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Abstract

In this paper we introduce a stochastic integral with respect to the solution X of the fractional heat equation on $[0,1]$, interpreted as a divergence operator. This allows to use the techniques of the Malliavin calculus in order to establish an Itô-type formula for the process X.

Keywords: heat equation, fractional Brownian motion, Itô's formula.

MSC: 60H15, 60H07, 60G15

1 Introduction

In the last past years, a great amount of effort has been devoted to a proper definition of stochastic PDEs driven by a general noise. For instance, the case of stochastic heat and wave equations in $\mathbb{R}^n$ driven by a Brownian motion in time, with some mild conditions on its spatial covariance, has been considered e.g. in [4, 12, 9], leading to some optimal results. More recently, the case of SPDEs driven by a fractional Brownian motion has been analyzed in [3, 16] in the linear case, or in [7, 13] for the non-linear situation.

In this context, it seems natural to investigate the basic properties (Hölderianity, behavior of the density, invariant measures, numerical approximations, etc) of these objects.

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And indeed, in case of an equation driven by a Brownian motion, a lot of effort has been made in this direction (let us cite [15, 9, 8] among others). On the other hand, results concerning SPDEs driven by a fractional Brownian motion are rather scarce (see however [11] for a result on SPDEs with irregular coefficients, and [14] for a study of the Hölder regularity of solutions).

This article proposes then to go further into the study of processes defined by fractional PDEs, and we will establish a Itô-type formula for a random function X on $[0,T]\times[0,1]$ defined as the solution to the heat equation with an additive fractional noise. More specifically, we will consider X as the solution to the following equation:

$$\partial_t X(t,x) = \Delta X(t,x) + B(dt,dx), \quad (t,x)\in[0,T]\times[0,1], \tag{1}$$

with Dirichlet boundary conditions and null initial condition. In equation (1), the driving noise B will be considered as a fractional Brownian motion in time, with Hurst parameter $H>1/2$, and as a white noise in space (notice that some more general correlations in space could have been considered, as well as the case $1/3<H<1/2$, but we have restrained ourselves to this simple situation for sake of conciseness).

Then, for X solution to (1), $t\in[0,T]$, $x\in[0,1]$ and a C_b^2-function $f:\mathbb{R}\to\mathbb{R}$, we will prove that $f(X(t,x))$ can be decomposed into:

$$f(X(t,x)) = f(0) + \int_0^t\int_0^1 \left(M^*_{t,x} f'(X)\right)(s,y)W(ds,dy) + \frac{1}{2}\int_0^t f''(X(s,x))K_x(ds), \tag{2}$$

where in the last formula, $M^*_{t,x}$ is an operator based on the heat kernel G_t on $[0,1]$ and the covariance function of B, W is a space-time white noise, and K_x is the function defined on $[0,T]$ by:

$$K_x(s) = H(2H-1)\int_0^s\int_0^s G_{2s-v_1-v_2}(x,x)|v_1-v_2|^{2H-2}dv_1dv_2.$$

Notice also that, in (2), the stochastic integral has to be interpreted in the Skorohod sense (see Theorem 3.13 for a precise statement).

As mentioned above, once the existence and uniqueness of the solution to (1) is established, it certainly seems to be a natural question to ask whether an Itô-type formula is available for the process we have produced. Furthermore, this kind of result can also yield a better understanding of some properties of the process itself, such as the distribution of hitting times, as shown in [5]. It is also worth mentioning at this point that formula (2) will be obtained thanks to some Gaussian tools inspired by the case of the fractional Brownian motion itself. This is due to the fact that X can be represented by the convolution

$$X(t,x) = \int_0^t\int_0^1 M_{t,s}(x,y)W(ds,dy) \tag{3}$$

of a certain kernel M on $[0,T]\times[0,1]$, defined at (19), with respect to W. This kind of property has already been exploited in [6] for the case of the heat equation driven by a space-time white noise, but let us stress here two differences with respect to this latter reference:

1. On the one hand, an important step of our computations will be to obtain the representation (3) itself (see Corollary 3.3) and to give some reasonable bounds on the kernel M and its derivatives.

2. On the other hand, the little gain in regularity we have in the current situation with respect to [6] will allow us to obtain a formula for $t \mapsto f(X(t,x))$, while in the latter reference, we had to restrict ourselves to a change of variable formula for

$$t \mapsto \int_0^1 f(X(t,x))\psi(x)\,dx,$$

for a continuous function ψ.

Let us say now a few words about the method we have used in order to get our result: as mentioned above, the first step in our approach consists in establishing the representation (3). This representation, together with the properties of the kernel M, suggest that the differential of X should be of the form

$$X(dt,x) = \left[\int_0^t \int_0^1 \partial_t M_{t,s}(x,y) W(ds,dy)\right] dt. \tag{4}$$

This formula is of course ill-defined, since $(s,y) \mapsto \partial_t M_{t,s}(x,y)$ is not a L^2-function on $[0,t] \times [0,1]$, but it holds true for a regularization M^ε of M. We will then obtain easily an Itô type formula for the process X^ε corresponding to M^ε, where the differential (4) appears. Therefore, the main step in our calculations will be to study the limit of the regularized Itô formula when $\varepsilon \to 0$. Notice that this approach is quite different (and from our point of view more intuitive) from the one adopted in [1, 6], where the quantity $E[f(X(t,x))I_n(\varphi)]$ was evaluated for an arbitrary multiple integral $I_n(\varphi)$ with respect to W.

Our paper is divided as follows: at Section 2, we will describe precisely the noise and the equation under consideration, and we will give some basic properties of the process X. Section 3 is devoted to the derivation of our Itô-type formula: at Section 3.1 we obtain the representation (3) for X, the regularized formula is given at Section 3.2, and eventually the limiting procedure is carried out at Sections 3.3 and 3.4. In the sequel of the paper, c will designate a positive constant whose exact value can change from line to line.

2 Preliminary definitions

In this section we introduce the framework that will be used in this paper: we will define precisely the noise which will be considered, then give a brief review of some Malliavin calculus tools, and eventually introduce the fractional heat equation.

2.1 Noise under consideration

Throughout the article, we will consider a complete probability space $(\Omega, \mathcal{F}, P)$ on which we define a noise that will be a fractional Brownian motion with Hurst parameter $H > 1/2$

in time, and a Brownian motion in space. More specifically, we define a zero mean Gaussian field $B = \{B(s,x) : s \in [0,T],\ x \in [0,1]\}$ of the form

$$B(t,x) = \int_0^t \int_0^x K_H(t,s) W(ds,dy). \tag{5}$$

Here W is a two-parameter Wiener process and K_H is the kernel of the fractional Brownian motion (fBm) with Hurst parameter $H \in (\frac{1}{2}, 1)$. Namely, for $0 \le s \le t \le T$, we have

$$K_H(t,s) = C_H s^{\frac{1}{2}-H} \int_s^t (u-s)^{H-\frac{3}{2}} u^{H-\frac{1}{2}} du,$$

where C_H is a constant whose exact value is not important for our aim. Observe that the standard theory of martingale measures introduced in [17] easily yields the existence of the integral (5).

