Carleman estimates and controllability results for the one-dimensional heat equation with *BV* coefficients

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Abstract

We derive global Carleman estimates for one-dimensional linear parabolic equations $\partial_t \pm \partial_x(c\partial_x)$ with a coefficient of bounded variations. These estimates are obtained by approximating c by piecewise constant coefficients, c_ε, and passing to the limit in the Carleman estimates associated to the operators defined with c_ε. Such estimates yields observability inequalities for the considered linear parabolic equation, which, in turn, yield controllability results for classes of *semi-linear* equations.

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Introduction and settings

We consider the elliptic operator A formally defined by $-\partial_x(c\partial_x)$ on $L^2(\Omega)$ in the one-dimensional bounded domain $\Omega = (0,1) \subset \mathbb{R}$. The diffusion coefficient c is assumed to be of bounded variations (BV). The domain of A is given by

$$D(A) = \{u \in H_0^1(\Omega);\ c\partial_x u \in H^1(\Omega)\},$$

i.e., we consider Dirichlet boundary conditions.

We let $T > 0$. We shall use the following notations $Q = (0,T) \times \Omega$, $\Gamma = \{0,1\}$, and $\Sigma = (0,T) \times \Gamma$.

We shall first study the following linear parabolic problems,

$$\text{(1)} \qquad \begin{cases} \partial_t y \pm Ay = f & \text{in } Q, \\ y(0,x) = y_0(x) \text{ (resp. } y(T,x) = y_T(x)) & \text{in } \Omega, \end{cases}$$

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for $y_0 \in L^2(\Omega)$ and $f \in L^2(Q)$.

Here, we show that we can achieve *global* Carleman estimates for the operators $\partial_t \pm A$, in Q, with an interior observation region $(0,T)\times\omega$, where $\omega \Subset \Omega$ with a non-empty interior, and such that c is of class $\mathscr{C}^1$ in some open subset of ω.

With a Carleman estimate for $\partial_t + \partial_x(c\partial_x)$ at hand, we treat the problem of the null controllability for semi-linear parabolic systems of the form

$$\begin{cases} \partial_t y - \partial_x(c\partial_x y) + \mathscr{G}(y, \partial_x y) = 1_\omega v & \text{in } Q, \\ y(t,x) = 0 & \text{on } \Sigma, \\ y(0,x) = y_0(x) & \text{in } \Omega, \end{cases} \tag{2}$$

where $\mathscr{G} : \mathbb{R}^2 \to \mathbb{R}$ is locally Lipschitz and $\mathscr{G}(0,0) = 0$. In this case, we have

$$\mathscr{G}(y_1, y_2) = y_1\, g(y_1, y_2) + y_2\, G(y_1, y_2), \quad y_1, y_2 \in \mathbb{R}.$$

with g and G in $L^\infty_{\text{loc}}(\mathbb{R}^2)$. We shall assume

Assumption 1. The functions g and G satisfy

$$\lim_{|(y_1,y_2)|\to\infty} \frac{|g(y_1,y_2)|}{\ln^{3/2}(1+|y_1|+|y_2|)} = 0, \qquad \lim_{|(y_1,y_2)|\to\infty} \frac{|G(y_1,y_2)|}{\ln^{1/2}(1+|y_1|+|y_2|)} = 0. \tag{3}$$

Under such an assumption we shall prove the complete null controllability for system (2), i.e., that for all positive time T and for all $y_0 \in L^2(\Omega)$, there exists a control $v \in L^\infty(Q)$ such that the solution satisfies $y(T) = 0$. We also prove the controllability of system (2) in the case where the control acts through one of the boundary conditions, at 0 or 1. Then, we need not require the coefficient c to be of class $\mathscr{C}^1$ in some inner region of Ω. More generally, we can address the question of the controllability to the trajectories.

A null controllability result for a *linear* parabolic equation with *BV* coefficients was proven in [12]. The proof relies on Russell's method [19]. However, the question of the existence of a Carleman-type observability estimate was open. The present article, providing a Carleman estimate allows to treat the case of semilinear equations following the (fix-point) method of [2, 11] (generalized in [7]). For a review of the role played by Carleman estimates in establishing controllability results for parabolic equations we refer to [10].

Carleman estimates for parabolic equations in several dimensions with smooth coefficients were proven in [13]. The proof is based on the construction of suitable weight functions β whose gradient is non-zero in the complement of the observation region. In particular the function β is chosen to be smooth. In [8], the authors treat the case of piecewise regular coefficients and introduce non-smooth weight functions assuming that they satisfy the *same transmission condition as the solution*. To obtain observability, they have to add some assumption on the monotonicity of the coefficients. In the one-dimensional case, this monotonicity assumption was relaxed in [4, 3], by introducing additional requirements on the non-smooth weight function β. In several dimensions, the existence of a Carleman estimate when the monotonicity condition is not satisfied is an open question.

The Carleman estimates derived here for the operator $\partial_t \pm \partial_x(c\partial_x)$ are obtained through a limiting process from the Carleman estimates associated for $\partial_t \pm \partial_x(c_\varepsilon\partial_x)$, for c_ε piecewise constant converging to c. The main issue in this limiting process is to keep both the weight functions and constants in the Carleman estimate under control. Section 2 of the present article is devoted to this question.

The approximation of the BV coefficient c by some piecewise coefficient c_ε is closely related to numerical methods. The techniques developed here could also be applied in the numerical analysis of discrete type estimates of the form of Carleman estimates.

