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On the similarity solutions for a steady MHD equation

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Abstract

In this paper, we investigate the similarity solutions for a steady laminar incompressible boundary layer equations governing the magnetohydrodynamic (MHD) flow near the forward stagnation point of two-dimensional and axisymmetric bodies. This leads to the study of a boundary value problem involving a third order autonomous ordinary differential equation. Our main results are the existence, uniqueness and nonexistence for concave or convex solutions.

1 Introduction

Boundary layer flow of an electrically conducting fluid over moving surfaces emerges in a large variety of industrial and technological applications. It has been investigated by many researchers. Wu [1] has studied the effects of suction or injection on a steady two-dimensional MHD boundary layer flow on a flat plate, Takhar et al. [2] studied a MHD asymmetric flow over a semi-infinite moving surface and numerically obtained the solutions. An analysis of heat and mass transfer characteristics in an electrically conducting fluid over a linearly stretching sheet with variable wall temperature was investigated by Vajravelu and Rollins [3]. In [4] Muhapatra and Gupta treated the steady two-dimensional stagnation-point flow of an incompressible viscous electrically conducting fluid towards a stretching surface, the flow being permeated by a uniform transverse magnetic field. For more details see also [5], [6], [7], [8] and the references therein.

Motivated by the above works, we aim here to give analytical results about the third order non-linear autonomous differential equation

\[ f''' + \frac{m+1}{2} f f'' + m(1 - f'^2) + M(1 - f') = 0 \quad \text{on } [0, \infty) \] (1)

accompanied by the boundary conditions

\[ f(0) = a, \quad f'(0) = b, \quad f'(\infty) = 1 \] (2)

where \( a, b, m, M \in \mathbb{R} \) and \( f'(\infty) := \lim_{t \to \infty} f'(t) \). Equation (1) is very interesting because it contains many known equations as particular cases. Let us give some examples.

Setting \( M = 0 \) in (1), leads to the well-known Falkner-Skan equation (see [9], [10], [11] and the references therein). Equation (1) reduces to equation that arises when considering the mixed

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convection in a fluid saturated porous medium near a semi-infinite vertical flat plate with prescribed
temperature studied by many authors, we refer the reader to [12],[13],[14],[15] and the references therein.
The case \( M = m = 0 \) is refereed to the Blasius equation introduced in [16] and studied by several authors
(see for example [17],[18],[19]). Recently, the case \( m = -1 \) have been studied in [20] the authors show
existence of “pseudo-similarity” solution, provided that the plate is permeable with suction. Mention
may be made also to [21], where the authors show existence of an infinite number of similarity solutions
for the case of a non-Newtonian fluid.
More recently, some results have been obtained by Brighi and Hoernel [22], about the more general
equation
\[
f''' + ff'' + g(f') = 0 \quad \text{on} \quad [0, \infty)
\] (3)
with the boundary conditions
\[
f(0) = \alpha, \quad f'(0) = \beta, \quad f'(\infty) = \lambda
\] (4)
where \( \alpha, \beta, \lambda \in \mathbb{R} \) and \( g \) is a given function. Guided by the analysis of [22] we shall prove that problem (1)-(2) admits a unique concave or a unique convex solution for \( m > -1 \) according to the values of \( M \).