Note that it is natural to interpret the left-hand side of (5) as the stochastic integral

$$B(1_{[0,t]\times[0,x]}) := \int_0^t \int_0^x B(ds,dy). \tag{6}$$

The domain of this Wiener integral is then extended as follows: let $\mathcal{H}$ be the Hilbert space defined as the completion of the step functions with respect to the inner product

$$\begin{aligned} \langle 1_{[0,s]}, 1_{[0,t]} \rangle_{\mathcal{H}} &= \langle K_H(t,\cdot), K_H(s,\cdot) \rangle_{L^2([0,T])} \\ &= H(2H-1) \int_0^t \int_0^s |u-r|^{2H-2} du dr. \end{aligned} \tag{7}$$

Thus, by Alòs and Nualart [2], the kernel K_H allows to construct an isometry $K^*_{H,T}$ from $\mathcal{H} \times L^2([0,1])$ (denoted by $\mathcal{H}_T$ for short) into $L^2([0,T]\times[0,1])$ such that, for $0 \le s < t \le T$,

$$\begin{aligned} \left(K^*_{H,T} 1_{[0,t]\times[0,x]}\right)(s,y) &= K_H(t,s) 1_{[0,x]}(y) \\ &= 1_{[0,x]}(y) \int_s^T 1_{[0,t]}(r) \partial_r K_H(r,s) dr. \end{aligned}$$

Therefore the Wiener integral (6) can be extended into an isometry $\varphi \mapsto B(\varphi)$ from $\mathcal{H}_T$ into a subspace of $L^2(\Omega)$ so that, for any $\varphi \in \mathcal{H}_T$,

$$B(\varphi) = \int_0^T \int_0^1 (K^*_{H,T}\varphi)(s,y) W(ds,dy). \tag{8}$$

Then, for two elements φ and ψ of $\mathcal{H}_T$, the covariance between $B(\varphi)$ and $B(\psi)$ is given by

$$E\left[B(\varphi)B(\psi)\right] = H(2H-1) \int_0^T \int_0^T \int_0^1 \varphi(s,y) |s-r|^{2H-2} \psi(r,y)\, ds dr dy. \tag{9}$$

Notice that an element of $\mathcal{H}_T$ could possibly not be a function. Hence, as the in fBm case, we will deal with the Banach space $|\mathcal{H}_T|$ of all the measurable functions φ :

$[0,T] \times [0,1] \to \mathbb{R}$ such that

$$\begin{aligned}||\varphi||_{|\mathcal{H}_T|} &= H(2H-1)\int_0^T\int_0^T\int_0^1 |\varphi(r,y)||u-r|^{2H-2}|\varphi(u,y)|dydudr \\ &= \int_0^1\int_0^T\left(\int_s^T |\varphi(r,y)|\partial_r K_H(r,s)dr\right)^2 dsdy < \infty.\end{aligned}$$

It is then easy to see that $L^2([0,T]\times[0,1]) \subset |\mathcal{H}_T| \subset \mathcal{H}_T$.

2.2 Malliavin calculus tools

The goal of this section is to recall the basic definitions of the Malliavin calculus which will allow us to define the divergence operator with respect to W. For a more detailed presentation, we recommend Nualart [10].

Let $\mathcal{S}$ be the family of all smooth functionals of the form

$$F = f(W(s_1,y_1),\ldots,W(s_n,y_n)), \quad \text{with} \quad (s_i,y_i) \in [0,T]\times[0,1],$$

where $f \in C_b^\infty(\mathbb{R}^n)$ (i.e., f and all its partial derivatives are bounded). The derivative of this kind of smooth functional is the $L^2([0,T]\times[0,1])$-valued random variable

$$DF = \sum_{i=1}^n \frac{\partial f}{\partial x_i}(W(s_1,y_1),\ldots,W(s_n,y_n))1_{[0,s_i]\times[0,y_i]}.$$

It is then well-known that D is a closeable operator from $L^2(\Omega)$ into $L^2(\Omega\times[0,T]\times[0,1])$. Henceforth, to simplify the notation, we also denote its closed extension by D. Consequently D has an adjoint δ, which is also a closed operator, characterized via the duality relation

$$E\left(F\delta(u)\right) = E\left(\langle DF,u\rangle_{L^2([0,T]\times[0,1])}\right),$$

with $F \in \mathcal{S}$ and $u \in \mathrm{Dom}(\delta) \subset L^2(\Omega\times[0,T]\times[0,1])$. The operator δ has been considered as a stochastic integral because it is an extension of the Itô integral with respect to W that allows us to integrate anticipating processes (see, for instance, [10]). According to this fact, we will sometimes use the notational convention

$$\delta(u) = \int_0^T\int_0^1 u_{s,y}W(ds,dy).$$

Notice that the operator δ (or Skorohod integral) has the following property: Suppose that F is a random variable in $\mathrm{Dom}(D)$ and that u is Skorohod integrable (i.e., $u \in \mathrm{Dom}(\delta)$), such that $E(F^2\int_0^T\int_0^1(u(s,y))^2dyds) < \infty$. Then

$$\int_0^T\int_0^1 Fu(s,y)W(ds,dy) = F\int_0^T\int_0^1 u(s,y)W(ds,dy) - \int_0^T\int_0^1 (D_{s,y}F)u(s,y)dyds, \tag{10}$$

in the sense that $(Fu) \in \mathrm{Dom}(\delta)$ if and only if the right-hand side is in $L^2(\Omega)$.

2.3 Heat equation

This paper is concerned with the solution X to the following stochastic heat equation on $[0,1]$, with Dirichlet boundary conditions and null initial condition:

$$\begin{cases} \partial_t X(t,x) = \Delta X(t,x) + B(dt,dx), & (t,x) \in [0,T] \times [0,1] \\ X(0,x) = 0, \quad X(t,0) = X(t,1) = 0. \end{cases} \tag{11}$$

It is well-known (see [16]) that equation (11) has a unique solution, which is given explicitly by

$$X(t,x) = \int_0^t \int_0^1 G_{t-s}(x,y) B(ds,dy), \tag{12}$$

where

$$G_t(x,y) = \frac{1}{\sqrt{4\pi t}} \sum_{n=-\infty}^{\infty} \left[\exp\left(-\frac{(y-x-2n)^2}{4t} \right) - \exp\left(-\frac{(y+x-2n)^2}{4t} \right) \right] \tag{13}$$

stands for the Dirichlet heat kernel on $[0,1]$ with Dirichlet boundary conditions. Let us recall here some elementary but useful identities for the heat kernel G:

Lemma 2.1. *The following relations hold true for the heat kernel G given by (13):*

$$\int_0^1 G_t(x,y)dy = 1, \quad G_t(x,y) \le \frac{c_1}{t^{1/2}} \exp\left(-\frac{c_2(x-y)^2}{t} \right),$$

and

$$|\partial_t G_t(x,y)| \le \frac{c_3}{t^{3/2}} \exp\left(-\frac{c_4(x-y)^2}{t} \right),$$

for some positive constants c_1, c_2, c_3 and c_4. Furthermore, G can be decomposed into

$$G_t(x,y) = G_{1,t}(x,y) + R_t(x,y), \tag{14}$$

where

$$G_{1,t}(x,y) = \frac{1}{\sqrt{4\pi t}} \left[\exp\left(-\frac{(y-x)^2}{4t} \right) - \exp\left(-\frac{(y+x)^2}{4t} \right) - \exp\left(-\frac{(y+x-2)^2}{4t} \right) \right],$$

and $R_t(x,y)$ is a smooth bounded function on $[0,T] \times [0,1]^2$.

Let us recall now some basic properties of the process X defined by (11) and (12), starting with its integrability.

Lemma 2.2. *The process defined on $[0,T] \times [0,1]$ by (12) satisfies*

$$\sup_{t \in [0,T], x \in [0,1]} E\left[|X(t,x)|^2 \right] < \infty.$$

Proof. We have, according to (9) and Lemma 2.1, that

$$\begin{aligned} E\left[|X(t,x)|^2\right] &= c_H \int_{[0,t]^2} \frac{dsdu}{|s-u|^{2-2H}} \int_0^1 G_{t-s}(x,y)G_{t-u}(x,y)dy \\ &\leq c \int_{[0,t]^2} \frac{dsdu}{(t-s)^{1/2}|s-u|^{2-2H}} \int_0^1 G_{t-u}(x,y)dy \\ &= c \int_{[0,t]^2} \frac{dsdu}{(t-s)^{1/2}|s-u|^{2-2H}}, \end{aligned}$$

and the last integral is finite by elementary arguments. □

One can go further in the study of X, and show the following regularity result (see also [14]):

Proposition 2.3. *Let X be the solution to (11). Then, for $t_1, t_2 \in [0,T]$ and $x \in [0,1]$, we have*

$$E\left[|X(t_2,x) - X(t_1,x)|^2\right] \leq c|t_2 - t_1|^{2\gamma},$$

for any $\gamma < H - 1/4$. In particular, for any $T > 0$ and $x \in [0,1]$, the function $t \in [0,T] \mapsto X(t,x)$ is γ-Hölder continuous for any $\gamma < H - 1/4$.