The outline of the article is as follows. In Section 1, we recall the Carleman estimate obtained in [4, 3] for piecewise continuous coefficients (Theorem 1.2) and especially the form of the weight functions in the estimate (Lemma 1.1). (The results of this section are not essential as we revisit the arguments used to prove them in the following section.) In Section 2, we construct limit weight functions by approaching the BV coefficient c by piecewise constant coefficients c_ε (Lemma 2.3). In Theorem 2.8, we prove a Carleman estimate associated to $\partial_t \pm \partial_x(c\partial_x)$ by proving that the constants in the Carleman estimate of $\partial_t \pm \partial_x(c_\varepsilon\partial_x)$ can be taken uniform with respect to the parameter ε (Proposition 2.4) and passing to the limit in each term of the estimate. In Section 3, we derive a Carleman estimate for the linear system (1) with the r.h.s., f, in $L^2(0,T,H^{-1}(\Omega))$. This estimate is needed for the analysis of the controllability of the semilinear system (2), which is carried out in Section 4.

In this article, when the constant C is used, its value may change from one line to the other. If we want to keep track of the value of a constant we shall use another letter. We denote the jump of a function ρ, at some point $x \in (0,1)$, by $[\rho]_x := \rho(x^+) - \rho(x^-)$, with the conventions $[\rho]_1 = -\rho(1^-)$ and $[\rho_0] = \rho(0^+)$.

1 Carleman estimate in the case of a piecewise $\mathscr{C}^1$ coefficient

In the case of a piecewise-$\mathscr{C}^1$ diffusion coefficient c, we denote its singularities by $a_1, \dots, a_{n-1}$, with $0 = a_0 < a_1 < a_2 < \cdots < a_{n-1} < a_n = 1$. We first introduce a particular type of weight function to be used in the Carleman estimate. Let $j \in \{0, \dots, n-1\}$ be fixed in the sequel and $\omega_0 \Subset \mathcal{O} \Subset (a_j, a_{j+1})$ be non-empty open sets. We have the following lemma [4, 3].

Lemma 1.1. *There exists a function $\widetilde{\beta} \in \mathscr{C}(\overline{\Omega})$ satisfying*

$$\begin{aligned}
&\widetilde{\beta}_{|[a_i,a_{i+1}]} \in \mathscr{C}^2([a_i, a_{i+1}]), \quad i = 0, \dots, n-1,\\
&\widetilde{\beta} > 0 \text{ in } \Omega, \quad \widetilde{\beta} = 0 \text{ on } \Gamma, \quad (\widetilde{\beta}_{|[a_j,a_{j+1}]})' \neq 0 \text{ in } [a_j, a_{j+1}] \setminus \omega_0,\\
&(\widetilde{\beta}_{|[a_i,a_{i+1}]})' \neq 0, \ i \in \{1, \dots, n\}, \ i \neq j,\\
&\widetilde{\beta}' > 0 \text{ on the l.h.s. of } \omega_0, \quad \widetilde{\beta}' < 0 \text{ on the r.h.s. of } \omega_0,
\end{aligned}$$

and the function $\widetilde{\beta}$ satisfies the following trace properties, for some $\alpha > 0$,

$$(A_i u, u) \geq \alpha |u|^2, \quad u \in \mathbb{R}^2, \tag{1.1}$$

with the matrices A_i, defined by

$$A_i = \begin{pmatrix} [\widetilde{\beta}']_{a_i} & \widetilde{\beta}'(a_i^+)[c\widetilde{\beta}']_{a_i} \\ \widetilde{\beta}'(a_i^+)[c\widetilde{\beta}']_{a_i} & \widetilde{\beta}'(a_i^+)[c\widetilde{\beta}']_{a_i}^2 + [c^2(\widetilde{\beta}')^3]_{a_i} \end{pmatrix}, \quad i = 1, \dots, n-1.$$

Figure 1 illustrates a typical shape for the function $\widetilde{\beta}$.

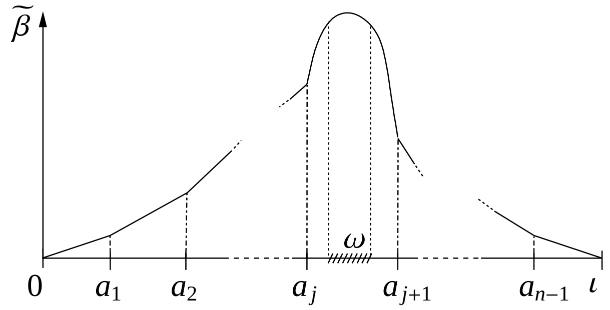

Figure 1: Sketch of a typical shape for the function $\widetilde{\beta}$ for an 'observation' in (a_j, a_{j+1}).

