Sharp estimates of bounded solutions to a second order forced equation with structural damping

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Résumé. A l'aide d'inégalités différentielles, on établit une estimation essentiellement optimale pour la norme dans $L^\infty(\mathbb{R}, D(A))$ de l'unique solution bornée de $u'' + cAu' + A^2u = f(t)$ lorsque $A = A^* \geq \lambda I$ est un opérateur borné ou non sur un espace de Hilbert réel H et λ, c sont des constantes positives, tandis que $f \in L^\infty(\mathbb{R}, H)$.

Abstract. By using differential inequalities, an essentially optimal $L^\infty(\mathbb{R}, D(A))$ bound of the unique bounded solution of $u'' + cAu' + A^2u = f(t)$ is obtained whenever $A = A^* \geq \lambda I$ is a bounded or unbounded linear operator on a real Hilbert space H and λ, c are positive constants, while $f \in L^\infty(\mathbb{R}, H)$.

Keywords: second order equation, bounded solution, structural damping

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Introduction

Let H be a real Hilbert space. In the sequel we denote by (u, v) the inner product of two vectors u, v in H and by $|u|$ the H-norm of u. Let $A : D(A) \to H$ a possibly unbounded self-adjoint linear operator such that

$$\exists \lambda > 0, \forall u \in D(A), \quad (Au, u) \geq \lambda |u|^2$$

We consider the largest possible number satisfying the above inequality, in other terms

$$\lambda_1 = \inf_{u \in D(A), |u|=1} (Au, u)$$

We also introduce

$$V = D(A)$$

endowed with the norm given by

$$\forall u \in V, \quad \|u\| = |Au|$$

It is clear that the norm just defined on V is equivalent to the graph norm of A as a consequence of our coerciveness assumption on A.

Given $f \in L^\infty(\mathbb{R}, H)$ the second order evolution equation

$$u'' + cAu' + A^2 u = f(t) \tag{1}$$

is well-known to have a unique bounded solution

$$u \in C_b(\mathbb{R}, V) \cap C_b^1(\mathbb{R}, H) \tag{2}$$

Indeed, putting (1) in the equivalent form

$$u' = v; \quad v' + cAv + A^2 u = f$$

and introducing the contraction semi-group $S(t)$ generated on $V \times H$ by the system

$$u' - v = v' + cAv + A^2 u = 0$$

since S is exponentially damped on $V \times H$, we have (cf.eg. [3])

$$\forall t \in \mathbb{R}, \quad [u(t), u'(t)] = \int_0^\infty S(\tau)[0, f(t-\tau)] d\tau = \int_{-\infty}^t S(t-s)[0, f(s)] ds \tag{3}$$

Assuming

$$\|S(t)\| \le Me^{-\delta t}$$

we also get the estimate

$$\forall t \in \mathbb{R}, \quad \|u(t)\| \le \{\|u(t)\|^2 + |u'(t)|^2\}^{1/2} \le \frac{M}{\delta}\|f\|_{L^\infty(R,H)}$$

In the previous work [6] we obtained a a close-to-optimal $L^\infty(\mathbb{R}, D(A^{1/2})$ bound of the unique bounded solution of $u'' + cu' + Au = f(t)$ by extending in the Hilbert space setting some methods devised for the second order scalar ODE $u'' + cu' + \omega^2 u = f(t)$ for which the optimal bound is known, cf [5]. However in [6] we do not recover what would be an exact generalization of the scalar case, we lose a factor 2 or a factor $\sqrt{2}$ depending on the position of c compared to $2\sqrt{\lambda_1}$. In the case of equation (1) where the constant damping is replaced by the so-called *structural damping* (cf. [2]), the equation looks more comparable to the scalar case in the sense that the ratio of the square of the dissipation over the eigenvalue is the same for all elementary modes. It turns out that an essentially optimal bound can then be obtained by a suitable modification of the methods from [5, 6].

The plan of the paper is the following: Section 1 contains the statement of the main result. Sections and 2 and 3 are devoted to the proofs. In Section 4 we give an example of application to the size of attractors of some nonlinear plate equations in a bounded domain.

1- Main result.