Proof. Assume $t_1 < t_2$. We then have

$$X(t_2,x) - X(t_1,x) = A(t_1,t_2,x) + B(t_1,t_2,x),$$

with

$$A(t_1,t_2,x) = \int_0^{t_1} \int_0^1 \left[G_{t_2-s}(x,y) - G_{t_1-s}(x,y)\right] B(ds,dy)$$

and

$$B(t_1,t_2,x) = \int_{t_1}^{t_2} \int_0^1 G_{t_2-s}(x,y)B(ds,dy).$$

Hence

$$E\left[|X(t_2,x) - X(t_1,x)|^2\right] \leq 2\left(E\left[A^2(t_1,t_2,x)\right] + E\left[B^2(t_1,t_2,x)\right]\right). \tag{15}$$

We first note that (9) and Lemma 2.1 imply

$$\begin{aligned} & E\left[B^2(t_1,t_2,x)\right] \\ &= c_H \int_{t_1}^{t_2} \int_{t_1}^{t_2} duds|s-u|^{2H-2} \int_0^1 G_{t_2-s}(x,y)G_{t_2-u}(x,y)dy \\ &\leq c \int_{t_1}^{t_2} ds(t_2-s)^{-1/2} \int_{t_1}^{t_2} |s-u|^{2H-2}du \\ &\leq c(t_2-t_1)^{2H-\frac{1}{2}}. \end{aligned} \tag{16}$$

Now we will concentrate on the estimate on $E[A^2(t_1,t_2,x)]$. By (9), we have

$$E\left[A^2(t_1,t_2,x)\right] = c_H \int_0^{t_1} \int_0^{t_1} \frac{duds}{|s-u|^{2-2H}} C_x(s,u), \tag{17}$$

with $C_x(s,u)$ defined by

$$C_x(s,u) = \int_0^1 \left[G_{t_2-s}(x,y) - G_{t_1-s}(x,y)\right]\left[G_{t_2-u}(x,y) - G_{t_1-u}(x,y)\right] dy.$$

Thus, invoking Lemma 2.1, we obtain that, for a given $\alpha < 1/2$,

$$C_x(s,u) \leq c\frac{(t_2-t_1)^{2\alpha}}{(t_1-u)^{3\alpha/2}(t_1-s)^{3\alpha/2}} D_x(s,u),$$

where

$$D_x(s,u) = \int_0^1 |G_{t_2-s}(x,y) - G_{t_1-s}(x,y)|^{1-\alpha}|G_{t_2-u}(x,y) - G_{t_1-u}(x,y)|^{1-\alpha} dy.$$

It is then easily seen that $D_x(s,u)$ can be bounded by a sum of terms of the form

$$F_x(s,u) = \int_0^1 G_{\sigma-s}^{1-\alpha}(x,y) G_{\tau-u}^{1-\alpha}(x,y) dy,$$

with $\sigma, \tau \in \{t_1, t_2\}$. This latter expression can be bounded in the following way:

$$\begin{aligned} F_x(s,u) &\leq \left(\int_0^1 G_{\sigma-s}^{2(1-\alpha)}(x,y)dy\right)^{1/2} \left(\int_0^1 G_{\tau-u}^{2(1-\alpha)}(x,y)dy\right)^{1/2} \\ &\leq \frac{c}{(t_1-s)^{1/4-\alpha/2}(t_1-u)^{1/4-\alpha/2}}. \end{aligned}$$

We have thus obtained that

$$E\left[A^2(t_1,t_2,x)\right] \leq c(t_2-t_1)^{2\alpha} \int_0^{t_1}\int_0^{t_1} \frac{duds}{|s-u|^{2-2H}(t_1-s)^{1/4+\alpha}(t_1-u)^{1/4+\alpha}}.$$

Now thanks the change of variable $v = \frac{u-s}{t_1-s}$, the latter integral is finite whenever $\alpha < H - 1/4$, which, together with (15) and (16), ends the proof. $\square$

3 Itô's formula for the heat equation

Let us turn to the main aim of this paper, namely the Itô-type formula for the process X introduced in (12). The strategy of our computations can be briefly outlined as follows: first we will try to represent X as a convolution of a certain kernel M with respect to W, with reasonable bounds on M. Then we will be able to establish our Itô's formula for a smoothed version of X, involving a regularized kernel M^ε for $\varepsilon > 0$, by applying the usual Itô formula. Our main task will then be to study the limit of the quantities we will obtain as $\varepsilon \to 0$.

3.1 Differential of X

Before getting a suitable expression for the differential of X, let us see how to represent this process as a convolution with respect to W.

3.1.1 Representation of X

The expressions (7) and (8) lead to the following result (see [2]).

Lemma 3.1. *Let φ be a function in $|\mathcal{H}_T|$. Then*

$$\int_0^t \int_0^1 \varphi(s,y)B(ds,dy) = \int_0^t \int_0^1 [K^*_{H,T}1_{[0,t]}\varphi](u,y)W(du,dy),$$

with

$$[K^*_{H,T}1_{[0,t]}\varphi](u,y) = 1_{[0,t]}(u)\int_u^t \varphi(r,y)\partial_r K_H(r,u)dr.$$

Remark 3.2. *This result could also have been obtained by some heuristic arguments. Indeed, a formal way to write (5) is to say that, for $t>0$ and $y\in[0,1]$, the differential $B(t,dy)$ is defined as*

$$B(t,dy) = \int_0^t K_H(t,s)W(ds,dy).$$

Thus, if we differentiate formally this expression in time, since $K_H(t,t)=0$, we obtain

$$\partial_t B(t,dy) = \left[\int_0^t \partial_t K_H(t,s)W(ds,dy)\right]dt.$$

Since $\partial_t K_H(t,s)$ is not a L^2-function, the last equality has to be interpreted in the following way: if φ is a deterministic function, then

$$\begin{aligned}\int_0^t \int_0^1 \varphi(s,y)B(ds,dy) &= \int_0^t \int_0^1 \varphi(s,y)\left[\int_0^s \partial_s K_H(s,u)W(du,dy)\right]ds\\ &= \int_0^t \int_0^1 W(du,dy)\left[\int_u^t \varphi(s,y)\partial_s K(s,u)ds\right],\end{aligned}$$

which recovers the result of Lemma 3.1.

We can now easily get the announced representation for X:

Corollary 3.3. *The solution X to (11) can be written as*

$$X(t,x) = \int_0^t \int_0^1 M_{t,s}(x,y)W(ds,dy), \tag{18}$$

with

$$M_{t,s}(x,y) = \int_s^t G_{t-u}(x,y)\partial_u K_H(u,s)du. \tag{19}$$

Proof. The result is an immediate consequence of the proof of Proposition 2.3 and Lemma 3.1. □

3.1.2 Some bounds on M

The kernel M will be algebraically useful in order to obtain our Itô's formula, and we will proceed to show now that it behaves similarly to the heat kernel G. To do so, let us first state the following technical lemma:

Lemma 3.4. *Let f be defined on $0 < r < t \le T$ by*

$$f(r,t) = \int_r^t (t-u)^{-1/2}(u-r)^{-\alpha} \exp\left(-\frac{\kappa x^2}{t-u}\right) du,$$

for a constant $\kappa > 0$, $x \in [0,2]$ and $\alpha \in (0,1)$. Then, there exist some constants $c_1, c_2, c_3, c_4 > 0$ such that

$$f(r,t) \le c_1 (t-r)^{-(\alpha-1/2)} \exp\left(-\frac{c_2 x^2}{t-r}\right) \tag{20}$$

and

$$\partial_t f(r,t) \le c_3 (t-r)^{-(\alpha+1/2)} \exp\left(-\frac{c_4 x^2}{t-r}\right). \tag{21}$$

