y) \simeq k.$$

Let $F : k(\mathbb{Z}\Delta) \to T$ and $G : k(\mathbb{Z}\Delta) \to T$ be covering functors satisfying the AR-property. Suppose that $F$ and $G$ agree up to isomorphism on the objects of $k(\mathbb{Z}\Delta)$. Then $F$ and $G$ are isomorphic as $k$-linear functors.

**Proof.** Using lemma 5.3.1, we have just to construct an isomorphism between the functors restricted to $\Delta$. Let $\alpha : x \to y$ be an arrow of $\Delta$. Using theorem 5.1.1 and the hypothesis, we have the following isomorphisms:

$$\text{Hom}_T(Fx, Fy) \simeq \bigoplus_{Fz=Fy} \text{Hom}_{k(\mathbb{Z}\Delta)}(x, z) \simeq \bigoplus_{r \in \mathbb{Z}} \text{Hom}_{k(\mathbb{Z}\Delta)}(x, \varphi^r y) \simeq k$$

and then

$$\text{Hom}_T(Gx, Gy) \simeq \text{Hom}_T(Fx, Fy) \simeq k.$$ 

Thus there exists a scalar $\lambda$ such that $Go = \lambda F\alpha$. This scalar does not vanish since $F$ and $G$ are covering functors. As $\Delta$ is a tree, we can find some $\lambda_x$ for $x \in \Delta$ by induction such that

$$Go = \lambda_x \lambda_y^{-1} F\alpha.$$ 

Now it is easy to check that $\Phi_{x} = \lambda_x Id_{Fx}$ is the functor isomorphism.

This lemma gives us an isomorphism between the functors $F$ and $F \circ \varphi$. Moreover, using the same argument, one can show that the covering functor $F$ is an $S$-functor and a $\tau$-functor.

For each Dynkin tree $\Delta$ we can determine the automorphisms $\varphi$ which satisfy this combinatorial property. Using the preceding lemma and the equivalence criterion we deduce the following theorem:

**Theorem 6.3.2.** Let $T$ be a finite triangulated category with AR-quiver $\Gamma = \mathbb{Z}\Delta/G$. Let $\varphi$ be a generator of $G$. If one of these cases holds,

- $\Delta = A_n$ with $n$ odd and $G$ is generated by $\tau^r$ or $\varphi = \tau^r \phi$ with $r \geq n-1$ and $\phi = \tau^{n-2} S$;
- $\Delta = A_n$ with $n$ even and $G$ is generated by $\rho^r$ with $r \geq n-1$ and $\rho = \tau^2 S$;
Theorem 7.0.5. The following result which is proved in section 7.3:

- $\Delta = \mathbb{D}_n$ with $n \geq 5$ and $G$ is generated by $\tau^r$ or $\tau^r \phi$ with $r \geq n - 2$ and $\phi$ as in theorem 2.2.2.
- $\Delta = \mathbb{D}_4$ and $G$ is generated by $\phi \tau^r$, where $r \geq 2$ and $\phi$ runs over $\sigma_3$;
- $\Delta = \mathbb{E}_6$ and $G$ is generated by $\tau^r$ or $\tau^r \phi$ where $r \geq 5$ and $\phi$ is as in theorem 2.2.2.
- $\Delta = \mathbb{E}_7$ and $G$ is generated by $\tau^r$, $r \geq 8$;
- $\Delta = \mathbb{E}_8$ and $G$ is generated by $\tau^r$, $r \geq 14$.

Then $T$ is standard, i.e. the categories $T$ and $k(\Gamma)$ are equivalent as $k$-linear categories.

Corollary 6.3.3. A finite maximal $d$-Calabi-Yau (see [19, 8]) triangulated category $T$, with $d \geq 2$, is standard, i.e. there exists a $k$-linear equivalence between $T$ the orbit category $D^b(\text{mod} \, k \Delta)/\tau^{-1}S^d$ where $\Delta$ is Dynkin of type $A$, $D$ or $E$.

7. Algebraic case

For some automorphism groups $G$, we know the $k$-linear structure of $T$. But what about the triangulated structure? We can only give an answer adding hypothesis on the triangulated structure. In this section, we distinguish two cases:

If $T$ is locally finite, not finite, we have the following theorem which is proved in section 7.3.

Theorem 7.0.4. Let $T$ be a connected locally finite triangulated category with infinitely many indecomposables. If $T$ is the base of a tower of triangulated categories [18], then $T$ is triangle equivalent to $D^b(\text{mod} \, k \Delta)$ for some Dynkin diagram $\Delta$.

Now if $T$ is a finite standard category which is algebraic, i.e. $T$ is triangle equivalent to $E$ for some $k$-linear Frobenius category $E$ ([20, 3.6]), then we have the following result which is proved in section 7.3.

Theorem 7.0.5. Let $T$ be a finite triangulated category, which is connected, algebraic and standard. Then, there exists a Dynkin diagram $\Delta$ of type $\mathbb{A}$, $\mathbb{D}$ or $\mathbb{E}$ and an auto-equivalence $\Phi$ of $D^b(\text{mod} \, k \Delta)$ such that $T$ is triangle equivalent to the orbit category $D^b(\text{mod} \, k \Delta)/\Phi$.

This theorem combined with corollary 6.3.3 yields the following result (compare to [18, Cor 8.4]):

Corollary 7.0.6. If $T$ is a finite algebraic maximal $d$-Calabi-Yau category with $d \geq 2$, then $T$ is triangle equivalent to the orbit category $D^b(\text{mod} \, k \Delta)/S^d\nu^{-1}$ for some Dynkin diagram $\Delta$.

7.1. $\partial$-functor. We recall the following definition from [18] and [30].

Definition 7.1.1. Let $\mathcal{H}$ be an exact category and $T$ a triangulated category. A $\partial$-functor $(I, \partial) : \mathcal{H} \rightarrow T$ is given by:

- an additive $k$-linear functor $I : \mathcal{H} \rightarrow T$;
- for each conflation $\epsilon : X \xrightarrow{i} Y \xrightarrow{p} Z$ of $\mathcal{H}$, a morphism $\partial \epsilon : IZ \rightarrow SIX$ functorial in $\epsilon$ such that $I \xrightarrow{Ii} IY \xrightarrow{Ip} IZ \xrightarrow{\partial \epsilon} SIX$ is a triangle of $T$.

For each exact category $\mathcal{H}$, the inclusion $I : \mathcal{H} \rightarrow D^b(\mathcal{H})$ can be completed to a $\partial$-functor $(I, \partial)$ in a unique way. Let $T$ and $T'$ be triangulated categories. If $(F, \varphi) : T \rightarrow T'$ is an $S$-functor and $(I, \partial) : \mathcal{H} \rightarrow T$ is a $\partial$-functor, we say that $F$ respects $\partial$ if $(F \circ I, \varphi(F\partial)) : \mathcal{H} \rightarrow T'$ is a $\partial$-functor. Obviously each triangle functor respects $\partial$. 