Choosing a function $\widetilde{\beta}$, as in the previous lemma, we introduce $\beta = \widetilde{\beta} + K$ with $K = m\|\widetilde{\beta}\|_\infty$ and $m > 1$. For $\lambda > 0$ and $t \in (0, T)$, we define the following weight functions

$$\varphi(x,t) = \frac{e^{\lambda\beta(x)}}{t(T-t)}, \qquad \eta(x,t) = \frac{e^{\lambda\overline{\beta}} - e^{\lambda\beta(x)}}{t(T-t)}, \tag{1.2}$$

with $\overline{\beta} = 2m\|\widetilde{\beta}\|_\infty$ (see [8],[10]). We next set

$$\aleph = \Big\{ q \in \mathscr{C}(Q,\mathbb{R});\, q_{|[0,T]\times[a_i,a_{i+1}]} \in \mathscr{C}^2([0,T]\times[a_i,a_{i+1}]),\ i = 0,\dots,n-1,$$
$$q_{|\Sigma} = 0,\ \text{ and } q \text{ satisfies } (\mathrm{TC}_n), \text{ for all } t \in (0,T) \Big\},$$

with

$$q(a_i^-) = q(a_i^+), \quad c(a_i^-)\partial_x q(a_i^-) = c(a_i^+)\partial_x q(a_i^+),\ i = 1,\dots,n-1. \tag{TC_n}$$

The following global Carleman estimate is proven in [4, 3].

Theorem 1.2. *Let $\omega_0 \Subset \mathcal{O} \Subset (a_j, a_{j+1})$ be non-empty open sets. There exists $\lambda_1 = \lambda_1(\Omega, \mathcal{O}) > 0$, $s_1 = s_1(\lambda_1, T) > 0$ and a positive constant $C = C(\Omega, \mathcal{O})$ so that the following estimate holds*

$$\begin{aligned} s^{-1} \iint_Q e^{-2s\eta}\varphi^{-1} \left(|\partial_t q|^2 + |\partial_x(c\partial_x q)|^2\right) dxdt \\ + s\lambda^2 \iint_Q e^{-2s\eta}\varphi\, |\partial_x q|^2\, dxdt + s^3\lambda^4 \iint_Q e^{-2s\eta}\varphi^3\, |q|^2\, dxdt \\ \le C\left[s^3\lambda^4 \iint_{(0,T)\times\mathcal{O}} e^{-2s\eta}\varphi^3\, |q|^2\, dxdt + \iint_Q e^{-2s\eta}\, |\partial_t q \pm \partial_x(c\partial_x q)|^2\, dxdt \right], \end{aligned} \tag{1.3}$$

for $s \ge s_1$, $\lambda \ge \lambda_1$ and for all $q \in \aleph$.

Remark 1.3. By a density argument, we see that the Carleman estimate (1.3) remains valid for q (weak) solution to

$$\begin{cases} \partial_t q \pm \partial_x(c\partial_x q) = f & \text{in } Q, \\ q = 0 & \text{on } \Sigma, \\ q(T,x) = q_T(x) \text{ (resp. } q(0,x) = q_0(x)) & \text{in } \Omega, \end{cases}$$

with $f \in L^2(Q)$ and q_T (resp. q_0) in $L^2(\Omega)$.

2 Carleman estimates in the case of a *BV* coefficient

To obtain a Carleman estimate in the case of more general non-smooth coefficients, such as *BV* coefficients, we shall first revisit the conditions imposed on the weight function $\widetilde{\beta}$ in Lemma 1.1. Since the conditions imposed on $\widetilde{\beta}$ will only make use of its derivative, we shall sometimes employ β in place of $\widetilde{\beta}$ here, as they only differ by a constant (see the definition of β in (1.2) above). We shall use a limiting process to obtain a Carleman estimate in the case of a *BV* coefficient making use of estimate (1.3) in the case of a piecewise-$\mathscr{C}^1$ coefficients.

We first consider a piecewise-$\mathscr{C}^1$ diffusion coefficient, c, with a discontinuity at $a \in (0, 1)$. Defining a function β, as in the Lemma 1.1, we then define the matrix A as

$$A = \begin{pmatrix} [\beta']_a & \beta'(a^+)[c\beta']_a \\ \beta'(a^+)[c\beta']_a & \beta'(a^+)[c\beta']_a^2 + [c^2(\beta')^3]_a \end{pmatrix}.$$

This symmetric matrix is positive definite if and only if $[\beta']_a > 0$ and $\det(A) > 0$. We now set

$$Y = \frac{c(a^+)}{c(a^-)}, \quad X = \frac{\beta'(a^-)}{\beta'(a^+)},$$

and write

$$A = \begin{pmatrix} \beta'(a^+)(1 - X) & c(a^-)(\beta'(a^+))^2(Y - X) \\ c(a^-)(\beta'(a^+))^2(Y - X) & c^2(a^-)(\beta'(a^+))^3\left((Y - X)^2 + (Y^2 - X^3)\right) \end{pmatrix},$$

which yields $\det(A) = c^2(a^-)(\beta'(a^+))^4\, P_Y(X)$ with

$$P_Y(X) = (1 - X)(Y^2 - X^3) - X(Y - X)^2.$$

In the case $Y = 1$, there is actually no discontinuity for the coefficient c at the considered point. An inspection of the proof of the Carleman estimate (1.3) in [3] shows that with $X = 1$, i.e. $\partial_x\beta$ continuous at a, the integrals over $(0, T)$ at the point a vanish in the course of the proof of the estimate.

We now place ourselves in the case $Y \neq 1$ and $\beta' < 0$, i.e., on the r.h.s. of the open set ω_0 (see Lemma 1.1). There, $[\beta']_a > 0$ is equivalent to $X > 1$. The polynomial function P_Y can be made positive for X sufficiently large, since its leading coefficient is positive. Here, we shall in fact give *explicit* sufficient conditions on X for this to be satisfied.