Proof. Recall that, in the remainder of the paper, κ stands for a positive constant which can change from line to line. Notice also that (20) is easy to see due to

$$f(r,t) \le \exp\left(-\frac{\kappa x^2}{t-r}\right) \int_r^t (t-u)^{-1/2}(u-r)^{-\alpha} dr.$$

Now we will concentrate on (21): let us perform the change of variable $v = \frac{u-r}{t-r}$. This yields

$$f(r,t) = (t-r)^{-(\alpha-1/2)} \int_0^1 (1-v)^{-1/2} v^{-\alpha} \exp\left(-\frac{\kappa x^2}{(1-v)(t-r)}\right) dv,$$

and thus

$$\partial_t f(r,t) = g_1(r,t) + g_2(r,t),$$

with

$$g_1(r,t) = \kappa x^2 (t-r)^{-(\alpha+3/2)} \int_0^1 (1-v)^{-3/2} v^{-\alpha} \exp\left(-\frac{\kappa x^2}{(1-v)(t-r)}\right) dv$$

and

$$g_2(r,t) = \left(\frac{1}{2} - \alpha\right) (t-r)^{-(\alpha+1/2)} \int_0^1 (1-v)^{-1/2} v^{-\alpha} \exp\left(-\frac{\kappa x^2}{(1-v)(t-r)}\right) dv.$$

Therefore, thanks to the fact that $u \mapsto u e^{-u}$ is a bounded function on $\mathbb{R}_+$, we have

$$\begin{aligned} g_1(r,t) &\le c (t-r)^{-(\alpha+1/2)} \int_0^1 (1-v)^{-1/2} v^{-\alpha} \exp\left(-\frac{\kappa x^2}{2(1-v)(t-r)}\right) dv \\ &\le c (t-r)^{-(\alpha+1/2)} \exp\left(-\frac{\kappa x^2}{2(t-r)}\right) \int_0^1 (1-v)^{-1/2} v^{-\alpha} dv, \end{aligned}$$

which is an estimate of the form (20). Finally, it is easy to see that

$$g_2(r,t) \leq c(t-r)^{-(\alpha+1/2)} \exp\left(-\frac{\kappa x^2}{2(t-r)}\right) \int_0^1 (1-v)^{-1/2} v^{-\alpha} dv,$$

which completes the proof.

□

We are now ready to prove our bounds on M:

Proposition 3.5. *Let M be the kernel defined at (19). Then, for some strictly positive constants $c_5, c_6, c_7, c_8 > 0$, we have*

$$\begin{aligned} & M_{t,s}(x,y) \\ \leq \;& c_5 (t-s)^{-(1-H)} \left(\frac{t}{s}\right)^{H-1/2} \left[\exp\left(-\frac{c_6(x-y)^2}{t-s}\right) + \exp\left(-\frac{c_6(x+y-2)^2}{t-s}\right)\right] \end{aligned}$$

and

$$\begin{aligned} & |\partial_t M_{t,s}(x,y)| \\ \leq \;& c_7 (t-s)^{-(2-H)} \left(\frac{t}{s}\right)^{H-1/2} \left[\exp\left(-\frac{c_8(x-y)^2}{t-s}\right) + \exp\left(-\frac{c_8(x+y-2)^2}{t-s}\right)\right]. \end{aligned}$$

Proof. First of all, we will use the decomposition (14), which allows to write

$$M_{t,s}(x,y) = \int_s^t G_{1,t-u}(x,y) \partial_u K_H(u,s) du + \int_s^t R_{t-u}(x,y) \partial_u K_H(u,s) du.$$

Now the result is an immediate consequence of Lemma 3.4 applied to $\alpha < \frac{3}{2} - H$, the only difference being the presence of the term $(u/s)^{H-1/2}$, which can be bounded by $(t/s)^{H-1/2}$ each time it appears. This yields the desired result.

□

3.1.3 Differential of X

With the representation (18) in hand, we can now follow the heuristic steps in Remark 3.2 in order to get a reasonable definition of the differential of X in time. That is, we can write formally that

$$X(dt,x) = \left[\int_0^t \int_0^1 \partial_t M_{t,s}(x,y) W(ds,dy)\right] dt,$$

which means that if $\varphi : [0,T] \times [0,1] \to \mathbb{R}$ is a smooth enough function, we have

$$\begin{aligned} \int_0^T \varphi(t,x) X(dt,x) &= \int_0^T \varphi(t,x) \left[\int_0^t \int_0^1 \partial_t M_{t,s}(x,y) W(ds,dy)\right] dt \\ &= \int_0^T \int_0^1 W(ds,dy) \left[\int_s^T \varphi(t,x) \partial_t M_{t,s}(x,y) dt\right]. \end{aligned}$$

Note that this expression may not be convenient because it does not take advantage of the continuity of φ. But, by Proposition 3.5, we can write

$$\int_s^T \varphi(t,x)\partial_t M_{t,s}(x,y)dt = \int_s^T (\varphi(t,x)-\varphi(s,x))\partial_t M_{t,s}(x,y)dt + \varphi(s,x)M_{T,s}(x,y).$$

Here again, we can formalize these heuristic considerations into the following:

Definition 3.6. *Let $\varphi : \Omega \times [0,T] \times [0,1] \to \mathbb{R}$ be a measurable process. We say that φ is integrable with respect to X if the mapping*

$$(s,y) \mapsto [M^*_{T,x}\varphi](s,y) := \int_s^T \left(\varphi(t,x)-\varphi(s,x)\right) \partial_t M_{t,s}(x,y)dt + \varphi(s,x)M_{T,s}(x,y) \quad (22)$$

belongs to Dom(δ), for almost all $x \in [0,1]$. In this case we set

$$\int_0^T \varphi(t,x)X(dt,x) = \int_0^T \int_0^1 [M^*_{T,x}\varphi](s,y)W(ds,dy).$$

Remark 3.7. *Just like in the case of the fractional Brownian motion [1] or of the heat equation driven by the space-time white noise [6], one can show that $\int_0^T \varphi(t,x)X(dt,x)$ can be interpreted as a divergence operator for the Wiener space defined by X.*

Remark 3.8. *It is easy to see that Proposition 3.5 implies that $\varphi : [0,T] \to \mathbb{R}$ is integrable with respect to X if it is β-Hölder continuous in time with $\beta > 1-H$.*

3.2 Regularized version of Itô's formula

The representation (18) of X also allows us to define a natural regularized version X^ε of X, depending on a parameter $\varepsilon > 0$, such that $t \mapsto X^\varepsilon(t,x)$ will be a semi-martingale. Indeed, set, for $\varepsilon > 0$,

$$M^\varepsilon_{t,s}(x,y) = \int_s^t G_{t-u+\varepsilon}(x,y)\partial_u K_H(u+\varepsilon,s)du,$$

and

$$X^\varepsilon(t,x) = \int_0^t \int_0^1 M^\varepsilon_{t,s}(x,y)W(ds,dy). \quad (23)$$

We will also need a regularized operator $M^{\varepsilon,*}_{t,x}$ (see (22)), defined naturally by

$$\left[M^{\varepsilon,*}_{t,x}\varphi\right](s,y) = \int_s^t (\varphi(r,x)-\varphi(s,x))\partial_r M^\varepsilon_{r,s}(x,y)dr + \varphi(s,x)M^\varepsilon_{t,s}(x,y).$$

Our strategy in order to get an Itô type formula for X will then be the following:

1. Apply the usual Itô formula to the semi-martingale $t \mapsto X^\varepsilon(t,x)$.

2. Rearrange terms in order to get an expression in terms of the operator $M^{\varepsilon,*}_{t,x}$.

3. Study the limit of the different terms obtained through Steps 1 and 2, as $\varepsilon \to 0$.

The current section will be devoted to the elaboration of Steps 1 and 2.