**Proposition 7.1.1.** Let $\mathcal{H}$ be a $k$-linear hereditary abelian category and let $(I, \partial) : \mathcal{H} \to T$ be a $\partial$-functor. Then there exists a unique (up to isomorphism) $k$-linear $S$-functor $F : D^b(\mathcal{H}) \to T$ which respects $\partial$.

**Proof.** On $\mathcal{H}$ (which can be seen as a full subcategory of $D^b(\mathcal{H})$), the functor $F$ is uniquely determined. We want $F$ to be an $S$-functor, so $F$ is uniquely determined on $S^n\mathcal{H}$ for $n \in \mathbb{Z}$ too. Since $\mathcal{H}$ is hereditary, each object of $D^b(\mathcal{H})$ is isomorphic to a direct sum of stalk complexes, i.e. complexes concentrated in a single degree. Thus, the functor $F$ is uniquely determined on the objects. Now, let $X$ and $Y$ be stalk complexes of $D^b(\mathcal{H})$ and $f : X \to Y$ a non-zero morphism. We can suppose that $X$ is in $\mathcal{H}$ and $Y$ is in $S^n\mathcal{H}$. If $n \neq 0, 1$, $f$ is necessarily zero. If $n = 0$, then $f$ is a morphism in $\mathcal{H}$ and $Ff$ is uniquely determined. If $n = 1$, $f$ is an element of $\text{Ext}^1_{\mathcal{H}}(X, S^{-1}Y)$, so gives us a conflation $\epsilon : S^{-1}Y \overset{i}{\to} E \overset{p}{\to} X$ in $\mathcal{H}$. The functor $F$ respects $\partial$, thus $Ff$ has to be equal to $\varphi \circ \partial \epsilon$ where $\varphi$ is the natural isomorphism between $SFS^{-1}Y$ and $FY$. Since $\partial$ is functorial, $F$ is a functor. The result follows. \[\]

A priori this functor is not a triangle functor. We recall a theorem proved by B. Keller [18, cor 2.7].

**Theorem 7.1.2.** Let $\mathcal{H}$ be a $k$-linear exact category, and $T$ be the base of a tower of triangulated categories $\{I, \partial\}$. Let $(I, \partial) : \mathcal{H} \to T$ be a $\partial$-functor such that for each $n < 0$, and all objects $X$ and $Y$ of $\mathcal{H}$, the space $\text{Hom}_T(I, \mathcal{H}, S^nIX)$ vanishes. Then there exists a unique triangle functor $F : D^b(\mathcal{H}) \to T$ such that the following diagram commutes up to isomorphism of $\partial$-functors:

\[
\begin{array}{c}
\mathcal{H} \\
(1, \partial)
\end{array} \rightarrow \begin{array}{c} D^b(\mathcal{H}) \\
F
\end{array} \rightarrow \begin{array}{c} T
\end{array}
\]

From theorem 7.1.2 and the proposition above we deduce the following corollary:

**Corollary 7.1.3.** ([compare to [29] ] Let $T$, $\mathcal{H}$ and $(I, \partial) : \mathcal{H} \to T$ be as in theorem 7.1.3. If $\mathcal{H}$ is hereditary, then the unique functor $F : D^b(\mathcal{H}) \to T$ which respects $\partial$ is a triangle functor.

**7.2. Proof of theorem 7.0.4.** Let $F$ be the $k$-linear equivalence constructed in theorem 7.1.3 between an algebraic triangulated category $T$ and $D^b(\mathcal{H})$ where $\mathcal{H} = \text{mod} k\Delta$ and $\Delta$ is a simply laced Dynkin graph. As we saw in section 5, the covering functor is an $S$-functor.

The category $\mathcal{H}$ is the heart of the standard $t$-structure on $D^b(\mathcal{H})$. The image of this $t$-structure through $F$ is a $t$-structure on $T$. Indeed, $F$ is an $S$-equivalence, so the conditions (i) and (ii) from [3] Def 1.3.1 hold obviously. And since $\mathcal{H}$ is hereditary, for an object $X$ of $D^b(\mathcal{H})$, the morphism $X \to S_{t\leq 0}X$ of the triangle

\[
\tau_{t\leq 0}X \rightarrow X \rightarrow \tau_{t\geq 0}X \rightarrow S_{t\leq 0}X
\]

vanishes. Thus the image of this triangle through $F$ is a triangle of $T$ and condition (iii) of [3] Def 1.3.1 holds. Then we get a $t$-structure on $T$ whose heart is $\mathcal{H}$.

It results from [3, Prop 1.2.4] that the inclusion of the heart of a $t$-structure can be uniquely completed to a $\partial$-functor. Thus we obtain a $\partial$-functor $(F_0, \partial) : \mathcal{H} \to T$ with $F_0 = F|_{\mathcal{H}}$.

The functor $F$ is an $S$-equivalence. Thus for each $n < 0$, and all objects $X$ and $Y$ of $\mathcal{H}$, the space $\text{Hom}_T(FX, S^nFY)$ vanishes. Now we can apply theorem 7.1.2.
and we get the following commutative diagram:

\[
\begin{array}{ccc}
\mathcal{H} & \xrightarrow{G} & \mathcal{D}^b(\mathcal{H}), \\
\downarrow (F_0, \delta) & & \downarrow F \\
\mathcal{T} & \xrightarrow{F} & \mathcal{T}
\end{array}
\]

where \( F \) is the \( S \)-equivalence and \( G \) is a triangle functor. Note that a priori \( F \) is an \( S \)-functor which does not respect \( \delta \). The functors \( F|_{\mathcal{T}} \) and \( G|_{\mathcal{T}} \) are isomorphic. The functor \( F \) is an \( S \)-functor thus we have an isomorphism \( F|_{\mathcal{T}} \simeq G|_{\mathcal{T}} \) for each \( n \in \mathbb{Z} \). Thus the functor \( G \) is essentially surjective. Since \( \mathcal{H} \) is the category \( \text{mod} \ k \Delta \), to show that \( G \) is fully faithful, we have just to show that for each \( p \in \mathbb{Z} \),

\[
\text{Hom}_{\mathcal{D}^b(\mathcal{H})} \rightarrow \text{Hom}_{\mathcal{T}} (GA, S^p GA)
\]

where \( A \) is the free module \( k \Delta \). For \( p = 0 \), this is clear because \( A \) is in \( \mathcal{H} \). And for \( p \neq 0 \) both sides vanish.

