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On polynomial Torus Knots

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Abstract

We show that no torus knot of type \((2, n)\), \(n > 3\) odd, can be obtained from a polynomial embedding \(t \mapsto (f(t), g(t), h(t))\) where \((\deg(f), \deg(g)) \leq (3, n + 1)\). Eventually, we give explicit examples with minimal lexicographic degree.

keywords: Knot theory, polynomial curves, torus knots, parametrized space curve
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1 Introduction

The study of non compact knots began with Vassiliev [V]. He proved that any non-compact knot type can be obtained from a polynomial embedding \(t \mapsto (f(t), g(t), h(t))\), \(t \in \mathbb{R}\). The proof uses Weierstrass approximation theorem on a compact interval, the degrees of the polynomials may be quite large, and the plane projections of the polynomial knots quite complicated.

Independently, Shastri [Sh] gave a detailed proof of this theorem, he also gave simple polynomial parametrizations of the trefoil and of the figure eight knot.

This is what motivated A. Ranjan and Rama Shukla [RS] to find small degree parametrizations of the simplest knots, the torus knots of type \((2, n)\), \(n\) odd, denoted by \(K_n\). They proved that these knots can be attained from polynomials of degrees \((3, 2n - 2, 2n - 1)\). In particular, they obtain a parametrization of the trefoil \(K_3\) analogous to Shastri’s one. They also asked the natural question which is to find the minimal degrees of the polynomials representing a general torus knot of a given type (there is an analogous question in Vassiliev’s paper [V]).

The number of crossings of a plane projection of \(K_n\) is at least \(n\) (Bankwitz theorem, see [Re]). It is not difficult to see, using Bézout theorem, that this plane curve cannot be parametrized by polynomials of degrees smaller than \((3, n + 1)\).

Naturally, Rama Mishra ([Mi]) asked whether it was possible to parametrize the knot \(K_n\) by polynomials of degrees \((3, n + 1, m)\) when \(n \equiv 1\), or \(0 \mod 3\).

In this paper, we shall prove the following result

**Theorem.** If \(n \neq 3\) is odd, the torus knot \(K_n\) cannot be represented by polynomials of degrees \((3, n + 1, m)\).

Our method is based on the fact that all plane projections of \(K_n\) with the minimal number \(n\) of crossings have essentially the same diagram. This is a consequence of the now solved classical
Tait’s conjectures \([\text{Mu}, \text{Ka}, \text{Pr}, \text{MT}]\). This allows us to transform our problem into a problem of real polynomial algebra.

As a conclusion, we give explicit parametrizations of \(K_3, K_5\) and \(K_7\). By our result, they are of minimal degrees. We also give an explicit parametrization of \(K_9\) with a plane projection possessing the minimal number of crossing points. This embedding is of smaller degree than those already known.

2 The principal result

If \(n\) is odd, the torus knot \(K_n\) of type \((2, n)\) is the boundary of a Moebius band twisted \(n\) times (see \([\text{Re}, \text{Ka}, \text{St}]\)). The recently proved Tait’s conjectures allow us to characterize plane projections of \(K_n\) with the minimal number of crossings.

**Lemma 1** Let \(C\) be a plane curve with \(n\) crossings parametrized by \(C(t) = (x(t), y(t))\). If \(C\) is the projection of a knot \(K_n\) then there exist real numbers \(s_1 < \cdots < s_n < t_1 < \cdots < t_n\), such that \(C(s_i) = C(t_i)\).

**Proof.** Let \(C\) be a plane projection of a knot of type \(K_n\) with the minimal number \(n\) of crossings.

Using the Murasugi’s theorem B \((\text{Mu})\) which says that a minimal projection of a prime alternating knot is alternating, we see that \(C\) is alternating.

Then the Tait’s flyping conjecture, proved by Menasco and Thistlethwaite \((\text{MT}, \text{Pr})\), asserts that \(C\) is related to the standard diagram of \(K_n\) by a sequence of flypes. Let us recall that a flype is a transformation most clearly described by the following picture.

![Figure 1: \(K_n, n = 3, 5, 7\).](image1.png)

The standard diagram \(S_0\) of \(K_n\) has the property (cf \([\text{Re}]\)) that there exist real numbers \(s_1 < \cdots < s_n < t_1 < \cdots < t_n\) such that \(S_0(s_i) = S_0(t_i)\). It is alternating.

Let \(S\) be a diagram with real parameters \(s_1 < \cdots < s_n < t_1 < \cdots < t_n\) such that \(S(s_i) = S(t_i)\), and let us perform a flype of a part \(B\) of \(S\).