Observe that $P_Y(Y) = Y^2(1 - Y)^2$. In the case $Y > 1$, we can thus choose $X = Y$ and the desired conditions on the function β are satisfied. This choice corresponds to that made in [8] since in this case we have $c(a^-)\partial_x\beta(a^-) = c(a^+)\partial_x\beta(a^+)$.

In the case $Y < 1$, the previous choice, $X = Y$, is not possible as it would yield a negative definite quadratic form A. Observe, however, that $P_Y(2 - Y) = Y^2(1 - Y)^2$. In the case $0 < Y < 1$, we can thus choose $X = 2 - Y$. Observe also that $P_Y(1/Y) > 0$, which makes $X = 1/Y$ an alternative choice.

Remark 2.1. Note that the proposed choices are not optimal but yield easy-to-handle conditions to compute an adapted weight function β. We can actually show that there exists $g(Y) \geq 1$, defined for $Y > 0$, with $g(Y) > 1$ if $Y \neq 1$ such that $P_Y(X) > 0$ if and only if $X > g(Y)$. Figure 2 compares the proposed solution to the optimal one.

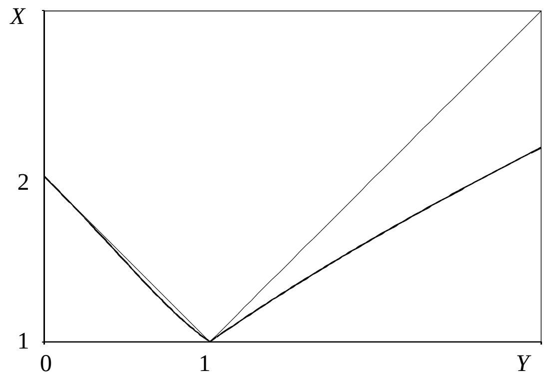

Figure 2: Graph of the optimal solution $g(Y)$ (thick) and graph of the proposed solution (thin) in the case $\beta' < 0$.

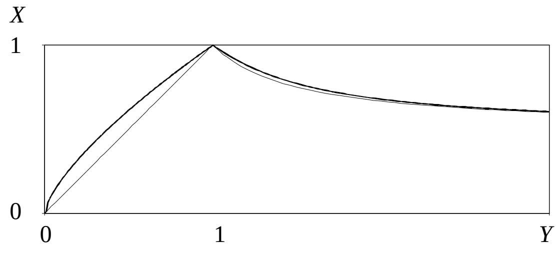

Figure 3: Graph of the optimal solution $h(Y)$ (thick) and graph of the proposed solution (thin) in the case $\beta' > 0$.

In the case $\beta' > 0$, i.e., on the l.h.s. of the open set ω_0, we now need $0 < X < 1$ to satisfy $[\beta']_a > 0$. We can make the following choices: $X = Y$ if $Y < 1$ and $X = \frac{Y}{2Y-1}$ if $Y > 1$. Figure 3 compares the proposed solution to the optimal one (here $P_Y(X) > 0$ if and only if $0 < X < h(Y)$ for some function h satisfying $h(Y) < 1$ if $Y \neq 1$). Note that $X = \frac{Y}{2Y-1}$, actually yields $\frac{1}{X} = 2 - \frac{1}{Y}$, which makes the connexion with the proposed choice in the case $\beta' < 0$ above. In fact, we have $P_Y(\frac{Y}{2Y-1}) = \frac{Y^2(Y-1)^2}{(2Y-1)^4}$.

We now consider a diffusion coefficient c, of bounded variations, yet $\mathscr{C}^1$ on $\overline{O}$, with O an open subset of Ω, $O \Subset \Omega$. We also assume $0 < c_{min} \leq c \leq c_{max}$. Without any loss of generality we may assume $O = (x_0, x_1)$, with $0 < x_0 < x_1 < 1$. We also let $\omega_0 \Subset O$. We denote the total variations of c on $[0, x_0]$ and $[x_1, 1]$ by $\vartheta_0 = V_0^{x_0}(c), \quad \text{and } \vartheta_1 = V_{x_1}^1(c)$.

Let $\varepsilon > 0$. There exists a function c_ε, *piecewise-constant* on $(0, x_0) \cup (x_1, 1)$, and smooth on O such that (see e.g. [5])

$$\|c - c_\varepsilon\|_{L^\infty(\Omega)} \leq \varepsilon, \quad V_0^{x_0}(c_\varepsilon) \leq \vartheta_0, \quad \text{and } V_{x_1}^1(c_\varepsilon) \leq \vartheta_1, \quad \|c_\varepsilon - c\|_{\mathscr{C}^1(\overline{O})} \leq \varepsilon.$$

We denote by $a_1, \ldots, a_n$ the points of discontinuity of c_ε in the interval $[x_1, 1]$. We then have

$$\sum_{i=1}^{n} |c_\varepsilon(a_i^+) - c_\varepsilon(a_i^-)| \leq \vartheta_1.$$

Let $Y_i = c_\varepsilon(a_i^+)/c_\varepsilon(a_i^-)$ and X_i, $i = 1, \ldots, n$, be defined according to what is described above, i.e.,

$$X_i = Y_i, \text{ if } Y_i > 1, \quad \text{and } X_i = 2 - Y_i, \text{ if } Y_i < 1,$$

as we are on the r.h.s. of ω_0. We define the *piecewise-constant* function $\gamma_{1,\varepsilon}$ as

$$\gamma_{1,\varepsilon}(x) := \gamma_{1,\varepsilon}(1) \prod_{x < a_j} X_j, \quad x \notin \{a_1, \ldots, a_n\}, \tag{2.1}$$

for some fixed $\gamma_{1,\varepsilon}(1) < 0$. Observe that $X_i = \frac{\gamma_{1,\varepsilon}(a_i^-)}{\gamma_{1,\varepsilon}(a_i^+)}$, $i = 1, \ldots, n$.