Thus \( G \) is a triangle equivalence between \( \mathcal{D}^b(\mathcal{H}) \) and \( \mathcal{T} \).

7.3. Finite algebraic standard case. For a small dg category \( A \), we denote by \( \mathcal{C}A \) the category of dg \( A \)-modules, by \( \mathcal{D}A \) the derived category of \( A \) and by \( \text{per} A \) the perfect derived category of \( A \), i.e. the smallest triangulated subcategory of \( \mathcal{D}A \) which is stable under passage to direct factors and contains the free \( A \)-modules \( A(?, A) \), where \( A \) runs through the objects of \( A \). Recall that a small triangulated category is algebraic if it is triangle equivalent to \( \text{per} A \) for a dg category \( A \). For two small dg categories \( A \) and \( B \), a triangle functor \( \text{per} A \to \text{per} B \) is algebraic if it is isomorphic to the functor \( F_X = L \otimes_{k \Delta} X \) associated with a dg bimodule \( X \), i.e. an object of the derived category \( \mathcal{D}(A^{op} \otimes B) \).

Let \( \Phi \) be an algebraic autoequivalence of \( \mathcal{D}^b(\mathcal{H}) \) such that the orbit category \( \mathcal{D}^b(\mathcal{H})/\Phi \) is triangulated. Let \( Y \) be a dg \( k \Delta \times k \Delta \)-bimodule such that \( \Phi = F_Y \). In section 9.3 of [19], it was shown that there is a canonical triangle equivalence between this orbit category and the perfect derived category of a certain small dg category. Thus, the orbit category is algebraic, and endowed with a canonical triangle equivalence to the perfect derived category of a small dg category. Moreover, by the construction in [loc. cit.], the projection functor

\[
\pi : \mathcal{D}^b(\text{mod} k \Delta) \to \mathcal{D}^b(\text{mod} k \Delta)/\Phi
\]

is algebraic.

The proof of theorem 7.0.5 is based on the following universal property of the triangulated orbit category \( \mathcal{D}^b(\text{mod} k \Delta)/\Phi \). For the proof, we refer to section 9.3 of [19].

**Proposition 7.3.1.** Let \( B \) be a small dg category and

\[
F_X = L \otimes_{k \Delta} X : \mathcal{D}^b(\text{mod} k \Delta) \to \text{per} B
\]

an algebraic triangle functor given by a dg \( k \Delta \times A \)-bimodule \( X \). Suppose that there is an isomorphism between \( Y L \otimes_{k \Delta} X \) and \( X \) in the derived bimodule category \( \mathcal{D}(k \Delta^{op} \otimes B) \). Then the functor \( F_X \) factors, up to isomorphism of triangle functors, through the projection

\[
\pi : \mathcal{D}^b(\text{mod} k \Delta) \to \mathcal{D}^b(\text{mod} k \Delta)/\Phi.
\]

Moreover, the induced triangle functor is algebraic.

Let us recall a lemma of Van den Bergh [21]:
Lemma 7.3.2. Let $Q$ be a quiver without oriented cycles and $\mathcal{A}$ be a dg category. We denote by $k(Q)$ the category of paths of $Q$ and by $\text{Can}: \mathcal{C}\mathcal{A} \to \mathcal{D}\mathcal{A}$ the canonical functor. Then we have the following properties:

a) Each functor $F: k(Q) \to \mathcal{D}\mathcal{A}$ lifts, up to isomorphism, to a functor $\tilde{F}: k(Q) \to \mathcal{C}\mathcal{A}$ which verifies the following property: For each vertex $j$ of $Q$, the induced morphism

$$\bigoplus_i \tilde{F}i \to \tilde{F}j,$$

where $i$ runs through the immediate predecessors of $j$, is a monomorphism which splits as a morphism of graded $\mathcal{A}$-modules.

b) Let $F$ and $G$ be functors from $k(Q)$ to $\mathcal{C}\mathcal{A}$, and suppose that $F$ satisfies the property of a). Then any morphism of functors $\varphi: \text{Can} \circ F \to \text{Can} \circ G$ lifts to a morphism $\tilde{\varphi}: F \to G$.

Proof. a) For each vertex $i$ of $Q$, the object $Fi$ is isomorphic in $\mathcal{D}\mathcal{A}$ to its cofibrant resolution $X_i$. Thus for each arrow $\alpha: i \to j$, $F$ induces a morphism $f_\alpha: X_i \to X_j$ which can be lifted to $\mathcal{C}\mathcal{A}$ since the $X_i$ are cofibrant. Since $Q$ has no oriented cycle, it is easy to choose the $f_\alpha$ such that the property is satisfied.

b) For each vertex $i$ of $Q$, we may assume that $Fi$ is cofibrant. Then we can lift $\varphi_i: \text{Can} \circ Fi \to \text{Can} \circ Gi$ to $\psi_i: Fi \to Gi$. For each arrow $\alpha$ of $Q$, the square

$$\begin{array}{ccc}
Fi & \rightarrow & Fj \\
\downarrow \psi_i & & \downarrow \psi_j \\
Gi & \rightarrow & Gj
\end{array}$$

is commutative in $\mathcal{D}\mathcal{A}$. Thus the square

$$\begin{array}{ccc}
\bigoplus_i Fi & \rightarrow & \bigoplus_j Fj \\
\downarrow (\psi_i) & & \downarrow (\psi_j) \\
\bigoplus_i Gi & \rightarrow & \bigoplus_j Gj
\end{array}$$

is commutative up to nullhomotopic morphism $h: \bigoplus_i Fi \to Gj$. Since the morphism $f: \bigoplus_i Fi \to Fj$ is split mono in the category of graded $\mathcal{A}$-modules, $h$ extends along $f$ and we can modify $\psi_j$ so that the square becomes commutative in $\mathcal{C}\mathcal{A}$. The quiver $Q$ does not have oriented cycles, so we can construct $\tilde{\varphi}$ by induction. □

Proof. (of theorem 7.0.3) The category $T$ is small and algebraic, thus we may assume that $T = \text{per}\ A$ for some dg category $\mathcal{A}$. Let $F: \mathcal{D}^b(\text{mod } k\Delta) \to T$ be the covering functor of theorem 5.1.1. Let $\Phi$ be an auto-equivalence of $\mathcal{D}^b(\text{mod } k\Delta)$ such that the AR-quot of the orbit category $\mathcal{D}^b(\text{mod } k\Delta)/\Phi$ is isomorphic (as translation quiver) to the AR-quot of $T$. We may assume that $\Phi = -\circ_{k\Delta} Y$ for an object $Y$ of $D(k\Delta^\text{op} \otimes k\Delta)$. The orbit category $\mathcal{D}^b(\text{mod } k\Delta)/\Phi$ is algebraic, thus it is $\text{per}\ B$ for some dg category $B$.

