Borel Ranks and Wadge Degrees of Omega Context Free Languages

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Abstract. We determine completely the Borel hierarchy of the class of context free ω-languages, showing that, for each recursive non null ordinal α, there exist some $\mathbf{\Sigma}^0_\alpha$-complete and some $\mathbf{\Pi}^0_\alpha$-complete ω-languages accepted by Büchi 1-counter automata.

Keywords: 1-counter Büchi automata; context free ω-languages; Cantor topology; topological properties; Borel hierarchy; Borel ranks; Wadge degrees.

1 Introduction

Languages of infinite words accepted by finite automata were first studied by Büchi to prove the decidability of the monadic second order theory of one successor over the integers. The theory of the so called regular ω-languages is now well established and has found many applications for specification and verification of non-terminating systems; see [Tho90,Sta97,PP04] for many results and references. More powerful machines, like pushdown automata, Turing machines, have also been considered for the reading of infinite words, see Staiger's survey [Sta97] and the fundamental study [EH93] of Engelfriet and Hoogeboom on $\mathbf{X}$-automata, i.e. finite automata equipped with a storage type $\mathbf{X}$. A way to study the complexity of ω-languages is to study their topological complexity, and particularly to locate them with regard to the Borel and the projective hierarchies. On one side all ω-languages accepted by *deterministic* $\mathbf{X}$-automata with a Muller acceptance condition are Boolean combinations of $\mathbf{\Pi}^0_2$-sets hence $\mathbf{\Delta}^0_3$-sets, [Sta97,EH93]. This implies, from Mc Naughton's Theorem, that all regular ω-languages, which are accepted by deterministic Muller automata, are also $\mathbf{\Delta}^0_3$-sets. On the other side, for *non deterministic* finite machines, the question, posed by Lescow and Thomas in [LT94], naturally arises: what is the topological complexity of ω-languages accepted by automata equipped with a given storage type $\mathbf{X}$? It is well known that every ω-language accepted by a Turing machine (hence also by a $\mathbf{X}$-automaton) with a Muller acceptance condition is an analytic set. In previous papers, we proved that there are context free ω-languages, accepted by Büchi or Muller pushdown automata, of every finite Borel rank, of infinite Borel rank, or even being analytic but non Borel sets, [DFR01,Fin01b,Fin03a,Fin03b].

In this paper we determine completely the Borel hierarchy of ω-languages accepted by $\mathbf{X}$-automata, for every storage type $\mathbf{X}$ such that 1-counter automata can be simulated by $\mathbf{X}$-automata. In particular, we show that, for every recursive non-null ordinal α, there are some $\mathbf{\Sigma}^0_\alpha$-complete and some $\mathbf{\Pi}^0_\alpha$-complete ω-languages accepted by real time 1-counter Büchi automata, hence also in the class CFL_ω of context free ω-languages.
We think that the surprising result obtained in this paper is of interest for both logicians working on hierarchies arising in recursion theory or in descriptive set theory, and also for computer scientists working on questions connected with non-terminating systems, like the construction of effective strategies in infinite games, [Tho02,Wal00].
The paper is organized as follows. In Section 2 we define multicounter automata which will be a useful tool in the sequel. Recall on Borel hierarchy is given in Section 3. In Section 4 is studied the Borel hierarchy of ω-languages accepted by real time 8-counter automata. Our main result is proved in Section 5.

2 Multicounter automata

We assume the reader to be familiar with the theory of formal (ω)-languages [Tho90,Sta97]. We shall use usual notations of formal language theory.
When Σ is a finite alphabet, a *non-empty finite word* over Σ is any sequence $x = a_1 \ldots a_k$, where $a_i \in \Sigma$ for $i = 1, \ldots, k$, and k is an integer ≥ 1. The *length* of x is k, denoted by $|x|$. The *empty word* has no letter and is denoted by λ; its length is 0. For $x = a_1 \ldots a_k$, we write $x(i) = a_i$ and $x[i] = x(1) \ldots x(i)$ for $i \leq k$ and $x[0] = \lambda$. $\Sigma^\star$ is the *set of finite words* (including the empty word) over Σ.
The *first infinite ordinal* is ω. An ω-*word* over Σ is an ω -sequence $a_1 \ldots a_n \ldots$, where for all integers $i \geq 1$, $a_i \in \Sigma$. When σ is an ω-word over Σ, we write $\sigma = \sigma(1)\sigma(2) \ldots \sigma(n) \ldots$, where for all i, $\sigma(i) \in \Sigma$, and $\sigma[n] = \sigma(1)\sigma(2) \ldots \sigma(n)$ for all $n \geq 1$ and $\sigma[0] = \lambda$.
The *prefix relation* is denoted $\sqsubseteq$: a finite word u is a *prefix* of a finite word v (respectively, an infinite word v), denoted $u \sqsubseteq v$, if and only if there exists a finite word w (respectively, an infinite word w), such that $v = u.w$. The *set of* ω-*words* over the alphabet Σ is denoted by Σ^ω. An ω-*language* over an alphabet Σ is a subset of Σ^ω.