Lemma 3.9. *Let $\varepsilon > 0$. Then, the process $t \mapsto X^\varepsilon(t,x)$ has bounded variations on $[0,T]$, for all $x \in [0,1]$.*

Proof. The Fubini theorem for W and the semigroup property of G imply

$$X^\varepsilon(t,x) = \int_0^t \int_0^1 G_{t-u+\frac{\varepsilon}{2}}(x,z) \left(\int_0^u \int_0^1 G_{\varepsilon/2}(z,y) \partial_u K_H(u+\varepsilon, s) W(ds,dy) \right) dzdu,$$

and notice that this integral is well-defined due to Kolmogorov's continuity theorem. Therefore, since $t \mapsto G_{t-u+\varepsilon/2}(x,z)$ is also a C^1-function on $[u,T]$, we obtain that X^ε is differentiable with respect to $t \in [0,T]$, and

$$\begin{aligned}
&\partial_t X^\varepsilon(t,x) \\
&\quad = \int_0^t \int_0^1 \partial_t G_{t-u+\frac{\varepsilon}{2}}(x,z) \left(\int_0^u \int_0^1 G_{\varepsilon/2}(z,y) \partial_u K_H(u+\varepsilon, s) W(ds,dy) \right) dzdu \\
&\qquad + \int_0^1 G_{\varepsilon/2}(x,z) \left(\int_0^t \int_0^1 G_{\varepsilon/2}(z,y) \partial_t K_H(t+\varepsilon, s) W(ds,dy) \right) dz,
\end{aligned}$$

which is a continuous process on $[0,T] \times [0,1]$, invoking Kolmogorov's continuity theorem again in a standard manner.

□

An immediate consequence of the previous lemma is the following:

Corollary 3.10. *Let $t \in [0,T]$, $x \in [0,1]$ and $\varepsilon > 0$. Then,*

$$\begin{aligned}
\partial_t X^\varepsilon(t,x) &= \int_0^t \int_0^1 \left(\int_s^t \partial_t G_{t-u+\varepsilon}(x,y) \partial_u K_H(u+\varepsilon, s) du \right) W(ds,dy) \\
&\quad + \int_0^t \int_0^1 G_\varepsilon(x,y) \partial_t K_H(t+\varepsilon, s) W(ds,dy) \\
&= \int_0^t \int_0^1 \partial_t M^\varepsilon_{t,s}(x,y) W(ds,dy).
\end{aligned}$$

Proof. The result follows from Fubini's theorem for W and from the semigroup property of G.

□

Now we are ready to establish our regularized Itô's formula in order to carry out Steps 1 and 2 of this section.

Proposition 3.11. *Let f be a regular function in $C_b^2(\mathbb{R})$, $\varepsilon > 0$, and X^ε the process defined by (23). Then, for $t \in [0,T]$ and $x \in [0,1]$, $M^{\varepsilon,*}_{t,x} f'(X^\varepsilon)$ belongs to $Dom(\delta)$ and*

$$f(X^\varepsilon(t,x)) = f(0) + \mathbf{A}_{\mathbf{1},\varepsilon}(t,x) + \mathbf{A}_{\mathbf{2},\varepsilon}(t,x),$$

where

$$\mathbf{A}_{\mathbf{1},\varepsilon}(t,x)=\int_0^t\int_0^1\left(M_{t,x}^{\varepsilon,*}f'(X^\varepsilon)\right)(s,y)W(ds,dy)$$

is defined as a Skorohod integral, and

$$\mathbf{A}_{\mathbf{2},\varepsilon}(t,x)=\int_0^t f''(X^\varepsilon(s,x))K_{\varepsilon,x}(ds),$$

with

$$\begin{aligned}K_{\varepsilon,x}(s) &= \int_0^s dv_2\int_0^{v_2}dv_1 G_{2(s+\varepsilon)-v_1-v_2}(x,x)\Big\{H(2H-1)|v_1-v_2|^{2H-2}\\ &\quad -\partial^2_{v_1,v_2}\left(\int_{v_1}^{v_1+\varepsilon}K_H(v_1+\varepsilon,u)K_H(v_2+\varepsilon,u)du\right)\\ &\quad -\partial_{v_2}\left(K_H(v_1+\varepsilon,v_1)K_H(v_2+\varepsilon,v_1)\right)\Big\}. \qquad (24)\end{aligned}$$

Proof. By Corollary 3.10, we are able to apply the classical change of variable formula to obtain

$$f(X^\varepsilon(t,x))=f(0)+\int_0^t f'(X^\varepsilon(s,x))\left[\int_0^s\int_0^1\partial_s M_{s,u}^\varepsilon(x,y)W(du,dy)\right]ds. \qquad (25)$$

Moreover, the derivative of $f'(X^\varepsilon(s,x))$ in the Malliavin calculus sense is given by

$$D_{v,z}[f'(X^\varepsilon(s,x))]=M_{s,v}^\varepsilon(x,z)f''(X^\varepsilon(s,x))\mathbf{1}_{\{v\le s\}}.$$

Since the last quantity is bounded by $c_\varepsilon v^{\frac12-H}$ for $\varepsilon>0$, then invoking formula (10) for the Skorohod integral, we get

$$\begin{aligned}&f'(X^\varepsilon(s,x))\int_0^s\int_0^1\partial_s M_{s,u}^\varepsilon(x,y)W(du,dy)\\ &=\int_0^s\int_0^1 f'(X^\varepsilon(s,x))\partial_s M_{s,u}^\varepsilon(x,y)W(du,dy)\\ &\qquad + f''(X^\varepsilon(s,x))\int_0^s\int_0^1\left(\partial_s M_{s,u}^\varepsilon(x,y)\right)M_{s,u}^\varepsilon(x,y)dudy. \qquad (26)\end{aligned}$$

Denote for the moment the quantity $\int_0^s\int_0^1\left(\partial_s M_{s,u}^\varepsilon(x,y)\right)M_{s,u}^\varepsilon(x,y)dudy$ by $h_x(s)$. Then, combining (25) and (26), proceeding as the beginning of Section 3.1.3, and applying Fubini's theorem for the Skorohod integral, we have

$$f(X^\varepsilon(t,x))=f(0)+\mathbf{A}_{\mathbf{1},\varepsilon}(t,x)+\int_0^t f''(X^\varepsilon(s,x))h_x(s)ds. \qquad (27)$$

We can find now a simpler expression for $h_x(s)$. Indeed, since $M_{s,s}^\varepsilon(x,y)=0$, it is easily checked that

$$h_x(s)=\frac12\partial_s\left[\int_0^s\int_0^1\left(M_{s,u}^\varepsilon(x,y)\right)^2dudy\right]. \qquad (28)$$

Furthermore, the semigroup property for G yields

$$\begin{aligned}
&\int_0^s \int_0^1 \left(M^\varepsilon_{s,u}(x,y)\right)^2 dudy \\
&= \int_0^s du \int_u^s dv_1 \int_u^s dv_2 \int_0^1 dy\, G_{s+\varepsilon-v_1}(x,y) G_{s+\varepsilon-v_2}(x,y) \partial_{v_1} K_H(v_1+\varepsilon,u) \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \cdot \partial_{v_2} K_H(v_2+\varepsilon,u),
\end{aligned}$$

and this last expression is equal to

$$\begin{aligned}
&2\int_0^s du \int_u^s dv_1 \int_{v_1}^s dv_2 G_{2(s+\varepsilon)-v_1-v_2}(x,x) \left(\partial_{v_1} K_H(v_1+\varepsilon,u)\right) \partial_{v_2} K_H(v_2+\varepsilon,u) \\
&= 2\int_0^s dv_2 \int_0^{v_2} dv_1 G_{2(s+\varepsilon)-v_1-v_2}(x,x) \left(\int_0^{v_1} \left(\partial_{v_1} K_H(v_1+\varepsilon,u)\right) \partial_{v_2} K_H(v_2+\varepsilon,u) du \right).
\end{aligned} \tag{29}$$

But

$$\begin{aligned}
&\int_0^{v_1} \left(\partial_{v_1} K_H(v_1+\varepsilon,u)\right) \partial_{v_2} K_H(v_2+\varepsilon,u) du \\
&\quad = \partial_{v_2}\partial_{v_1} \left[\int_0^{v_1} K_H(v_1+\varepsilon,u) K_H(v_2+\varepsilon,u) du \right] \\
&\qquad -\partial_{v_2} \left[K_H(v_1+\varepsilon,v_1) K_H(v_2+\varepsilon,v_1) \right] \\
&\quad = H(2H-1)|v_1-v_2|^{2H-2} - \partial_{v_2}\partial_{v_1} \left[\int_{v_1}^{v_1+\varepsilon} K_H(v_1+\varepsilon,u) K_H(v_2+\varepsilon,u) du \right] \\
&\qquad -\partial_{v_2} \left[K_H(v_1+\varepsilon,v_1) K_H(v_2+\varepsilon,v_1) \right].
\end{aligned} \tag{30}$$

By putting together (29) and (30), we have thus obtained that

$$\frac{1}{2} \int_0^s \int_0^1 \left(M^\varepsilon_{s,u}(x,y)\right)^2 dudy = K_{\varepsilon,x}(s),$$

where $K_{\varepsilon,x}(s)$ is defined at (24). By plugging this equality into (27) and (28), the proof is now complete.