In a similar fashion, if $a_{n+1}, \ldots, a_{n+k}$ are the discontinuities of c_ε on $[0, x_0]$, we build the *piecewise-constant* function $\gamma_{0,\varepsilon}$ on $[0, x_0]$ as

$$\gamma_{0,\varepsilon}(x) := \gamma_{0,\varepsilon}(0) \prod_{x > a_j} \frac{1}{X_j}, \quad x \notin \{a_{n+1}, \ldots, a_{n+k}\}, \tag{2.2}$$

for some fixed $\gamma_{0,\varepsilon}(0) > 0$ and with $X_{n+1}, \ldots, X_{n+k}$ defined as described above, i.e.,

$$X_i = Y_i, \text{ if } Y_i < 1, \quad \text{and } X_i = \frac{Y_i}{2Y_i - 1}, \text{ if } Y_i > 1, \ i = n+1, \ldots, n+k.$$

We then have $X_i = \frac{\gamma_{0,\varepsilon}(a_i^-)}{\gamma_{0,\varepsilon}(a_i^+)}$, $i = n+1, \ldots, n+k$.

We define the functions $\widetilde{\beta}_{1,\varepsilon}(x) := \int_1^x \gamma_{1,\varepsilon}(y)\, dy$ and $\widetilde{\beta}_{0,\varepsilon}(x) := \int_0^x \gamma_{0,\varepsilon}(y)\, dy$, and we define a continuous function $\widetilde{\beta}_\varepsilon$ by $\beta_\varepsilon(x) = \beta_{0,\varepsilon}(x)$ in $[0, x_0]$ and $\beta_\varepsilon(x) = \beta_{1,\varepsilon}(x)$ in $[x_1, 1]$, and $\mathscr{C}^2$ on $\overline{O}$, such that $\widetilde{\beta}'_\varepsilon$ does not vanish outside ω_0. The precise definition of $\widetilde{\beta}_\varepsilon$ on O will be given below.

We observe that $\widetilde{\beta}_\varepsilon$ satisfies the conditions listed in Lemma 1.1. Hence, we obtain Carleman estimate (1.3) for the operator $\partial_t \pm \partial_x(c_\varepsilon \partial_x)$ with the associated weight functions η_ε and φ_ε: we introduce $\beta_\varepsilon = \widetilde{\beta}_\varepsilon + K_\varepsilon$ with $K_\varepsilon \geq m\|\widetilde{\beta}_\varepsilon\|_\infty$ and $m > 1$. For $\lambda > 0$ and $t \in (0, T)$, we define

$$\varphi_\varepsilon(x,t) = \frac{e^{\lambda\beta_\varepsilon(x)}}{t(T-t)}, \qquad \eta_\varepsilon(x,t) = \frac{e^{\lambda\bar{\beta}_\varepsilon} - e^{\lambda\beta_\varepsilon(x)}}{t(T-t)}, \quad \text{with } \bar{\beta}_\varepsilon = 2K_\varepsilon. \tag{2.3}$$

We now wish to pass to the limit in the Carleman estimate as c_ε converges to c in $L^\infty(\Omega)$. The remaining of this section is devoted to this question. We first need to control the behavior of β_ε, or rather its derivative, as ε goes to zero.

Lemma 2.2. *There exists $K > 0$ and $\varepsilon_0 > 0$ that depend solely on the diffusion coefficient $c \in BV(0,1)$ such that, for all $0 < \varepsilon \leq \varepsilon_0$, $V_0^{x_0}(\gamma_{0,\varepsilon}) \leq K\, \gamma_{0,\varepsilon}(0)$ and $V_{x_1}^1(\gamma_{1,\varepsilon}) \leq K\, |\gamma_{1,\varepsilon}(1)|$.*

Proof. We have $V_{x_1}^1(\gamma_{1,\varepsilon}) = |\gamma_{1,\varepsilon}(x_1) - \gamma_{1,\varepsilon}(1)|$ since $\gamma_{1,\varepsilon}$ is a non-decreasing function. Thus $V_{x_1}^1(\gamma_{1,\varepsilon}) = (X_1 \dots X_n - 1)|\gamma_{1,\varepsilon}(1)|$. We have

$$\sum_{i \in I_1} |c_\varepsilon(a_i^+) - c_\varepsilon(a_i^-)| + \sum_{i \in I_2} |c_\varepsilon(a_i^+) - c_\varepsilon(a_i^-)| \leq \vartheta_1,$$

with $i \in I_1$ if $c_\varepsilon(a_i^+) > c_\varepsilon(a_i^-)$ and $i \in I_2$ if $c_\varepsilon(a_i^+) < c_\varepsilon(a_i^-)$. Dividing by $c_\varepsilon(a_i^-)$ or $c_\varepsilon(a_i^+)$ accordingly, we obtain

$$\sum_{i \in I_1}(Y_i - 1) + \sum_{i \in I_2}(\frac{1}{Y_i} - 1) \leq \vartheta_1/(c_{min} - \varepsilon_0).$$