Definition 1. *Let k be an integer ≥ 1. A k-counter machine (k-CM) is a 4-tuple $\mathcal{M} = (K, \Sigma, \Delta, q_0)$, where K is a finite set of states, Σ is a finite input alphabet, $q_0 \in K$ is the initial state, and the transition relation Δ is a subset of $K \times (\Sigma \cup \{\lambda\}) \times \{0,1\}^k \times K \times \{0,1,-1\}^k$. The k-counter machine $\mathcal{M}$ is said to be real time iff: $\Delta \subseteq K \times \Sigma \times \{0,1\}^k \times K \times \{0,1,-1\}^k$, i.e. iff there are not any λ-transitions.*
If the machine $\mathcal{M}$ is in state q and $c_i \in \mathbb{N}$ is the content of the i^{th} counter $\mathcal{C}_i$ then the configuration (or global state) of $\mathcal{M}$ is the $(k+1)$-tuple $(q, c_1, \ldots, c_k)$.

For $a \in \Sigma \cup \{\lambda\}$, $q, q' \in K$ and $(c_1, \ldots, c_k) \in \mathbb{N}^k$ such that $c_j = 0$ for $j \in E \subseteq \{1, \ldots, k\}$ and $c_j > 0$ for $j \notin E$, if $(q, a, i_1, \ldots, i_k, q', j_1, \ldots, j_k) \in \Delta$ where

$i_j = 0$ for $j \in E$ and $i_j = 1$ for $j \notin E$, then we write:

$$a : (q, c_1, \ldots, c_k) \mapsto_{\mathcal{M}} (q', c_1 + j_1, \ldots, c_k + j_k)$$

$\mapsto^{\star}_{\mathcal{M}}$ is the transitive and reflexive closure of $\mapsto_{\mathcal{M}}$. (The subscript $\mathcal{M}$ will be omitted whenever the meaning remains clear).
Thus we see that the transition relation must satisfy:
if $(q, a, i_1, \ldots, i_k, q', j_1, \ldots, j_k) \in \Delta$ and $i_m = 0$ for some $m \in \{1, \ldots, k\}$, then $j_m = 0$ or $j_m = 1$ (but j_m may not be equal to -1).

Let $\sigma = a_1 a_2 \ldots a_n$ be a finite word over Σ. An sequence of configurations $r = (q_i, c^i_1, \ldots c^i_k)_{1 \leq i \leq p}$, for $p \geq n + 1$, is called a run of $\mathcal{M}$ on σ, starting in configuration $(p, c_1, \ldots, c_k)$, iff:

1. $(q_1, c^1_1, \ldots c^1_k) = (p, c_1, \ldots, c_k)$
2. *for each $i \geq 1$, there exists $b_i \in \Sigma \cup \{\lambda\}$ such that $b_i : (q_i, c^i_1, \ldots c^i_k) \mapsto_{\mathcal{M}} (q_{i+1}, c^{i+1}_1, \ldots c^{i+1}_k)$*
3. $a_1.a_2.a_3 \ldots a_n = b_1.b_2.b_3 \ldots b_p$

Let $\sigma = a_1 a_2 \ldots a_n \ldots$ be an ω-word over Σ. An ω-sequence of configurations $r = (q_i, c^i_1, \ldots c^i_k)_{i \geq 1}$ is called a run of $\mathcal{M}$ on σ, starting in configuration $(p, c_1, \ldots, c_k)$, iff:

1. $(q_1, c^1_1, \ldots c^1_k) = (p, c_1, \ldots, c_k)$
2. *for each $i \geq 1$, there exists $b_i \in \Sigma \cup \{\lambda\}$ such that $b_i : (q_i, c^i_1, \ldots c^i_k) \mapsto_{\mathcal{M}} (q_{i+1}, c^{i+1}_1, \ldots c^{i+1}_k)$ such that either $a_1 a_2 \ldots a_n \ldots = b_1 b_2 \ldots b_n \ldots$ or $b_1 b_2 \ldots b_n \ldots$ is a finite prefix of $a_1 a_2 \ldots a_n \ldots$*

The run r is said to be complete when $a_1 a_2 \ldots a_n \ldots = b_1 b_2 \ldots b_n \ldots$
For every such run, $\mathrm{In}(r)$ is the set of all states entered infinitely often during run r.
A complete run r of M on σ, starting in configuration $(q_0, 0, \ldots, 0)$, will be simply called "a run of M on σ".