2 Flow analysis

Let us suppose that an electrically conducting fluid (with electrical conductivity \( \sigma \)) in the presence of a
transverse magnetic field \( B(x) \) is flowing past a flat plate stretched with a power-law velocity. According
to [20],[23],[24], such phenomenon is described by the following equations
\[
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0,
\] (5)
\[
u \frac{\partial u}{\partial y} + \frac{\partial u}{\partial x} = u_e u e_x + \frac{\sigma B^2(x)}{\rho} (u_e - u).\] (6)
Here, the induced magnetic field is neglected. In a cartesian system of co-ordinates \((O, x, y)\), the variables
\( u \) and \( v \) are the velocity components in the \( x \) and \( y \) directions respectively. We will denote by \( u_e(x) = \gamma x^m, \gamma > 0 \) the external velocity, \( B(x) = B_0 x^{m-1/2} \) the applied magnetic field, \( m \) the power-law velocity
exponent, \( \rho \) the fluid density and \( \nu \) the kinematic viscosity.
The boundary conditions for problem (5)-(6) are
\[
u(x, 0) = u_w(x) = \alpha x^m, \quad v(x, 0) = v_w(x) = \beta x^{m-1/2}, \quad u(x, \infty) = u_e(x)
\] (7)
where \( u_w(x) \) and \( v_w(x) \) are the stretching and the suction (or injection) velocity respectively and \( \alpha, \beta \)
are constants. Recall that \( \alpha > 0 \) is referred to the suction, \( \alpha < 0 \) for the injection and \( \alpha = 0 \) for the
impermeable plate.
A little inspection shows that equations (5) and (6) accompanied by conditions (7) admit a similarity
solution. Therefore, we introduce the dimensional stream function \( \psi \) in the usual way to get the following
equation
\[
\frac{\partial \psi}{\partial y} \frac{\partial^2 \psi}{\partial x \partial y} - \frac{\partial \psi}{\partial x} \frac{\partial^2 \psi}{\partial y^2} = u_e u e_x + \nu \frac{\partial^3 \psi}{\partial y^3} + \frac{\sigma B^2(x)}{\rho} (u_e - u).
\] (8)
The boundary conditions become
\[
\frac{\partial \psi}{\partial y}(x, 0) = \alpha x^m, \quad \frac{\partial \psi}{\partial x}(x, 0) = -\beta x^{m-1/2}, \quad \frac{\partial \psi}{\partial y}(x, \infty) = \gamma x^m.
\] (9)
Defining the similarity variables as follows
\[ \psi(x, y) = x^{m+1} f(t) \sqrt{\nu \gamma} \quad \text{and} \quad t = x^{m-1} y \sqrt{\nu \gamma} \]
and substituting in equations (8) and (9) we get the following boundary value problem
\[
\begin{cases}
  f''' + \frac{m+1}{m} f'' + m(1-f') + M(1-f') = 0, \\
  f(0) = a, \quad f'(0) = b, \quad f'(\infty) = 1
\end{cases}
\]  
(10)
where \( a = \frac{2\beta}{(m+1)\sqrt{\nu \gamma}}, \quad b = \frac{\alpha}{\gamma} \) and \( M = \frac{\sigma R^2}{\gamma \rho} > 0 \) is the Hartmann number and the prime is for differentiating with respect to \( t \).

3 Various results

First, we give the following

Remark 1 Let \( b = 1 \), then the function \( f(t) = t + a \) is a solution of the problem (9) for any values of \( m \) and \( M \) in \( \mathbb{R} \). We cannot say much about the uniqueness of the previous solution, but if \( g \) is another solution with \( g''(0) = \gamma > 0 \) then, since \( g'(0) = g'(\infty) = 1 \) there exists \( t_0 > 0 \) such that \( g'(t_0) > 1, \quad g''(t_0) = 0 \) and \( g'''(t_0) \leq 0 \). However, from (9) we obtain that for \( m > 0 \) and \( M > 0 \), \( g'''(t_0) = -m(1-g''(t_0)) - M(1-g'(t_0)) > 0 \) and thus a contradiction.

Suppose now that \( f \) verifies the equation (9) only. We will now establish some estimations for the possible extremals of \( f' \).

Proposition 3.1 Let \( f \) be a solution of the equation (9) and \( t_0 \) be a minimum for \( f' \) (i.e. \( f''(t_0) = 0 \) and \( f'''(t_0) \geq 0 \)), if it exists. For such a point \( t_0 \) we have the following possibilities, according to the values of \( m \) and \( M \).