The functor $F_{|k\Delta}$ lifts by lemma 7.3.2 to a functor $\tilde{F}$ from $k\Delta$ to $\mathcal{C}\mathcal{A}$. This means that the object $X = \tilde{F}(k\Delta)$ has a structure of dg $k\Delta^\text{op} \otimes \mathcal{A}$-module. We denote by $X$ the image of this object in $D(k\Delta^\text{op} \otimes \mathcal{A})$.

The functors $F$ and $-\circ_{k\Delta} X$ become isomorphic when restricted to $k\Delta$. Moreover $-\circ_{k\Delta} X$ satisfies the AR-property since it is a triangulated functor. Thus by lemma 5.3.1 they are isomorphic as $k$-linear functors. So we have the following
diagram:

\[
\begin{array}{ccc}
D^b(\text{mod} \, k\Delta) & \xrightarrow{\mathcal{L}} & \text{per } A = T \\
\bigcirc & & \\
\xrightarrow{\mathcal{L}} & & \\
\bigcirc & & \\
\text{per } A = T & \xrightarrow{\mathcal{L}} & D^b(\text{mod} \, k\Delta) / \Phi = \text{per } B \\
\end{array}
\]

The category \(T\) is standard, thus there exists an isomorphism of \(k\)-linear functors:

\[
c : - \otimes_{k\Delta} X \xrightarrow{\mathcal{L}} - \otimes_{k\Delta} Y \otimes_{k\Delta} X.
\]

The functor \(- \otimes_{k\Delta} X\) restricted to the category \(k(\Delta)\) satisfies the property of \((a)\) of lemma 7.3.4. Thus we can apply \((b)\) and lift \(c|_{k(\Delta)}\) to an isomorphism \(\tilde{c}\) between \(X\) and \(Y \otimes_{k\Delta} X\) as \(\text{dg-}k\Delta^{\text{op}} \otimes A\)-modules.

By the universal property of the orbit category, the bimodule \(X\) endowed with the isomorphism \(\tilde{c}\) yields a triangle functor from \(D^b(\text{mod} \, k\Delta)/\Phi\) to \(T\) which comes from a bimodule \(Z\) in \(D(B^{\text{op}} \otimes A)\).

The functor \(- \otimes_{k\Delta} Z\) is essentially surjective. Let us show that it is fully faithful. For \(M\) and \(N\) objects of \(D^b(\text{mod} \, k\Delta)\) we have the following commutative diagram:

\[
\begin{array}{ccc}
\bigoplus_{n \in \mathbb{Z}} \text{Hom}_D(M, \Phi^n N) & \xrightarrow{\mathcal{L}} & \text{Hom}_T(FM, FN), \\
\bigoplus_{n \in \mathbb{Z}} \text{Hom}_D(M, \Phi^n N) & \xrightarrow{\mathcal{L}} & \text{Hom}_T(FM, FN), \\
\end{array}
\]

where \(D\) means \(D^b(\text{mod} \, k\Delta)\). The two diagonal morphisms are isomorphisms, thus so is the horizontal morphism. This proves that \(- \otimes_{k\Delta} Z\) is a triangle equivalence between the orbit category \(D^b(\text{mod} \, k\Delta)/\Phi\) and \(T\). □

8. Triangulated structure on the category of projectives

Let \(k\) be a algebraically closed field and \(\mathcal{P}\) a \(k\)-linear category with split idempotents. The category \(\text{mod } \mathcal{P}\) of contravariant finitely presented functors from \(\mathcal{P}\) to \(\text{mod } k\) is exact. As the idempotents split, the projectives of \(\text{mod } \mathcal{P}\) coincide with the representables. Thus the Yoneda functor gives a natural equivalence between \(\mathcal{P}\) and \(\text{proj } \mathcal{P}\). Assume besides that \(\text{mod } \mathcal{P}\) has a structure of Frobenius category. The stable category \(\text{mod } \mathcal{P}\) is a triangulated category, we write \(\Sigma\) for the suspension functor.

Let \(S\) be an auto-equivalence of \(\mathcal{P}\). It can be extended to an exact functor from \(\text{mod } \mathcal{P}\) to \(\text{mod } \mathcal{P}\) and thus to a triangle functor of \(\text{mod } \mathcal{P}\). The aim of this part is to find a necessary condition on the functor \(S\) such that the category \((\mathcal{P}, S)\) has a triangulated structure. Heller already showed [14, thm 16.4] that if there exists an isomorphism of triangle functors between \(S\) and \(\Sigma^3\), then \(\mathcal{P}\) has a pretriangulated structure. But he did not succeed in proving the octahedral axiom. We are going to impose a stronger condition on the functor \(S\) and prove the following theorem:
Theorem 8.0.3. Assume there exists an exact sequence of exact functors from \( \text{mod}\mathcal{P} \) to \( \text{mod}\mathcal{P} \):

\[
0 \longrightarrow \text{Id} \longrightarrow X^0 \longrightarrow X^1 \longrightarrow X^2 \longrightarrow S \longrightarrow 0,
\]

where the \( X^i, i = 0, 1, 2 \), take values in \( \text{proj}\mathcal{P} \). Then the category \( \mathcal{P} \) has a structure of triangulated category with suspension functor \( S \).

For an \( M \) in \( \text{mod}\mathcal{P} \), denote \( T_M : X^0 M \longrightarrow X^1 M \longrightarrow X^2 M \longrightarrow SX^0 M \) a standard triangle. A triangle of \( \mathcal{P} \) will be a sequence \( X : P \xrightarrow{u} Q \xrightarrow{v} R \xrightarrow{w} SP \) which is isomorphic to a standard triangle \( T_M \) for an \( M \) in \( \text{mod}\mathcal{P} \).

8.1. \textbf{S-complexes, \( \Phi \)-S-complexes and standard triangles}. Let \( Acp(\text{mod}\mathcal{P}) \) be the category of acyclic complexes with projective components. It is a Frobenius category whose projective-injectives are the contractible complexes, i.e. the complexes homotopic to zero. The functor \( Z^0 : Acp(\text{mod}\mathcal{P}) \rightarrow \text{mod}\mathcal{P} \) which sends a complex

\[
\cdots \longrightarrow X^{-1} \xrightarrow{x^{-1}} X^0 \xrightarrow{x^0} X^1 \xrightarrow{x^1} \cdots
\]

to the kernel of \( x^0 \) is an exact functor. It sends the projective-injectives to projective-injectives and induces a triangle equivalence between \( Acp(\text{mod}\mathcal{P}) \) and \( \text{mod}\mathcal{P} \).

