Continuum micromechanics: survey

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Abstract: The foundations of classical homogenization techniques, which aim at predicting the overall behavior of heterogeneous materials from that of their constituents, are reviewed. After introductory definitions and a methodological preamble, attention is focused on linear elasticity, for which the basic principles of estimating and bounding the overall properties are introduced and illustrated. In this context, special recourse is made for that to the solution of the inclusion and inhomogeneity problems as reported by Eshelby in 1957. Approaches proposed recently to account in a better way for the structural morphology of the considered materials are briefly mentioned. The case of linear elasticity with eigenstrains is then discussed: several applications, including heterogeneous thermoelasticity, can be investigated within this framework. Finally, outlines of nonlinear micromechanics are briefly reported from a historical point of view: from rate-independent elastoplasticity to nonlinear elasticity and viscoplasticity, examples of a fruitful interaction between the search for new estimates and the derivation of rigorous bounds are given and the crucial question of the description of intraphase heterogeneity is emphasized. Viscoelastic coupling and rate-dependent effects are briefly discussed in conclusion.

keywords: Micromechanics; Surveys; Heterogeneity; Material properties.

Introduction

The problem of the transition from the microscale to the macroscale in view of the prediction of the macroscopic mechanical constitutive behavior of materials has been the matter of intensive research for several decades. The microscale considered here refers to the inhomogeneous constitution of the considered materials and to the associated inhomogeneity of their mechanical characteristics. Whereas preliminary results in this field were obtained as early as in 1887 for the first ''law of mixtures'' (the well-known Voigt estimate) and in 1929 for another one (the Reuss estimate), the actual foundations of homogenization techniques are only about forty years old. Two main underlying motivations have been active. The first one originated from metal forming and from the will of a few pioneer material scientists to predict the plastic flow of polycrystals from plastic glide in single crystals: the Sachs model (Sachs 1928) as well as the popular Taylor model (1938) actually addressed very difficult problems of nonlinear micromechanics, though from what can be appreciated today as a rather empirical point of view. The preliminary development of composite materials progressively motivated a new interest in this field, mainly for linear elasticity, on the firm basis of continuum mechanics: the basic concepts of ''representative volume element'' with adequate boundary conditions and of ''homogeneous equivalent medium'' were defined and powerful variational approaches were achieved from which rigorous bounds, instead of approximate estimates, were derived. At the same time, the early Voigt, Reuss, and Taylor estimates were reassessed as bounds whereas the basic inclusion problem was given a general solution, opening the way to improved estimates. In the early 1960s the main basis of the so-called ''continuum micromechanics,'' according to Hill (1965), can be considered as founded.

New results in the field of linear elasticity have been continuously derived from that period, with special mention of refined bounding associated with an improved morphological description and of thorough theories for the special case of media with a periodic microstructure (see, e.g., Bensoussan et al. 1978; Sanchez-Palencia 1980; Suquet 1987). Nevertheless, it can be considered that the main novel developments during the last 30 years have been concerned with nonlinear continuum micromechanics. A striking feature of this period lies in the fruitful interaction which has developed between two different points of view, attached to the derivation of bounds and estimates, respectively: while they have been going forward independently for more than two decades, it has been possible at some stage to make comparisons between them in specific situations. Consequently, some unanimously adopted estimates were shown to violate rigorous bounds whereas variational procedures were used to obtain new estimates. Despite significant advances in this field, many problems still remain open today, especially for constitutive behavior which does not derive from one single potential. Another important new (or renewed) current field of research, especially promoted by the increasing facilities of numerical treatment and the associated ''multiscale'' approaches, is concerned with situations where the basic concept of ''representative volume element'' is failing and where internal length scales must be taken into account.

In this brief survey of continuum micromechanics, the stress is deliberately laid on the basic foundations and on the more classical and simpler case of linear elasticity, including the various situations for which eigenstrains or eigenstresses are present. The main questions and approaches concerning nonlinear micromechanics are only reported shortly and qualitatively while the limits of the considered theories are briefly discussed in conclusion.

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Foundations of Continuum Micromechanics

Scale Separation

Classical homogenization techniques aim at replacing an actual inhomogeneous complex body by a fictitious homogeneous one which behaves globally in the same way. Continuum micromechanics is mainly concerned with statistically homogeneous materials for which it is possible to define a ''representative volume element'' (RVE) and an ''equivalent homogeneous medium'' (EHM) which are equivalent to each other from a mechanical point of view. This means that their overall responses to any mechanical loading have to be the same. In other words, the EHM must be such that the stress and strain fields $\mathbf{\Sigma}(\underline{X})$ and $\mathbf{E}(\underline{X})$ derived at the macroscale by solving the boundary value problem of a homogeneous body constituted by this fictitious homogeneous material are the average values, over any RVE centered at $\underline{X}$, of the local stress and strain fields $\boldsymbol{\sigma}(\underline{x})$, and $\boldsymbol{\varepsilon}(\underline{x})$ which would have been derived at the microscale if the considered microstructural inhomogeneities could have been accounted for in such a calculation.

This requires at least that, for the mechanical behavior under investigation, the characteristic length, say d, of the considered inhomogeneities and deformation mechanisms to be much smaller than the size, say l, of the studied volume element, so that this element could be considered as representative of the studied material whatever its location in the macroscopically (or statistically) homogeneous body. Moreover, l must be sufficiently smaller than the characteristic dimensions L of the whole body and than the fluctuation length λ of its prescribed mechanical loading so as to make possible the use of the classical integral and differential tools of structural analysis. In addition, the smallest characteristic length d has to be compatible with the use of the basic concepts (strain and stress tensors, etc.) of continuum mechanics. To sum up, we must have

$$d_0 \ll d \ll l \ll L, \quad l \ll \lambda \tag{1}$$

where d_0 = lower length bound under which continuum mechanics is no more valid. Formula (1) gives only necessary scale separation conditions for such a problem not to be meaningless: these conditions may be not sufficient when percolation phenomena or long-range correlation effects are prominent.

Homogenization Methodology

Once the scales have been defined, the RVE's *description*, for both its geometrical and mechanical characteristics, has first to be achieved. A basic difference must be emphasized at this point between two kinds of situations. The first one is concerned with materials whose microstructure can be completely described; this is especially the case for media with a periodic microstructure. For such materials, deterministic approaches, which are not basically different from those used to solve classical boundary-value problems of continuum mechanics, have been developed within a rigorous mathematical framework; approximate numerical treatments, resorting to the popular ''unit-cell'' method, have also been finalized. This case is not considered in the sequel and attention is focused on heterogeneous materials with a random microstructure. For such materials, a statistical (or stochastic) approach is necessary.

This needs first the specification of the considered mechanical ''phases'' and the determination of their geometric and mechanical characteristics, and then the statistical description of the spatial distribution (or the ''morphology'') of the phases: the texture function and the correlation functions of the lattice orientations for a polycrystal, the volume fraction, and the shape and orientation distribution of the particles or fibers for a composite material, etc. The crucial point is that, from a practical point of view, the spatial distribution of the phases cannot be completely determined, even in a statistical sense. Consequently, without appropriate assumptions or approximations, the constitutive behavior of the EHM cannot be determined unambiguously.