For any \((a, b, c) \in A \times B \times C\) we have

\[s_a < s_b < s < s_c < t_a < t_b < t < t_c.\]
After the flype on $B$, we have new parameters corresponding to the crossing points satisfying
\[ s'_a < s' < s'_b < s'_c < t'_a < t' < t'_b < t'_c. \]

The transformed diagram $S'$ has the same property: there exist real parameters $s_1 < \cdots < s_n < t_1 < \cdots < t_n$, such that $S'(s_i) = S'(t_i)$.

So then, after any sequence of flypes, the transformed diagram will have the same property. \(\square\)

In this paper we shall consider polynomial knots, that is to say, polynomial embeddings $\mathbb{R} \rightarrow \mathbb{R}^3$, $t \mapsto (x(t), y(t), z(t))$. Polynomial knots are non-compact subsets of $\mathbb{R}^3$. The closure of a polynomial knot in the one point compactification $S^3$ of the space $\mathbb{R}^3$ is an ordinary knot (see [Va, Sh, RS] and figures at the end).

**Lemma 2** Let $C$ be a plane polynomial curve with $n$ crossings parametrized by $C(t) = (x(t), y(t))$. Suppose that $C$ is the projection of $K_n$ and $\deg x(t) \leq \deg y(t)$. Then we have $\deg x(t) \geq 3$. If $\deg x(t) = 3$, then $\deg y(t) \geq n + 1$.

**Proof.** $x(t)$ must be non-monotonic, so $\deg x(t) \geq 2$. Suppose that $x(t)$ is of degree 2. Then $x(t_i) = x(s_i)$ implies that $t_i + s_i$ is constant, and so the parameter values corresponding to the crossing points are ordered as
\[ s_1 < \cdots < s_n < t_n < \cdots < t_1. \]
We have a contradiction according to lemma 1.

Suppose now that $\deg x(t) = 3$. The crossing points of the curve $C$ correspond to parameters $(s, t)$, $s \neq t$, that are common points of the curves of degrees 2 and $\deg y(t) - 1$:
\[
\frac{x(t) - x(s)}{t - s} = 0, \quad \frac{y(t) - y(s)}{t - s} = 0.
\]
By Bézout theorem ([Fi]), the number of such points are at most $2 \times (\deg y(t) - 1)$. $(s, t)$ and $(t, s)$ are distinct points and correspond to the same crossing point. So, the curve $C$ has at most $\deg y(t) - 1$ crossing points, and this implies that $\deg y(t) \geq n + 1$. \(\square\)

### 3 Proof of the main result

Our proof makes use of Chebyshev (monic) polynomials.
3.1 Chebyshev Polynomials

**Definition 1** If $t = 2 \cos \theta$, let $T_n(t) = 2 \cos(n\theta)$ and $V_n(t) = \frac{\sin((n+1)\theta)}{\sin \theta}$.

**Remark 1** $T_n$ and $V_n$ are both monic and have degree $n$. We have

$$V_0 = 1, \quad V_1 = t, \quad V_{n+1} = tV_n - V_{n-1}. \tag{1}$$

We have also

$$T_0 = 2, \quad T_1 = t, \quad T_{n+1} = tT_n - T_{n-1}. \quad \text{For } n \geq 2,$$

let $V_n = t^n + a_nt^{n-2} + b_nt^{n-4} + \cdots$. Using recurrence formula, we get

$$a_{n+1} = a_n - 1, \quad b_{n+1} = b_n - a_n - 1$$

so by induction,

$$V_n = t^n - (n - 1)t^{n-2} + \frac{1}{2}(n - 2)(n - 3)t^{n-4} + \cdots. \tag{2}$$

We shall also need the following lemmas which will be proved in the next paragraph.

**Lemma A.** Let $s \neq t$ be real numbers such that $T_3(s) = T_3(t)$. For any integer $k$, we have

$$\frac{T_k(t) - T_k(s)}{t - s} = 2\frac{\sin k\pi}{3} V_{k-1}(s + t).$$

**Lemma B.** Let $n \geq 3$ be an integer.

Let $s_1 < s_2 < \cdots < s_n$ and $t_1 < \cdots < t_n$ be real numbers such that $T_3(s_i) = T_3(t_i)$. Let $u_i = t_i + s_i$.