- For \( m < 0 \)
  - if \( M < -2m \), then \( -1 - \frac{M}{m} \leq f'(t_0) \leq 1 \),
  - if \( M = -2m \), then \( f'(t_0) = 1 \),
  - if \( M > -2m \), then \( 1 \leq f'(t_0) \leq -1 - \frac{M}{m} \).
- For \( m = 0 \)
  - if \( M < 0 \), then \( f'(t_0) \leq 1 \),
  - if \( M > 0 \), then \( 1 \leq f'(t_0) \).
- For \( m > 0 \)
  - if \( M < -2m \), then \( f'(t_0) \leq 1 \) or \( -1 - \frac{M}{m} \leq f'(t_0) \),
  - if \( M > -2m \), then \( 1 \leq f'(t_0) \) or \( f'(t_0) \leq -1 - \frac{M}{m} \).

Proof. Let \( t_0 \) be a maximum of \( f' \) with \( f \) a solution of (9). Using the equation (9) and the fact that \( f''(t_0) = 0 \), we obtain that
\[
f'''(t_0) + m(1-f'^2(t_0)) + M(1-f'(t_0)) = 0.
\]
Setting \( p(x) = m(1-x^2) + M(1-x) \), we have that \( f'''(t_0) \geq 0 \) leads to \( g(f'(t_0)) \leq 0 \) and the results follows. Let us remark that in both cases \( m = M = 0 \) and \( m > 0, M = -2m \) we cannot deduce anything about \( f'(t_0) \). ■

Proposition 3.2 Let \( f \) be a solution of the equation (9) and \( t_0 \) be a maximum for \( f' \) (i.e. \( f''(t_0) = 0 \) and \( f'''(t_0) \leq 0 \)), if it exists. For such a point \( t_0 \) we have the following possibilities, according to the values of \( m \) and \( M \).

- For \( m < 0 \)
  - if \( M < -2m \), then \( f'(t_0) \leq -1 - \frac{M}{m} \) or \( f'(t_0) \geq 1 \),
  - if \( M > -2m \), then \( f'(t_0) \leq 1 \) or \( f'(t_0) \geq -1 - \frac{M}{m} \).
We proceed as in the previous Proposition, but this time, with the condition $g(f'(t_0)) \geq 0$. Let us remark that in both of cases $m < 0$, $M = -2m$ and $m = M = 0$ we cannot deduce anything about $f'(t_0)$. ■

We will now use the two previous Propositions to deduce results about the possible extremals for $f'$ with $f$ a solution of the problem (1)-(2).

**Theorem 1** Let $f$ be a solution of the problem (1)-(2), $t_0$ be a minimum for $f'$ (i.e. $f''(t_0) = 0$ and $f'''(t_0) \geq 0$), if it exists, and $t_1$ be a maximum for $f'$ (i.e. $f''(t_1) = 0$ and $f'''(t_1) \leq 0$), if it exists. For such points $t_0$ and $t_1$, we have the following possibilities for the values of $f'$.

- **For $m < 0$**
  - if $M < -2m$, then $-1 - \frac{M}{m} \leq f'(t_0) \leq 1 \leq f'(t_1)$,
  - if $M = -2m$, then $f'(t_0) = 1$,
  - if $M > -2m$, then $1 \leq f'(t_0) \leq -1 - \frac{M}{m} \leq f'(t_1)$.

- **For $m = 0$**
  - if $M < 0$, then $f'(t_0) \leq 1 \leq f'(t_1)$,
  - if $M > 0$, then $f'$ cannot vanish.

- **For $m > 0$**
  - if $M < -2m$, then $f'(t_0) \leq 1 \leq f'(t_1) \leq -1 - \frac{M}{m}$,
  - if $M = -2m$, then $f'(t_1) = 1$,
  - if $M > -2m$, then $f'(t_0) \leq -1 - \frac{M}{m} \leq f'(t_1) \leq 1$.

**Proof.** Taking into account the fact that $f' \to 1$ for large $t$ and combining Propositions 1, 2 and Proposition 3 lead to the results. ■

**Remark 2** A consequence of the previous Theorem is that, for $m = 0$ and $M > 0$ all the solutions of the problem (1)-(2) have to be concave or convex everywhere.