Definition 2. *A Büchi k-counter automaton is a 5-tuple $\mathcal{M} = (K, \Sigma, \Delta, q_0, F)$, where $\mathcal{M}' = (K, \Sigma, \Delta, q_0)$ is a k-counter machine and $F \subseteq K$ is the set of accepting states. The ω-language accepted by $\mathcal{M}$ is*

$$L(\mathcal{M}) = \{\sigma \in \Sigma^\omega \mid \text{ there exists a run } r \text{ of } \mathcal{M} \text{ on } \sigma \text{ such that } \mathrm{In}(r) \cap F \neq \emptyset\}$$

Definition 3. *A Muller k-counter automaton is a 5-tuple $\mathcal{M} = (K, \Sigma, \Delta, q_0, \mathcal{F})$, where $\mathcal{M}' = (K, \Sigma, \Delta, q_0)$ is a k-counter machine and $\mathcal{F} \subseteq 2^K$ is the set of accepting sets of states. The ω-language accepted by $\mathcal{M}$ is*

$$L(\mathcal{M}) = \{\sigma \in \Sigma^\omega \mid \text{ there exists a run } r \text{ of } \mathcal{M} \text{ on } \sigma \text{ such that } \exists F \in \mathcal{F} \ \mathrm{In}(r) = F\}$$

The class of Büchi k-counter automata will be denoted $\mathbf{BC}(k)$.
The class of real time Büchi k-counter automata will be denoted $\mathbf{r}$-$\mathbf{BC}(k)$.
The class of ω-languages accepted by Büchi k-counter automata will be denoted

$\mathbf{BCL}(k)_\omega$.
The class of ω-languages accepted by real time Büchi k-counter automata will be denoted $\mathbf{r}$-$\mathbf{BCL}(k)_\omega$.

It is well known that an ω-language is accepted by a (real time) Büchi k-counter automaton iff it is accepted by a (real time) Muller k-counter automaton [EH93]. Notice that it cannot be shown without using the non determinism of automata and this result is no longer true in the deterministic case.
Remark that 1-counter automata introduced above are equivalent to pushdown automata whose stack alphabet is in the form $\{Z_0, A\}$ where Z_0 is the bottom symbol which always remains at the bottom of the stack and appears only there and A is another stack symbol. The pushdown stack may be seen like a counter whose content is the integer N if the stack content is the word $Z_0.A^N$.
In the model introduced here the counter value cannot be increased by more than 1 during a single transition. However this does not change the class of ω-languages accepted by such automata. So the class $\mathbf{BCL}(1)_\omega$ is equal to the class $\mathbf{1}$-$\mathbf{ICL}_\omega$, introduced in [Fin01c], and it is a strict subclass of the class $\mathbf{CFL}_\omega$ of context free ω-languages accepted by Büchi pushdown automata.

3 Borel hierarchy

We assume the reader to be familiar with basic notions of topology which may be found in [Mos80,LT94,Kec95,Sta97,PP04]. There is a natural metric on the set Σ^ω of infinite words over a finite alphabet Σ which is called the *prefix metric* and defined as follows. For $u, v \in \Sigma^\omega$ and $u \neq v$ let $\delta(u, v) = 2^{-l_{\mathrm{pref}(u,v)}}$ where $l_{\mathrm{pref}(u,v)}$ is the first integer n such that the $(n+1)^{st}$ letter of u is different from the $(n+1)^{st}$ letter of v. This metric induces on Σ^ω the usual Cantor topology for which *open subsets* of Σ^ω are in the form $W.\Sigma^\omega$, where $W \subseteq \Sigma^\star$. A set $L \subseteq \Sigma^\omega$ is a *closed set* iff its complement $\Sigma^\omega - L$ is an open set. Define now the *Borel Hierarchy* of subsets of Σ^ω:

Definition 4. *For a non-null countable ordinal α, the classes $\mathbf{\Sigma}^0_\alpha$ and $\mathbf{\Pi}^0_\alpha$ of the Borel Hierarchy on the topological space Σ^ω are defined as follows:*
$\mathbf{\Sigma}^0_1$ is the class of open subsets of Σ^ω, $\mathbf{\Pi}^0_1$ is the class of closed subsets of Σ^ω, and for any countable ordinal $\alpha \geq 2$:
$\mathbf{\Sigma}^0_\alpha$ is the class of countable unions of subsets of Σ^ω in $\bigcup_{\gamma<\alpha} \mathbf{\Pi}^0_\gamma$.
$\mathbf{\Pi}^0_\alpha$ is the class of countable intersections of subsets of Σ^ω in $\bigcup_{\gamma<\alpha} \mathbf{\Sigma}^0_\gamma$.

For a countable ordinal α, a subset of Σ^ω is a Borel set of *rank* α iff it is in $\mathbf{\Sigma}^0_\alpha \cup \mathbf{\Pi}^0_\alpha$ but not in $\bigcup_{\gamma<\alpha}(\mathbf{\Sigma}^0_\gamma \cup \mathbf{\Pi}^0_\gamma)$.

There are also some subsets of Σ^ω which are not Borel. In particular the class of Borel subsets of Σ^ω is strictly included into the class $\mathbf{\Sigma}^1_1$ of *analytic sets* which are obtained by projection of Borel sets, see for example [Sta97,LT94,PP04,Kec95] for more details. The (lightface) class Σ^1_1 of *effective analytic sets* is the class of sets which are obtained by projection of arithmetical sets. It is well known that

a set $L \subseteq \Sigma^\omega$, where Σ is a finite alphabet, is in the class Σ_1^1 iff it is accepted by a Turing machine with a Büchi or Muller acceptance condition [Sta97].
We now define completeness with regard to reduction by continuous functions. For a countable ordinal $\alpha \geq 1$, a set $F \subseteq \Sigma^\omega$ is said to be a $\mathbf{\Sigma}_\alpha^0$ (respectively, $\mathbf{\Pi}_\alpha^0$, $\mathbf{\Sigma}_1^1$)-*complete set* iff for any set $E \subseteq Y^\omega$ (with Y a finite alphabet): $E \in \mathbf{\Sigma}_\alpha^0$ (respectively, $E \in \mathbf{\Pi}_\alpha^0$, $E \in \mathbf{\Sigma}_1^1$) iff there exists a continuous function $f : Y^\omega \to \Sigma^\omega$ such that $E = f^{-1}(F)$. $\mathbf{\Sigma}_n^0$ (respectively $\mathbf{\Pi}_n^0$)-complete sets, with n an integer ≥ 1, are thoroughly characterized in [Sta86].

4 Borel hierarchy of ω-languages in r-BCL(8)$_\omega$

It is well known that if $L \subseteq \Sigma^\omega$ is accepted by a Turing machine with a Büchi acceptance condition and is a Borel set of rank α, then α is smaller than ω_1^{CK}, where ω_1^{CK} is the first non-recursive ordinal, usually called the Church-Kleene ordinal. Moreover for every non null countable ordinal $\alpha < \omega_1^{\mathrm{CK}}$, there exist some $\mathbf{\Sigma}_\alpha^0$-complete and some $\mathbf{\Pi}_\alpha^0$-complete sets in the class Σ_1^1 of ω-languages accepted by Turing machines with a Büchi acceptance condition.
On the other hand it is well known that every Turing machine can be simulated by a (non real time) 2-counter automaton. Thus for every non null countable ordinal $\alpha < \omega_1^{\mathrm{CK}}$, there exist some $\mathbf{\Sigma}_\alpha^0$-complete and some $\mathbf{\Pi}_\alpha^0$-complete ω-languages in the class $\mathbf{BCL}(2)_\omega$. We shall prove the following proposition.

Proposition 5. *For every non null countable ordinal $\alpha < \omega_1^{\mathrm{CK}}$, there exist some $\mathbf{\Sigma}_\alpha^0$-complete and some $\mathbf{\Pi}_\alpha^0$-complete ω-languages in the class* $\mathbf{r}$-$\mathbf{BCL}(8)_\omega$.

In order to prove this result, we first state the two following lemmas.