We have

$$\sum_{i=1}^{n} u_i^2 \leq n + 4, \quad \sum_{i=1}^{n} u_i^4 \leq n + 22.$$  

3.2 Proof of the theorem

**Proof.** We shall prove this result by reducing it to a contradiction. Suppose the plane curve $C$ parametrized by $x = P(t), \quad y = Q(t)$ where $\deg P = 3, \quad \deg Q = n + 1$ is a plane projection of $K_n$.

By translation on $t$, one can suppose that $P(t) = t^3 - \alpha t + \beta$. If the polynomial $P$ was monotonic, $C$ would have no crossings, which is absurd. Therefore $\alpha > 0$. Dividing $t$ by $\rho = \sqrt{3}/\sqrt{\alpha}$, one has $P(t) = \rho^3(t^3 - 3t) + \mu$. By translating the origin and scaling $x$, one can now suppose that $P(t) = t^3 - 3t = T_3(t)$.

By translating the origin and scaling $y$, we can also suppose that $Q(t)$ is monic and write

$$P(t) = T_3(t), \quad Q(t) = T_{n+1}(t) + a_n T_n(t) + \cdots + a_1 T_1(t).$$

By Bézout theorem, the curve $C$ has at most $(3 - 1)(n + 1 - 1)/2 = n$ double points. As it has at least $n$ crossings, we see that it has exactly $n$ crossings and therefore is a minimal crossing diagram of $K_n$. According to the lemma, there exist real numbers $s_1 < \cdots < s_n, \quad t_1 < \cdots < t_n, \quad s_i < t_i$ such that $P(s_i) = P(t_i), \quad Q(s_i) = Q(t_i)$. Let $u_i = t_i - s_i, \quad 1 \leq i \leq n$. We have

$$\frac{Q(t_i) - Q(s_i)}{t_i - s_i} = \frac{T_{n+1}(t_i) - T_{n+1}(s_i)}{t_i - s_i} + \sum_{k=1}^{n} a_k \frac{T_k(t_i) - T_k(s_i)}{t_i - s_i}. \quad \text{for } n \geq 3.$$
so by lemma A, \( u_1, \ldots, u_n \) are the distinct roots of the polynomial

\[
R(u) = \varepsilon_{n+1} V_n(u) + \sum_{k=1}^{n} a_k \varepsilon_k V_{k-1}(u),
\]

where

\[
\varepsilon_k = \frac{2}{\sqrt{3}} \sin \frac{k\pi}{6}.
\]

**Remark 2** Note that \( \varepsilon_k = V_{k-1}(1) \) is the 6-period sequence \( \varepsilon_0 = 0, \varepsilon_1 = 1, \varepsilon_2 = 1, 0, -1, -1, \ldots \).

We have to consider several cases.

\( \triangleright \) **Case** \( n \equiv 2 \mod 3 \). \( \varepsilon_{n+1} = 0 \) and \( R(u) \) has degree at most \( n - 1 \). This is a contradiction.

\( \triangleright \) **Case** \( n \equiv 1 \mod 3 \). In this case, \( n \equiv 1 \mod 6 \) and \( \varepsilon_{n+1} = \varepsilon_n = 1, \varepsilon_{n-1} = 0 \). Thus \( R(u) \) can be written as

\[
R(u) = V_n(u) + a_n V_{n-1}(u) - a_{n-2} V_{n-3}(u) - \cdots + a_2 V_1(u) + a_1 = u^n + a_n u^{n-1} - (n - 1) u^{n-2} + \cdots.
\]

Using equation 2, we get

\[
\sum_{1 \leq i \leq n} u_i = -a_n, \quad \sum_{1 \leq i \leq j \leq n} u_i u_j = -(n - 1),
\]

and then

\[
\sum_{i=1}^{n} u_i^2 = \left( \sum_{i=1}^{n} u_i \right)^2 - 2 \sum_{1 \leq i < j \leq n} u_i u_j = a_n^2 + 2(n - 1) \geq 2(n - 1).
\]

According to lemma B we also have \( \sum_{i=1}^{n} u_i^2 \leq n + 4 \), we get a contradiction for \( n > 6 \).