In other words, for a given description of the RVE, the *concentration* (or ''localization'') problem, which is concerned with the mechanical modeling of the interactions between the phases and the associated derivation of the local fields $\boldsymbol{\sigma}(\underline{x})$, $\boldsymbol{\varepsilon}(\underline{x})$,..., within the RVE from the knowledge of the macroscopic quantities $\mathbf{\Sigma}$, $\mathbf{E}$,..., cannot be solved in general. One can then proceed along two routes: either one makes additional assumptions or approximations in order to derive some *estimates* for the overall behavior or one looks for *bounds* for the overall mechanical properties. Such estimates are more or less pertinent according to the adequacy of the underlying assumptions; as for the bounds, they are derived by considering all the EHMs which are consistent with the limited available information on the phase distribution and by computing the extremal properties which could be reached in this way. This can be done by using variational approaches which deal with admissible strain or stress fields, i.e., compatible strain or equilibrated stress fields obeying the displacement or tension boundary conditions on the RVE, respectively: this needs an adequate definition of these conditions.

As a matter of fact, the original concentration problem is not a ''well-posed'' boundary-value problem since the detailed conditions on the boundary are not known. So, this problem is first transformed into a simpler one by assuming homogeneous boundary conditions on the RVE (Hill 1963; 1967; Hashin 1983). According to the *homogeneous stress boundary conditions*, surface tractions $\underline{T}^g$ are supposed to be prescribed at the boundary and given by

$$\underline{T}^g = \mathbf{\Sigma} \cdot \underline{n} \tag{2}$$

where the constant tensor $\mathbf{\Sigma}$ = known macroscopic stress tensor and $\underline{n}$ = unit outward normal at the boundary. This approximation is valid far enough from the boundary, i.e., almost everywhere in the RVE as long as conditions (1) are fulfilled. From Eq. (2), one proves that $\mathbf{\Sigma}$ equals the volume average stress $\langle \boldsymbol{\sigma} \rangle$ in the RVE, denoted by V; in fact, for any equilibrated (i.e., divergence free) stress field $\boldsymbol{\sigma}^*(\underline{x})$ which obeys Eq. (2), one has

$$\langle \boldsymbol{\sigma}^* \rangle = (1/V) \int_V \boldsymbol{\sigma}^*(\underline{x}) \mathrm{d}V = \mathbf{\Sigma} \tag{3}$$

Similarly, *homogeneous strain boundary conditions* are associated to prescribed displacements $\underline{u}^g$ at the boundary given by

$$\underline{u}^g = \mathbf{E} \cdot \underline{x} \tag{4}$$

where $\mathbf{E}$ = macroscopic (constant) strain tensor and $\underline{x}$ belongs to the boundary. This approximation is satisfactory as long as $d \ll l$. One gets $\mathbf{E} = \langle \boldsymbol{\varepsilon} \rangle$ since, for any compatible (i.e., derived from a displacement field) strain field $\boldsymbol{\varepsilon}'(\underline{x})$ obeying Eq. (4), one gets

$$\langle \boldsymbol{\varepsilon}' \rangle = (1/V) \int_V \boldsymbol{\varepsilon}'(\underline{x}) \mathrm{d}V = \mathbf{E} \tag{5}$$

For the homogeneous boundary conditions (2) or (4), it is easy to

prove, in addition, the Hill lemma which consists in a similar average equation for the strain energy density $\boldsymbol{\sigma}^*:\boldsymbol{\varepsilon}'(=\sigma^*_{ij}\varepsilon'_{ij})$, namely

$$\langle\boldsymbol{\sigma}^*:\boldsymbol{\varepsilon}'\rangle=\langle\boldsymbol{\sigma}^*\rangle:\langle\boldsymbol{\varepsilon}'\rangle=\boldsymbol{\Sigma}:\mathbf{E} \tag{6}$$

where $\boldsymbol{\sigma}^*(\underline{x})$ is equilibrated; $\boldsymbol{\varepsilon}'(\underline{x})$ is compatible; and either the former or the latter obeys homogeneous boundary conditions (note that this result can also be proved with more general, including periodic, boundary conditions). In both cases, one gets finally, in view of estimating or bounding the overall properties, the approximate or trial strain and stress fields, respectively

$$\boldsymbol{\varepsilon}'(\underline{x})=.\mathscr{A}'(\mathbf{E}) \quad \text{or} \quad \boldsymbol{\sigma}^*(\underline{x})=.\mathscr{B}^*(\Sigma) \tag{7}$$

where $.\mathscr{A}'$ and $.\mathscr{B}^*$=strain and stress concentration operators, respectively.

The last step consists in the *homogenization* procedure itself, by combining the local constitutive equations [say $\boldsymbol{\sigma}(\boldsymbol{\varepsilon})$ or $\boldsymbol{\varepsilon}(\boldsymbol{\sigma})$], the average Eqs. (3) or (5) and the concentration relations (7). This leads, according to the boundary conditions (2) or (4), to

$$\mathbf{E}^*(\boldsymbol{\Sigma})=\langle\boldsymbol{\varepsilon}^*(\boldsymbol{\sigma}^*)\rangle=\langle\boldsymbol{\varepsilon}^*[.\mathscr{B}^*(\Sigma)]\rangle \tag{8}$$

or to

$$\boldsymbol{\Sigma}'(\mathbf{E})=\langle\boldsymbol{\sigma}'(\boldsymbol{\varepsilon}')\rangle=\langle\boldsymbol{\sigma}'[.\mathscr{A}'(\mathbf{E})]\rangle \tag{9}$$

While in principle, Eqs. (8) and (9) are not equivalent, they tend to be so when d/l tends to zero (Hill 1967; Mandel 1972), as assumed in the sequel, according to Eq. (1). On the contrary, for periodic media and d of the order of l, Eqs. (8) and (9) would lead to substantial differences (Suquet 1987).

Linear Elasticity

In this section, a natural initial state is considered, i.e., no eigenstrains or eigenstresses are present. Due to the linearity of the constitutive equations and to the unicity of the solution, Eqs. (7) must be linear and homogeneous with respect to $\mathbf{E}$ or $\boldsymbol{\Sigma}$. This means that the strain or stress concentration operators $.\mathscr{A}'$ and $.\mathscr{B}^*$ reduce to simple fourth-order tensors fields $\mathbf{A}'(\underline{x})$ and $\mathbf{B}^*(\underline{x})$, respectively

$$\boldsymbol{\varepsilon}'(\underline{x})=\mathbf{A}'(\underline{x}):\mathbf{E} \quad \text{or} \quad \boldsymbol{\sigma}^*(\underline{x})=\mathbf{B}^*(\underline{x}):\Sigma \tag{10}$$

with, because of Eqs. (3) and (5), $\langle\mathbf{A}'\rangle=\langle\mathbf{B}^*\rangle=\mathbf{I}$ ($\mathbf{I}$ is the fourth-order symmetric unit tensor). With the local constitutive equations

$$\boldsymbol{\sigma}=\mathbf{c}_r:\boldsymbol{\varepsilon} \quad \text{or} \quad \boldsymbol{\varepsilon}=\mathbf{s}_r:\boldsymbol{\sigma} \tag{11}$$

where the elastic moduli $\mathbf{c}_r$ and compliances $\mathbf{s}_r$ are known and uniform in every phase (r), Eqs. (8) and (9) become

