Wreath products, nilpotent orbits and symplectic deformations
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Abstract

We recover the wreath product $X := \text{Sym}^2(\mathbb{C}^2/\pm 1)$ as a transversal slice to a nilpotent orbit in $\mathfrak{sp}_6$. By using deformations of Springer resolutions, we construct a symplectic deformation of symplectic resolutions of $X$. AMS Classification: 14E15, 14M17

0. Introduction

Let $H \subset \text{Sp}(2n)$ be a finite sub-group and $X := \mathbb{C}^{2n}/H$ the quotient symplectic variety. Given a projective symplectic resolution $Z \rightarrow X$, it was shown in [GK] that there exists a symplectic deformation of (1) over $B := H^2(Z, \mathbb{C})$, i.e. a morphism $\Pi : Z \rightarrow X$ over $B$ such that over the origin $0 \in B$, $\Pi_0 : Z_0 \rightarrow X_0$ is the resolution (1), and over a generic point $b \in B$, $Z_b, X_b$ are symplectic smooth varieties isomorphic under $\Pi_b$, where $\Pi_b$ is the restriction of $\Pi$ to the fibers over $b$. The proof of this theorem is based on the infinitesimal and formal deformations of $\pi$ developed in [KV] and the globalization is obtained by using the expanding $\mathbb{C}^\ast$-action on $X$. As noted already in [GK], this deformation is very similar to the deformation of the Springer resolution of nilpotent cones given by Grothendieck’s simultaneous resolution ([S]). However, the construction of symplectic deformations in general is rather implicit. The purpose of this note is to provide some explicit examples of such deformations.

A class of important examples of symplectic resolutions is given by Hilbert-Chow morphisms ([Wan]): $\text{Hilb}^n(\mathbb{C}^2/\Gamma) \rightarrow \text{Sym}^n(\mathbb{C}^2/\Gamma)$, where $\Gamma \subset SL(2)$
is a finite sub-group and $\mathbb{C}^2/\Gamma \rightarrow \mathbb{C}^2/\Gamma$ is the minimal resolution. The simplest case is $n = 1$. It can be shown ([Sti]) that a transverse slice of the sub-regular nilpotent orbit in the nilpotent cone has $ADE$ singularities, then Grothendieck’s simultaneous resolution provides symplectic deformations of the minimal resolution (see also [GR] section 3).

The next simple case is $n = 2$ and $\Gamma = \pm 1$, i.e. the resolution $\pi : \text{Hilb}^2(T^*\mathbb{P}^1) \rightarrow \text{Sym}^2(\mathbb{C}^2/\pm 1)$. Our aim of this note is to construct a symplectic deformation of the resolution $\pi$. The key idea is to recover $\pi$ as a slice of some Springer resolution. More precisely, let us consider the following two nilpotent orbits in $\mathfrak{sp}_6$:

$$\mathcal{O}_{[2,2,2]} := \text{Sp}_6 \cdot \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad \mathcal{O}_{[4,2]} := \text{Sp}_6 \cdot \begin{pmatrix} 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \end{pmatrix},$$

then their closures in $\mathfrak{sp}_6$ are given by:

$$\overline{\mathcal{O}}_{[2,2,2]} := \{ A \in \mathfrak{sp}_6 \mid A^2 = 0 \}, \quad \overline{\mathcal{O}}_{[4,2]} := \{ A \in \mathfrak{sp}_6 \mid A^4 = 0 \}.$$

We will prove that the wreath product $\text{Sym}^2(\mathbb{C}^2/\pm 1)$ is in fact isomorphic to the intersection of a transverse slice of the nilpotent orbit $\mathcal{O}_{[2,2,2]}$ with the nilpotent orbit closure $\overline{\mathcal{O}}_{[4,2]}$ in $\mathfrak{sp}_6$. The singular variety $\overline{\mathcal{O}}_{[4,2]}$ admits exactly two symplectic resolutions. By restricting them to the transverse slice, we recover exactly the two symplectic resolutions of $\text{Sym}^2(\mathbb{C}^2/\pm 1)$. Using deformations of Springer resolutions (e.g. [Fu]), we construct a symplectic deformation of $\pi$.