\( \triangleright \) **Case** \( n \equiv 0 \mod 3 \). In this last case we have \( n \equiv 3 \mod 6 \), so \( \varepsilon_{n+1} = -1, \varepsilon_n = 0 \) and \( \varepsilon_{n-1} = 1 \), so

\[
-R(u) = V_n(u) - a_{n-1} V_{n-2}(u) - a_{n-2} V_{n-3}(u) + \cdots - a_2 V_1(u) - a_1.
\]

Let \( \sigma_i \) be the coefficients of

\[
-R(u) = u^n + \sum_{k=1}^{n} (-1)^k \sigma_k u^{n-k}.
\]

From the equation 2, we see that

\[
\sigma_1 = 0, \quad \sigma_2 = -(a_{n-1} + n - 1), \quad \sigma_4 = (n - 3) a_{n-1} + \frac{(n - 2)(n - 3)}{2}.
\]

Let \( S_k \) be the Newton sums \( \sum_{i=1}^{n} u_i^k \) of the roots of the polynomial \( R \). Using the classical Newton formulas ([FS]), we obtain

\[
S_1 = \sigma_1 = 0, \quad S_2 = \sigma_1^2 - 2\sigma_2 = -2\sigma_2, \quad S_4 = 2\sigma_2^2 - 4\sigma_4,
\]

and then

\[
S_4 = 2(a_{n-1} + 2)^2 + 6n - 18 \geq 6n - 18.
\]

By the lemma B, we deduce that \( 22 + n \geq 6n - 18 \), i.e. \( n \leq 8 \) so \( n = 3 \). \( \square \)
3.3 Proof of lemmas A and B

We shall use the following lemma

**Lemma 3 (Lissajous ellipse)** Let \( s \neq t \) be complex numbers such that

\[
T_3(t) = T_3(s).
\]

There exists a complex number \( \alpha \) such that

\[
s = 2 \cos(\alpha + \pi/3), \quad t = 2 \cos(\alpha - \pi/3).
\]

Furthermore, \( \alpha \) is real if and only if \( s \) and \( t \) are both real, and then \( t > s \) if and only if \( \sin \alpha > 0 \).

**Proof.** We have

\[
\frac{T_3(t) - T_3(s)}{t - s} = t^2 + s^2 + st - 3.
\]  

Then, if \( T_3(t) = T_3(s), t \neq s, \) we get

\[
\frac{3}{2}(t + s)^2 + \frac{1}{2}(t - s)^2 = 2(t^2 + s^2 + st) = 6.
\]

That means

\[
\left[ \frac{t + s}{2} \right]^2 + \left[ \frac{t - s}{2\sqrt{3}} \right]^2 = 1.
\]

Then there exists a complex number \( \alpha \) such that

\[
\cos \alpha = \frac{t + s}{2}, \quad \sin \alpha = \frac{t - s}{2\sqrt{3}},
\]

that is

\[
t = 2 \cos(\alpha - \pi/3), \quad s = 2 \cos(\alpha + \pi/3).
\]

\( \alpha \) is real if and only if \( \cos \alpha \) and \( \sin \alpha \) are both real that is to say, iff \( s \) and \( t \) are real. In this case: \( t > s \Leftrightarrow \sin \alpha > 0 \). \( \square \)

In order to prove lemma B, we shall use the following lemma which describes the geometrical configuration. Let us denote \( s(\alpha) = 2 \cos(\alpha + \pi/3) \) and \( t(\alpha) = 2 \cos(\alpha - \pi/3) \).

**Lemma 4** Let \( \alpha, \alpha' \in [0, \pi] \) be such that \( s(\alpha) < s(\alpha'), \) and \( t(\alpha) < t(\alpha') \). Then \( \alpha > \alpha' \) and

\[
\frac{2\pi}{3} > \frac{\alpha + \alpha'}{2} > \frac{\pi}{3}.
\]

**Proof.** We have

\[
2 \cos \alpha = s(\alpha) + t(\alpha) < s(\alpha') + t(\alpha') = 2 \cos \alpha'
\]

so \( \alpha > \alpha' \).

\[
t(\alpha') - t(\alpha) = 4 \sin(\frac{\alpha + \alpha'}{2} - \frac{\pi}{3}) \cdot \sin(\frac{\alpha - \alpha'}{2}) > 0,
\]

\[
s(\alpha') - s(\alpha) = 4 \sin(\frac{\alpha + \alpha'}{2} + \frac{\pi}{3}) \cdot \sin(\frac{\alpha - \alpha'}{2}) > 0.
\]

From \( \alpha > \alpha' \) we get \( 0 < \frac{\alpha + \alpha'}{2} - \frac{\pi}{3} \) and \( \frac{\alpha + \alpha'}{2} + \frac{\pi}{3} < \pi \), that is to say

\[
\frac{\pi}{3} < \frac{\alpha + \alpha'}{2} < \frac{2\pi}{3}.
\]
Proof of lemma B.

