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Existence and a priori estimate for elliptic problems with subquadratic gradient dependent terms

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Abstract. In this Note we prove an a priori estimate and the existence of a solution for a class of nonlinear elliptic problems whose model is $-\operatorname{div} A(x)Du + \alpha_0 u = \gamma|Du|^q + f(x)$, when $1 < q < 2$ and $f \in L^m(\Omega)$ for some suitable m . The main interest of the result lies in the a priori estimate, the complete proof of which is given in the Note. © XXXX Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

Existence et estimation a priori pour des problèmes elliptiques avec des termes sous quadratiques par rapport au gradient

Résumé. Dans cette Note nous démontrons une estimation a priori et l'existence d'une solution pour une classe de problèmes non linéaires dont le modèle est $-\operatorname{div} A(x)Du + \alpha_0 u = \gamma|Du|^q + f(x)$, où $1 < q < 2$ et où $f \in L^m(\Omega)$ pour un m convenable. L'intérêt principal du résultat réside dans l'estimation a priori, dont la démonstration complète est donnée dans la Note. © XXXX Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

Version française abrégée

Dans cette Note, nous démontrons l'existence d'une solution de

$$u \in H_0^1(\Omega), \quad -\operatorname{div} A(x)Du + a(x)u = H(x, u, Du) \quad \text{in } \mathcal{D}'(\Omega), \quad (1)$$

quand Ω est un ouvert borné de $\mathbb{R}^N$, $N \geq 3$, A une matrice bornée coercive, a une fonction bornée non négative (voir les hypothèses (2), (3) de la version anglaise) et quand $H : \Omega \times \mathbb{R} \times \mathbb{R}^N \rightarrow \mathbb{R}$ est une fonction de Carathéodory qui vérifie

$$|H(x, s, \xi)| \leq \gamma|\xi|^q + f(x) \quad \text{avec } \gamma \geq 0, 1 + \frac{2}{N} \leq q < 2, f \in L^m(\Omega) \text{ et } m = \frac{N}{q'}. \quad (4)$$

Note présentée par XXX XXX

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Un exemple modèle de fonction H qui vérifie (4) est donné par $H(x, u, Du) = c(x)Du + d(x)|Du|^r + f(x)$, où $c \in L^N(\Omega)^N$, $0 \leq r \leq q < 2$, $d \in L^s(\Omega)$ pour un s convenable et $f \in L^p(\Omega)$ avec $p \geq m = \frac{N}{q'}$.

Le problème (1) a fait l'objet de nombreux travaux quand H est à croissance quadratique ou linéaire par rapport à Du . Quand $q = 2$, la condition (4) devient $f \in L^{\frac{N}{2}}(\Omega)$; sous cette hypothèse, l'existence d'une solution de (1) qui vérifie de plus la condition $e^{\gamma|u|} - 1 \in H_0^1(\Omega)$ a été démontrée dans [10] si $\alpha_0 = 0$ et dans [8] si $\alpha_0 > 0$ (voir aussi [4], [5] dans le cas où $q = 2$ et $m > \frac{N}{2}$). Quant à lui, le cas $q = 1$ est classique, même si l'opérateur est non coercif lorsque γ est grand; cette difficulté a été résolue dans [7]. Mais à notre connaissance, le cas $1 < q < 2$ est resté ouvert jusqu'à maintenant, à l'exception de l'article récent [9] (voir aussi [6]).

Dans cette Note, nous nous restreignons au cas où $q \geq 1 + \frac{2}{N}$. En effet, dans ce cas, m vérifie $m = \frac{N}{q'} \geq \frac{2N}{N+2} = (2^*)'$, et on a donc $f \in H^{-1}(\Omega)$, ce qui permet de chercher des solutions dans $H_0^1(\Omega)$. Dans notre futur article [11], nous traiterons aussi le cas $1 < q < 1 + \frac{2}{N}$ par la méthode que nous utilisons ici; les résultats obtenus sont similaires même si les solutions, qui ne sont plus dans $H_0^1(\Omega)$, doivent être définies au sens des solutions renormalisées (ou d'entropie).