Let Σ be an alphabet having at least two letters, E be a new letter not in Σ, S be an integer ≥ 1, and $\theta_S : \Sigma^\omega \to (\Sigma \cup \{E\})^\omega$ be the function defined, for all $x \in \Sigma^\omega$, by:

$$\theta_S(x) = x(1).E^S.x(2).E^{S^2}.x(3).E^{S^3}.x(4)\ldots x(n).E^{S^n}.x(n+1).E^{S^{n+1}}\ldots$$

Lemma 6. *Let Σ be an alphabet having at least two letters and let $L \subseteq \Sigma^\omega$ be a $\mathbf{\Sigma}_\alpha^0$-complete (respectively, $\mathbf{\Pi}_\alpha^0$-complete) subset of Σ^ω for some ordinal $\alpha \geq 2$. Then the ω-language $\theta_S(L)$ is a $\mathbf{\Sigma}_\alpha^0$-complete (respectively, $\mathbf{\Pi}_\alpha^0$-complete) subset of $(\Sigma \cup \{S\})^\omega$.*

Lemma 7. *Let Σ be an alphabet having at least two letters and let $L \subseteq \Sigma^\omega$ be an ω-language in the class* $\mathbf{BCL}(2)_\omega$. *Then there exists an integer $S \geq 1$ such that $\theta_S(L)$ is in the class* $\mathbf{r}$-$\mathbf{BCL}(8)_\omega$.

5 Borel hierarchy of ω-languages in r-BCL(1)$_\omega$

We shall firstly prove the following result.

Proposition 8. *Let $k \geq 2$ be an integer. If, for some ordinal $\alpha \geq 2$, there is a $\mathbf{\Sigma}^0_\alpha$-complete (respectively, $\mathbf{\Pi}^0_\alpha$-complete) ω-language in the class* $\mathbf{r}$-$\mathbf{BCL}(k)_\omega$*, then there is some $\mathbf{\Sigma}^0_\alpha$-complete (respectively, $\mathbf{\Pi}^0_\alpha$-complete) ω-language in the class* $\mathbf{r}$-$\mathbf{BCL}(1)_\omega$.

To simplify the exposition of the proof of this result, firstly, we are going to sketch the proof for $k = 2$. Next we shall explain the modifications to do in order to infer the result for the integer $k = 8$ which is in fact the only case we shall need in the sequel. (However our main result will show that the proposition is true for every integer $k \geq 2$).
For that purpose we define first a coding of ω-words over a finite alphabet Σ by ω-words over the alphabet $\Sigma \cup \{A, B, 0\}$ where A, B and 0 are new letters not in Σ. We shall code an ω-word $x \in \Sigma^\omega$ by the ω-word $h(x)$ defined by

$$h(x) = A.0^6.x_1.B.0^{6^2}.A.0^{6^2}.x_2.B.0^{6^3}.A.0^{6^3}.x_3.B \ldots B.0^{6^n}.A.0^{6^n}.x_n.B \ldots$$

This coding defines a mapping $h : \Sigma^\omega \rightarrow (\Sigma \cup \{A, B, 0\})^\omega$. The function h is continuous because for all ω-words $x, y \in \Sigma^\omega$ and each positive integer n, it holds that $\delta(x, y) < 2^{-n} \rightarrow \delta(h(x), h(y)) < 2^{-n}$.

Lemma 9. *Let Σ be a finite alphabet and $(h(\Sigma^\omega))^- = (\Sigma \cup \{A, B, 0\})^\omega - h(\Sigma^\omega)$. If $\mathcal{L} \subseteq \Sigma^\omega$ is $\mathbf{\Sigma}^0_\alpha$-complete (respectively, $\mathbf{\Pi}^0_\alpha$-complete), for a countable ordinal $\alpha \geq 2$, then $h(\mathcal{L}) \cup h(\Sigma^\omega)^-$ is a $\mathbf{\Sigma}^0_\alpha$-complete (respectively, $\mathbf{\Pi}^0_\alpha$-complete) subset of $(\Sigma \cup \{A, B, 0\})^\omega$.*

In order to apply Lemma 9, we want now to prove that if $L(\mathcal{A}) \subseteq \Sigma^\omega$ is accepted by a real time 2-counter automaton $\mathcal{A}$ with a Büchi acceptance condition then $h(L(\mathcal{A})) \cup h(\Sigma^\omega)^-$ is accepted by a 1-counter automaton with a Büchi acceptance condition. We firstly prove the following lemma.