(Recall that $c \geq c_{min} > 0$; here we take $0 < \varepsilon \leq \varepsilon_0 < c_{min}$.) Note that if $0 < Y < 1$ then $X = 2 - Y < 1/Y$. We thus obtain $\sum_{i=1}^n (X_i - 1) \leq \vartheta_1/(c_{min} - \varepsilon_0)$. Finally, since $X_1, \dots, X_n > 1$, we write

$$X_1 \dots X_n \leq e^{X_1 - 1} \dots e^{X_n - 1} = e^{\sum_{i=1}^n (X_i - 1)} \leq e^{\vartheta_1/(c_{min} - \varepsilon_0)},$$

which concludes the proof for $\gamma_{1,\varepsilon}$.

For $\gamma_{0,\varepsilon}$ we have $V_0^{x_0}(\gamma_{0,\varepsilon}) = (\frac{1}{X_{n+1} \dots X_{n+k}} - 1)\gamma_{0,\varepsilon}(0)$. This time, if $Y > 1$ then

$$\frac{1}{X} - 1 = \frac{2Y - 1}{Y} - 1 = \frac{Y - 1}{Y} < Y - 1.$$

Thus, we obtain $\sum_{i=n+1}^{n+k}(\frac{1}{X_i} - 1) \leq \vartheta_0/(c_{min} - \varepsilon_0)$, and accordingly

$$\frac{1}{X_{n+1} \dots X_{n+k}} \leq e^{\frac{1}{X_{n+1}} - 1} \dots e^{\frac{1}{X_{n+k}} - 1} = e^{\sum_{i=n+1}^{n+k}(\frac{1}{X_i} - 1)} \leq e^{\vartheta_0/(c_{min} - \varepsilon_0)}.$$

∎

By Helly's theorem [15, 5], up to a subsequence, the functions $\gamma_{0,\varepsilon}$ (resp. $\gamma_{1,\varepsilon}$) converge *everywhere* to a function γ_0 (resp. γ_1) as ε goes to 0. (We take for instance $\varepsilon = \frac{1}{n+1}$ but shall not write it explicitly for the sake of concision.) Moreover, these two functions satisfy

$$V_0^{x_0}(\gamma_0) \leq K\, \gamma_{0,\varepsilon}(0) = K\gamma_0(0), \quad \text{and } V_{x_1}^1(\gamma_1) \leq K\, |\gamma_{1,\varepsilon}(1)| = K\, |\gamma_1(1)|.$$

The functions $\gamma_{0,\varepsilon}$ (resp. $\gamma_{1,\varepsilon}$) are bounded in $L^\infty(0, x_0)$ (resp. $L^\infty(x_1, 1)$) uniformly w.r.t. ε. Thus, by dominated convergence, the associated functions $\widetilde{\beta}_{0,\varepsilon}$ and $\widetilde{\beta}_{1,\varepsilon}$ converge everywhere to the continuous functions $\widetilde{\beta}_0(x) := \int_0^x \gamma_0(y)dy$, and $\widetilde{\beta}_1(x) := \int_1^x \gamma_1(y)dy$.

We define $\widetilde{\beta}$ on Ω by $\widetilde{\beta}(x) = \widetilde{\beta}_0(x)$ in $[0, x_0]$, $\widetilde{\beta}(x) = \widetilde{\beta}_1(x)$ in $[x_1, 1]$, and we design $\widetilde{\beta}_\varepsilon$ and $\widetilde{\beta}$ to be $\mathscr{C}^2$ on $\overline{O}$ and such that

$$|\widetilde{\beta}'_\varepsilon(x)| \geq \min(\widetilde{\beta}'(0), |\widetilde{\beta}'(1)|), \text{ and } |\widetilde{\beta}'(x)| \geq \min(\widetilde{\beta}'(0), |\widetilde{\beta}'(1)|), \quad \text{in } \Omega \setminus \omega_0, \tag{2.4}$$

and such that $\widetilde{\beta}_{\varepsilon|_O}$ converges to $\widetilde{\beta}_{|_O}$ in $\mathscr{C}^2(\overline{O})$. We have thus obtained the following lemma.

Lemma 2.3. *Let $\omega_0 \Subset O \Subset \Omega$, be open sets, $O = (x_0, x_1)$. Let c in $BV(\Omega)$ be of class $\mathscr{C}^1$ in $\overline{O}$ with $0 < c_{min} \leq c \leq c_{max}$. Let c_ε be piecewise-constant on $\Omega \setminus O$, and smooth on O such that*

$$\|c - c_\varepsilon\|_{L^\infty(\Omega)} \leq \varepsilon, \quad V_0^{x_0}(c_\varepsilon) \leq \vartheta_0, \quad \textit{and } V_{x_1}^1(c_\varepsilon) \leq \vartheta_1, \quad \|c_\varepsilon - c\|_{\mathscr{C}^1(\overline{O})} \leq \varepsilon.$$

There exist weight functions $\widetilde{\beta}_\varepsilon$ that satisfy the properties listed in Lemma 1.1 for the associated coefficient c_ε, and are uniformly bounded in $L^\infty(\Omega)$, with derivatives uniformly bounded in $L^\infty(\Omega)$ and piecewise-constant on $\Omega \setminus O$. Furthermore, $\widetilde{\beta}_\varepsilon$ converges everywhere in $\overline{\Omega}$ to a function $\widetilde{\beta}$ which is in $\mathscr{C}(\overline{\Omega})$ and $\widetilde{\beta}_{\varepsilon|_O}$ can be chosen uniformly bounded in $\mathscr{C}^2(\overline{O})$ and the functions $\widetilde{\beta}_\varepsilon$ and $\widetilde{\beta}$ satisfy (2.4).