THÉORÈME 1. – *Supposons que l'on a, outre (2), (3) et (4), l'une des deux hypothèses suivantes :*

ou bien $\alpha_0 > 0$, ou bien $\alpha_0 = 0$ et $\gamma^{\frac{1}{q-1}} \|f\|_{L^m(\Omega)} < C_0 \alpha^{\frac{q}{q-1}}$,

où C_0 est une constante qui dépend seulement de N et q .

Alors il existe au moins une solution u de (1) qui de plus vérifie

$$|u|^\sigma \in H_0^1(\Omega) \quad \text{avec} \quad \sigma = \frac{(N-2)(q-1)}{2(2-q)}. \quad (7)$$

De plus, toute solution de (1) qui vérifie la condition de régularité (7) vérifie l'estimation a priori

$$\|u\|_{H_0^1(\Omega)} + \| |u|^\sigma \|_{H_0^1(\Omega)} \leq M, \quad (8)$$

où M dépend seulement de N , q , $|\Omega|$, α , α_0 , γ et f .

Quand $\alpha_0 > 0$, la constante M qui apparaît dans l'estimation a priori (8) dépend de la fonction f , non seulement par l'intermédiaire de sa norme $\|f\|_{L^m(\Omega)}$, mais aussi par l'intermédiaire du nombre k^* défini par (17). Cependant la constante M est bornée quand f varie dans un ensemble de fonctions qui sont bornées et équi-intégrables dans $L^m(\Omega)$.

La relation entre les paramètres q et m imposée dans (4) est naturelle, car les conditions nécessaires pour l'existence d'une solution de (1) démontrées dans [1], [12] conduisent dans le cadre adopté ici à $m = \frac{N}{q'}$. De plus ces résultats montrent qu'une condition sur la taille des données est nécessaire pour l'existence d'une solution de (1) quand $\alpha_0 = 0$.

L'exposant σ défini par (7) vérifie $\sigma \geq 1$ quand $q \geq 1 + \frac{2}{N}$.

Il existe des solutions de (1) qui ne vérifient pas la condition de régularité (7). Un exemple classique en est $u(x) = C_q (|x|^{-\frac{2-q}{q-1}} - 1)$, qui, lorsque $q > 1 + \frac{2}{N}$, vérifie, pour un choix convenable de C_q , $u \in H_0^1(\Omega)$, $-\Delta u = |Du|^q$ in $\mathcal{D}'(B_1)$ et $|u|^\rho \in H_0^1(\Omega)$ pour tout $\rho < \sigma$, mais ne vérifie pas $|u|^\sigma \in H_0^1(\Omega)$. Par contre, des résultats d'unicité pour les solutions de (1) qui vérifient (7) ont été récemment démontrés dans [2].

Dans notre futur article [11], nous donnerons un certain nombre de généralisations du Théorème 1. Comme nous l'avons déjà dit, nous traiterons tout l'intervalle $1 < q < 2$. Nous étudierons aussi le cas des conditions aux limites de Neumann et de Fourier, le cas des ouverts non bornés, et celui de fonctions f appartenant à des espaces de Lorentz ou à des espaces de Sobolev négatifs. Tout cela sera présenté dans le cadre d'opérateurs non linéaires pseudomonotones de type Leray-Lions définis dans l'espace $W^{1,p}(\Omega)$. Nous considérerons plus tard l'analogue parabolique du problème (1).