Lemma 10. *Let Σ be a finite alphabet and h be the coding of ω-words over Σ defined as above. Then $h(\Sigma^\omega)^- = (\Sigma \cup \{A, B, 0\})^\omega - h(\Sigma^\omega)$ is accepted by a real time 1-counter Büchi automaton.*

We would like now to prove that if $L(\mathcal{A}) \subseteq \Sigma^\omega$ is accepted by a real time 2-counter automaton $\mathcal{A}$ with a Büchi acceptance condition then $h(L(\mathcal{A}))$ is in $\mathbf{BCL}(1)_\omega$. We cannot show this, so we are firstly going to define another ω-language $\mathcal{L}(\mathcal{A})$ accepted by a 1-counter Büchi automaton and we shall prove that $h(L(\mathcal{A})) \cup h(\Sigma^\omega)^- = \mathcal{L}(\mathcal{A}) \cup h(\Sigma^\omega)^-$.
We shall need the following notion. Let $N \geq 1$ be an integer such that $N = 2^x.3^y.N_1$ where x, y are positive integers and $N_1 \geq 1$ is an integer which is neither divisible by 2 nor by 3. Then we set $P_2(N) = x$ and $P_3(N) = y$. So $2^{P_2(N)}$ is the greatest power of 2 which divides N and $2^{P_3(N)}$ is the greatest power of 3 which divides N.
Let then a 2-counter Büchi automaton $\mathcal{A} = (K, \Sigma, \Delta, q_0, F)$ accepting the ω-language $L(\mathcal{A}) \subseteq \Sigma^\omega$. The ω-language $\mathcal{L}(\mathcal{A})$ is the set of ω-words over the alphabet $\Sigma \cup \{A, B, 0\}$ in the form

$$A.u_1.v_1.x_1.B.w_1.z_1.A.u_2.v_2.x_2.B.w_2.z_2.A \ldots A.u_n.v_n.x_n.B.w_n.z_n.A \ldots$$

where, for all integers $i \geq 1$, $v_i, w_i \in 0^+$, $u_i, z_i \in 0^\star$, $|u_1| = 5$, $|u_{i+1}| = |z_i|$ and there is a sequence $(q_i)_{i\geq 0}$ of states of K and integers $j_i, j'_i \in \{-1; 0; 1\}$, for $i \geq 1$, such that for all integers $i \geq 1$:

$$x_i : (q_{i-1}, P_2(|v_i|), P_3(|v_i|)) \mapsto_{\mathcal{A}} (q_i, P_2(|v_i|) + j_i, P_3(|v_i|) + j'_i)$$

and

$$|w_i| = |v_i|.2^{j_i}.3^{j'_i}$$

Moreover some state $q_f \in F$ occurs infinitely often in the sequence $(q_i)_{i\geq 0}$.
Notice that the state q_0 of the sequence $(q_i)_{i\geq 0}$ is also the initial state of $\mathcal{A}$.

Lemma 11. *Let $\mathcal{A}$ be a real time 2-counter Büchi automaton accepting ω-words over the alphabet Σ and $\mathcal{L}(\mathcal{A}) \subseteq (\Sigma \cup \{A, B, 0\})^\omega$ be defined as above. Then $\mathcal{L}(\mathcal{A})$ is accepted by a 1-counter Büchi automaton $\mathcal{B}$.*

Lemma 12. *Let $\mathcal{A}$ be a real time 2-counter Büchi automaton accepting ω-words over the alphabet Σ and $\mathcal{L}(\mathcal{A}) \subseteq (\Sigma \cup \{A, B, 0\})^\omega$ be defined as above. Then $L(\mathcal{A}) = h^{-1}(\mathcal{L}(\mathcal{A}))$, i.e. $\forall x \in \Sigma^\omega \quad h(x) \in \mathcal{L}(\mathcal{A}) \longleftrightarrow x \in L(\mathcal{A})$.*

Proof. Let $\mathcal{A}$ be a real time 2-counter Büchi automaton accepting ω-words over the alphabet Σ and $\mathcal{L}(\mathcal{A}) \subseteq (\Sigma \cup \{A, B, 0\})^\omega$ be defined as above. Let $x \in \Sigma^\omega$ be an ω-word such that $h(x) \in \mathcal{L}(\mathcal{A})$. So $h(x)$ may be written

$$h(x) = A.0^6.x_1.B.0^{6^2}.A.0^{6^2}.x_2.B.0^{6^3}.A.0^{6^3}.x_3.B \ldots B.0^{6^n}.A.0^{6^n}.x_n.B \ldots$$

and also

$$h(x) = A.u_1.v_1.x_1.B.w_1.z_1.A.u_2.v_2.x_2.B.w_2.z_2.A \ldots A.u_n.v_n.x_n.B.w_n.z_n.A \ldots$$