□

3.3 Itô's formula

We are now ready to perform the limiting procedure which will allow to go from Proposition 3.11 to the announced Itô formula. To this end we will need the following technical result, which states that the modulus of continuity of $t \mapsto X^\varepsilon(t,x)$ can be bounded from below by any $\nu < H - 1/4$, independently of ε.

Proposition 3.12. *Let X^ε be given by (23). Then for $t_1, t_2 \in [0,T]$ and $x \in [0,1]$, there is a positive constant c (independent of ε) such that*

$$E\left(|X^\varepsilon(t_2,x) - X^\varepsilon(t_1,x)|^2\right) \le c|t_2-t_1|^{2\nu},$$

for any $\nu < H - \frac{1}{4}$.

Proof. Suppose that $t_1 < t_2$. Then

$$\begin{aligned} E\left(|X^\varepsilon(t_2,x) - X^\varepsilon(t_1,x)|^2\right) &\leq 2\int_0^{t_1}\int_0^1 \left(M^\varepsilon_{t_2,s}(x,y) - M^\varepsilon_{t_1,s}(x,y)\right)^2 dyds \\ &\quad +2\int_{t_1}^{t_2}\int_0^1 \left(M^\varepsilon_{t_2,s}(x,y)\right)^2 dyds. \end{aligned} \tag{31}$$

Now using the fact that $\partial_u K_H(u,s) > 0$, we have

$$\begin{aligned} &\int_{t_1}^{t_2}\int_0^1 \left(M^\varepsilon_{t_2,s}(x,y)\right)^2 dyds \\ &= \int_{t_1}^{t_2}\int_0^1 \left(\int_{s+\varepsilon}^{t_2+\varepsilon} G_{t_2+2\varepsilon-u}(x,y)\partial_u K_H(u,s)du\right)^2 dyds \\ &\leq \int_0^{t_2+\varepsilon}\int_0^1 \left(\int_s^{t_2+\varepsilon} 1_{[t_1+\varepsilon,t_2+\varepsilon]}(u) G_{t_2+2\varepsilon-u}(x,y)\partial_u K_H(u,s)du\right)^2 dyds \\ &= H(2H-1)\int_{t_1+\varepsilon}^{t_2+\varepsilon}\int_{t_1+\varepsilon}^{t_2+\varepsilon}\int_0^1 |u-v|^{2H-2} G_{t_2+2\varepsilon-u}(x,y) G_{t_2+2\varepsilon-v}(x,y) dydudv \\ &\leq c(t_2-t_1)^{2H-\frac{1}{2}}, \end{aligned} \tag{32}$$

where the last inequality follows as in (16).

On the other hand, it is not difficult to see that

$$\begin{aligned} &\int_0^{t_1}\int_0^1 \left(M^\varepsilon_{t_2,s}(x,y) - M^\varepsilon_{t_1,s}(x,y)\right)^2 dyds \\ &\leq 2\int_0^{t_1}\int_0^1 \left(\int_{s+\varepsilon}^{t_1+\varepsilon} \left[G_{t_2+2\varepsilon-u}(x,y) - G_{t_1+2\varepsilon-u}(x,y)\right]\partial_u K_H(u,s)du\right)^2 dyds \\ &\quad +2\int_0^{t_1}\int_0^1 \left(\int_{t_1+\varepsilon}^{t_2+\varepsilon} G_{t_2+2\varepsilon-u}(x,y)\partial_u K_H(u,s)du\right)^2 dyds \\ &= B_1 + B_2. \end{aligned} \tag{33}$$

Observe now that we can proceed as in (32) to obtain

$$B_2 \leq c(t_2-t_1)^{2H-\frac{1}{2}}, \tag{34}$$

and it is also readily checked that

$$\begin{aligned} B_1 &\leq 2\int_0^{t_1+2\varepsilon}\int_0^1 \left(\int_s^{t_1+2\varepsilon} |G_{t_2+2\varepsilon-u}(x,y) - G_{t_1+2\varepsilon-u}(x,y)|\partial_u K_H(u,s)du\right)^2 dyds \\ &= 2H(2H-1)\int_0^{t_1+2\varepsilon}\int_0^{t_1+2\varepsilon}\int_0^1 |u-v|^{2H-2}|G_{t_2+2\varepsilon-u}(x,y) - G_{t_1+2\varepsilon-u}(x,y)| \\ &\qquad \cdot|G_{t_2+2\varepsilon-v}(x,y) - G_{t_1+2\varepsilon-v}(x,y)|dydudv. \end{aligned}$$

Finally, the proof follows combining (17), and (31)-(34).

□

Let us state now the main result of this paper.

Theorem 3.13. *Let X be the process defined by (12) and $f \in C_b^2(\mathbb{R})$. Then, for $t \in [0,T]$ and $x \in [0,1]$, the process $M_{t,x}^* f'(X)$ belongs to Dom(δ) and*

$$f(X(t,x)) = f(0) + \mathbf{A_1}(t,x) + \mathbf{A_2}(t,x),$$

where

$$\mathbf{A_1}(t,x) = \int_0^t \int_0^1 \left(M_{t,x}^* f'(X) \right)(s,y) W(ds,dy)$$

and

$$\mathbf{A_2}(t,x) = \frac{1}{2} \int_0^t f''(X(s,x)) K_x(ds),$$

with

$$K_x(s) = H(2H-1) \int_0^s \int_0^s G_{2s-v_1-v_2}(x,x) |v_1 - v_2|^{2H-2} dv_1 dv_2.$$

As mentioned before, in order to prove this theorem, we use the regularized Itô formula of Proposition 3.11, and we only need to study the convergence of the terms $\mathbf{A}_{\mathbf{1},\varepsilon}$ and $\mathbf{A}_{\mathbf{2},\varepsilon}$ appearing there. However, this analysis implies long and tedious calculations. This is why we have chosen to split the proof of our theorem into a series of lemmas which will be given in the next section.

3.4 Proof of the main result

The purpose of this section is to present some technical results whose combination provides us the proof of our Itô's formula given at Theorem 3.13. We begin with the convergence $\mathbf{A}_{\mathbf{2},\varepsilon} \to \mathbf{A_2}$, for which we provide first a series of lemmas.

Lemma 3.14. *Let L_1^ε be the function defined on $[0,T]$ by*

$$L_1^\varepsilon(s) = \int_0^s dv_2 \int_0^{v_2} dv_1 \, G_{2(s+\varepsilon)-v_1-v_2}(x,x) K_H(v_1+\varepsilon, v_1) \partial_{v_2} K_H(v_2+\varepsilon, v_1).$$

Then $s \mapsto \partial_s L_1^\varepsilon(s)$ converges to 0 in $L^1([0,T])$, as $\varepsilon \downarrow 0$.

Proof. Note that by (14) we only need to study the convergence of $\partial_s L_{11}^\varepsilon(s)$, where

$$L_{11}^\varepsilon(s) = \int_0^s dv_2 \int_0^{v_2} dv_1 \, \frac{K_H(v_1+\varepsilon, v_1)}{\sqrt{2(s+\varepsilon) - v_1 - v_2}} \partial_{v_2} K_H(v_2+\varepsilon, v_1). \tag{35}$$

Indeed, this term will show us the technique and the difficulties for the remaining terms.