1. Introduction, main result and comments

In this Note we prove the existence of a solution of

$$u \in H_0^1(\Omega), \quad -\operatorname{div} A(x)Du + a(x)u = H(x, u, Du) \quad \text{in } \mathcal{D}'(\Omega), \quad (1)$$

when Ω is an open bounded set of $\mathbb{R}^N$ ($N \geq 3$), A a bounded coercive matrix, a a bounded nonnegative function, i.e.,

$$A \in L^\infty(\Omega)^{N \times N}, \quad A \geq \alpha I, \quad \alpha > 0, \quad (2)$$

$$a \in L^\infty(\Omega), \quad a \geq \alpha_0, \quad \alpha_0 \geq 0, \quad (3)$$

and when $H : \Omega \times \mathbb{R} \times \mathbb{R}^N \rightarrow \mathbb{R}$ is a Carathéodory function which satisfies

$$|H(x, s, \xi)| \leq \gamma |\xi|^q + f(x) \quad \text{with } \gamma \geq 0, \quad 1 + \frac{2}{N} \leq q < 2, \quad f \in L^m(\Omega) \quad \text{and } m = \frac{N}{q'}. \quad (4)$$

A model example of function H which satisfies (4) is $H(x, u, Du) = c(x)Du + d(x)|Du|^r + f(x)$ with $c \in L^N(\Omega)^N$, $0 \leq r \leq q < 2$, $d \in L^s(\Omega)$ for a suitable s and $f \in L^p(\Omega)$ with $p \geq m = \frac{N}{q'}$.

There is a wide literature concerning problem (1) when H has a quadratic or a linear growth with respect to Du . When $q = 2$, condition (4) becomes $f \in L^{\frac{N}{2}}(\Omega)$; in this case existence of a solution of (1) which satisfies the further regularity $e^{\gamma|u|} - 1 \in H_0^1(\Omega)$ has been proved in [10] if $\alpha_0 = 0$ and in [8] if $\alpha_0 > 0$ (previous references for the case where $q = 2$ and $m > \frac{N}{2}$ are, e.g., [4], [5]). On the other hand, the case $q = 1$ is classical, but exhibits an important difficulty when γ is large, due to the fact that the operator is then non coercive; this problem has been solved in [7]. As far as we know, the case $1 < q < 2$ has been left open until now, except for the very recent paper [9] (see also [6]).

In the present Note, we restrict ourselves to the case where $q \geq 1 + \frac{2}{N}$. Indeed in this case m satisfies $m = \frac{N}{q'} \geq \frac{2N}{N+2} = (2^*)'$, hence $f \in H^{-1}(\Omega)$. This allows us to look for solutions which belong to $H_0^1(\Omega)$. In our forthcoming paper [11] we will also treat the case $1 < q < 1 + \frac{2}{N}$ by the method used in the present Note and obtain very similar results, except for the fact that the solution is no more in $H_0^1(\Omega)$ and has to be defined as a renormalized (or entropy) solution.

Theorem 1. – Assume (2), (3), (4), and one of the two following hypotheses:

$$\text{either } \alpha_0 > 0, \quad (5)$$

$$\text{or } \alpha_0 = 0 \quad \text{and} \quad \gamma^{\frac{1}{q-1}} \|f\|_{L^m(\Omega)} < C_0 \alpha^{\frac{q}{q-1}}, \quad (6)$$

where C_0 is a constant which depends only on N and q .

Then there exists at least one solution u of (1) which further satisfies

$$|u|^\sigma \in H_0^1(\Omega) \quad \text{with} \quad \sigma = \frac{(N-2)(q-1)}{2(2-q)}. \quad (7)$$

Moreover, every solution of (1) which satisfies the regularity requirement (7) satisfies the estimate

$$\|u\|_{H_0^1(\Omega)} + \| |u|^\sigma \|_{H_0^1(\Omega)} \leq M, \quad (8)$$

where M depends only on N , q , $|\Omega|$, α , α_0 , γ and f .

Remark 2. – When $\alpha_0 > 0$, the constant M which appears in the a priori estimate (8) depends on the function f , not only through its norm $\|f\|_{L^m(\Omega)}$, but also through the number k^* defined by (17) below. However, the constant M is bounded when f varies in a set of functions which are bounded and equi-integrable in $L^m(\Omega)$.