where, for all integers $i \geq 1$, $v_i, w_i \in 0^+$, $u_i, z_i \in 0^\star$, $|u_1| = 5$, $|u_{i+1}| = |z_i|$ and there is a sequence $(q_i)_{i\geq 0}$ of states of K and integers $j_i, j'_i \in \{-1; 0; 1\}$, for $i \geq 1$, such that for all integers $i \geq 1$:

$$x_i : (q_{i-1}, P_2(|v_i|), P_3(|v_i|)) \mapsto_{\mathcal{A}} (q_i, P_2(|v_i|) + j_i, P_3(|v_i|) + j'_i)$$

and

$$|w_i| = |v_i|.2^{j_i}.3^{j'_i}$$

some state $q_f \in F$ occurring infinitely often in the sequence $(q_i)_{i\geq 0}$.
In particular, $u_1 = 0^5$ and $u_1.v_1 = 0^6$ thus $|v_1| = 1 = 2^0.3^0$. We can prove by induction on the integer $i \geq 1$ that, for all integers $i \geq 1$, $|w_i| = |v_{i+1}| = 2^{P_2(|w_i|)}.3^{P_3(|w_i|)}$. Moreover, setting $c_1^i = P_2(|v_i|)$ and $c_2^i = P_3(|v_i|)$, we can prove that for each integer $i \geq 1$ it holds that

$$x_i : (q_{i-1}, c_1^i, c_2^i) \mapsto_{\mathcal{A}} (q_i, c_1^{i+1}, c_2^{i+1})$$

But there is some state $q_f \in K$ which occurs infinitely often in the sequence $(q_i)_{i\geq 1}$. This implies that $(q_{i-1}, c_1^i, c_2^i)_{i\geq 1}$ is a successful run of $\mathcal{A}$ on x thus $x \in L(\mathcal{A})$.
Conversely it is easy to see that if $x \in L(\mathcal{A})$ then $h(x) \in \mathcal{L}(\mathcal{A})$. This ends the proof of Lemma 12. □

Remark 13. *The simulation, during the reading of $h(x)$ by the 1-counter Büchi automaton $\mathcal{B}$, of the behaviour of the real time 2-counter Büchi automaton $\mathcal{A}$ reading x, can be achieved, using a coding of the content (c_1, c_2) of two counters by a single integer $2^{c_1}.3^{c_2}$ and the* **special shape** *of ω-words in $h(\Sigma^\omega)$ which allows the propagation of the counter value of $\mathcal{B}$. This will be sufficient here, because of the previous lemmas, and in particular of the fact that $h(\Sigma^\omega)^-$ is in the class* $\mathbf{r}$-$\mathbf{BCL}(1)_\omega$*. and we can now end the proof of Proposition 8.*

End of Proof of Proposition 8. Let $\alpha \geq 2$ be an ordinal. Assume that there is a $\mathbf{\Sigma}^0_\alpha$-complete (respectively, $\mathbf{\Pi}^0_\alpha$-complete) ω-language $L(\mathcal{A}) \subseteq \Sigma^\omega$ which is accepted by a real time 2-counter Büchi automaton $\mathcal{A}$. By Lemma 9, $h(\mathcal{L}) \cup h(\Sigma^\omega)^-$ is a $\mathbf{\Sigma}^0_\alpha$-complete (respectively, $\mathbf{\Pi}^0_\alpha$-complete) subset of $(\Sigma \cup \{A, B, 0\})^\omega$. On the other hand Lemma 12 states that $L(\mathcal{A}) = h^{-1}(\mathcal{L}(\mathcal{A}))$ and this implies that $h(L(\mathcal{A})) \cup h(\Sigma^\omega)^- = \mathcal{L}(\mathcal{A}) \cup h(\Sigma^\omega)^-$. But we know by Lemmas 10 and 11 that the ω-languages $h(\Sigma^\omega)^-$ and $\mathcal{L}(\mathcal{A})$ are in the class $\mathbf{BCL}(1)_\omega$ thus their union is also accepted by a 1-counter Büchi automaton. Therefore $h(L(\mathcal{A})) \cup h(\Sigma^\omega)^-$ is a $\mathbf{\Sigma}^0_\alpha$-complete (respectively, $\mathbf{\Pi}^0_\alpha$-complete) ω-language in the class $\mathbf{BCL}(1)_\omega$.