Definition 8.1.1. An object of \( Acp(\text{mod}\mathcal{P}) \) is called an \( S \)-complex if it is \( S \)-periodic, i.e. if it has the following form:

\[
\cdots \longrightarrow P \xrightarrow{u} Q \xrightarrow{v} R \xrightarrow{w} SP \xrightarrow{Su} SQ \longrightarrow \cdots.
\]

The category \( S\text{-comp} \) of \( S \)-complexes with \( S \)-periodic morphisms is a non full subcategory of \( Acp(\text{mod}\mathcal{P}) \). It is a Frobenius category. The projective-injectives are the \( S \)-contractibles, i.e. the complexes homotopic to zero with an \( S \)-periodic homotopy. Using the functor \( Z^0 \), we get an exact functor from \( S\text{-comp} \) to \( \text{mod}\mathcal{P} \) which induces a triangle functor:

\[
Z^0 : S\text{-comp} \longrightarrow \text{mod}\mathcal{P}.
\]

Fix a sequence as in theorem 8.0.3. Clearly, it induces for each object \( M \) of \( \text{mod}\mathcal{P} \), a functorial isomorphism in \( \text{mod}\mathcal{P} \), \( \Phi_M : \Sigma^3 M \longrightarrow SM \).

Let \( Y \) be an \( S \)-complex,

\[
\cdots \longrightarrow P \xrightarrow{u} Q \xrightarrow{v} R \xrightarrow{w} SP \xrightarrow{Su} SQ \longrightarrow \cdots.
\]

Let \( M \) be the kernel of \( u \). Then \( Y \) induces an isomorphism \( \theta \) (in \( \text{mod}\mathcal{P} \)) between \( \Sigma^3 M \) and \( SM \). If \( \theta \) is equal to \( \Phi_M \), we will say that \( X \) is a \( \Phi \)-\( S \)-complex.

Let \( M \) be an object of \( \text{mod}\mathcal{P} \). The standard triangle \( T_M \) can be see as a \( \Phi \)-\( S \)-complex:

\[
\cdots \longrightarrow X^0 M \longrightarrow X^1 M \longrightarrow X^2 M \longrightarrow SX^0 M \longrightarrow SX^1 M \longrightarrow \cdots.
\]

The functor \( T \) which sends an object \( M \) of \( \text{mod}\mathcal{P} \) to the \( S \)-complex \( T_M \) is exact since the \( X^i \) are exact. It satisfies the relation \( Z^0 \circ T \simeq \text{Id}_{\text{mod}\mathcal{P}} \). Moreover, as it preserves the projective-injectives, it induces a triangle functor:

\[
T : \text{mod}\mathcal{P} \rightarrow S\text{-comp}.
\]
8.2. Properties of the functors $Z^0$ and $T$.

**Lemma 8.2.1.** An $S$-complex which is homotopy-equivalent to a $\Phi$-$S$-complex is a $\Phi$-$S$-complex.

*Proof.* Let $X : P \xrightarrow{u} Q \xrightarrow{v} R \xrightarrow{w} SP$ be an $S$-complex homotopy-equivalent to the $\Phi$-$S$-complex $X' : P' \xrightarrow{u'} Q' \xrightarrow{v'} R' \xrightarrow{w'} SP'$. Let $M$ be the kernel of $u$ and $M'$ the kernel of $u'$. By assumption, there exists a $S$-periodic homotopy equivalence $f$ from $X$ to $X'$, which induces a morphism $g = Z^0 f : M \to M'$. Thus, we get the following commutative diagram:

![Diagram](https://example.com/diagram.png)

The morphism $g$ is an isomorphism of $\text{mod } P$ since $f$ is an isomorphism of $\text{S-comp}$. Thus the morphisms $\Sigma^3 g$ and $Sg$ are isomorphisms of $\text{mod } P$. The following equality in $\text{mod } P$

$$\theta = (Sg)^{-1} \Phi_{M'} \Sigma^3 g = \Phi_M$$

shows that the complex $X$ is a $\Phi$-$S$-complex. \qed

**Lemma 8.2.2.** Let

$$X : P \xrightarrow{u} Q \xrightarrow{v} R \xrightarrow{w} SP \quad \text{and} \quad X' : P' \xrightarrow{u'} Q' \xrightarrow{v'} R' \xrightarrow{w'} SP'$$

be two $\Phi$-$S$-complexes. Suppose that we have a commutative square:

$$P \xrightarrow{u} Q \xrightarrow{v} R \xrightarrow{w} SP$$

Then, there exists a morphism $f^2 : R \to R'$ such that $(f^0, f^1, f^2)$ extends to an $S$-periodic morphism from $X$ to $X'$.

*Proof.* Let $M$ be the kernel of $u$, $M'$ be the kernel of $u'$ and $f : M \to M'$ be the morphism induced by the commutative square. As $R$ and $R'$ are projective-injective objects, we can find a morphism $g^2 : R \to R'$ such that the following square commutes:

$$Q \xrightarrow{v} R \xrightarrow{w} SP$$

$$Q' \xrightarrow{v'} R'$$

\[ \square \]
The morphism $g^2$ induces a morphism $g : SM \to SM'$ such that the following square is commutative in $\text{mod} \mathcal{P}$:

$$
\begin{array}{c}
\Sigma^3 M \\
\Phi_M \\
\Sigma^3 f \\
\end{array}
\xrightarrow{egin{array}{c}
\Phi_M \\
\downarrow g \\
\end{array}}
\begin{array}{c}
SM \\
\downarrow \\
SM' \\
\end{array}
$$

Thus the morphisms $Sf$ and $g$ are equal in $\text{mod} \mathcal{P}$, i.e. there exists a projective-injective $I$ of $\text{mod} \mathcal{P}$ and morphisms $\alpha : SM \to I$ and $\beta : I \to SM'$ such that $g = Sf = \beta \alpha$. Let $p$ (resp. $p'$) be the epimorphism from $R$ onto $SM$ (resp. from $R'$ onto $SM'$). Then, as $I$ is projective, $\beta$ factors through $p'$.

We put $f^2 = g^2 - \gamma \alpha p$. Then obviously, we have the equalities $f^2 v = v' f^1$ and $w' f^2 = Sf^0 w$. Thus the morphism $(f^0, f^1, f^2)$ extends to a morphism of $S$-comp.

Proposition 8.2.3. The functor $Z^0 : \Phi \text{-S-comp} \to \text{mod} \mathcal{P}$ is full and essentially surjective. Its kernel is an ideal whose square vanishes.

Proof. The functor $Z^0$ is essentially surjective since we have the relation $Z^0 \circ T = \text{Id}_{\text{mod} \mathcal{P}}$.