Our main result is the following

Theorem 1.1. *The bounded solution u of (1) satisfies the estimate*

$$\forall t \in \mathbb{R}, \quad \|u(t)\| \le \frac{\max\{1, \frac{2}{c}\}}{\lambda_1}\|f\|_{L^\infty(R,H)} \tag{1.1}$$

Moreover if $c \le 2$ we have

$$\forall t \in \mathbb{R}, \quad |u'(t)| \le (1 + \frac{2}{c})\frac{1}{\lambda_1}\|f\|_{L^\infty(R,H)} \tag{1.2}$$

and if $c \ge 2$

$$\forall t \in \mathbb{R}, \quad |u'(t)| \le \frac{4}{\lambda_1(c + \sqrt{c^2 - 4})}\|f\|_{L^\infty(R,H)} \tag{1.3}$$

2- The case of a small damping.

This section is devoted to the proof of Theorem 1.1 when $c \le 2$. Under this condition we can use following variant of the energy functional already used in [4, 5, 6]:

$$\Phi(t) = |A^{\frac{1}{2}}u'(t)|^2 + |A^{\frac{3}{2}}u(t)|^2 + c(Au(t), Au'(t)) \tag{2.1}$$

which is well-defined at least when $f \in L^\infty(\mathbb{R}, D(A))$ for instance since then

$$u \in L^\infty(\mathbb{R}, D(A^2)) \cap W^{1,\infty}(\mathbb{R}, D(A)) \cap W^{2,\infty}(\mathbb{R}, H)$$

Assuming moreover $f \in L^\infty(\mathbb{R}, D(A^2))$ we can differentiate Φ in the classical sense and we find

$$\Phi' = (u'' + A^2u, 2Au') + c|Au'|^2 + c(A^2u, u'')$$

$$= (2Au', f - cAu') + c|Au'|^2 + c(A^2u, f - A^2u - cAu')$$

$$= -c[|Au'(t)|^2 + |A^2u(t)|^2 + c(A^2u(t), Au'(t))] + (f, 2Au' + cA^2u)$$

hence

$$\Phi' = -c\Psi + (f, 2Au' + cA^2u) \tag{2.2}$$

with

$$\Psi(t) := |Au'(t)|^2 + |A^2u(t)|^2 + c(A^2u(t), Au'(t)) \tag{2.3}$$

We claim

$$\forall t \in \mathbb{R}, \quad \Psi(t) \ge \lambda_1\Phi(t) \tag{2.4}$$

Indeed setting for t fixed $w = A^{\frac{1}{2}}u'(t)$, $z = A^{\frac{3}{2}}u(t)$ we have

$$\Psi(t) := |Au'(t)|^2 + |A^2u(t)|^2 + c(A^2u(t), Au'(t)) = |A^{\frac{1}{2}}w|^2 + |A^{\frac{1}{2}}w|^2 + c(Aw, z)$$

$$= |A^{\frac{1}{2}}(w + \frac{c}{2}z)|^2 + (1 - \frac{c^2}{4})|A^{\frac{1}{2}}w|^2 \ge \lambda_1(|w + \frac{c}{2}z|^2 + (1 - \frac{c^2}{4})|w|^2 = \lambda_1\Phi(t)$$

On the other hand

$$|2Au' + cA^2u|^2 = 4|Au'|^2 + 4c(A^2u, Au') + c^2|A^2u|^2 \le 4\Psi$$

hence, using

$$(f, 2Au' + cA^2u) \le \frac{2}{c}|f|^2 + \frac{c}{8}|2Au' + cA^2u|^2 \le \frac{2}{c}|f|^2 + \frac{c}{2}\Psi$$

we deduce from (2.3) and (2.4) the inequality

$$\Phi' \leq -\frac{c}{2}\Psi + \frac{2}{c}|f|^2 \leq -\frac{c}{2}\lambda_1\Phi + \frac{2}{c}|f|^2$$

In particular, since Φ is bounded we find

$$\forall t \in \mathbb{R}, \quad \Phi(t) \leq \frac{4}{\lambda_1 c^2}\|f\|_\infty^2$$

which means

$$\forall t \in \mathbb{R}, \quad |A^{\frac{1}{2}}u'(t)|^2 + |A^{\frac{3}{2}}u(t)|^2 + c(Au(t), Au'(t)) \leq \frac{4}{\lambda_1 c^2}\|f\|_\infty^2 \qquad (2.5)$$