It is somewhat surprising that we can recover the wreath product $\text{Sym}^2(\mathbb{C}^2/\pm 1)$ from nilpotent orbits, although the interplay between nilpotent orbits and Hilbert schemes has been noticed in [Man], where a transverse slice to the nilpotent orbit $\mathcal{O}_{2m-n,n}(n \leq m)$ in the nilpotent cone of $\mathfrak{sl}_{2m}$ is recovered as an open subset (whose complement is of codimension 1 when $n \geq 2$) of the Hilbert scheme $\text{Hilb}^n(A_{2m})$, for some singular surface $A_{2m}$. Here $\mathcal{O}_{2m-n,n}$ consists of nilpotent matrices $A \in \mathfrak{sl}_{2m}$ whose Jordan form has only two blocks, with sizes $2m - n$ and $n$ respectively. It would be very interesting to recover other wreath products as a transverse slice to nilpotent orbits, which would in turn reveal more the mysterious relationships between Hilbert-Chow
resolutions and Springer resolutions, although the two objects are studied in usual separately.

1. The transverse slice

Let \( \mathfrak{g} \) be a simple Lie algebra and \( G \) its adjoint group. For any nilpotent element \( x \in \mathfrak{g} \), by the theorem of Jacobson-Morozov, there exists an \( \mathfrak{sl}_2 \)-triplet \((x, y, h)\). Then \( S = x + \mathfrak{g}^y \) is a transverse slice to the nilpotent orbit \( G \cdot x \) in \( \mathfrak{g} \), and the morphism \( G \times S \to \mathfrak{g} \) is smooth ([Slo], Section 7.4). Here \( \mathfrak{g}^y := \{ z \in \mathfrak{g} \mid [z, y] = 0 \} \).

From now on, let \( \mathfrak{g} = \mathfrak{sp}_6 \), and consider the following \( \mathfrak{sl}_2 \)-triplet associated to the nilpotent orbit \( O_{[2, 2, 2]} \):

\[
x_0 = \begin{pmatrix} 0 & I \\ I & 0 \\ 0 & 0 \end{pmatrix}, \quad y_0 = \begin{pmatrix} 0 & 0 \\ 0 & I \\ I & 0 \end{pmatrix}, \quad h_0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix},
\]

(2)

where \( I \) is the \( 3 \times 3 \) identity matrix. Note that \( O_{[2, 2, 2]} = \mathfrak{sp}_6 \cdot x_0 \).

The transverse slice to the orbit \( O_{[2, 2, 2]} \) is given by

\[
S = x_0 + \mathfrak{g}^{y_0} = \left\{ \begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix} \mid Z_1 + Z_1^T = 0, Z_2 = Z_2^T \right\} \subset \mathfrak{sp}_6.
\]

We choose the following parameters for \( Z_1 \) and \( Z_2 \):

\[
Z_1 = \begin{pmatrix} 0 & a_3/2 & -a_2/2 \\ -a_3/2 & 0 & a_1/2 \\ a_2/2 & -a_1/2 & 0 \end{pmatrix}, \quad Z_2 = \begin{pmatrix} x_1 & y_1 & y_2 \\ y_1 & x_2 & y_3 \\ y_2 & y_3 & x_3 \end{pmatrix}.
\]

Note that \( O_{[2, 2, 2]} \subset \overline{O}_{[4, 2]} \), and the codimension is is 4. Let \( T \) be the scheme intersection \( S \cap \overline{O}_{[4, 2]} \). The variety \( \overline{O}_{[4, 2]} \) is normal and the morphism \( G \times T \to \overline{O}_{[4, 2]} \) is smooth. It follows that \( T \) is normal. As easily seen, a matrix \( A \in S \) is in \( T \) if and only if \( \text{rk}(A) \leq 4 \) and \( \text{tr}(A) = \text{tr}(A^2) = \text{tr}(A^3) = \text{tr}(A^4) = 0 \).