Let us show that $Z^0$ is full. Let

$$
X : P \xrightarrow{u} Q \xrightarrow{v} R \xrightarrow{w} SP \\
X' : P' \xrightarrow{u'} Q' \xrightarrow{v'} R' \xrightarrow{w'} SP'
$$

be two $\Phi$-$S$-complexes. Let $M$ (resp. $M'$) be the kernel of $u$ (resp. $u'$). As $P, Q, P'$ and $Q'$ are projective-injective, there exist morphisms $f^0 : P \to P'$ and $f^1 : Q \to Q'$ such that the following diagram commutes:

$$
\begin{array}{c}
M \\
\downarrow f \\
\end{array}
\xrightarrow{egin{array}{c}
\downarrow f^0 \\
\end{array}}
\begin{array}{c}
P \\
\downarrow \downarrow f^1 \\
M' \rightarrow P' \rightarrow Q'.
\end{array}
$$

Now the result follows from lemma 8.2.2.
Now let \( f : X \to X' \) be a morphism in the kernel of \( Z^0 \). Up to homotopy, we can suppose that \( f \) has the following form:

\[
\begin{array}{ccc}
P & \overset{u}{\longrightarrow} & Q \overset{v}{\longrightarrow} R \overset{w}{\longrightarrow} SP \\
0 & \downarrow & 0 & \downarrow f^2 & \downarrow 0 \\
P' & \overset{u'}{\longrightarrow} & Q' \overset{v'}{\longrightarrow} R' \overset{w'}{\longrightarrow} SP'.
\end{array}
\]

As the composition \( w'f^2 \) vanishes and as \( Q' \) is projective-injective, \( f^2 \) factors through \( v' \). For the same argument, \( f^2 \) factors through \( w \). If \( f \) and \( f' \) are composable morphisms of the kernel of \( Z^0 \), we get the following diagram:

\[
\begin{array}{ccc}
P & \overset{u}{\longrightarrow} & Q \overset{v}{\longrightarrow} R \overset{w}{\longrightarrow} SP \\
0 & \downarrow & 0 & \downarrow h^2 & \downarrow 0 & \downarrow \phi^2 \\
P' & \overset{u'}{\longrightarrow} & Q' \overset{v'}{\longrightarrow} R' \overset{w'}{\longrightarrow} SP' \\
0 & \downarrow & 0 & \downarrow h'^2 & \downarrow 0 & \downarrow \phi'^2 \\
P'' & \overset{u''}{\longrightarrow} & Q'' \overset{v''}{\longrightarrow} R'' \overset{w''}{\longrightarrow} SP''.
\end{array}
\]

The composition \( f'f \) vanishes obviously. \( \square \)

**Corollary 8.2.4.** A \( \Phi \)-\( S \)-complex morphism \( f \) which induces an isomorphism \( Z^0(f) \) in \( \mod \mathcal{P} \) is an homotopy-equivalence.

This corollary comes from the previous theorem and from the following lemma.

**Lemma 8.2.5.** Let \( F : \mathcal{C} \to \mathcal{C}' \) be a full functor between two additive categories. If the kernel of \( F \) is an ideal whose square vanishes, then \( F \) detects isomorphisms.

**Proof.** Let \( u \in \text{Hom}_\mathcal{C}(A, B) \) be a morphism in \( \mathcal{C} \) such that \(Fu \) is an isomorphism. Since the functor \( F \) is full, there exists \( v \in \text{Hom}_\mathcal{C}(B, A) \) such that \( Fv = (Fu)^{-1} \). The morphism \( w = uv - Id_B \) is in the kernel of \( F \), thus \( w^2 \) vanishes. Then the morphism \( v(Id_B - w) \) is a right inverse of \( u \). In the same way we show that \( u \) has a left inverse, so \( u \) is an isomorphism. \( \square \)

**Proposition 8.2.6.** The category of \( \Phi \)-\( S \)-complexes is equivalent to the category of \( S \)-complexes which are homotopy-equivalent to standard triangles.

**Proof.** Since standard triangles are \( \phi \)-\( S \)-complexes, each \( S \)-complex that is homotopy equivalent to a standard triangle is a \( \Phi \)-\( S \)-complex (lemma 8.2.3).

Let \( X : P \overset{u}{\longrightarrow} Q \overset{v}{\longrightarrow} R \overset{w}{\longrightarrow} SP \) be a \( \Phi \)-\( S \)-complex. Let \( M \) be the kernel of \( u \). Then there exist morphisms \( f^1 : P \to X^0M \) and \( f^1 : Q \to X^1M \) such that the following diagram is commutative:

\[
\begin{array}{ccc}
M & \longrightarrow & P \overset{u}{\longrightarrow} Q \\
\downarrow & & \downarrow f^0 & \downarrow f^1 \\
M & \longrightarrow & X^0M \longrightarrow X^1M.
\end{array}
\]

We can complete (lemma 8.2.3) \( f \) into an \( S \)-periodic morphism from \( X \) in \( T_M \). The morphism \( f \) satisfies \( Z^0f = Id_M \), so \( Z^0(T_M) \) and \( Z^0(X) \) are equal in \( \mod \mathcal{P} \). By the corollary, \( T_M \) and \( X \) are homotopy-equivalent. Thus the inclusion functor \( T \) is essentially surjective. \( \square \)
These two diagrams summarize the results of this section:

8.3. Proof of theorem 8.0.3. We are going to show that the \( \Phi-S \)-complexes form a system of triangles of the category \( \mathcal{P} \). We use triangle axioms as in [22].

**TR0:** For each object \( M \) of \( \mathcal{P} \), the \( S \)-complex \( \xymatrix{ 0 & M 
\ar [r] & 0 \ar [r] & \Sigma M } \) is homotopy-equivalent to the zero complex, so is a \( \Phi-S \)-complex.

**TR1:** Let \( u : P \to Q \) be a morphism of \( \mathcal{P} \), and let \( M \) be its kernel. We can find morphisms \( f^0 \) and \( f^1 \) so as to obtain a commutative square:

We form the following push-out:

It induces a triangle morphism of the triangular category \( \text{mod}\mathcal{P} \):

\[
\begin{array}{c}
\text{Coker } a \\
\downarrow \gamma \\
\Sigma \text{Coker } a \\
\Sigma \gamma \\
\text{Coker } u \\
\end{array}
\]

\[
\begin{array}{c}
\text{R} \\
\downarrow \gamma \\
\Sigma \text{R} \\
\Sigma \gamma \\
\text{S} \\
\end{array}
\]
The morphism $\gamma$ is an isomorphism in $\text{mod } P$ since $\text{Coker } a$ and $\text{Coker } u$ are canonically isomorphic to $\Sigma^2 M$ in $\text{mod } P$. By the five lemma, $X^2 M \rightarrow R$ is an isomorphism in $\text{mod } P$. Since $X^2 M$ is projective-injective, so is $R$. Thus the complex $P \xrightarrow{u} Q \xrightarrow{v} R \xrightarrow{w} SP$ is an $S$-complex. Then we have to see that it is a $\Phi$-$S$-complex. Let $\theta$ be the isomorphism between $SM$ and $\Sigma^3 M$ induced by this complex. We write $\alpha$ (resp. $\beta$) for the canonical isomorphism in $\text{mod } P$ between $\Sigma^2 M$ and $\text{Coker } a$ (resp. $\text{Coker } u$). From the commutative diagram:

we deduce the equality $\theta = (\Sigma \beta)^{-1} \gamma \Sigma \alpha \Phi_M = \Phi_M$ in $\text{mod } P$. The constructed $S$-complex is a $\Phi$-$S$-complex.