We shall now revisit the proof of Carleman estimate (1.3) and check that the constants, C, s_1 and λ_1, can be chosen uniformly w.r.t. ε with the properties of $\widetilde{\beta}_\varepsilon$ listed in Lemma 2.3. Note that in the definitions of φ_ε and η_ε, in (2.3), the constants K_ε and $\overline{\beta}_\varepsilon$ can actually be chosen uniformly w.r.t. ε by Lemma 2.3.

Proposition 2.4. *Let $c \in BV(0, 1)$ be $\mathscr{C}^1$ in $\overline{O}$. Let c_ε and β_ε be defined as above. The constant C on the r.h.s. of the Carleman estimate (1.3) for the operators $\partial_t \pm \partial_x(c_\varepsilon \partial_x)$ and the constants s_1 and λ_1 can be chosen uniformly w.r.t. ε for $0 < \varepsilon \leq \varepsilon_0$, with ε_0 sufficiently small.*

Proof. We treat the case of the operator $\partial_t + \partial_x(c_\varepsilon \partial_x)$. The proof is similar for $\partial_t - \partial_x(c_\varepsilon \partial_x)$. Call $a_1, \dots, a_{n-1}$ the discontinuities of c_ε, with $a_0 = 0 < a_1 < \dots, a_{n-1} < a_n = 1$. We choose $0 < \varepsilon_0 < c_{min}$ and thus $0 < c_{min} - \varepsilon_0 \leq c_\varepsilon \leq c_{max} + \varepsilon_0$.

In the derivation of Carleman estimate (1.3) (see [3]) we consider $s > 0$, $\lambda > 1$ and $q \in \aleph_\varepsilon$ with

$$\aleph_\varepsilon = \Big\{ q \in \mathscr{C}(Q, \mathbb{R}); q_{|[0,T]\times[a_i,a_{i+1}]} \in \mathscr{C}^2([0, T] \times [a_i, a_{i+1}]),\ i = 0, \dots, n-1,$$
$$q_{|\Sigma} = 0, \text{ and } q \text{ satisfies } (\mathrm{TC}_{\varepsilon,n}), \text{ for all } t \in (0, T) \Big\},$$

with

$$q(a_i^-) = q(a_i^+), \quad c_\varepsilon(a_i^-)\partial_x q(a_i^-) = c_\varepsilon(a_i^+)\partial_x q(a_i^+),\ i = 1, \dots, n-1. \tag{$\mathrm{TC}_{\varepsilon,n}$}$$

We set $\psi_\varepsilon = e^{-s\eta_\varepsilon} q$. Since q satisfies transmission conditions (TC_n) we have

$$\psi_\varepsilon(t, a_i^-) = \psi_\varepsilon(t, a_i^+), \tag{2.5}$$
$$[c_\varepsilon \partial_x \psi_\varepsilon(t, .)]_{a_i} = s\lambda\varphi_\varepsilon(t, a_i)\, \psi_\varepsilon(t, a_i)[c_\varepsilon \beta'_\varepsilon]_{a_i}, \quad i = 1, \dots, n-1. \tag{2.6}$$

In each $(0,T)\times(a_i,a_{i+1})$, $i=0,\dots,n-1$, the function ψ_ε satisfies $M_1\psi_\varepsilon+M_2\psi_\varepsilon=f_s$, with

$$\begin{aligned}M_1\psi_\varepsilon &= \partial_x(c_\varepsilon\partial_x\psi_\varepsilon)+s^2\lambda^2\varphi_\varepsilon^2(\beta_\varepsilon')^2c_\varepsilon\psi_\varepsilon+s(\partial_t\eta_\varepsilon)\psi_\varepsilon,\\ M_2\psi_\varepsilon &= \partial_t\psi_\varepsilon-2s\lambda\varphi_\varepsilon c_\varepsilon\beta_\varepsilon'\partial_x\psi_\varepsilon-2s\lambda^2\varphi_\varepsilon c_\varepsilon(\beta_\varepsilon')^2\psi_\varepsilon,\\ f_s &= e^{-s\eta_\varepsilon}f+s\lambda\varphi_\varepsilon(c_\varepsilon\beta_\varepsilon')'\psi_\varepsilon-s\lambda^2\varphi_\varepsilon c_\varepsilon(\beta_\varepsilon')^2\psi_\varepsilon.\end{aligned}$$

We have

$$\|M_1\psi_\varepsilon\|^2_{L^2(Q')}+\|M_2\psi_\varepsilon\|^2_{L^2(Q')}+2(M_1\psi_\varepsilon,M_2\psi_\varepsilon)_{L^2(Q')}=\|f_s\|^2_{L^2(Q')}. \tag{2.7}$$

where $Q'=(0,T)\times\Omega'$, with $\Omega'=(\cup_{i=0}^{n-1}(a_i,a_{i+1}))$. With the same notations as in [8, Theorem 3.3], we write $(M_1\psi_\varepsilon,M_2\psi_\varepsilon)_{L^2(Q')}$ as a sum of 9 terms I_{ij}, $1\le i,j\le 3$, where I_{ij} is the inner product of the ith term in the expression of $M_1\psi_\varepsilon$ and the jth term in the expression of $M_2\psi_\varepsilon$ above. For the computation of the terms I_{ij} see [3].