## 4 The concave and convex solutions

In this section we will first prove that, under some hypotheses, the problem (1)-(2) admits a unique concave solution or a unique convex solution for $m > -1$. Then, we will give some nonexistence results about the concave or convex solutions for $m \in \mathbb{R}$ according to the values of $M$. To this aim, we will use the fact that, if $f$ is a solution of the problem (1)-(2), then the function $h$ defined by

$$f(t) = \sqrt{\frac{2}{m+1}} h\left(\sqrt{\frac{m+1}{2}} t\right)$$

with $m > -1$, is a solution of the equation

$$h''' + hh'' + g(h') = 0$$

on $[0, \infty)$, with the boundary conditions

$$h(0) = \sqrt{\frac{m+1}{2}} a, \quad h'(0) = b, \quad h'(<\infty) = 1$$

and where

$$g(x) = \frac{2m}{m+1} (1-x^2) + \frac{2M}{m+1} (1-x).$$

In the remainder of this section we will make intensive use of the results found in the paper [22] by Brighi and Hoernel.
Remark 3 It is immediate that for any \( a \in \mathbb{R} \), if \( b < 1 \) there is no concave solutions of the problem (1)-(2) and if \( b > 1 \) there is no convex solutions of the problem (1)-(2).

4.1 Concave solutions

Let us begin with the two following results about existence, uniqueness and nonexistence of concave or convex solutions for the problem (1)-(2).

Theorem 2 Let \( a \in \mathbb{R} \) and \( b > 1 \). Then, there exists a unique concave solution of the problem (1)-(2) in the two following cases

- \( -1 < m \leq 0 \) and \( M > -m(b+1) \),
- \( m > 0 \) and \( M \geq -2m \).

Moreover, there exists \( a < l < \sqrt{a^2 + 4 \frac{k-1}{m+1}} \) such that \( \lim_{t \to \infty} \{ f(t) - (t+l) \} = 0 \) and for all \( t \geq 0 \) we have \( t + a \leq f(t) \leq t + l \).

Proof. Let \( f \) be a solution of the problem (1)-(2) with \( m > -1 \) and consider the function \( h \) that is defined by (11) and that verifies (12)-(13). Then, as \( g(1) = 0 \) for the function \( g \) defined by (14), using Theorem 1 of \([22]\) we get that the problem (12)-(13) admits a unique concave solution \( h \) for every \( a \in \mathbb{R} \) and \( b > 1 \) if and only if \( g(x) < 0 \) for all \( x \in (1,b) \). Noticing that \( g(x) \leq - \frac{m}{m+1} \) for all \( x \in (1,b) \), we obtain from (1) that \( g(x) < 0 \) for all \( x \) near infinity because \( f'(t) \to 1 \) as \( t \to \infty \). Using the fact that \( \frac{m+1}{m} f'' > 0 \) near infinity, we obtain from (1) that

\[
f'''(x) = -m(1-f'^2) - M(1-f')
\]

near infinity. As the polynomial function \(-m(1-x^2) - M(1-x)\) is negative for all \( x \) in \([1,\infty)\) if \( m \leq -1 \) and \( M \leq -2m \), we get that \( f''' < 0 \) near infinity because \( f' > 1 \) everywhere. This is a contradiction, so concave solutions cannot exist in this case. Consider now \( m > -1 \) and \( h \) a solution of the problem (12)-(13). Let us define the function \( \hat{g} \) by \( \hat{g}(x) = g(x) - x^2 + x \), a simple calculation leads to

\[
\hat{g}(x) = \frac{1}{m+1} \left(-3m+1\right)x^2 + (m+1-2M)x + 2(m+M).
\]

Then, the Theorem 2 from \([22]\) tells us that problem (12)-(13) admits no concave solutions for \( a \leq 0 \) if \( \forall x \in [1,b], \hat{g}(x) \geq 0 \) and \( -a + \max_{x \in [1,b]} \hat{g}(x) > 0 \). These conditions lead to the results for problem (1)-(2) with \( m > -1 \).

The results from Theorem 2 and Theorem 3 are summarized in the Figure 1 in which the plane \((m,M)\) contains three disjoint regions \( A, B \) and \( C \) are defined as

- \( A \): Existence of a unique concave solution for \( m > -1 \), \( b > 1 \) and \( a \in \mathbb{R} \),
- \( B \): No concave solutions for \( m > -1 \), \( b > 1 \) and \( a \leq 0 \),
- \( C \): No concave solutions for \( m \leq -1 \), \( b > 1 \) and \( a \in \mathbb{R} \).
4.2 Convex solutions

We will now give existence, uniqueness and non-existence results for the convex solutions of the problem (1)-(2).