In particular

$$\forall t \in \mathbb{R}, \quad \lambda_1|Au(t)|^2 + c(Au(t), Au'(t)) \leq \frac{4}{c^2}\|f\|_\infty^2$$

and this means

$$\frac{c}{2}(|Au(t)|^2)' + \lambda_1|Au(t)|^2 \leq \frac{4}{\lambda_1 c^2}\|f\|_\infty^2$$

Along with boundedness of $Au(t)$ in H on $\mathbb{R}$ this implies

$$\forall t \in \mathbb{R}, \quad |Au(t)|^2 \leq \frac{4}{c^2\lambda_1^2}\|f\|_\infty^2$$

therefore (1.1) is proved when f is smooth. The general case follows at once by density. Finally from (2.5) we deduce

$$|A^{\frac{1}{2}}u'(t) + \frac{c}{2}A^{\frac{3}{2}}u(t)|^2 \leq |A^{\frac{1}{2}}u'(t)|^2 + |A^{\frac{3}{2}}u(t)|^2 + c(Au(t), Au'(t)) \leq \frac{4}{\lambda_1 c^2}\|f\|_\infty^2$$

hence

$$|A^{\frac{1}{2}}u'(t) + \frac{c}{2}A^{\frac{3}{2}}u(t)| \leq \frac{2}{c\sqrt{\lambda_1}}\|f\|_\infty$$

therefore

$$|u'(t) + \frac{c}{2}Au(t)| \leq \frac{2}{c\lambda_1}\|f\|_\infty$$

and finally

$$|u'(t)| \leq (1 + \frac{2}{c})\frac{1}{\lambda_1}\|f\|_\infty$$

as claimed.

Remark 2.1. If $c < 2$, inequality (2.5) implies in fact $u \in L^\infty(\mathbb{R}, D(A^{3/2})) \cap W^{1,\infty}(\mathbb{R}, D(A^{1/2}))$ for any $f \in L^\infty(\mathbb{R}, H)$. Actually, for any $c > 0$ it follows from [7] that $S(t)$ is *analytic* on $V \times H$ and then for all $\eta > 0$

$$u \in C_b(\mathbb{R}, D(A^{2-\eta})) \cap C_b^1(\mathbb{R}, D(A^{1-\eta}))$$

3- The case of a large damping

This section is devoted to the proof of Theorem 1.1 for $c \geq 2$. We shall use the following simple lemma.

Lemma 3.1 *Let $B = B^* \geq 0$ be a possibly unbounded linear operator on H such that $B \geq \eta I$ with $\eta > 0$. Then for each $f \in L^\infty(\mathbb{R}, H)$ the unique mild solution u bounded on $\mathbb{R}$ with values in H of*

$$u' + Bu = f \tag{3.1}$$

takes its values in $D(B^{\frac{1}{2}})$ and we have

$$\forall t \in \mathbb{R}, \quad |B^{\frac{1}{2}} u(t)| \leq \frac{1}{\sqrt{\eta}} \|f\|_{L^\infty(R,H)} \tag{3.2}$$

Proof. Assume first that f is smooth and let u be the (smooth) bounded solution u of (3.1) on $\mathbb{R}$ We have for almost all $t \in J$

$$\frac{d}{dt}|B^{\frac{1}{2}}u|^2 + 2(Bu, Bu) = 2(f, Bu) \leq |f|^2 + (Bu, Bu)$$

hence

$$\frac{d}{dt}|B^{\frac{1}{2}}u|^2 + \eta|B^{1/2}u|^2 \leq \frac{d}{dt}|B^{\frac{1}{2}}u|^2 + |Bu|^2 \leq |f|^2$$

from which (3.2) follows immediately. The result follows by density for any $f \in L^\infty(\mathbb{R}, H)$.