Notice that \( \text{rk}(A) \leq 4 \) is equivalent to \( \text{rk}(Z_2 - Z_1^2) \leq 1 \). The matrix \( Z_2 - Z_1^2 \) is symmetric, so this is equivalent to the existence of \( u = (u_1, u_2, u_3) \in \mathbb{C}^3 \) such that \( Z_2 - Z_1^2 = u^T u \), from which we can substitute the variables \( x_i, y_j \) by \( u_k \). Remark that \( u \) and \( -u \) give the same \( Z_2 \), so we should quotient by the following action of \( Z_2 \): \( u \mapsto -u \).

Now a direct calculus shows that \( \text{tr}(A) = \text{tr}(A^3) = 0 \). That \( \text{tr}(A^2) = 0 \) is equivalent to \( \sum_{i=1}^{3} u_i^2 = \sum_{i=1}^{3} a_i^2 \), and \( \text{tr}(A^4) = 2 \text{tr}(Z_1^4) + 2 \text{tr}(Z_2^4) + \)
12 \text{tr}(Z_1^2Z_2) = 0 \text{ is equivalent to } \sum_{i=1}^{3} a_i u_i = 0, \text{ which gives:}

\[ T = \left\{ (a_1, a_2, a_3, u_1, u_2, u_3) \mid \sum_i a_i^2 = \sum_i a_i^2, \sum_i a_i u_i = 0 \right\}/\mathbb{Z}_2, \]

where the action of $\mathbb{Z}_2$ is given by

\[(a_1, a_2, a_3, u_1, u_2, u_3) \mapsto (a_1, a_2, a_3, -u_1, -u_2, -u_3).\]

Consider the following two nilpotent orbits in $\mathfrak{sp}_6$:

\[ \mathcal{O}_{[3,3]} = \{ A \in \mathfrak{sp}_6 \mid A^3 = 0, \text{rk}(A) = 4 \}, \]

\[ \mathcal{O}_{[4,1,1]} = \{ A \in \mathfrak{sp}_6 \mid A^4 = 0, \text{rk}(A) = 3, A^2 \neq 0 \}. \]

Then $\mathcal{O}_{[2,2,2]} = \overline{\mathcal{O}_{[3,3]}} \cap \overline{\mathcal{O}_{[4,1,1]}}$ and $\overline{\mathcal{O}_{[3,3]}} \subset \overline{\mathcal{O}_{[4,2]}} \supset \overline{\mathcal{O}_{[4,1,1]}}$. The relationship of inclusions can be resumed in the following diagram:

\[ \overline{\mathcal{O}_{[4,2]}} \]

\[ \overline{\mathcal{O}_{[4,1,1]}} \]

\[ \overline{\mathcal{O}_{[3,3]}} \]

\[ \overline{\mathcal{O}_{[2,2,2]}} \]

The intersection of $T$ with the two orbit closures $\overline{\mathcal{O}_{[3,3]}}, \overline{\mathcal{O}_{[4,1,1]}}$ is exactly the singular locus of $T$, which is defined by the following

\[ T \cap \overline{\mathcal{O}_{[3,3]}} = \left\{ \sum_i a_i^2 = 0, u^T u = -4Z_1^2 \right\}, \]

\[ T \cap \overline{\mathcal{O}_{[4,1,1]}} = \left\{ \sum_i a_i^2 = 0, u_1 = u_2 = u_3 = 0 \right\}. \]

Both are isomorphic to the surface $\mathbb{C}^2/\pm 1$ with an isolated $A_1$-singularity. The intersection $T \cap \mathcal{O}_{[2,2,2]} = x_0$ is just a point.