$$\mathbf{E}^*=\langle\boldsymbol{\varepsilon}^*\rangle=\langle\mathbf{s}:\boldsymbol{\sigma}^*\rangle=\langle\mathbf{s}:\mathbf{B}^*\rangle:\boldsymbol{\Sigma}=\mathbf{S}^*:\Sigma \tag{12}$$

$$\boldsymbol{\Sigma}'=\langle\boldsymbol{\sigma}'\rangle=\langle\mathbf{c}:\boldsymbol{\varepsilon}'\rangle=\langle\mathbf{c}:\mathbf{A}'\rangle:\mathbf{E}=\mathbf{C}':\mathbf{E} \tag{13}$$

where $\mathbf{S}^*$ and $\mathbf{C}'$ can be viewed either as estimates or as bounds (through the use of the minimum principles of linear elasticity, i.e., the theorems of potential and complementary energy) for the overall *effective* compliances and moduli $\mathbf{S}^{\text{eff}}$ and $\mathbf{C}^{\text{eff}}$, respectively. Thanks to Hill's lemma, the energetic definition of these tensors, namely

$$\langle\boldsymbol{\sigma}:\mathbf{s}:\boldsymbol{\sigma}\rangle=\Sigma:\mathbf{S}^{\text{eff}}:\Sigma \tag{14}$$

$$\langle\boldsymbol{\varepsilon}:\mathbf{c}:\boldsymbol{\varepsilon}\rangle=\mathbf{E}:\mathbf{C}^{\text{eff}}:\mathbf{E} \tag{15}$$

can be proved to be equivalent to the direct one

$$\mathbf{E}=\langle\boldsymbol{\varepsilon}\rangle=\langle\mathbf{s}:\boldsymbol{\sigma}\rangle=\mathbf{S}^{\text{eff}}:\boldsymbol{\Sigma} \tag{16}$$

$$\boldsymbol{\Sigma}=\langle\boldsymbol{\sigma}\rangle=\langle\mathbf{c}:\boldsymbol{\varepsilon}\rangle=\mathbf{C}^{\text{eff}}:\mathbf{E} \tag{17}$$

Estimates for Effective Moduli and Compliances

Estimates for $\mathbf{C}^{\text{eff}}$ or $\mathbf{S}^{\text{eff}}$ can be obtained whenever $\mathbf{A}'$ or $\mathbf{B}^*$ are estimated. The simple choice $\mathbf{A}'=\mathbf{I}$ or $\mathbf{B}^*=\mathbf{I}$ leads to the popular direct or inverse "laws of mixtures," i.e., either to $\mathbf{C}^{\text{LM}}=\langle\mathbf{c}\rangle$ or to $\mathbf{S}^{\text{LM}}=\langle\mathbf{s}\rangle$. Various other estimates have been derived by use of Eshelby's solution of the inhomogeneity problem (Eshelby 1957). Consider an ellipsoidal inhomogeneity (H) with the moduli $\mathbf{c}_H$, perfectly bonded to an infinite homogeneous elastic matrix (the "reference medium") with the moduli $\mathbf{C}^0$ (or compliances $\mathbf{S}^0$) subjected to the homogeneous strain $\mathbf{E}$ or stress Σ at infinity. The stress and strain fields $\boldsymbol{\sigma}_H$ and $\boldsymbol{\varepsilon}_H$ in (H) are found to be uniform and given by

$$\begin{aligned}&\boldsymbol{\varepsilon}_H=[\mathbf{I}+\mathbf{P}^0_H:(\mathbf{c}_H-\mathbf{C}^0)]^{-1}:\mathbf{E}, \quad \boldsymbol{\sigma}_H=\mathbf{c}_H:\boldsymbol{\varepsilon}_H\\ &P^0_{Hijkl}=-\left(\int_H G^0_{ik}(\underline{x}-\underline{x}')\mathrm{d}V'\right)_{lj_{(ij)(kl)}} \qquad \underline{x}\in H\end{aligned} \tag{18}$$

where $(ij)(kl)$=symmetrization with respect to (ij) and (kl) and $\mathbf{G}^0(\underline{x}-\underline{x}')$=Green tensor for the infinite medium with moduli $\mathbf{C}^0$ (i.e., the tensor which correlates the displacement at $\underline{x}$ to the unit point force applied at $\underline{x}'$). The tensor $\mathbf{P}^0$ is closely related to the Eshelby tensor $\mathbf{S}^{0\ \text{Esh}}$ (namely, $\mathbf{S}^{0\ \text{Esh}}_H=\mathbf{P}^0_H:\mathbf{C}^0$). For a sphere and isotropic elasticity, one has

$$\begin{aligned}&\mathbf{S}^{0\ \text{Esh}}_{\text{Sph}}=\alpha^0\mathbf{J}+\beta^0\mathbf{K} \quad \mathbf{I}=\mathbf{J}+\mathbf{K} \quad J_{ijkl}=(1/3)\delta_{ij}\delta_{kl}\\ &\alpha^0=\frac{3k^0}{3k^0+4\mu^0} \quad \beta^0=\frac{6(k^0+2\mu^0)}{5(3k^0+4\mu^0)}\end{aligned} \tag{19}$$

with μ^0 and k^0=shear and bulk moduli of the reference medium, respectively.

Inclusion-based estimates for the overall moduli can be defined from this solution by estimating the average mechanical state in each phase (r) of the RVE subjected to the macroscopic strain $\mathbf{E}$ as that of an ellipsoidal inhomogeneity (H_r) with the same moduli $\mathbf{c}_r$, embedded in an infinite matrix with arbitrary moduli $\mathbf{C}^0$ subjected to some adequate uniform strain at infinity $\mathbf{E}^0$. The shape and orientation of (H_r) can be specified from what is known about the geometry of phase (r) whereas $\mathbf{E}^0$ is determined by the average equation $\langle\boldsymbol{\varepsilon}\rangle=\mathbf{E}$. This results in the following set of equations:

$$\begin{aligned}&\boldsymbol{\varepsilon}_r=[\mathbf{I}+\mathbf{P}^0_r:(\mathbf{c}_r-\mathbf{C}^0)]^{-1}:\mathbf{E}^0\\ &\mathbf{E}=\langle\boldsymbol{\varepsilon}\rangle=\langle[\mathbf{I}+\mathbf{P}^0:(\mathbf{c}-\mathbf{C}^0)]^{-1}\rangle:\mathbf{E}^0\\ &\mathbf{C}^{\text{est}}_0=\langle\mathbf{c}:[\mathbf{I}+\mathbf{P}^0:(\mathbf{c}-\mathbf{C}^0)]^{-1}\rangle:\langle[\mathbf{I}+\mathbf{P}^0:(\mathbf{c}-\mathbf{C}^0)]^{-1}\rangle^{-1}\end{aligned} \tag{20}$$

As for the moduli $\mathbf{C}^0$, they can be chosen at will in order to express at best the specific morphology of the considered material. Several classical estimates correspond to special choices for $\mathbf{C}^0$ or $\mathbf{S}^0$: the law of mixtures is recovered with vanishing values for $\mathbf{C}^0$ or $\mathbf{S}^0$; the Mori-Tanaka model (Mori and Tanaka 1973; Benveniste 1987), which is devoted to particle reinforced composites for small volume fractions of particles, is associated to the choice $\mathbf{C}^0=\mathbf{C}^{\text{mat}}$, i.e., the elastic moduli of the matrix phase, expressing the fact that the continuous matrix plays a prominent morphological role; the self-consistent estimate (Hershey 1954; Kröner 1958) corresponds to the choice $\mathbf{C}^0=\mathbf{C}^{\text{SC}}$, with $\mathbf{C}^{\text{SC}}$ the searched overall moduli, which is adequate for materials, such as polycrystals, whose phases are dispersed in the RVE so that none of them plays any specific morphological role. In the latter case,

Eqs. (20) for $\mathbf{C}^{SC}$ become integral equations, since $\mathbf{P}^0=\mathbf{P}^{SC}$ and $\mathbf{C}^0=\mathbf{C}^{SC}$ are not known in advance.