We can now easily show that there is a $\mathbf{\Sigma}^0_\alpha$-complete (respectively, $\mathbf{\Pi}^0_\alpha$-complete) ω-language in the class $\mathbf{r}$-$\mathbf{BCL}(1)_\omega$, using the two following facts. (1) $h(\Sigma^\omega)^-$ is accepted by a *real time* 1-counter Büchi automaton; (2) $\mathcal{L}(\mathcal{A})$ is accepted by a (non real time) 1-counter Büchi automaton $\mathcal{B}$ but at most 5 consecutive λ-transitions can occur during the reading of an ω-word x by $\mathcal{B}$ (see the proof of Lemma 11 in the full version of this paper).

In order to prove Proposition 8 for the integer $k = 8$, we reason in a similar way. We first replace the integer $6 = 2.3$ by the product of the eight first prime numbers:

$$K = 2.3.5.7.11.13.17.19 = 9699690$$

and the mapping h by the mapping h_K, defined for all $x \in \Sigma^\omega$, by:

$$h_K(x) = A.0^K.x_1.B.0^{K^2}.A.0^{K^2}.x_2.B.0^{K^3}.A.0^{K^3}.x_3.B \ldots B.0^{K^n}.A.0^{K^n}.x_n.B \ldots$$

We define also, for every 8-counter Büchi automaton $\mathcal{A}$, an ω-language $\mathcal{L}(\mathcal{A})$, accepted by a 1-counter Büchi automaton, such that $L(\mathcal{A}) = h_K^{-1}(\mathcal{L}(\mathcal{A}))$.
The essential change is that now the content $(c_1, c_2, \ldots, c_8)$ of eight counters is coded by the product $2^{c_1}.3^{c_2}.\ldots.(17)^{c_7}.(19)^{c_8}$.
Details will be included in the full version of this paper. □

From the results of Section 4 and Proposition 8, we can now state the following result.

Theorem 14. *Let $\mathcal{C}$ be a class of ω-languages such that:*

$$\mathbf{r}\text{-}\mathbf{BCL}(1)_\omega \subseteq \mathcal{C} \subseteq \Sigma^1_1.$$

(a) If $L \in \mathcal{C}$ is a Borel set of rank α, then α is smaller than ω_1^{CK}.
(b) For every non null countable ordinal $\alpha < \omega_1^{\mathrm{CK}}$, there exists some $\mathbf{\Sigma}^0_\alpha$-complete and some $\mathbf{\Pi}^0_\alpha$-complete ω-languages in the class $\mathcal{C}$.

The Wadge hierarchy is a great refinement of the Borel hierarchy, [Dup01,Wad83]. Looking carefully at the proofs given in this paper, we can easily show the following strengthening of Theorem 14 (see the proof in the full version of this paper).

Theorem 15. *The Wadge hierarchy of the class* $\mathbf{r}$-$\mathbf{BCL}(1)_\omega$*, hence also of the class* CFL_ω*, or of every class* $\mathcal{C}$ *such that* $\mathbf{r}$-$\mathbf{BCL}(1)_\omega \subseteq \mathcal{C} \subseteq \Sigma^1_1$*, is the Wadge hierarchy of the class* Σ^1_1 *of* ω*-languages accepted by Turing machines with a Büchi acceptance condition.*

6 Concluding remarks

We have completely determined the Borel hierarchy of classes $\mathbf{r}$-$\mathbf{BCL}(1)_\omega$ and CFL_ω and showed that their Wadge hierarchy is also the Wadge hierarchy of the class Σ^1_1. The methods used in this paper are different from those used in previous papers on context free ω-languages [Fin01b,Fin01a,Fin03a,Fin03b], where we gave an inductive construction of some $\mathbf{\Delta}^0_\omega$ context free ω-languages of a given Borel rank or Wadge degree, using work of Duparc on the Wadge hierarchy of $\mathbf{\Delta}^0_\omega$ Borel sets, [Dup01]. However it will be possible to combine both methods for the effective construction of ω-languages in the class $\mathbf{r}$-$\mathbf{BCL}(1)_\omega$, and of 1-counter Büchi automata accepting them, of a given Wadge degree among the ε_ω degrees obtained in [Fin01a] for $\mathbf{\Delta}^0_\omega$ context free ω-languages.
Finally we mention that in a forthcoming paper we apply similar methods to the study of topological properties of infinitary rational relations and we prove that their Wadge and Borel hierarchies are equal to the corresponding hierarchies of of the classes $\mathbf{r}$-$\mathbf{BCL}(1)_\omega$, CFL_ω or Σ^1_1, [Fin04].

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