Proof of Theorem 1.1.continued. We introduce

$$\alpha = \frac{c + \sqrt{c^2 - 4}}{2}; \quad \beta = \frac{c - \sqrt{c^2 - 4}}{2} = \frac{1}{\alpha}$$

For each $f \in L^\infty(\mathbb{R}, H)$ there is a unique bounded solution v of

$$v' + \alpha A v = f$$

As a consequence of Lemma 3.1, we have $v \in L^\infty(\mathbb{R}, D(A^{\frac{1}{2}})$ with

$$\forall t \in \mathbb{R}, \quad |A^{\frac{1}{2}} v(t)| \leq \frac{1}{\alpha\sqrt{\lambda_1}} \|f\|_{L^\infty(R,H)} \tag{3.3}$$

Since v is bounded there is a unique bounded solution $u \in L^\infty(\mathbb{R}, D(A^{\frac{1}{2}})$ of

$$u' + \beta A u = v$$

As a consequence of Lemma 3.1 applied with f replaced by $A^{\frac{1}{2}}v$, we have $u \in L^\infty(\mathbb{R}, D(A)$ with

$$\forall t \in \mathbb{R}, \quad |Au(t)| \leq \frac{1}{\beta\sqrt{\lambda_1}} \|A^{\frac{1}{2}}v\|_{L^\infty(R,H))} \leq \frac{1}{\lambda_1}\|f\|_{L^\infty(R,H)} \tag{3.4}$$

Now when $f \in L^\infty(R, D(A)))$ we have

$$u'' = v' - \beta Au' = f - \alpha Av - \beta Au' = f - \beta Au' - \alpha A(u' + \beta Au) = f - Au' - A^2u$$

Finally, u is the bounded solution of

$$u'' + Au' + A^2u = f$$

The result extends by density to the general case. To estimate the norm of u' it is now sufficient to write

$$|u'| = |v - \beta Au| \leq \frac{1}{\sqrt{\lambda_1}}|A^{\frac{1}{2}}v| + \beta|Au| \leq \frac{2\beta}{\lambda_1}\|f\|_\infty = \frac{4}{\lambda_1(c + \sqrt{c^2 - 4})}\|f\|_{L^\infty(R,H)}$$

The proof of Theorem 1.1 is now complete.

4- Application.

Let (Ω, μ) be a finitely measured space and A a positive definite self-adjoint linear operator as in the introduction on $H = L^2(\Omega, d\mu)$. Assuming $\mu(\Omega) < \infty$, let us consider a bounded function $F : \mathbb{R} \to [-a, +a]$ with $a > 0$ and let $u \in C_b(\mathbb{R}, V) \cap C_b^1(\mathbb{R}, H) \cap W^{2,\infty}(\mathbb{R}, V')$ be a solution of

$$u'' + cAu' + A^2u = F(u) \tag{4.1}$$

Then if $c \geq 2$ we have

$$\forall t \in \mathbb{R}, \quad \|u(t)\| \leq \frac{a}{\lambda_1}\mu(\Omega)^{1/2} \tag{4.2}$$

and if $c \leq 2$ we have

$$\forall t \in \mathbb{R}, \quad \|u(t)\| \leq \frac{2a}{c\lambda_1}\mu(\Omega)^{1/2} \tag{4.3}$$

For instance, let Ω be a bounded open domain in $\mathbb{R}^N$ and $b \geq 0, c > 0, \alpha \in \mathbb{R}$. We consider the problem

$$u'' + \Delta^2u - c\Delta u' = \alpha \sin u \tag{4.4}$$

with the boundary conditions

$$u = \Delta u = 0 \quad \text{on } \partial\Omega \tag{4.5}$$

It is well known that problem (4.4)-(4.5) has a compact attractor $\mathcal{A}$. Our result gives an upper bound of the size of the $u-$projection of $\mathcal{A}$ since the attractor is just the union of the ranges of bounded solutions. More precisely we have

$$\mathcal{A} \subset \left\{(u,v) \in (H^2(\Omega)\cap H_0^1(\Omega)) \times L^2(\Omega),\, |\Delta u|_{L^2(\Omega)} \leq \frac{\max\{1, \frac{2}{c}\}|\alpha|}{\lambda_1}|\Omega|^{1/2}\right\} \tag{4.6}$$

We conjecture that for $c = 2$ this result is optimal.

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