2. The wreath product

Now we consider the simplest wreath product $\text{Sym}^2(\mathbb{C}^2/\pm 1) = \mathbb{C}^4/H$, where $H$ is the subgroup of $\text{Sp}(4)$ generated by the following elements:

\[ \sigma(x_1, x_2, y_1, y_2) = (y_1, y_2, x_1, x_2), \quad \tau(x_1, x_2, y_1, y_2) = (-x_1, -x_2, y_1, y_2). \]
To write down equations for this affine normal variety, we put
\[ a_1 = -i(x_1^2 + y_1^2 + x_2^2 + y_2^2)/2, \quad a_2 = (x_1^2 + y_1^2 - x_2^2 - y_2^2)/2, \quad a_3 = x_1x_2 + y_1y_2, \]
\[ u_1 = x_1y_1 + x_2y_2, \quad u_2 = i(x_1y_1 - x_2y_2), \quad u_3 = i(x_1y_2 + x_2y_1). \]

The functions \( a_1, a_2, a_3 \) are \( H \)-invariant, and the action of \( H \) restricts to a \( \mathbb{Z}_2 \)-action on \((u_1, u_2, u_3)\) given by \((u_1, u_2, u_3) \mapsto (-u_1, -u_2, -u_3)\). Now it is straight-ward to check that
\[ \mathbb{C}^4/H \simeq \{(a_1, a_2, a_3, u_1, u_2, u_3) \mid \sum a_i^2 = \sum u_i^2, \sum a_i u_i = 0\}/\mathbb{Z}_2, \]
where the \( \mathbb{Z}_2 \)-action is given by \((a_1, a_2, a_3, u_1, u_2, u_3) \mapsto (a_1, a_2, a_3, -u_1, -u_2, -u_3)\).

This gives the following proposition.

**Proposition 1.** The transverse slice \( T \) is isomorphic to the wreath product \( X := \text{Sym}^2(\mathbb{C}^2/\pm 1) \).

The singular locus of the wreath product \( X \) has two components: one is the diagonal \( \Delta \) and the other will be denoted by \( \Xi \). One sees that the isomorphism between \( T \) and \( X \) sends \( T \cap \mathcal{O}_{[3,3]} \) to \( \Delta \) and \( T \cap \mathcal{O}_{[4,1,1]} \) to \( \Xi \).

A symplectic resolution of \( X \) is given by the composition:
\[ \pi : \text{Hilb}^2(T^*\mathbb{P}^1) \to \text{Sym}^2(T^*\mathbb{P}^1) \to \text{Sym}^2(\mathbb{C}^2/\pm 1) = X. \]

The central fiber of \( \pi \) contains a \( \mathbb{P}^2 \), so we can blow up this \( \mathbb{P}^2 \) and then blow down along the other direction, i.e. we can perform a Mukai flop, which gives another symplectic resolution \( \pi^+ : \text{Hilb}^2(T^*\mathbb{P}^1) \to X \). One sees that \( \pi^{-1}(0) \subset \pi^{-1}(\Delta - \{0\}) \), but \( \pi^{-1}(0) \) is not contained in \( \pi^{-1}(\Xi - \{0\}) \), so \( \Delta \) and \( \Xi \) are not symmetric with respect to \( \pi \). For \( \pi^+ \), it changes the role of \( \Delta \) and \( \Xi \). It is known that any projective symplectic resolutions of \( X \) is isomorphic to \( \pi \) or \( \pi^+ \) (for details, see ([FN], [Fuj])).