Bounds for Effective Moduli and Compliances

Bounds for $\mathbf{C}^{eff}$ or $\mathbf{S}^{eff}$ can be derived with use of the minimum principle for elastic media. For homogeneous boundary conditions, they read

$$\langle \boldsymbol{\sigma}^*:\mathbf{s}:\boldsymbol{\sigma}^*\rangle \geqslant \langle \boldsymbol{\sigma}:\mathbf{s}:\boldsymbol{\sigma}\rangle \tag{21}$$

$$\langle \boldsymbol{\varepsilon}':\mathbf{c}:\boldsymbol{\varepsilon}\rangle \geqslant \langle \boldsymbol{\varepsilon}:\mathbf{c}:\boldsymbol{\varepsilon}\rangle \tag{22}$$

so that, by combining Eqs. (10), (14) and (15), (21) and (22), we get

$$\boldsymbol{\Sigma}:(\langle {}^T\mathbf{B}^*:\mathbf{s}:\mathbf{B}^*\rangle - \mathbf{S}^{eff}):\boldsymbol{\Sigma}\geqslant 0 \quad \forall \boldsymbol{\Sigma} \tag{23}$$

$$\mathbf{E}:(\langle {}^T\mathbf{A}':\mathbf{c}:\mathbf{A}'\rangle - \mathbf{C}^{eff}):\mathbf{E}\geqslant 0 \quad \forall \mathbf{E} \tag{24}$$

where the tensor ${}^T\mathbf{A}$ is transposed from $\mathbf{A}$ (${}^TA_{ijkl}=A_{klij}$).

Bounds can be derived from Eqs. (23) and (24) whenever admissible strain or stress fields, $\boldsymbol{\varepsilon}'(\underline{x})$ or $\boldsymbol{\sigma}^*(\underline{x})$, are found which make this derivation possible from the known statistical description only. When the phase moduli and volume fractions are known, uniform fields can be chosen, namely, $\boldsymbol{\sigma}^*=\boldsymbol{\Sigma}$ (so that $\mathbf{B}^*=\mathbf{I}$) or $\boldsymbol{\varepsilon}'=\mathbf{E}$, ($\mathbf{A}'=\mathbf{I}$), which leads to the well-known Reuss and Voigt bounds

$$\boldsymbol{\Sigma}:(\langle \mathbf{s}\rangle - \mathbf{S}^{eff}):\boldsymbol{\Sigma}\geqslant 0 \quad \forall \boldsymbol{\Sigma} \ \text{(Reuss)} \tag{25}$$

$$\mathbf{E}:(\langle \mathbf{c}\rangle - \mathbf{C}^{eff}):\mathbf{E}\geqslant 0 \quad \forall \mathbf{E} \ \text{(Voigt)} \tag{26}$$

If nothing more is known about the space distribution of the phases, $\mathbf{S}^{eff}$ and $\mathbf{C}^{eff}$ are arbitrarily anisotropic, so that Eqs. (25) and (26) lead to a rather loose bounding of the 21 corresponding elastic constants. On the other hand, if the symmetry of the global anisotropy is known in advance, the Reuss and Voigt bounds are more efficient, but they are certainly not optimal. If the inhomogeneous material is known to be isotropic at the micro and the macro scales, the overall shear and bulk moduli μ^{eff} and k^{eff} are bounded by

$$\langle \mu^{-1}\rangle^{-1} \leqslant \mu^{eff} \leqslant \langle \mu\rangle \quad \text{and} \quad \langle k^{-1}\rangle^{-1} \leqslant k^{eff} \leqslant \langle k\rangle \tag{27}$$

Note that the laws of mixtures, which estimate any overall parameter as the mean value of the corresponding local ones, are only valid here as bounding properties. This is no more the case for other elastic constants, such as the Young modulus or the Poisson ratio, and it is completely wrong for anisotropic elasticity.

In addition, when the spatial distribution of the phases is known to be isotropic, the Voigt-Reuss bounding can be improved, as shown by Hashin and Shtrikman (1963). The corresponding derivation is based on an optimization procedure with respect to an infinite set of admissible trial strain fields obtained as the solution of the following problem: an infinite elastic body with uniform moduli $\mathbf{C}^0$ is subjected to an arbitrary distribution of fictitious body forces $\underline{f}(\underline{x})$ derived from a *polarization stress field* $\mathbf{p}(\underline{x})$ by $\underline{f}=\underline{\text{div}}\,\mathbf{p}$. The associated strain field can be found by using the same Green techniques as those mentioned above for the solution of the inhomogeneity problem; for polarization stress fields which are piecewise constant per phase, $\mathbf{p}_r$, the property of an isotropic distribution of the phases can be used to derive explicit bounds which have still to be optimized through an adequate choice of $\mathbf{p}(\underline{x})$ and $\mathbf{C}^0$. The corresponding bounding for local and global isotropy reads as follows:

$$\langle \mu(\mu^- + \beta^-\delta\mu^-)^{-1}\rangle \cdot \langle (\mu^- + \beta^-\delta\mu^-)^{-1}\rangle^{-1} \leqslant \mu^{eff}$$
$$\leqslant \langle \mu(\mu^+ + \beta^+\delta\mu^+)^{-1}\rangle \cdot \langle (\mu^+ + \beta^+\delta\mu^+)^{-1}\rangle^{-1} \tag{28}$$

$$\langle k(k^- + \alpha^-\delta k^-)^{-1}\rangle \cdot \langle (k^- + \alpha^-\delta k^-)^{-1}\rangle^{-1} \leqslant k^{eff}$$
$$\leqslant \langle k(k^+ + \alpha^+\delta k^+)^{-1}\rangle \cdot \langle (k^+ + \alpha^+\delta k^+)^{-1}\rangle^{-1} \tag{29}$$

where $\mu^- = \inf(\mu)$; $k^- = \inf(k)$; $\mu^+ = \sup(\mu)$; $k^+ = \sup(k)$; $\delta\mu^\pm = \mu - \mu^\pm$; $\delta k^\pm = k - k^\pm$; and $\beta^\pm$ and $\alpha^\pm$ are defined from Eq. (19). When k and μ are "well ordered" i.e., μ^- and k^- refer to the same "softest" phase and μ^+ and k^+ to the same "stiffest" phase), the Hashin-Shtrikman bounds belong to the set of estimates defined by Eq. (20). This result is an illustration of the strong connection which exists between heterogeneity and polarization stresses $\mathbf{p}(\underline{x})$, which can be viewed as eigenstresses (associated with eigenstrains $\boldsymbol{\varepsilon}^T$ through $\mathbf{p}(\underline{\mathbf{x}}) = -\mathbf{C}^0:\boldsymbol{\varepsilon}^T(\underline{x})$), as well as between Eshelby's problems of the heterogeneity and of the inclusion (see hereafter).