Remark 3. – The link imposed in (4) between the parameters q and m is natural. Consider indeed the model problem $u \in H_0^1(\Omega)$, $-\Delta u + u = \gamma|Du|^q + f(x)$ in $\mathcal{D}'(\Omega)$, with $1 < q < 2$ and $f \in L^m(\Omega)$ for some m . Since $f \in L^m(\Omega)$, one expects, in view of the $W^{2,p}$ regularity result, that $u \in W^{2,m}(\Omega)$, which in turns implies, by reading the equation, that $|Du|^q \in L^m(\Omega)$. But $W^{2,m}(\Omega) \subset W^{1,m^*}(\Omega)$ implies $|Du| \in L^{m^*}(\Omega)$. This leads to $qm = m^*$, namely $m = \frac{N}{q}$ as required. On the other hand, the necessary conditions obtained in [1], [12] for the existence of a solution of (1), when specialized to the present setting, lead to the condition $m = \frac{N}{q}$. Moreover, these papers show that a condition like (6) on the size of the data is necessary in order to have the existence of a solution of (1) when $\alpha_0 = 0$.

Remark 4. – The exponent σ defined by (7) satisfies $\sigma \geq 1$ when $q \geq 1 + \frac{2}{N}$.

Remark 5. – There exist solutions of (1) which do not satisfy the regularity (7). A well-known example is the function $u(x) = C_q(|x|^{-\frac{2-q}{q-1}} - 1)$, which, when $q > 1 + \frac{2}{N}$, satisfies, for a suitable choice of C_q , $u \in H_0^1(\Omega)$, $-\Delta u = |Du|^q$ in $\mathcal{D}'(B_1)$ and $|u|^\rho \in H_0^1(\Omega)$ for every $\rho < \sigma$, but does not satisfy $|u|^\sigma \in H_0^1(\Omega)$. In contrast, uniqueness results for the solutions of (1) which satisfy the regularity requirement (7) have been recently proved in [2].

Remark 6. – In our forthcoming paper [11], we will present many extensions of Theorem 1. As said before, we will consider, as far as the growth of $H(x, u, Du)$ with respect to $|Du|$ is concerned, the full range $1 < q < 2$. We will also treat the case of Neumann's and Robin's boundary conditions, the case of unbounded domains and the case of functions f in some Lorentz spaces and in some negative Sobolev spaces. All of this will be done for general nonlinear pseudomonotone operators of Leray-Lions type defined in $W^{1,p}(\Omega)$. We will consider the parabolic analogue of problem (1) later.

2. Proof of Theorem 1

In this proof, C_0, C_1, C_2, C_3 and the generic constant C will denote different positive constants which depend only on N and q .

We first prove the second part of Theorem 1, namely the a priori estimate (8).

Let u be any solution of (1) which satisfies the regularity requirement (7). In equation (1) we can take as test function any function $v \in H_0^1(\Omega) \cap L^\infty(\Omega)$, but also $v = |u|^{2\sigma-2}u$ (recall that $\sigma \geq 1$): to prove this assertion, take $v = |T_n(u)|^{2\sigma-2}T_n(u)$, where T_n is the truncation at height n , and pass to the limit using (4) and (7). We actually use a slight modification of the latest test function: given $k > 0$, we choose as test function in (1) $v = |G_k(u)|^{2\sigma-2}G_k(u)$, where $G_k(s) = s - T_k(s)$. We obtain

$$\begin{aligned} \alpha(2\sigma - 1) \int_{\Omega} |G_k(u)|^{2\sigma-2} |DG_k(u)|^2 dx + \alpha_0 \int_{\Omega} |u| |G_k(u)|^{2\sigma-1} dx &\leq \\ &\leq \gamma \int_{\Omega} |DG_k(u)|^q |G_k(u)|^{2\sigma-1} dx + \int_{\Omega} |f| |G_k(u)|^{2\sigma-1} dx. \end{aligned} \quad (9)$$