3. Springer resolutions

The nilpotent orbit closure \( \mathcal{O}_{[4,2]} \) admits exactly two symplectic resolutions, given by Springer maps:
\[ T^*(G/P_1) \xrightarrow{\phi_1} \mathcal{O}_{[4,2]} \xrightarrow{\phi_2} T^*(G/P_2), \quad (3) \]
where $P_1$ (resp. $P_2$) is the standard parabolic sub-group of $G$ with flag type \([1,2,2,1]\) (resp. \([2,1,1,2]\)). The matrix forms of $P_1$ and $P_2$ are as follows:

$$
P_1 = \{ \begin{pmatrix} * & * & * & * & * & * \\ 0 & * & * & * & * & * \\ 0 & 0 & * & 0 & 0 & 0 \\ 0 & 0 & 0 & * & * & * \\ 0 & 0 & 0 & * & * & * \end{pmatrix}, \quad P_2 = \{ \begin{pmatrix} * & * & * & * & * & * \\ * & * & * & * & * & * \\ 0 & 0 & * & * & * & * \\ 0 & 0 & 0 & * & * & * \\ 0 & 0 & 0 & 0 & * \end{pmatrix} \}.
$$

The restrictions of $\phi_1, \phi_2$ to the pre-image of the transverse slice $T$ give two projective symplectic resolutions of $T$.

$$\text{Hilb}^2(T^*\mathbb{P}^1) \simeq Z_1 \overset{\pi_1}{\longrightarrow} T \overset{\pi_2}{\longleftarrow} Z_2 \simeq \text{Hilb}^2(T^*\mathbb{P}^1).$$

**Proposition 2.** The two symplectic resolutions $\pi_1, \pi_2$ are related by a Mukai flop, in particular, they are not isomorphic. Furthermore $\pi_1 = \pi$ and $\pi_2 = \pi^+$.

**Proof.** We will calculate the central fiber over the point $x_0 = T \cap O_{[2,2,2]}$ (c.f. (3)) under the maps $\pi_1$ and $\pi_2$. Let $\{e_i, 1 \leq i \leq 6\}$ be the natural basis of $\mathbb{C}^6$ and the symplectic form is $\omega = \sum_{i=1}^{3} e_i^* \wedge e_{i+3}^*$. Notice that $\text{Im}(x_0) = \text{Ker}(x_0) = \mathbb{C}\langle e_1, e_2, e_3 \rangle =: K$ is Lagrangian.

It is easy to see that

$$\pi_1^{-1}(x_0) = \{ \text{flags } (F_1 \subset F_2) \mid x_0F_2 \subset F_1 \subset K, F_2 = F_2^\perp, \dim F_1 = 1 \}.$$

Since $x_0F_2 \subset F_1$ is of dimension 1, one has two possibilities:

(i). $\dim(K \cap F_2) = 2$, then $F_1 = x_0F_2 \subset x_0F_1^\perp$. Suppose that $F_1$ is generated by $\sum_{i=1}^{3} a_ie_i$, then $x_0F_1^\perp = \{ \sum_{i=1}^{3} b_ie_i \mid \sum_{i=1}^{3} a_ib_i = 0 \}$. The condition $F_1 \subset x_0F_1^\perp$ is equivalent to $\sum_{i=1}^{3} a_i^2 = 0$, which is a $\mathbb{P}^1$ inside $\mathbb{P}(K)$. The condition for $F_2$ is just $x_0F_1^\perp \subset F_2 \subset x_0^{-1}F_1$ which is a $\mathbb{P}^1$. So finally this component is a $\mathbb{P}^1$-bundle over $\mathbb{P}^1$.

(ii). $F_2 = K$, then this is isomorphic to $\mathbb{P}(K)$. The two components intersect at a curve $C_1 \simeq \mathbb{P}^1$ inside $\mathbb{P}(K)$.