**Theorem 4** Let \( a \in \mathbb{R} \) and \( 0 \leq b < 1 \). Then, there exists a unique convex solution of the problem (1)-(2) in the following cases
- \(-1 < m < 0 \) and \( M \geq -2m \),
- \( m \geq 0 \) and \( M > -m(b+1) \).
Moreover, there exists \( l > a \) such that \( \lim_{t \to \infty} \{ f(t) - (t+l) \} = 0 \) and for all \( t \geq 0 \) we have \( t+a \leq f(t) \leq t+l \).

**Proof.** We proceed the same way as for Theorem 2, but with the condition that \( g(x) > 0 \) for all \( x \) in \([b,1)\). We conclude by using first the Theorem 3 from [22], then the Proposition 2 from [24]. \( \blacksquare \)

**Theorem 5** Let \( 0 \leq b < 1 \). Then, there are no convex solutions of the problem (1)-(2) in the following cases
- \( a \in \mathbb{R}, m \leq -1 \) and \( M \leq -m(b+1) \),
- \( a \leq 0, -1 < m < -\frac{1}{3} \) and \( M \leq \frac{(3m+1)b+2m}{2} \),
- \( a < 0, m = -\frac{1}{3} \) and \( M \leq \frac{1}{2} \),
- \( a \leq 0, m > -\frac{1}{3} \) and \( M \leq -\frac{5m+1}{2} \).

**Proof.** For \( m > -1 \) and \( a \leq 0 \), the proof is the same as the previous one, but this time we need that \( \forall x \in [b,1], \hat{g}(x) \leq 0 \) and \( -a + \max_{x \in [b,1]} \hat{g}(x) > 0 \), according to the Theorem 4 from [22]. Consider now \( m \leq -1, a \in \mathbb{R} \) and let \( f \) be a convex solution of the problem (1)-(2). We have that \( b \leq f'(t) < 1, f'' > 0, f''' < 0 \) everywhere and that \( f(t) > 0 \) for \( t \) large enough because \( f'(t) \to 1 \) as \( t \to \infty \). According to equation (1), we have that
\[
 f''' = \frac{m+1}{2} ff'' - m(1-f'^2) - M(1-f')
\]
with \(-\frac{m+1}{2} ff'' > 0 \) near infinity. As the polynomial function \(-m(1-x^2) - M(1-x)\) is positive for all \( x \) in \([b,1]\) if \( m \leq -1 \) and \( M \leq -m(b+1) \), we get that \( f''' > 0 \) near infinity because \( b \leq f' < 1 \). This is a contradiction, thus convex solutions cannot exist in this case. \( \blacksquare \)

The results from Theorem 4 and Theorem 5 are summarized in the Figure 2 in which the plane \((m,M)\) contains three disjoints regions A, B and C that corresponds to
- A: Existence of an unique convex solution for \( m > -1, \ 0 \leq b < 1 \) and \( a \in \mathbb{R} \),
- B: No convex solutions for \( m > -1, \ 0 \leq b < 1 \) and \( a \leq 0 \),
- C: No convex solutions for \( m \leq -1, \ 0 \leq b < 1 \) and \( a \in \mathbb{R} \).

Figure 2

5 Conclusion

In this paper, we have shown the existence of a unique concave or a unique convex solution of the problem (1)-(2) for \( m > -1 \), according to the values of \( M \). We also have obtained nonexistence results for \( m \in \mathbb{R} \) and related values of \( M \), as well as some clues about the possible behavior of \( f' \). This paper is a first work on this problem, there is still much left to do because of its complexity. Notice that the case \( M = -2m \) plays a particular role in the problem (1)-(2), because it is the only one for which we are able to predict the possible changes of concavity for \( f \). Its study will be the subject of a forthcoming paper.

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