**TR2:** Let $X : P \xrightarrow{u} Q \xrightarrow{v} R \xrightarrow{w} SP$ be a $\Phi$-$S$-complex. It is homotopy-equivalent to a standard triangle $T_M$. Thus the $S$-complex

$$X' : Q \xrightarrow{-v} R \xrightarrow{-w} SP \xrightarrow{-Su} SQ$$

is homotopy-equivalent to $T_M[1]$. Since $T$ is a triangle functor, the objects $T_{SM}$ and $T_M[1]$ are isomorphic in the stable category $S\text{-comp}$, i.e. they are homotopy-equivalent. Thus, by lemma 8.2.1, $T_M[1]$ is a $\Phi$-$S$-complex and then so is $X'$.

**TR3:** This axiom is a direct consequence of lemma 8.2.2.

**TR4:** Let $X$ and $X'$ be two $\Phi$-$S$-complexes and suppose we have a commutative diagram:

Let $M$ (resp. $M'$) be the kernel of $u$ (resp. $u'$), and $g : M \rightarrow M'$ the induced morphism. The morphism $Tg : T_M \rightarrow T_{M'}$, induces a $S$-complex morphism $\tilde{g} = (g^0, g^1, g^2)$ between $X$ and $X'$.

We are going to show that we can find a morphism $f^2 : R \rightarrow R'$ such that $(f^0, f^1, f^2)$ can be extended in an $S$-complex morphism that is homotopic to $\tilde{g}$. As $(g^0, g^1)$ and $(f^0, f^1)$ induce the same morphism $g$ in the kernels, we have some morphisms $h^1 : Q \rightarrow P'$ and $h^2 : R \rightarrow Q'$ such that $f^0 - g^0 = h^1 u$ and $f^1 - g^1 = u'h^1 + h^2 v$. We put $f^2 = g^2 + v'h^2$. We have the following equalities:

$$
\begin{align*}
    f^2 v &= g^2 v + v'h^2 v \\
    &= v'(g^1 + h^2 v) \\
    &= v'(f^1 - u'h^1) \\
    &= v'f^1 - \gamma v
\end{align*}
$$

and

$$
\begin{align*}
    w' f^2 &= u'g^2 \\
    &= (Sf^0)w \\
    &= (Sf^0 - Sh^1 Su)w
\end{align*}
$$
Thus \((f_0, f_1, f_2)\) can be extended to an \(S\)-periodic morphism \(\tilde{f}\) which is \(S\)-homotopic to \(\tilde{g}\). Their respective cones \(C(f)\) and \(C(g)\) are isomorphic as \(S\)-complexes. Moreover, since \(\tilde{g}\) is a composition of \(Tg : T_M \to T_{M'}\) with homotopy-equivalences, the cones \(C(\tilde{g})\) and \(C(Tg)\) are homotopy-equivalent.

In \(\text{mod} \mathcal{P}\), we have a triangle

\[
\begin{array}{c}
\text{M} \\
\downarrow^g \\
\text{M}' \\
\downarrow^C(\tilde{g}) \\
\Sigma \text{M}
\end{array}
\]

Since \(T\) is a triangle functor, the sequence

\[
\begin{array}{c}
T_M \\
\downarrow^{Tg} \\
T_{M'} \\
\downarrow^{TC(\tilde{g})} \\
T_{\Sigma \text{M}}
\end{array}
\]

is a triangle in \(S\text{-comp}\). But we know that

\[
\begin{array}{c}
T_M \\
\downarrow^{Tg} \\
T_{M'} \\
\downarrow^{C(Tg)} \\
T_M[1]
\end{array}
\]

is a triangle in \(S\text{-comp}\). Thus the objects \(C(Tg)\) and \(TC(\tilde{g})\) are isomorphic in \(S\text{-comp}\), i.e., homotopy-equivalent. Thus, the cone \(C(\tilde{f})\) of \(\tilde{f}\) is a \(\Phi\)-\(S\)-complex by lemma 8.2.1.

9. Application to the deformed preprojective algebras

In this section, we apply the theorem 8.0.3 to show that the category of finite dimensional projective modules over a deformed preprojective algebra of generalized Dynkin type (see [4]) is triangulated. This will give us some examples of non standard triangulated categories with finitely many indecomposables.

9.1. Preprojective algebra of generalized Dynkin type. Recall the notations of [4]. Let \(\Delta\) be a generalized Dynkin graph of type \(\mathbb{A}_n, \mathbb{D}_n (n \geq 4), \mathbb{E}_n (n = 6, 7, 8), \) or \(\mathbb{L}_n\). Let \(Q_\Delta\) be the following associated quiver:

\[
\begin{array}{c}
\Delta = \mathbb{A}_n (n \geq 1) : \\
0 \xrightarrow{a_0} \frac{a_1}{\mathfrak{m}_0} 1 \xrightarrow{a_1} 2 \cdots \frac{a_n-2}{\mathfrak{m}_{n-2}} n - 2 \xrightarrow{a_n-2} n - 1 \\
\Delta = \mathbb{D}_n (n \geq 4) : \\
0 \xrightarrow{a_0} \frac{a_1}{\mathfrak{m}_0} \xrightarrow{a_2} \frac{a_3}{\mathfrak{m}_2} 3 \cdots \frac{a_{n-2}}{\mathfrak{m}_{n-2}} n - 2 \xrightarrow{a_{n-2}} n - 1 \\
\Delta = \mathbb{E}_n (n = 6, 7, 8) : \\
1 \xrightarrow{a_1} \frac{a_2}{\mathfrak{m}_1} 2 \xrightarrow{a_2} \frac{a_3}{\mathfrak{m}_2} 3 \xrightarrow{a_3} \frac{a_4}{\mathfrak{m}_3} 4 \cdots \frac{a_{n-2}}{\mathfrak{m}_{n-2}} n - 2 \xrightarrow{a_{n-2}} n - 1 \\
\Delta = \mathbb{L}_n (n \geq 1) : \\
\epsilon = \sum_{a_0=1} a_i, \quad \text{for each vertex } i \text{ of } Q_\Delta.
\end{array}
\]

The preprojective algebra \(P(\Delta)\) associated to the graph \(\Delta\) is the quotient of the path algebra \(kQ_\Delta\) by the relations:

\[
\sum_{a_0=1} a_i, \quad \text{for each vertex } i \text{ of } Q_\Delta.
\]
The following proposition is classical [4, prop 2.1].