The fiber of $x_0$ under $\pi_2$ consists of flags $(F_1 \subset F_2)$ such that $x_0F_2 \subset F_1 \subset K, F_2 = F_2^\perp$ and $\dim F_1 = 2$. Since $F_1 \subset K \cap F_2$, so $\dim(K \cap F_2) \geq 2$. There are two cases:

(i). $\dim(K \cap F_2) = 2$, then $F_1 = K \cap F_2$ and $x_0F_2 \subset F_1$. This gives that $x_0F_2 = x_0F_1^\perp \subset F_1$. Suppose $F_1$ is generated by $\sum_{i=1}^{3} a_ie_i, \sum_{i=1}^{3} b_ie_i$. Then
we have \( x_0 F^\perp_1 = \{ \sum_i c_i e_i | \sum_i a_i c_i = \sum_i b_i c_i = 0 \} \). The condition \( x_0 F^\perp_1 \subset F_1 \) is equivalent to the existence of \( (y, y') \neq (0,0) \) such that \( y(\sum_i a_i^2) + y'\left( \sum_i a_i b_i \right) = 0 \) and \( y(\sum_i a_i b_i) + y'(\sum_i b_i^2) = 0 \). So the condition for \( F_1 \) is \( (\sum_i a_i^2)(\sum_i b_i^2) = (\sum_i a_i b_i)^2 \). Under the Plücker embedding \( \mathbb{P}(\wedge^2 F_1) \to \mathbb{P}(\wedge^2 K) \simeq \mathbb{P}(K^*) \), one sees that this is a conic in \( \mathbb{P}^2 \). The condition for \( F_2 \) turns to be \( F_1 \subset F_2 \subset F^\perp_1 \). So this component is a \( \mathbb{P}^1 \)-bundle over \( \mathbb{P}^1 \).

(ii). \( K = F_2 \), then \( F_1 \subset K \), this component is just \( \mathbb{P}(K^*) \). The two components intersects at a \( C_2 \simeq \mathbb{P}^1 \) inside \( \mathbb{P}(K^*) \).

Now it is clear that the two resolutions are different and are related by the Mukai flop along the component \( \mathbb{P}(K^*) \), and \( C_1, C_2 \) are dual conics.

Now we will identify \( \pi_1, \pi_2 \) with \( \pi, \pi^+ \). By definition, we have

\[
\pi_1^{-1}(T \cap \mathcal{O}_{[3,3]}) = \{ (F_1 \subset F_2, z) | z F_2 \subset F_1 \subset Ker(z), F_2 = F^\perp_2, \dim F_1 = 1 \},
\]

where \( z \) is in \( T \cap \mathcal{O}_{[3,3]} \). Consider the elements \( z_t \in T \cap \mathcal{O}_{[3,3]} (t \in \mathbb{C}) \) given by

\[
z_t = \begin{pmatrix}
tB & I \\
-3t^2 B^2 & tB
\end{pmatrix}, \text{ with } B = \begin{pmatrix}
0 & \sqrt{-1} & 1 \\
-\sqrt{-1} & 0 & 0 \\
1 & 0 & 0
\end{pmatrix}.
\]

One has \( Ker(z_t) = \mathbb{C}\langle e_1 + \sqrt{-1} e_5 + te_6, e_2 - \sqrt{-1} e_3 \rangle \). When \( t \) goes to 0, \( Ker(z_t) \) goes to \( \mathbb{C}\langle e_1, e_2 - \sqrt{-1} e_3 \rangle \), thus the limit of \( \pi_1^{-1}(z_t) \) will be \( \mathbb{P}(\mathbb{C}\langle e_1, e_2 - \sqrt{-1} e_3 \rangle) \subset \mathbb{P}(K) \), which is not a point. This shows that \( \pi_1 = \pi \) by the description of \( \pi \) and \( \pi^+ \) in section 2. 