The first term of the left-hand side of (9) is nothing but

$$\alpha(2\sigma - 1) \int_{\Omega} |G_k(u)|^{2\sigma-2} |DG_k(u)|^2 dx = \alpha C \int_{\Omega} |D(|G_k(u)|^\sigma)|^2 dx. \quad (10)$$

We then estimate the first term of the right-hand side of (9); Hölder's inequality and the same computation as in (10) yield

$$\begin{aligned} \gamma \int_{\Omega} |DG_k(u)|^q |G_k(u)|^{2\sigma-1} dx &= \gamma \int_{\Omega} |DG_k(u)|^q |G_k(u)|^{\frac{2\sigma-2}{2}q} |G_k(u)|^{2\sigma-1-(\sigma-1)q} dx \leq \\ &\leq \gamma C \left(\int_{\Omega} |D(|G_k(u)|^\sigma)|^2 dx \right)^{\frac{q}{2}} \left(\int_{\Omega} |G_k(u)|^{(2\sigma-1-(\sigma-1)q)\frac{2}{2-q}} dx \right)^{1-\frac{q}{2}}. \end{aligned}$$

But $(2\sigma - 1 - (\sigma - 1)q) \frac{2}{2-q} = \sigma 2^*$, and Sobolev's embedding yields

$$\gamma \int_{\Omega} |DG_k(u)|^q |G_k(u)|^{2\sigma-1} dx \leq \gamma C \|D(|G_k(u)|^\sigma)\|_{L^2(\Omega)}^{q+(1-\frac{q}{2})2^*}. \quad (11)$$

We finally estimate the second term of the right-hand side of (9) by (note that $|u| > k$ when $G_k(u) \neq 0$)

$$\begin{aligned} \int_{\Omega} |f| |G_k(u)|^{2\sigma-1} dx &= \int_{\{|f| \leq \alpha_0 |u|\}} |f| |G_k(u)|^{2\sigma-1} dx + \int_{\{|f| > \alpha_0 |u|\}} |f| |G_k(u)|^{2\sigma-1} dx \leq \\ &\leq \alpha_0 \int_{\Omega} |u| |G_k(u)|^{2\sigma-1} dx + \int_{\{|f| > \alpha_0 k\}} |f| |G_k(u)|^{2\sigma-1} dx. \end{aligned} \quad (12)$$

The first term of the right-hand side of (12) is absorbed by the second term of the left-hand side of (9). Using Hölder's inequality, and since $(2\sigma - 1)m' = \sigma 2^*$, we estimate the second term by

$$\begin{aligned} \int_{\{|f| > \alpha_0 k\}} |f| |G_k(u)|^{2\sigma-1} dx &\leq \|f \chi_{\{|f| > \alpha_0 k\}}\|_{L^m(\Omega)} \| |G_k(u)|^{2\sigma-1} \|_{L^{m'}(\Omega)} \leq \\ &\leq C \|f \chi_{\{|f| > \alpha_0 k\}}\|_{L^m(\Omega)} \|D(|G_k(u)|^\sigma)\|_{L^2(\Omega)}^{\frac{2^*}{m'}}. \end{aligned} \quad (13)$$

Set

$$Y_k = \|D(|G_k(u)|^\sigma)\|_{L^2(\Omega)}.$$

Using (10), (11), (12) and (13) in (9) yields, for two positive constants C_1 and C_2

$$\alpha C_1 Y_k^2 \leq \gamma C_2 Y_k^{q+(1-\frac{1}{q})2^*} + \|f \chi_{\{|f| > \alpha_0 k\}}\|_{L^m(\Omega)} Y_k^{\frac{2^*}{m'}}.$$

Dividing by $Y_k^{\frac{2^*}{m'}}$, and using $2 - \frac{2^*}{m'} = \frac{2^*}{N}(q' - 2)$ and $q + (1 - \frac{q}{2})2^* - \frac{2^*}{m'} = \frac{2^*}{N}(q' - q)$, we finally obtain

$$\forall k \geq 0, \quad \alpha C_1 Y_k^{\frac{2^*}{N}(q'-2)} - \gamma C_2 Y_k^{\frac{2^*}{N}(q'-q)} \leq \|f \chi_{\{|f| > \alpha_0 k\}}\|_{L^m(\Omega)}. \quad (14)$$