"Point" Versus "Pattern" Approaches

The Hashin-Shtrikman bounds have been derived for more general cases, especially when the phase distribution is no more isotropic but has an ellipsoidal symmetry (Willis 1977). They can also be integrated in the so-called "systematic theory" developed by Kröner (1977) for elastic random materials. This theory is based on the solution of the general equation for inhomogeneous elasticity by use of Green techniques and on the description of random media by correlation functions of their elastic moduli. The effective moduli (or compliances) can be written as formal infinite series whose general term of order n is associated with the n-point correlation functions of the elastic moduli. By truncation of the series at any rank, bounds for the effective moduli (or compliances) are derived: they are closer and closer when the rank of truncation increases. In the case of "graded disordered" materials, which satisfy some property of the correlation functions up to a given rank n, the Voigt-Reuss bounds are recovered for $n=1$, the Hashin-Shtrikman bounds for $n=2$, and the self-consistent estimate for $n\to\infty$, which corresponds to "perfectly disordered" materials (Kröner 1978).

This theory is essential for a thorough understanding of the elasticity of random media, but it is of limited practical help. Besides some mathematical difficulties in the effective computation of bounds, the use of (point) correlation functions happens not to be very convenient in many cases: their experimental identification can be performed only up to limited orders while some major primary morphological characteristics, such as the phase connectedness or disconnectedness, need quite high orders to be accounted for efficiently. Fifteen years before Kröner's theory, this crucial point of the geometrical description of "matrix/inclusion"-type morphologies had been given a quite attractive tentative solution by Hashin (1962) according to a completely different point of view: the "composite spheres (or cylinders) assemblage" (CSA, say, or CCA) is made of spherical particles (or cylindrical parallel fibers) surrounded by a concentric matrix layer whose thickness is such that the phase volume fraction of each composite element equals the prescribed volume fractions of the composite; these similar composite elements have a variable size going to zero so as to fill up the whole space and to maintain the connectedness of the matrix; they are distributed in an isotropic (or transversely isotropic) manner according to the case (particles or fibers).

Unfortunately, the mechanical treatment of such assemblages proved to be disappointing: Hashin's bounds for the CSA [or Hashin-Rosen's bounds (1964) for the CCA] for the shear (or transverse shear) modulus could be very distant from each other, even more than the Hashin-Shtrikman's ones which consider *any* isotropic (or transversely isotropic) distribution instead of the very specific ones expressed by the CSA (or the CCA). Later, this approach was resumed by Christensen and Lo (1979) in view of estimating the overall shear modulus: by transferring to this case the basic idea of the self-consistent procedure, they proposed, after Kerner (1956), to consider one composite sphere (or cylinder) embedded in an infinite matrix with the effective (unknown) moduli and to determine these effective moduli through a self-consistent energy condition. The results, as compared with those derived from several micromechanics models of the same class (Christensen 1990), were more satisfying; but the morphological meaning of this ''three-phase model'' (or ''generalized self-consistent scheme'') was less clear than the initial one.

Going back to Hashin's initial approach, one can save the basic idea of finite composite elements and improve the geometrical description of their spatial distribution. This leads to combine a deterministic description of small, but finite, well-chosen ''composite patterns'' and a statistical representation of their distribution: in many cases, essential morphological features can be expressed in this way much more directly and easily than by using point correlation functions. According to this ''pattern approach'' (Stolz and Zaoui 1991), a given multiphase material is decomposed, after some morphological analysis, into N patterns or ''morphological phases'' (λ) consisting of identical composite representative domains $D_{\lambda l}$. The general Hashin-Shtrikman procedure is then applied with a polarization stress field $\mathbf{p}_\lambda(\underline{y})$ which is no more uniform within the morphological phases (λ) but has identical values at homologous points $\underline{y}$ of the domains of the same pattern. With help of adequate definitions of strain and stress averages over homologous points in each pattern and use of the Green techniques, the polarization stress field can be optimized and new Hashin-Shtrikman-type bounds can be obtained if the distribution of the pattern centers is isotropic or ellipsoidal. Like for the classical ''point'' Hashin-Shtrikman bounds, these bounds can be obtained from the solution of N elementary inclusion-matrix problems where each domain $D_{\lambda i}$ is embedded in an infinite homogeneous matrix with extremal moduli. These problems can be solved numerically (Bornert et al. 1996) in the general case. For the CSA and isotropy or for the CCA and transverse isotropy, an analytical solution can be derived (Hervé et al. 1991) and extended to n-layered spheres or cylinders according to Hervé and Zaoui (1993, 1995). They improve significantly on classical Hashin or Hashin-Shtrikman bounds. Similarly, new generalized ''pattern-based'' self-consistent estimates can be obtained (Bornert 1996).

Linear Elasticity with Eigenstrains

A number of interesting physical situations can be analyzed from an extension of the foregoing developments to the case where eigenstrains $\boldsymbol{\varepsilon}^T(\underline{x})$ or eigenstresses $\mathbf{p}(\underline{x})$ are present in the initial state. A typical situation of this kind is concerned with thermal strains $\boldsymbol{\varepsilon}^{\text{th}}(\underline{x})$, which can derive, for instance, from a uniform temperature variation ΔT in a heterogeneous material whose constituents (r) have different elastic moduli $\mathbf{c}_r$ and different tensors of thermal expansion $\boldsymbol{\alpha}_r$ (or coefficients α_r in case of local isotropy). It may be useful to predict the overall thermal strain $\mathbf{E}^{\text{th}}$ as well as the effective tensors of thermal expansion $\boldsymbol{\alpha}^{\text{eff}}$ (or coefficients α^{eff} in case of global isotropy). This cannot be derived by direct averaging of the local quantities since, while the total strain field $\boldsymbol{\varepsilon}(\underline{x})=\boldsymbol{\varepsilon}^{\text{el}}(\underline{x})+\boldsymbol{\varepsilon}^{\text{th}}(\underline{x})$ is still compatible, so that the overall strain tensor $\mathbf{E}$ is the volume average of $\boldsymbol{\varepsilon}(\underline{x})$, neither the elastic nor the thermal strain fields are generally compatible; consequently, $\mathbf{E}^{\text{th}}$ and $\boldsymbol{\alpha}^{\text{eff}}$ do not reduce to $\langle\boldsymbol{\varepsilon}^{\text{th}}\rangle$ and $\langle\boldsymbol{\alpha}\rangle$, respectively.

Similar questions arise when electric or magnetic effects as well as various environmental phenomena (moisture, oxidation, corrosion, etc.) are considered in heterogeneous elastic media, i.e., whenever stress-independent eigenstrains are superimposed on the elastic strains; prestresses or eigenstresses can be dealt with in the same manner, e.g., when gas (Fen-Chong et al. 1999) or liquid (Dormieux and Maghous 2000) pressure is present in porous elastic materials. Though stress dependent, even plastic strains can also be considered as long as they are fixed.