4. Symplectic deformations

A deformation of the symplectic resolutions \( \phi_i, i = 1, 2 \) (cf. (3)) can be constructed as follows (43). Let \( \mathfrak{c}_i \) be the center of the Levi sub-algebra of \( \mathfrak{p}_i := \text{Lie}(F_i) \) and \( \mathfrak{u}_i \) the nil-radical of \( \mathfrak{p}_i \). The vector space \( \mathcal{V}_i := \mathfrak{c}_i + \mathfrak{u}_i \) is a flat family over \( \mathfrak{c}_i \). Let \( \mathcal{Y}_i \) be the closed sub-variety

\[
\mathcal{Y}_i := \{ (z, v) \in \mathfrak{c}_i \times \mathcal{V}_i | v \in G \cdot (z + u_i) \}.
\]

Now consider the morphism \( \Phi_i : G \times^{P_i} \mathcal{V}_i \to \mathcal{Y}_i \) given by

\[
g \ast (z + u) \mapsto (z, g \cdot (z + u)),
\]

where \( g \in G, z \in \mathfrak{c}_i \) and \( u \in \mathfrak{u}_i \). Notice that if \( z \neq 0 \), then \( z + u_i = P_i \cdot z \), so this morphism is well-defined. One can show that \( \Phi_i \) is birational and it
gives a family of morphisms over \( c_i \). When \( z \in c_i \) is generic, in the sense that the stabilizer \( G^z \) of \( z \) is exactly the Levi sub-group \( L_i \) of \( P_i \), then

\[
\Phi_i^z : G \times P_i (z + u_i) \simeq G \times P_i (P_i \cdot z) \to Y_i^z = G \cdot z \simeq G/L_i
\]

is an isomorphism. Notice that \( Y_i^z \) is a semi-simple orbit, thus it is symplectic. When \( z = 0 \), the map \( \Phi_i^0 \) is just the Springer resolution \( \phi_i \). In other words, \( \Phi_i \) is a symplectic deformation of \( \phi_i \) with base \( c_i \).

Let \( T_i \) be the intersection \((c_i \times S) \cap Y_i \) and \( Z_i \) its pre-image under the morphism \( \Phi_i \), which gives a map \( Z_i \xrightarrow{\Pi_i} T_i \) over \( c_i \). Now we will show that the family \( \psi_i : Z_i \to c_i \) is smooth. Recall that the map \( G \times S \to g \) is smooth, so is \( G \times (S \cap (G \cdot V_i)) \to G \cdot V_i \) (Section 5.1). The morphism \( \Phi_i \) is \( G \)-equivariant, so \( G \times Z_i \to G \times P_i V_i \) is smooth. Notice that the map \( G \times Z_i \to c_i \) is smooth, so is the composition \( G \times Z_i \to c_i \). The projection \( G \times Z_i \to Z_i \) is smooth, which implies the smoothness of \( Z_i \to c_i \).

An immediately corollary is that \( Z_i \) is smooth and for any \( 0 \neq z \in c_i \), generic, the intersection \( S \cap G \cdot z \) is smooth and symplectic, which deforms \( \text{Sym}^2(\mathbb{C}^2/\pm 1) \). So \( Z_i \xrightarrow{\Pi_i} T_i \) gives a symplectic deformation of the symplectic resolution \( \pi_1 : \text{Hilb}^2(T^*\mathbb{P}^1) \to \text{Sym}^2(\mathbb{C}^2/\pm 1) \).

5. Universal Poisson deformations

Now we will show that our picture is similar to that of Brieskorn (see section 3 [GK]). We will only consider \( \phi_1 \). To simplify the notations, we will write \( P \) (resp. \( \phi, L \) etc.) instead of \( P_1 \) (resp. \( \phi_1, L_1 \) etc.).

Fix a maximal torus \( U \) in \( G \) and a Cartan sub-algebra \( \mathfrak{h} \) in \( \mathfrak{g} = \mathfrak{sp}_6 \). Coordinates in \( \mathfrak{h} \) are denoted by \((h_1, h_2, h_3)\). We define the Weyl group of \( L \) to be \( W(L) := N_L(U)/U \), where \( N_L(U) \) is the normalizer of \( U \) in \( L \). The partial Weyl group of \( P \) is \( W^P := N_G(L)/L \). Then \( W^P \) is naturally isomorphic to the quotient \( N_{W(G)}(W(L))/W(L) \), where \( W(G) \) is the Weyl group of \( G \).