Define the function $F : \mathbb{R}^+ \rightarrow \mathbb{R}$ by

$$F(Y) = \alpha C_1 Y^{\frac{2^*}{N}(q'-2)} - \gamma C_2 Y^{\frac{2^*}{N}(q'-q)}.$$

Then (14) is equivalent to

$$\forall k \geq 0, \quad F(Y_k) \leq \|f \chi_{\{|f| > \alpha_0 k\}}\|_{L^m(\Omega)}. \quad (15)$$

Since $q < 2$, F is a concave function with a unique maximizer Z^* and maximum F^* , where Z^* and F^* are given by

$$Z^* = C_3 \left(\frac{\alpha}{\gamma}\right)^{\frac{N}{2^*(2-q)}} \quad \text{and} \quad F^* = F(Z^*) = C_0 \frac{\alpha^{\frac{q}{q-1}}}{\gamma^{\frac{1}{q-1}}}.$$

Inequality (15) is non trivial only if its right-hand side is strictly smaller than F^* .

Here we split the proof into two cases.

(i) If $\alpha_0 = 0$, hypothesis (6) is nothing but $\|f\|_{L^m(\Omega)} < F^*$. Then equation $F(Y) = \|f\|_{L^m(\Omega)}$ has two roots Z_0^- and Z_0^+ , with $0 < Z_0^- < Z^* < Z_0^+$, and inequality (15) is equivalent to

$$\forall k \geq 0, \quad \text{either } Y_k \leq Z_0^- \quad \text{or} \quad Y_k \geq Z_0^+. \quad (16)$$

But since $|u|^\sigma \in H_0^1(\Omega)$, the function $k \rightarrow Y_k = \|D(|G_k(u)|^\sigma)\|_{L^2(\Omega)}$ is continuous and tends to zero when k tends to infinity. The alternative (16) then implies that $Y_k \leq Z_0^-$ for every k ; in particular, one has

$$Y_0 = \|D(|u|^\sigma)\|_{L^2(\Omega)} \leq Z_0^- < Z^*$$

(ii) If $\alpha_0 > 0$, we define k^* as

$$k^* = \inf \{k > 0 : \|f\chi_{\{|f|>\alpha_0 k\}}\|_{L^m(\Omega)} < F^*\}. \quad (17)$$

For every $\delta > 0$, one has $\|f\chi_{\{|f|>\alpha_0(k^*+\delta)\}}\|_{L^m(\Omega)} < F^*$, and equation $F(Y) = \|f\chi_{\{|f|>\alpha_0(k^*+\delta)\}}\|_{L^m(\Omega)}$ has two roots $Z_{k^*+\delta}^-$ and $Z_{k^*+\delta}^+$, with $0 < Z_{k^*+\delta}^- < Z^* < Z_{k^*+\delta}^+$. Inequality (15) implies that for every $k \geq k^* + \delta$, either $Y_k \leq Z_{k^*+\delta}^-$, or $Y_k \geq Z_{k^*+\delta}^+$. But the function $k \rightarrow Y_k$ is continuous and tends to zero when k tends to infinity. We conclude that for every $k \geq k^* + \delta$, one has $Y_k \leq Z_{k^*+\delta}^-$, and in particular that $Y_{k^*+\delta} = \|D(|G_{k^*+\delta}(u)|^\sigma)\|_{L^2(\Omega)} \leq Z_{k^*+\delta}^- < Z^*$. We then let δ tend to zero.