The main result which is easily derived from the analysis of such problems by repeated use of Hill's lemma (6), is the following:

$$\begin{gathered}\mathbf{E}^T=\langle\boldsymbol{\varepsilon}^{\mathbf{T}}\mathbf{:B}\rangle,\quad \mathbf{E}^{\text{el}}=\langle\boldsymbol{\varepsilon}^{\text{el}}\mathbf{:B}\rangle\\ \mathbf{P}=\langle\mathbf{p}\mathbf{:A}\rangle,\quad \mathbf{C}^{\text{eff}}\mathbf{:E}=\langle\boldsymbol{\varepsilon}\mathbf{:c:A}\rangle\end{gathered}\tag{30}$$

which leads directly to the effective tensors of thermal expansion $\boldsymbol{\alpha}^{\text{eff}}$ or to the effective ''tensors of thermal coefficients'' $\boldsymbol{\kappa}^{\text{eff}}$ (defined from the local tensors of thermal coefficients $\boldsymbol{\kappa}=\mathbf{c}:\alpha$).

$$\boldsymbol{\alpha}^{\text{eff}}=\langle\boldsymbol{\alpha}\mathbf{:B}\rangle,\quad \boldsymbol{\kappa}^{\text{eff}}=\langle\boldsymbol{\kappa}\mathbf{:A}\rangle\tag{31}$$

It is noteworthy that this result depends, through the strain or stress concentration tensors $\mathbf{A}$ or $\mathbf{B}$, only on the solution of the purely elastic problem (i.e., without eigenstrains)—which, by the way, proves that it is inconsistent to use the direct law of mixtures (i.e., $\mathbf{A}'=\mathbf{I}$) for estimating $\mathbf{C}^{\text{eff}}$ as $\langle\mathbf{c}\rangle$ and $\boldsymbol{\alpha}^{\text{eff}}$ as $\langle\boldsymbol{\alpha}\rangle$ simultaneously. For a two-phase material, these concentration tensors are not even needed if the overall elastic moduli or compliances are known (possibly from direct measurement instead of modelling), according to Levin's theorem (Levin 1967)

$$\begin{aligned}\boldsymbol{\alpha}^{\text{eff}}&=\langle\boldsymbol{\alpha}\rangle+(\mathbf{S}^{\text{eff}}-\langle\mathbf{s}\rangle):(\mathbf{s}_2-\mathbf{s}_1)^{-1}:(\boldsymbol{\alpha}_2-\boldsymbol{\alpha}_1)\\ \boldsymbol{\kappa}^{\text{eff}}&=\langle\boldsymbol{\kappa}\rangle+(\mathbf{C}^{\text{eff}}-\langle\mathbf{c}\rangle):(\mathbf{c}_2-\mathbf{c}_1)^{-1}:(\boldsymbol{\kappa}_2-\boldsymbol{\kappa}_1)\end{aligned}\tag{32}$$

Associated results can be derived for the average thermal stresses in both phases without additional knowledge of the concentration tensors.

In the general case, the local mechanical state within the phases can be estimated according to adequate models. Inclusion-based models have been developed from the solution of Eshelby's inclusion problem which refers to an infinite unloaded homogeneous elastic body with the moduli $\mathbf{C}^0$ an ellipsoidal part ($\mathbf{I}$) of which undergoes the uniform (''stress-free'') eigenstrain $\boldsymbol{\varepsilon}^T$ (or the eigenstress $\mathbf{p}^I=-\mathbf{C}^0:\boldsymbol{\varepsilon}^T$). The resulting strain and stress tensors in the inclusion are found to be uniform and given by

$$\boldsymbol{\varepsilon}^I=\mathbf{S}_I^{0\text{Esh}}:\boldsymbol{\varepsilon}^T=-\mathbf{P}_I^0:\mathbf{p}^I,\quad \boldsymbol{\sigma}^I=\mathbf{C}^0:(\boldsymbol{\varepsilon}^I-\boldsymbol{\varepsilon}^T)\tag{33}$$

where the tensors $\mathbf{S}_I^{0\text{Esh}}$ and $\mathbf{P}^0$ have already been defined in Eqs. (19) and (18) for the solution of Eshelby's inhomogeneity problem. If the inclusion is inhomogeneous, with the moduli $\mathbf{c}$, and if the matrix undergoes the uniform stress-free strain $\mathbf{E}^T$ and is subjected to the uniform strain $\mathbf{E}$ at infinity, the strain and stress tensors in (IH) are still uniform and given by

$$\begin{gathered}\boldsymbol{\varepsilon}^{IH}=[\mathbf{I}+\mathbf{P}_{IH}^0:(\mathbf{c}-\mathbf{C}^0)]^{-1}:[\mathbf{E}+\mathbf{P}_{IH}^0:(\mathbf{c}:\boldsymbol{\varepsilon}^T-\mathbf{C}^0:\mathbf{E}^T)]\\ \boldsymbol{\sigma}^{IH}=\mathbf{c}:(\boldsymbol{\varepsilon}^{IH}-\boldsymbol{\varepsilon}^T)\end{gathered}\tag{34}$$

These results can be used to derive sets of estimates in a way

similar to the one already reported when no eigenstrains are present. For example, for a polycrystal with isotropic (and then uniform) elasticity and an isotropic distribution of the constituents (e.g., the grain families (r) with the same lattice orientation) undergoing uniform isochoric eigenstrains $\boldsymbol{\varepsilon}_r^T$, the concentration equation associated with a self-consistent scheme will read, according to Eqs. (19) and (34)

$$\boldsymbol{\varepsilon}_r = \mathbf{E} + \mathbf{P}^0_{\text{Sph}} : \mathbf{C}^0 : (\boldsymbol{\varepsilon}_r^T - \mathbf{E}^T) = \mathbf{E} + \mathbf{S}^0_{\text{Sph}} : (\boldsymbol{\varepsilon}_r^T - \mathbf{E}^T)$$
$$= \mathbf{E} + \beta^0 (\boldsymbol{\varepsilon}_r^T - \mathbf{E}^T) \tag{35}$$

or

$$\boldsymbol{\sigma}_r = \boldsymbol{\Sigma} + 2\mu^0 (1 - \beta^0)(\mathbf{E}^T - \boldsymbol{\varepsilon}_r^T) \tag{36}$$

with $\mathbf{E}^T = \langle \boldsymbol{\varepsilon}^T \rangle$ (because of uniform isotropic elasticity, $\mathbf{B} = \mathbf{I}$) and $\boldsymbol{\Sigma} = \langle \boldsymbol{\sigma} \rangle$.