Let \( s_1 < \cdots < s_n \) and \( t_1 < \cdots < t_n \) be such that \( T_3(s_i) = T_3(t_i) \). Using lemmas 3 and 4 there are \( \alpha_1 > \cdots > \alpha_n \in [0, \pi] \) such that \( s_i = s(\alpha_i) \), \( t_i = t(\alpha_i) \) and we have

\[
\frac{2\pi}{3} > \frac{\alpha_1 + \alpha_2}{2} > \alpha_2 > \cdots > \alpha_{n-1} > \frac{\alpha_{n-1} + \alpha_n}{2} > \frac{\pi}{3}.
\]

At least two of the \( \alpha_i \)'s lie in the intervals \([0, \pi/2]\) or \([\pi/2, \pi]\). We have only two cases to consider: \( \pi > \alpha_1 > \alpha_2 \geq \frac{\pi}{2}, \) or \( \frac{\pi}{2} \geq \alpha_{n-1} > \alpha_n > 0. \)

On the other hand, we get the equality

\[
\cos^2 x + \cos^2 y = 1 - \cos^2(x + y) + 2 \cos x \cos y \cos(x + y).
\] (5)

\> **Case 1.** \( \frac{\pi}{2} \geq \alpha_{n-1} > \alpha_n > 0. \)

We get \( \cos \alpha_n \geq 0, \cos \alpha_{n-1} \geq 0 \) and \( \cos(\alpha_{n-1} + \alpha_n) < -\frac{1}{2} \) so eq. 5 becomes

\[
\cos^2 \alpha_{n-1} + \cos^2 \alpha_n \leq 1 - \cos^2(\alpha_{n-1} + \alpha_n) \leq \frac{3}{4}
\]

and

\[
\sum_{i=1}^{n} \cos^2 \alpha_i = \cos^2 \alpha_1 + \sum_{i=2}^{n-2} \cos^2 \alpha_i + (\cos^2 \alpha_{n-1} + \cos^2 \alpha_n) \leq 1 + (n - 3) \cdot \frac{1}{4} + \frac{3}{4} = \frac{1}{4}(n + 4),
\]

that is

\[
S_2 = \sum_{i=1}^{n} u_i^2 = \sum_{i=1}^{n} (2 \cos \alpha_i)^2 \leq n + 4.
\]

\> **Case 2.** \( \pi > \alpha_1 > \alpha_2 \geq \frac{\pi}{2}. \)

We get \( \cos \alpha_1 \leq 0, \cos \alpha_2 \leq 0 \) and \( \cos(\alpha_1 + \alpha_2) < -\frac{1}{2} \) so eq. 5 becomes

\[
\cos^2 \alpha_1 + \cos^2 \alpha_2 \leq 1 - \cos^2(\alpha_1 + \alpha_2) \leq \frac{3}{4}
\]

and similarly, we get

\[
S_2 = \sum_{i=1}^{n} (2 \cos \alpha_i)^2 \leq n + 4.
\]

Analogously, we get \( \cos^4 x + \cos^4 y \leq (\cos^2 x + \cos^2 y)^2 \), and we deduce:

\[
S_4 = \sum_{i=1}^{n} (2 \cos \alpha_i)^4 \leq n + 22.
\]
Proof of lemma A.  

Let $s < t$ be real numbers such that $T_3(s) = T_3(t)$. According to the ellipse lemma\[3\], there exists a real number $\alpha$ such that

$$t = 2 \cos(\alpha - \pi/3), \quad s = 2 \cos(\alpha + \pi/3).$$

We have $s + t = 2 \cos \alpha$ and $t - s = 4 \sin \frac{\pi}{3} \sin \alpha$, so

$$\frac{T_k(t) - T_k(s)}{t - s} = \frac{2(\cos k(\alpha - \pi/3) - \cos k(\alpha + \pi/3))}{4 \sin \frac{\pi}{3} \sin \alpha} = \frac{\sin k\alpha \cdot \sin \frac{k\pi}{3}}{\sin \alpha \cdot \sin \frac{\pi}{3}} \cdot \frac{\sin k\pi \sin \alpha \cdot \sin \pi}{2 \sqrt{3}} V_{k-1}(2 \cos \alpha).$$

\[\square\]