We will now proceed to a series of change of variables in order to get rid of the parameter s in the boundaries of the integrals defining L_{11}^ε: using first the change of variable $z = \frac{v_2 - v_1}{s - v_1}$, and then $\theta = v_1/s$, we can write

$$L_{11}^\varepsilon(s) = c_H s^{3-2H} \int_0^1 \left(\int_{s\theta}^{s\theta+\varepsilon} (u - s\theta)^{H-\frac{3}{2}} u^{H-\frac{1}{2}} du \right) \frac{(1-\theta)}{\theta^{2H-1}} \cdot \int_0^1 \frac{(\varepsilon + s\theta + zs(1-\theta))^{H-\frac{1}{2}}}{\sqrt{2\varepsilon + s(1-\theta)(2-z)}} (zs(1-\theta)+\varepsilon)^{H-\frac{3}{2}} dz d\theta.$$

Hence, the change of variable $v = u - s\theta$ leads to

$$L_{11}^{\varepsilon}(s) = c_H s^{3-2H} \int_0^1 \left(\int_0^{\varepsilon} v^{H-\frac{3}{2}} (v + s\theta)^{H-\frac{1}{2}} dv \right) \frac{(1-\theta)}{\theta^{2H-1}} \cdot \int_0^1 \frac{(\varepsilon + s\theta + zs(1-\theta))^{H-\frac{1}{2}}}{\sqrt{2\varepsilon + s(1-\theta)(2-z)}} (zs(1-\theta)+\varepsilon)^{H-\frac{3}{2}} dz d\theta.$$

Therefore, by differentiating this expression in s, we end up with a sum of the type

$$\partial_s L_{11}^{\varepsilon}(s) = \sum_{j=1}^{5} L_{11j}^{\varepsilon}(s),$$

where

$$L_{111}^{\varepsilon}(s) = c_H s^{2-2H} \int_0^1 \left(\int_0^{\varepsilon} v^{H-\frac{3}{2}} (v + s\theta)^{H-\frac{1}{2}} dv \right) \frac{(1-\theta)}{\theta^{2H-1}} \cdot \int_0^1 \frac{(\varepsilon + s\theta + zs(1-\theta))^{H-\frac{1}{2}}}{\sqrt{2\varepsilon + s(1-\theta)(2-z)}} (zs(1-\theta)+\varepsilon)^{H-\frac{3}{2}} dz d\theta,$$

and where the terms $L_{112}^{\varepsilon}, \ldots, L_{115}^{\varepsilon}$, whose exact calculation is left to the reader for sake of conciseness, are similar to L_{111}^{ε}.

Finally, we have

$$L_{111}^{\varepsilon}(s) \le c_H s^{-H} \left(\int_0^{\varepsilon} v^{H-\frac{3}{2}} dv \right) \left(\int_0^1 \frac{(1-\theta)^{H-1}}{\theta^{2H-1}} d\theta \right) \int_0^1 z^{H-\frac{3}{2}} dz \le c_H \varepsilon^{H-1/2} s^{-H},$$

and it is easily checked that this last term converges to 0 in $L^1([0,T])$. Furthermore, it can also be proved that $|L_{11j}^{\varepsilon}(s)| \le c L_{111}^{\varepsilon}(s)$ for $2 \le j \le 5$, which ends the proof. □

Lemma 3.15. *Let L_2^{ε} be the function defined on $[0,T]$ by*

$$L_2^{\varepsilon}(s) = \int_0^s dv_2 \int_0^{v_2} dv_1 \, G_{2(s+\varepsilon)-v_1-v_2}(x,x) \partial_{v_1} \left(\int_{v_1}^{v_1+\varepsilon} K_H(v_1+\varepsilon, u) \partial_{v_2} K_H(v_2+\varepsilon, u) du \right).$$

Then $s \mapsto \partial_s L_2^{\varepsilon}(s)$ converges to 0 in $L^1([0,T])$, as $\varepsilon \downarrow 0$.

Proof. As in the proof of Lemma 3.14 we only show the convergence of $\partial_s L_{21}^{\varepsilon}(s)$, where

$$L_{21}^{\varepsilon}(s) = \int_0^s dv_2 \int_0^{v_2} dv_1 \, \frac{1}{\sqrt{2(s+\varepsilon) - v_1 - v_2}} \partial_{v_1} \hat{L}(v_1, v_2),$$

with

$$\hat{L}(v_1, v_2) = \int_{v_1}^{v_1+\varepsilon} K_H(v_1+\varepsilon, u) \partial_{v_2} K_H(v_2+\varepsilon, u) du.$$

Towards this end, we will proceed again to a series of changes of variables in order to eliminate the parameter s from the boundaries of the integrals: notice first that the definition of K_H, and the change of variables $\theta = \frac{u-v_1}{r-v_1}$ and $z = r - v_1$ yield

$$\hat{L}(v_1, v_2) = c_H (v_2+\varepsilon)^{H-\frac{1}{2}} \int_0^{\varepsilon} (v_1+z)^{H-\frac{1}{2}} \int_0^1 (v_1+\theta z)^{1-2H} \frac{z^{H-\frac{1}{2}}}{(1-\theta)^{\frac{3}{2}-H}} \cdot (v_2+\varepsilon-v_1-\theta z)^{H-\frac{3}{2}} d\theta dz.$$

Thus

$$\begin{aligned}
\partial_{v_1}\hat{L}(v_1, v_2) = (v_2+\varepsilon)^{H-\frac{1}{2}} \Bigg[& c_H \int_0^{\varepsilon} (v_1+z)^{H-\frac{3}{2}} \int_0^1 (v_1+\theta z)^{1-2H} \frac{z^{H-\frac{1}{2}}}{(1-\theta)^{\frac{3}{2}-H}} \cdot (v_2+\varepsilon-v_1-\theta z)^{H-\frac{3}{2}} d\theta dz \\
& - c_H \int_0^{\varepsilon} (v_1+z)^{H-\frac{1}{2}} \int_0^1 (v_1+\theta z)^{-2H} \frac{z^{H-\frac{1}{2}}}{(1-\theta)^{\frac{3}{2}-H}} (v_2+\varepsilon-v_1-\theta z)^{H-\frac{3}{2}} d\theta dz \\
& + c_H \int_0^{\varepsilon} (v_1+z)^{H-\frac{1}{2}} \int_0^1 (v_1+\theta z)^{1-2H} \frac{z^{H-\frac{1}{2}}}{(1-\theta)^{\frac{3}{2}-H}} (v_2+\varepsilon-v_1-\theta z)^{H-\frac{5}{2}} d\theta dz \Bigg].
\end{aligned} \tag{36}$$

Hence, it is easily seen that L_{21}^{ε} is a sum of terms of the form

$$\begin{aligned}
Q_{\alpha,\beta,\nu}(s) &= \int_0^s dv_2 \int_0^{v_2} dv_1 \, \frac{(v_2+\varepsilon)^{H-\frac{1}{2}}}{\sqrt{2(s+\varepsilon)-v_1-v_2}} \int_0^{\varepsilon} (v_1+z)^{\alpha} \\
&\qquad \cdot \int_0^1 (v_1+\theta z)^{\beta} \frac{z^{H-\frac{1}{2}}}{(1-\theta)^{\frac{3}{2}-H}} (v_2+\varepsilon-v_1-\theta z)^{\nu} d\theta dz \\
&= s^2 \int_0^1 d\eta \int_0^1 du \frac{\eta(s\eta+\varepsilon)^{H-\frac{1}{2}}}{\sqrt{2(s+\varepsilon)-us\eta-s\eta}} \int_0^{\varepsilon} (us\eta+z)^{\alpha} \\
&\qquad \cdot \int_0^1 (us\eta+\theta z)^{\beta} \frac{z^{H-\frac{1}{2}}}{(1-\theta)^{\frac{3}{2}-H}} (s\eta+\varepsilon-s\eta u+\theta z)^{\nu} d\theta dz,
\end{aligned}$$

by applying the changes of variable $u = v_1/v_2$, and $\eta = v_2/s$. Differentiating this last relation, we are now able to compute $\partial_s L_{21}^{\varepsilon}(s)$, and see that this function goes to 0 as $\varepsilon \downarrow 0$ in $L^1([0,T])$, similarly to what we did in the proof of Lemma 3.14.