Outlines of Nonlinear Micromechanics

Rate-Independent Elastoplasticity

Except for pioneering contributions based on simple assumptions of uniform strain or stress and their variants, such as the Taylor model and its variants, nonlinear micromechanics was first addressed significantly through the self-consistent prediction of the overall elastoplastic response of polycrystals by Kröner (1961) on the basis of Eshelby's solution of the inclusion problem. He proposed to consider the plastic strain $\boldsymbol{\varepsilon}^p$ as a ''stress-free strain'' in the sense of Eshelby and to model the mechanical interaction between one phase r (i.e., one set of grains with the same lattice orientation) and all the other phases as the one between an ellipsoidal inclusion with the uniform plastic strain $\boldsymbol{\varepsilon}_r^P$ and the surrounding infinite matrix with the macroscopic (uniform) plastic strain $\mathbf{E}^P$. For isotropic elasticity with the moduli (μ, k), spherical inclusions and isochoric plastic strains, this corresponds to the problem solved by Eqs. (35) and (36), namely, to the concentration equation

$$\boldsymbol{\sigma}_r = \boldsymbol{\Sigma} + 2\mu(1 - \beta)(\mathbf{E}^P - \boldsymbol{\varepsilon}_r^P) \tag{37}$$

which is the so-called ''Kröner interaction law.''

This approach is open to the criticism that elastoplastic interactions cannot reduce to the elastic ones which are concerned with Eshelby's inclusion problem; actually, plastic flow is stress dependent and the plastic strain is not a ''stress-free strain'' when plastic flow is considered. As a matter of fact, predictions derived from Kröner's model proved to be very close to those derived from an assumption of uniform strain, which can be guessed from Eq. (37) where the plastic strain deviation is forced to be very small because of the high value of the elastic shear modulus μ. This means that the mechanical interactions between the phases are overestimated by this model which yields too ''stiff'' estimates. Four years later, Hill (1965) proposed an alternative use of Eshelby's solution: by linearization of the constitutive equations of elastoplasticity and use of the tangent (multibranch) local and global instantaneous moduli, denoted as $\mathbf{l}_r$ and $\mathbf{L}^{\text{SC}}$, respectively, the solution of the inhomogeneity (instead of the inclusion) problem can be integrated in the linear self-consistent concentration equation, as derived from Eq. (20), namely

$$\dot{\boldsymbol{\varepsilon}}_r = [\mathbf{I} + \mathbf{P}_r^{\text{SC}} : (\mathbf{l}_r - \mathbf{L}^{\text{SC}})]^{-1} : \dot{\mathbf{E}}^0$$
$$\dot{\mathbf{E}} = \langle \dot{\boldsymbol{\varepsilon}} \rangle = \langle [\mathbf{I} + \mathbf{P}^{\text{SC}} : (\mathbf{l} - \mathbf{L}^{\text{SC}})]^{-1} \rangle : \dot{\mathbf{E}}^0 \tag{38}$$
$$\mathbf{L}^{\text{SC}} = \langle \mathbf{l} : [\mathbf{I} + \mathbf{P}^{\text{SC}} : (\mathbf{l} - \mathbf{L}^{\text{SC}})]^{-1} \rangle : \langle [\mathbf{I} + \mathbf{P}^{\text{SC}} : (\mathbf{l} - \mathbf{L}^{\text{SC}})]^{-1} \rangle^{-1}$$

For similar and aligned ellipsoids, the concentration equation can be put in the more convenient form

$$\dot{\boldsymbol{\sigma}} = \dot{\boldsymbol{\Sigma}} - \mathbf{L}^* : (\dot{\boldsymbol{\varepsilon}} - \dot{\mathbf{E}}) \tag{39}$$

where Hill's ''constraint tensor'' $\mathbf{L}^*$ is given by

$$\mathbf{L}^* = (\mathbf{P}^{\text{SC}})^{-1} - \mathbf{L}^{\text{SC}} = \mathbf{L}^{\text{SC}} : [(\mathbf{S}^{\text{EshSC}})^{-1} - \mathbf{I}] \tag{40}$$

The fundamental interest of Hill's approach lies in the fact that it defines and applies the basic concept of *linearization* of the constitutive equations in view of *nonlinear* homogenization; consequently, the elastoplastic nature of the intergranular accommodation, which was reduced to an elastic one by Kröner's treatment, is restored and expressed through an incremental formulation of the concentration equation, in adequation with the flow theory of plasticity. Hill's incremental formulation makes the recourse to Eshelby's elastic problem of the inhomogeneity possible, whereas Kröner's assimilation of the plastic strains to eigenstrains of the inclusion problem constrains the solution of the concentration problem within an elastic framework. A number of applications of Hill's model to metal forming, initiated by Hutchinson (1970) and extended at finite strain by Iwakuma and Nemat-Nasser (1984) or Lipinski et al. (1990) have shown this model to be a significant improvement with respect to Kröner's model. The reason for that can be illustrated by the simplified isotropic version of Hill's model proposed by Berveiller and Zaoui (1979) within the deformation theory of plasticity, under the assumption of local and global proportional loading: for isotropic elasticity and phase distribution and isochoric plastic strain, the corresponding ''secant'' approach leads to the following concentration equation:

$$\boldsymbol{\sigma}_r = \boldsymbol{\Sigma} + 2\mu(1 - \beta)\alpha^{\text{ep}}(\mathbf{E}^P - \boldsymbol{\varepsilon}_r^P) \tag{41}$$

which only differs from Kröner's law by the scalar ''elastoplastic accommodation factor'' α^{ep}. This factor was shown to be equal to one in the elastic domain and to rapidly decrease by one or two orders of magnitude in the plastic regime, which allows the plastic strain deviation to increase significantly and the corresponding overall response to be far ''softer'' than the Taylor or Kröner one. That is the reason why Hill's approach rapidly gained acceptance so as to have been considered for a long time the standard for deriving nonlinear estimates. This is unfortunately no longer true today, because of the development of new variational approaches for nonlinear elasticity or viscoplasticity.

Nonlinear Elasticity and Viscoplasticity

The case of nonlinear elasticity or of ''nonhereditary'' viscoplasticity is considered now. Though the current response does not depend on the loading path, Hill's incremental formulation was adopted by Hutchinson (1976) for predicting the overall creep response of viscoplastic polycrystals: Hill's method was applied now to linearized relations between the stress rate tensor and the second time derivative of the strain tensor (instead of relations between the strain rate and stress rate tensors). Hutchinson proved in addition that the self-consistent prediction of power-law creep with the same exponent for all the slip systems can be reduced to a ''total'' treatment dealing directly with stresses and strain rates through the use of adequate secant creep compliances. About ten years later, a different treatment of the same problem was developed at finite strain by Molinari et al. (1987): since an incremental formulation was not necessary for such a nonhereditary behavior, they proposed to approximate the nonlinear local behavior at each stage by a linear relation between the stress and the strain

rate tensors by making use of the tangent creep compliances and of some adequate prestress or prestrain rate tensor. Initially given in some approximate isotropic form, this ''tangent model'' was fully implemented by Lebensohn and Tomé (1993) who observed this scheme to tend to the Reuss-type lower bound for high nonlinearities; it also suffers from several limitations, primary among which is the fact that it is restricted to (a specific) self-consistent type of model and to power-law creep. More recently, a more general approach based on the same tangent approximation of the nonlinear behavior of the constitutive phases has been proposed by Zaoui and Masson (2000) and extensively discussed by Masson et al. (2000). This ''affine'' formulation, which deals with the linearized constitutive equations as thermoelastic ones (see the section ''Linear Elasticity with Eigenstrains''), can be applied to any type of microstructure and to any form of stress/strain rate relation; it leads to predictions intermediate between the secant and the tangent approaches, without recourse to any appropriately fitted parameter.