4 Parametrized models of $K_3$, $K_5$, $K_7$ and $K_9$

We get parametrizations of $K_n$: $C = (x(t), y(t), z(t))$, with $n$ crossings obtained for parameter values satisfying the hypothesis of lemma\[4\]. According to lemma A, we choose $n$ distinct points $-1 \leq u_1 < \cdots < u_n \leq 1$. We look for $Q_1$ and $Q_2$ of minimal degrees, such that

$$R_1(t + s) = \frac{Q_1(t) - Q_1(s)}{t - s}, \quad R_2(t + s) = \frac{Q_2(t) - Q_2(s)}{t - s}$$

satisfy, for $i = 1, \ldots, n$,

$$R_1(u_i) = 0, \quad R_2(u_i) = (-1)^i.$$ 

We then choose $y(t) = Q_1(t)$ and $z(t) = Q_2(t)$. We also add some linear combinations of $T_6$ efficiently. We then obtain a knot whose projection is alternating, when $R_1$ has no more roots in $[-2, 2]$. As we have chosen symmetric $u_i$'s, all of our curves are symmetric with respect to the $y$-axis.

4.1 Parametrization of $K_3$

We can parametrize $K_3$ by $x = T_3(t), y = T_4(t), z = T_5(t)$. It is a Lissajous space curve (compare [Sh]). The plane curve $(T_3(t), T_4(t))$ has 3 crossing points. The plane curve $(T_3(t), T_5(t))$ has 4 crossing points corresponding to parameters $(s_i, t_i)$ with

$$s_1 < s_2 < s_3 < s_4 < t_2 < t_1 < t_4 < t_3$$

so there do not exist real numbers $s_1 < s_2 < s_3$, and $t_1 < t_2 < t_3$ such that $x(s_i) = x(t_i), \quad z(s_i) = z(t_i)$.

This example shows that our method cannot be generalized when the projections of $K_n$ have at least $n + 1$ crossing points.
4.2 Parametrization of $K_5$

Let us consider the curve of degree $(3, 7, 8)$:

\[
\begin{align*}
    x &= T_3(t), \\
    y &= T_8(t) - 2T_6(t) + 2.189T_4(t) - 2.170T_2(t), \\
    z &= T_7(t) - 0.56T_5(t) - 0.01348T_1(t).
\end{align*}
\]

The curve $(x(t), y(t))$ has exactly 5 double points when the projection $(x(t), z(t))$ has exactly 6. Note here that $\deg z(t) < \deg y(t)$. In conclusion we have found a curve of degree $(3, 7, 8)$. Using our theorem, we see that this curve has minimal degree. A. Ranjan and R. Mishra showed the existence of such an example ([RS, Mi]).

4.3 Parametrization of $K_7$

We choose

\[
\begin{align*}
    x &= T_3(t), \\
    y &= T_{10}(t) - 2.360T_8(t) + 4.108T_6(t) - 6.037T_4(t) + 7.397T_2(t), \\
    z &= T_{11}(t) + 3.580T_7(t) - 3.739T_5(t) - T_1(t).
\end{align*}
\]
The values of the parameters corresponding to the double points are obtained as intersection points between the ellipse

\[ t^2 + s^2 + st - 3 = 0 \]

and the curve of degree 9:

\[ \frac{(y(t) - y(s))}{(t - s)} = 0 \]

The curve \((x(t), y(t))\) has exactly 7 double points corresponding to \(\cos(\alpha) = \{\pm 1/2, \pm 3/10, \pm 2/10, 0\}\).

In conclusion we have found a curve of degree \((3, 10, 11)\). Using our theorem, this curve has minimal degree.

4.4 Parametrization of \(K_9\)

We choose polynomials of degree \((3, 13, 14)\).

\[
x = T_3(t),
\]

\[
y = T_{14}(t) - 4.516 T_{12}(t) + 12.16 T_{10}(t) - 24.46 T_8(t) + 39.92 T_6(t) - 55.30 T_4(t) + 66.60 T_2(t),
\]

\[
z = T_{13}(t) - 2.389 T_{11}(t) - 5.161 T_7(t) + 5.161 T_5(t) + 1.397 T_1(t).
\]

The curve \((x(t), y(t))\) has exactly 9 double points corresponding to \(\cos(\alpha) = \{\pm 1/2, \pm 3/10, \pm 2/10, \pm 1/10, 0\}\). One can prove that it is minimal under the assumption that the projection \((x(t), y(t))\) has exactly 9 double points.

Conclusion

We have found minimal degree polynomial curves for torus knots \(K_n, n = 3, 5, 7\). For degree 9, one can prove that it is minimal under the assumption that the projection \((x(t), y(t))\) has exactly 9 double points. We have similar constructions for higher degrees.

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Figure 7: Knot $K_9$

References