It is easy to see that \( W(L) \) is isomorphic to \( \mathbb{Z}_2 \), acting on \( \mathfrak{h} \) by \((h_1, h_2, h_3) \mapsto (h_1, -h_3, h_2)\). The center \( c \) of \( \text{Lie}(L) \) is naturally identified with the fixed point set \( \mathfrak{h}^{W(L)} \). The group \( W^P \) is isomorphic to \( \mathbb{Z}_2 \times \mathbb{Z}_2 \), which acts on \( \mathfrak{h}^{W(L)} \) by \((h_1, h_2, h_3) \mapsto (-h_1, h_2, h_2) \) and \((h_1, h_2, h_3) \mapsto (h_1, -h_2, -h_2)\), i.e. it is the sum of two copies of the sign representation of \( \mathbb{Z}_2 \).

Let \( T' \) be the intersection \( S \cap G \cdot (c + u) \), then we have a natural projection
$p : \mathcal{T} \rightarrow \mathcal{T}'$ and the following diagram is commutative:

$$
\begin{array}{c}
\mathcal{Z} \xrightarrow{\Pi} \mathcal{T} \\
\downarrow \psi \quad \downarrow \psi' \quad \downarrow \beta \\
\mathfrak{h}^W(L) \xrightarrow{id} \mathfrak{h}^W(L) \xrightarrow{\eta} \mathfrak{h}^W(L)/W^P,
\end{array}
\tag{4}
$$

where $\eta$ is the natural quotient map and $\beta$ is the restriction to $\mathcal{T}'$ of the adjoint quotient map $\mathfrak{g} \rightarrow \mathfrak{h}/W(G)$.

Claim: The second Poisson cohomology $HP^2(T)$ can be naturally identified with $\mathfrak{h}^W(L)/W^P$.

Let $H'$ be the semi-direct product of $\mathbb{Z}_2$ with $W^P$. Let $\mathbb{Z}_2$ acts on $\mathfrak{h}^W(L)$ by $(h_1, h_2, h_2) \mapsto (h_2, h_1, h_1)$, then it is easy to see that $(\mathfrak{h}^W(L) \oplus (\mathfrak{h}^W(L))^*)/H'$ is isomorphic to $T \simeq \operatorname{Sym}^2(\mathbb{C}^2/\pm 1)$. By [GK] (section 4), we have $HP^2(T)$ is naturally isomorphic to $HP^2(T \cap \overline{\mathfrak{O}_{[3,3]}}) \oplus HP^2(T \cap \overline{\mathfrak{O}_{[4,1,1]}})$. By Lemma 3.1 $HP^2(T \cap \overline{\mathfrak{O}_{[3,3]}})$ is naturally identified with $\mathbb{C}v_1/\mathbb{Z}_2$ and $HP^2(T \cap \overline{\mathfrak{O}_{[4,1,1]}})$ is identified with $\mathbb{C}v_2/\mathbb{Z}_2$, where $v_1 = (0, 1, 1), v_2 = (1, 0, 0)$ are two points in $\mathfrak{h}^W(L)$, and the group $\mathbb{Z}_2$ acts by sign representation.

Note that under this identification, $\mathfrak{h}^W(L)$ is identified with $H^2(\operatorname{Hilb}^2(T^*\mathbb{P}^1))$.

The first square in (4) is the symplectic deformation of $\pi$, the second square is Cartesian. For the vertical morphisms, $\psi$ is a universal Poisson deformation of $\operatorname{Hilb}^2(T^*\mathbb{P}^1)$, $\psi'$ is similar to the Calogero-Moser deformation of $T \simeq (\mathfrak{h}^W(L) \oplus (\mathfrak{h}^W(L))^*)/H'$, and $\beta$ is the universal Poisson deformation of $T$.

Remark 1. A diagram analogous to (4) can be constructed using the same method for any Springer resolution.

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