In both cases, we have proved that

$$\|D(|G_{k^*}(u)|^\sigma)\|_{L^2(\Omega)} \leq Z^* = C_3 \left(\frac{\alpha}{\gamma}\right)^{\frac{N}{2^*(2-q)}}, \quad (18)$$

where $k^* = 0$ when $\alpha_0 = 0$, and where k^* is defined by (17) when $\alpha_0 > 0$. When $k^* = 0$, inequality (18) is nothing but the second part of the a priori estimate (8). Note that the constant Z^* , which plays here a role similar to the constant M , depends only on N, q, α and γ , but that k^* depends on the function f itself.

We now prove the first part of the a priori estimate (8). Since one has $DG_{k_1^*}(u) = \chi_{\{|u| \geq k_1^*\}} Du$ and $|D(|G_{k^*}(u)|^\sigma)| = \sigma |G_{k^*}(u)|^{\sigma-1} |DG_{k^*}(u)|$, estimate (18) provides an estimate of $\|DG_{k_1^*}(u)\|_{L^2(\Omega)}$ for $k_1^* = k^* + 1$. We then use $v = T_{k_1^*}(u)$ in (1) and we get

$$\begin{aligned} \alpha \int_{\Omega} |DT_{k_1^*}(u)|^2 dx &\leq \int_{\Omega} (\gamma |Du|^q + f) |T_{k_1^*}(u)| dx \leq \gamma k_1^* \int_{\Omega} |Du|^q dx + k_1^* \|f\|_{L^1(\Omega)} \leq \\ &\leq \gamma k_1^* \int_{\Omega} |DT_{k_1^*}(u)|^q dx + \gamma k_1^* \int_{\Omega} |DG_{k_1^*}(u)|^q dx + k_1^* \|f\|_{L^1(\Omega)}, \end{aligned}$$

from which we deduce, using $q < 2$ and the estimate on $\|DG_{k_1^*}(u)\|_{L^2(\Omega)}$, an estimate on $\|DT_{k_1^*}(u)\|_{L^2(\Omega)}$, and thus the first part of the a priori estimate (8), with a constant which depends on k^* .

Finally combining the estimates on $\|DT_{k_1^*}(u)\|_{L^2(\Omega)}$ and $\|DG_{k_1^*}(u)\|_{L^2(\Omega)}$ and estimate (18) completes the proof of the second part of the a priori estimate (8). The constant which appears in this estimate depends on k^* .

We now pass to the proof of the existence of a solution of (1) which satisfies the regularity requirement (7). This proof is classical. One considers the approximation of (1) by the problem in which the function H is replaced by the function $H_\varepsilon = T_{\frac{1}{\varepsilon}}(H)$; this function H_ε satisfies (4) for every $\varepsilon > 0$. This equation has at least one solution u_ε which, by the weak maximum principle, belongs to $H_0^1(\Omega) \cap L^\infty(\Omega)$. Therefore u_ε satisfies the regularity requirement (7), and the a priori estimate (8) ensures that u_ε and $|u_\varepsilon|^\sigma$ are bounded in $H_0^1(\Omega)$. In view of the growth condition (4), $H_\varepsilon(x, u_\varepsilon, Du_\varepsilon)$ is then bounded in $L^{\frac{2}{q}}(\Omega)$. Let $u \in H_0^1(\Omega)$ be such that a subsequence (still denoted by ε) u_ε weakly converges to u in $H_0^1(\Omega)$. The bound of u_ε in $H_0^1(\Omega)$ implies that $-div A(x)Du_\varepsilon$ is bounded both in $H^{-1}(\Omega)$ and in $L^{\frac{2}{q}}(\Omega)$. These bounds imply (see, e.g., [3]) that, extracting if necessary a new subsequence, Du_ε converges to Du almost everywhere in Ω , which in turn implies that $H_\varepsilon(x, u_\varepsilon, Du_\varepsilon)$ converges to $H(x, u, Du)$ strongly in $L^s(\Omega)$ for every $s < \frac{2}{q}$. This result easily allows one to pass to the limit in the approximate equation, which proves the existence of a solution of (1) which satisfies (7).

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