**Proposition 9.1.1.** The preprojective algebra $P(\Delta)$ is finite dimensional and self-injective. Its Nakayama permutation $\nu$ is the identity for $\Delta = A_n, D_{2n}, E_7, E_8$ and $L_n$, and is of order 2 in all other cases.

9.2. Deformed preprojective algebras of generalized Dynkin type. Let us recall the definition of deformed preprojective algebra introduced by [4]. Let $\Delta$ be a graph of generalized Dynkin type. We define an associated algebra $R(\Delta)$ as follows:

$$
R(\mathbb{A}_n) = k; \\
R(\mathbb{D}_n) = k\langle x, y/(x^2, y^2, (x+y)^{n-2}) angle; \\
R(\mathbb{E}_n) = k\langle x, y/(x^2, y^3, (x+y)^{n-3}) angle; \\
R(\mathbb{L}_n) = k[x/(x^{2n})].
$$

Further, we fix an exceptional vertex in each graph as follows (with the notations of the previous section):

$$
\begin{align*}
0 & \text{ for } \Delta = \mathbb{A}_n \text{ or } \mathbb{I}_n, \\
2 & \text{ for } \Delta = \mathbb{D}_n, \\
3 & \text{ for } \Delta = \mathbb{E}_n.
\end{align*}
$$

Let $f$ be an element of the square $rad^2 R(\Delta)$ of the radical of $R(\Delta)$. The deformed preprojective algebra $P^f(\Delta)$ is the quotient of the path algebra $kQ_\Delta$ by the relations:

$$
\sum_{s_0=i} a_{s_0} \sigma_{s_0} \quad \text{for each non exceptional vertex } i \text{ of } Q,
$$

and

$$
\begin{align*}
\sigma_0 \sigma_0 + \sigma_1 a_1 + a_2 \sigma_2 + f(\sigma_0 a_0, \sigma_1 a_1), & \text{ for } \Delta = \mathbb{A}_n; \\
\sigma_0 \sigma_0 + \sigma_2 a_2 + a_3 \sigma_3 + f(\sigma_0 a_0, \sigma_2 a_2), & \text{ for } \Delta = \mathbb{D}_n; \\
\sigma_0 \sigma_0 + \sigma_2 a_2 + a_3 \sigma_3 + f(\sigma_0 a_0, \sigma_2 a_2), & \text{ for } \Delta = \mathbb{E}_n; \\
\epsilon^2 + a_0 \sigma_0 + \epsilon f(\epsilon), & \text{ for } \Delta = \mathbb{L}_n.
\end{align*}
$$

Note that if $f$ is zero, we get the preprojective algebra $P(\Delta)$.

9.3. Corollaries of [4]. The following proposition [4, prop 3.4] shows that the category $\text{proj} P^f(\Delta)$ of finite-dimensional projective modules over a deformed preprojective algebra satisfies the hypothesis of theorem 8.0.3.

**Proposition 9.3.1.** Let $A = P^f(\Delta)$ be a deformed preprojective algebra. Then there exists an exact sequence of $A$-$A$-bimodules

$$
0 \rightarrow 1 A_{\Phi^{-1}} \rightarrow P_2 \rightarrow P_1 \rightarrow P_0 \rightarrow A \rightarrow 0,
$$

where $\Phi$ is an automorphism of $A$ and where the $P_i$’s are projective as bimodules. Moreover, for each idempotent $e_i$ of $A$, we have $\Phi(e_i) = e_{\nu(i)}$.

So we can easily deduce the corollary:

**Corollary 9.3.2.** Let $P^f(\Delta)$ be a deformed preprojective algebra of generalized Dynkin type. Then the category $\text{proj} P^f(\Delta)$ of finite dimensional projective modules is triangulated. The suspension is the Nakayama functor.

Indeed, if $P_i = e_i A$ is a projective indecomposable, then $P_i \otimes_A A_{\Phi}$ is equal to $\Phi(e_i) A = e_{\nu(i)} A$ thus to $\nu(P_i)$.

Now we are able to answer to the question of the previous part and find a triangulated category with finitely many indecomposables which is not standard. The proof of the following theorem comes essentially from the theorem [4, thm 1.3].
Theorem 9.3.3. Let $k$ be an algebraically closed field of characteristic 2. Then there exist $k$-linear triangulated categories with finitely many indecomposables which are not standard.

Proof. By theorem [4, thm 1.3], we know that there exist basic deformed preprojective algebras of generalized Dynkin type $P_f(\Delta)$ which are not isomorphic to $P(\Delta)$. Thus the categories $\text{proj } P_f(\Delta)$ and $\text{proj } P(\Delta)$ can not be equivalent. But both are triangulated by corollary 9.3.2 and have the same AR-quiver $\mathbb{Z}\Delta/\tau = Q_\Delta$. □

Conversely, we have the following theorem:

Theorem 9.3.4. Let $T$ be a finite 1-Calabi-Yau triangulated category. Then $T$ is equivalent to $\text{proj } \Lambda$ as $k$-category, where $\Lambda$ is a deformed preprojective algebra of generalized Dynkin type.

Proof. Let $M_1, \ldots, M_n$ be representatives of the isoclasses of indecomposable objects of $T$. The $k$-algebra $\Lambda = \text{End}(\bigoplus_{i=1}^n M_i)$ is basic, finite-dimensional and selfinjective since $T$ has a Serre duality. It is easy to see that $T$ and $\text{proj } \Lambda$ are equivalent as $k$-categories.

Let $\text{mod } \Lambda$ be the category of finitely presented $\Lambda$-modules. It is a Frobenius category. Denote by $\Sigma$ the suspension functor of the triangulated category $\text{mod } \Lambda$. The category $T$ is 1-Calabi-Yau, that is to say that the suspension functor $S$ of the triangulated category $T$ and the Serre functor $\nu$ are isomorphic. But in $\text{mod } \Lambda$, the functors $S$ and $\Sigma^3$ are isomorphic. Thus, for each non projective simple $\Lambda$-module $M$ we have an isomorphism $\Sigma^3 M \simeq \nu M$. By [4, thm 1.2], we get immediately the result. □

References