□

Lemma 3.16. *Let L_3^{ε} be the function defined on $[0,T]$ by*

$$L_3^{\varepsilon}(s) = H(2H-1) \int_0^s dv_2 \int_0^{v_2} dv_1 \, G_{2(s+\varepsilon)-v_1-v_2}(x,x)(v_2-v_1)^{2H-2}.$$

Then $\partial_s L_3^{\varepsilon}(s)$ tends to $\frac{1}{2}K_x(ds)$ in $L^1([0,T])$, as $\varepsilon \downarrow 0$.

Proof. As in the proofs of Lemmas 3.14 and 3.15, we only need to use the change of variables $z = v_1/v_2$ and $\theta = v_2/s$.

□

Lemma 3.17. *Let X and X^ε be given in (18) and (23), respectively. Then $X^\varepsilon(\cdot,x)$ converges to $X(\cdot,x)$ in $L^2(\Omega\times[0,T])$ and, for $t\in[0,T]$, $X^\varepsilon(t,x)$ goes to $X(t,x)$ in $L^2(\Omega)$, as $\varepsilon\downarrow 0$.*

Proof. The result is an immediate consequence of the definitions of the processes $X^\varepsilon(\cdot,x)$ and $X(\cdot,x)$, the fact that $|M^\varepsilon_{t,s}(x,y)|\le c(t-s)^{H-1}s^{\frac{1}{2}-H}$ and of the dominated convergence theorem.

□

We are now ready to study the convergence of the term $\mathbf{A}_{\mathbf{2},\varepsilon}$:

Lemma 3.18. *Let $t\in[0,T]$ and $x\in[0,1]$. Then the random variable*

$$\begin{aligned}&B_2^\varepsilon(t,x)\\&:=H(2H-1)\int_0^t f''(X^\varepsilon(s,x))\partial_s\left(\int_0^s dv_2\int_0^{v_2}dv_1G_{2(s+\varepsilon)-v_1-v_2}(x,x)(v_2-v_1)^{2H-2}\right)ds\end{aligned}$$

converges to $\mathbf{A_2}(t,x)$ in $L^2(\Omega)$ as $\varepsilon\downarrow 0$.

Proof. Since f'' is a bounded function, then

$$\begin{aligned}&E\left(|B_2^\varepsilon(t,x)-\mathbf{A_2}(t,x)|^2\right)\\&\le c\int_0^t E\left((f''(X(s,x))-f''(X^\varepsilon(s,x)))^2\right)|\partial_sK_x(s)|ds\\&+c\left(\int_0^t|\partial_sK_x(s)-H(H-\frac{1}{2})\partial_s\int_0^s dv_2\int_0^{v_2}dv_1G_{2(s+\varepsilon)-v_1-v_2}(x,x)(v_2-v_1)^{2H-2}|ds\right)^2.\end{aligned}$$

Hence the result is a consequence of Lemmas 3.16 and 3.17, and the dominated convergence theorem. □

Now we study the convergence of $\mathbf{A}_{\mathbf{1},\varepsilon}$ to $\mathbf{A_1}$ in $L^2(\Omega)$.

Lemma 3.19. *Let X and X^ε be given in (18) and (23), respectively. Then, for $t\in[0,T]$ and $x\in[0,1]$,*

$$E\left(\int_0^t\int_0^1\left[(M^*_{t,x}f'(X))(s,y)-(M^{\varepsilon,*}_{t,x}f'(X^\varepsilon))(s,y)\right]^2dyds\right)\to 0$$

as $\varepsilon\downarrow 0$.

Proof. We first note that

$$E\left(\int_0^t\int_0^1\left[(M^*_{t,x}f'(X))(s,y)-(M^{\varepsilon,*}_{t,x}f'(X^\varepsilon))(s,y)\right]^2dyds\right)$$

can be bounded from above by:

$$
\begin{aligned}
& cE\left(\int_0^t\int_0^1\left[\int_s^t (f'(X(r,x)) - f'(X(s,x))\right.\right.\\
&\qquad\qquad \left.\left. - f'(X^\varepsilon(r,x)) + f'(X^\varepsilon(s,x)))\partial_r M_{r,s}(x,y)dr\right]^2 dyds\right)\\
& + cE\left(\int_0^t\int_0^1\left[\int_s^t (f'(X^\varepsilon(r,x)) - f'(X^\varepsilon(s,x)))(\partial_r M_{r,s}(x,y) - \partial_r M^\varepsilon_{r,s}(x,y))dr\right]^2 dyds\right)\\
& + cE\left(\int_0^t\int_0^1 [(f'(X(s,x)) - f'(X^\varepsilon(s,x)))M_{t,s}(x,y)]^2 dyds\right)\\
& + cE\left(\int_0^t\int_0^1 \left[f'(X^\varepsilon(s,x))(M_{t,s}(x,y) - M^\varepsilon_{t,s}(x,y))\right]^2 dyds\right)\\
& = c(B_1 + \ldots + B_4).
\end{aligned}
\tag{37}
$$

Next observe that

$$
\begin{aligned}
B_2 \;\le\; & \int_0^t\int_0^1\left[\int_s^t E\left((f'(X^\varepsilon(r,x)) - f'(X^\varepsilon(s,x)))^2\right)|\partial_r M_{r,s}(x,y) - \partial_r M^\varepsilon_{r,s}(x,y)|dr\right]\\
& \cdot\left[\int_s^t |\partial_\theta M_{\theta,s}(x,y) - \partial_\theta M^\varepsilon_{\theta,s}(x,y)|d\theta\right] dyds.
\end{aligned}
$$

Now notice that Proposition 3.12 and the inequality

$$
\begin{aligned}
& |\partial_r M^\varepsilon_{r,s}(x,y)|\\
& \;\le\; c\left(\frac{r+\varepsilon}{s}\right)^{H-\frac12}(r-s+\varepsilon)^{H-2}\left(\exp\left(-c_1\frac{(x-y)^2}{\varepsilon+(r-s)}\right) + \exp\left(-c_1\frac{(x+y-2)^2}{\varepsilon+(r-s)}\right)\right)
\end{aligned}
$$

imply, for β small enough, that

$$
E\left((f'(X^\varepsilon(r,x)) - f'(X^\varepsilon(s,x)))^2\right)|\partial_r M_{r,s}(x,y) - \partial_r M^\varepsilon_{r,s}(x,y)|
$$

goes to 0 as $\varepsilon \downarrow 0$ and that it is bounded by $cs^{\frac12-H}(r-s)^{3H-\frac52-\beta}$. Thus

$$
B_2 \to 0 \tag{38}
$$

because of the dominated convergence theorem.

Since f' is a bounded function, then

$$
B_4 \le c\int_0^t\int_0^1 \left(M_{t,s}(x,y) - M^\varepsilon_{t,s}(x,y)\right)^2 dyds,
$$

which goes to 0 due to the definition of M^ε and the dominated convergence theorem. Hence, by (37), and (38), we only need to show that $B_1 + B_3 \to 0$ as $\varepsilon \downarrow 0$ to finish the proof. This can been seen using Lemma 3.17 and proceeding as the beginning of this proof. □

Lemma 3.20. *Let X and X^ε be given by (18) and (23), respectively. Then, for $t \in [0,T]$ and $x \in [0,1]$, $M^*_{t,x} f'(X)$ belongs to Dom (δ). Moreover*

$$\delta\left(M^{\varepsilon,*}_{t,x} f'(X^\varepsilon)\right) \to \delta\left(M^*_{t,x} f'(X)\right)$$

as $\varepsilon \downarrow 0$ in $L^2(\Omega)$.

Proof. The result follows from Lemmas 3.14-3.19, and from the fact that δ is a closed operator. □

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