In the same field of viscoplasticity or nonlinear elasticity, a very different approach was developed in parallel according to a variational point of view; since the pioneering work of Talbot and Willis (1985), several contributions [Ponte Castañeda 1991; Suquet 1993; Willis 1994; see also Ponte Castañeda and Suquet (1998), for a general review] have provided rigorous Hashin-Shtrikman-type bounds for the effective dissipation potential, which, for the kind of nonlinearities exhibited by viscous materials, are upper bounds. It became then possible to compare these new bounds to the above estimates and it has been shown (Gilormini 1996) that all these approximate schemes, and especially the Hill-Hutchinson model which is the stiffest, may violate the bounds, at least for some particular combination of the parameters. This has motivated a novel interest in the development of ''variational estimates'', which would a priori conform with them (deBotton and Ponte Castañeda 1996; Nebozhyn et al. 2001) and of new formulations and procedures which would behave better with respect to these bounds. Let us quote especially the second-order procedure (Ponte Castañeda 1996) which is based on second-order Taylor expansions of the relevant potentials for the constituent phases; while similar to the affine formulation at the local level, it differs from it for the derivation of the overall response. The method has the distinctive advantage that it leads to estimates that are exact to second order in the heterogeneity contrast, but has the disadvantage that it exhibits a duality gap. It has recently been applied in the context of two-phase systems as well as polycrystals (Bornert et al. 2001) and the resulting estimates appear to be more accurate, particularly when compared to rigorous bounds, than earlier estimates such as those that are based on the Hill incremental or the Hutchinson total formulations.

The reason for that can be understood more easily by referring to the notion of the ''linear comparison composite'' (Ponte Castañeda 1991). According to this view, the actual nonlinear heterogeneous RVE is associated at any stage with a fictitious composite which, at each point, obeys the linear behavior derived from the chosen linearization procedure (let it be ''tangent,'' ''secant,'' ''total,'' ''affine,'' or of ''second-order,'' etc.) of the local constitutive equations; this ''comparison composite'' is continuously heterogeneous, so that the homogenization procedure cannot be achieved on it without adequate simplification. According to Hill's formulation, as well as to most of the current ones, this simplification consists in attributing to each phase, at any given stage, a homogeneous behavior referred to the average stress or strain (or strain rate). Due to the actual nonlinear behavior, which causes any inhomogeneity of the strain or stress field to provoke an equivalent inhomogeneity of the local moduli or compliances, this approximation amounts to neglecting the intraphase heterogeneity and, for the kind of nonlinearity exhibited by usual elastic or viscous materials, to overestimating the phases stiffness with more or less strong consequences on the overestimation of the overall stiffness, according to the chosen formulation. All current and future advances in this field are likely to depend on to what extent this intraphase heterogeneity of the linear comparison composite is accounted for in a better way.

Nonlinear Viscoelasticity and Elastoviscoplasticity

Viscoelasticity, whether it is linear or not, reveals the specific difficulty of the coupling between elasticity and viscosity which is responsible for a complex ''long-range memory effect'' which has to be captured by the micromechanical treatment: delayed phenomena are associated to the viscoelastic nature of the mechanical interactions between the constituents so that the overall behavior of heterogeneous viscoelastic media is much more complicated than the one of each phase. An illustration of that is given by the fact that a mixture of Maxwellian constituents is no more Maxwellian: while each phase has only one relaxation time, the simplest two-phase material exhibits, according to the classical self-consistent scheme, a complex continuous relaxation spectrum which is especially intricate if more general morphological situations or the existence of an interphase are taken into account (Rougier et al. 1993; Beurthey and Zaoui 2000). Additionally, the viscoelastic coupling is responsible for the fact that the concerned behavior cannot be described by one single potential, so that no variational approach from which bounds could be derived for an arbitrary loading path is available at the time being.

Another specific difficulty attached to nonlinear viscoelasticity or to rate-dependent elastoplasticity is associated with the fact that derivatives of different orders occur simultaneously in the constitutive equations, which makes the definition of tangent moduli or compliances problematical: actually, such a definition, which is straightforward for elasticity, viscosity, viscoplasticity or elastoplasticity, was the key to linearized treatments in what precedes. Fortunately, in the case of linear nonaging viscoelasticity, the Laplace technique can be used to transform, via the correspondence principle, a viscoelastic problem into a (symbolic) elastic one: this trick was already used more than two decades ago by Laws and McLaughlin (1978) for the self-consistent scheme but it is of no evident use as soon as nonlinearity is present.

This may be the reason why the development of the micromechanics of nonlinear viscoelastic (or rate-dependent elastoplastic) heterogeneous materials has been rather late and chaotic (Zaoui 1997). The first significant attempt in this field was made by Weng (1981) who proposed to adopt the Kröner model by arguing that, unlike the plastic strain for rate-independent elastoplastic materials, the viscoplastic strain would be a true stress-free strain in the sense of Eshelby. Using different approaches, Nemat-Nasser and Obata (1986) as well as Harren (1991) adopted the same Kröner-type framework in the context of finite strain. On the contrary, Zaoui and Raphanel (1991, 1993) proved that Hill's criticism of Kröner's treatment of rate-independent elastoplasticity was still valid for this case and a new model Zaoui and Masson (1998) was proposed for rate-dependent elastoplasticity, associated first to the linear self-consistent scheme (Rougier et al. 1994) and then, within the ''affine'' framework, to any linear transition model (Masson and Zaoui 1999; Pouya and Zaoui

1999; Masson et al. 2000). Unfortunately, no bounds are still available for this kind of behavior: any progress in this field would surely stimulate new advances.

Conclusion

As suggested by this paper, continuum micromechanics is now a well-developed scientific area and a number of problems and materials can be analyzed according to this approach. It is frequently claimed that homogenization techniques are restricted to the derivation of overall properties and are of no use for local analyses. This is a reducing statement: as a matter of fact, several modern micromechanical approaches such as the above-mentioned ''pattern approach,'' especially when they are coupled with developing micromechanical experimental investigations and numerical simulations, allow to tackle better and better local states and local effects. Nevertheless, many questions are still open and many fields have not yet been investigated significantly: this is especially the case for damage modeling for which, despite valuable advances already achieved (e.g., Kachanov 1992), important progress is likely to occur in the next future.

Note in conclusion that all the above-reported developments were concerned with problems and materials obeying the basic conditions (1) of ''macro-homogeneity.'' A number of important situations conflict with such conditions, because either d is of the order of d_0 (elasticity of metallic alloys, influence of vacancies, shearing of precipitates, precipitation hardening, etc.) or l is of the order of L (thin sheets, long range correlations, percolation effects, etc.) or of λ (short waves, stress gradients, etc.). In most such cases, one additional length scale, at least, would be needed in order to allow comparison of the standard size d either to a finite macroscopic length (distance to a free surface, fluctuation length of the macroscopic stress field, etc.) or to nonvanishing microscopic quantities (dislocation dissociation width, adequate mean free paths, precipitate size, etc.). This is a matter of current intensive research along several directions: multiscale approaches, second-gradient theories, generalized continua, etc. Answers to many open questions which are out of reach of classical micromechanical treatments are expected from such developments (influence of grain size on the yield stress, width of shear or persistent slip bands, instability and fracture analysis, etc.).

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