The term I_{11} follows as

$$I_{11}=\frac{1}{2}s\lambda\sum_{i=1}^{n-1}\int_0^T\partial_t\varphi_\varepsilon(t,a_i)[c_\varepsilon\beta_\varepsilon']_{a_i}|\psi_\varepsilon(t,a_i)|^2\,dt$$

The term I_{12} follows as

$$I_{12}=s\lambda^2\iint_{Q'}\varphi_\varepsilon(\beta_\varepsilon')^2\,|c_\varepsilon\partial_x\psi_\varepsilon|^2\,dxdt+X_{12}+s\lambda\sum_{i=0}^{n}\int_0^T\varphi_\varepsilon(t,a_i)\,[\beta_\varepsilon'\,|c_\varepsilon\partial_x\psi_\varepsilon|^2(t,.)]_{a_i}\,dt,$$

where $X_{12}=s\lambda\iint_{Q'}\varphi_\varepsilon(\beta_\varepsilon'')\,|c_\varepsilon\partial_x\psi_\varepsilon|^2\,dxdt$. The term I_{13} follows as

$$I_{13}=2s\lambda^2\iint_{Q'}|c_\varepsilon\partial_x\psi_\varepsilon|^2\varphi_\varepsilon(\beta_\varepsilon')^2dxdt+X_{13},$$

with

$$\begin{aligned}X_{13}=2s\lambda^2&\sum_{i=1}^{n-1}\int_0^T\varphi_\varepsilon(t,a_i)\psi_\varepsilon(t,a_i)\,[(\beta_\varepsilon')^2\,c_\varepsilon^2\partial_x\psi_\varepsilon(t,.)]_{a_i}\,dt\\ &+2s\lambda^3\iint_{Q'}c_\varepsilon^2(\partial_x\psi_\varepsilon)\psi_\varepsilon\varphi_\varepsilon(\beta_\varepsilon')^3dxdt+2s\lambda^2\iint_{Q'}c_\varepsilon(\partial_x\psi_\varepsilon)\psi_\varepsilon\varphi_\varepsilon(c_\varepsilon(\beta_\varepsilon')^2)'dxdt.\end{aligned}$$

The term I_{21} follows as

$$I_{21}=-s^2\lambda^2\iint_{Q'}c_\varepsilon\varphi_\varepsilon(\partial_t\varphi_\varepsilon)(\beta_\varepsilon')^2|\psi_\varepsilon|^2\,dxdt.$$

The term I_{22} follow as

$$\begin{aligned}I_{22}=3s^3\lambda^4&\iint_{Q'}\varphi_\varepsilon^3(\beta_\varepsilon')^4|c_\varepsilon\psi_\varepsilon|^2dxdt\\ &+s^3\lambda^3\sum_{i=1}^{n-1}\int_0^T\varphi_\varepsilon^3(t,a_i)|\psi_\varepsilon(t,a_i)|^2\,[c_\varepsilon^2(\beta_\varepsilon')^3]_{a_i}\,dt+X_{22},\end{aligned}$$

with $X_{22} = s^3\lambda^3 \iint_{Q'} \varphi_\varepsilon^3 (c_\varepsilon^2(\beta_\varepsilon')^3)'|\psi_\varepsilon|^2 \, dxdt$. The terms I_{23} and I_{31} follow as

$$I_{23} = -2s^3\lambda^4 \iint_{Q'} \varphi_\varepsilon^3(\beta_\varepsilon')^4|c_\varepsilon\psi_\varepsilon|^2 \, dxdt, \quad I_{31} = -\frac{s}{2}\iint_{Q'} (\partial_t^2\eta_\varepsilon)|\psi_\varepsilon|^2 \, dxdt.$$

The terms I_{32} is given by

$$\begin{aligned} I_{32} = s^2\lambda^2 \iint_{Q'} \varphi_\varepsilon(\beta_\varepsilon')^2 c_\varepsilon(\partial_t\eta_\varepsilon)|\psi_\varepsilon|^2 \, dxdt - s^2\lambda^2 \iint_{Q'} \varphi_\varepsilon(\partial_t\varphi_\varepsilon)(\beta_\varepsilon')^2 c_\varepsilon|\psi_\varepsilon|^2 \, dxdt \\ + s^2\lambda \iint_{Q'} \varphi_\varepsilon(c_\varepsilon\beta_\varepsilon')'(\partial_t\eta_\varepsilon)|\psi_\varepsilon|^2 \, dxdt \\ + s^2\lambda \sum_{i=1}^{n-1}\int_0^T \varphi_\varepsilon(t,a_i)(\partial_t\eta_\varepsilon)(t,a_i)|\psi_\varepsilon(t,a_i)|^2 \, [c_\varepsilon\beta_\varepsilon']_{a_i} \, dt. \end{aligned}$$

Finally, the term I_{33} follows as

$$I_{33} = -2s^2\lambda^2 \iint_{Q'} \varphi_\varepsilon c_\varepsilon(\partial_t\eta_\varepsilon)(\beta_\varepsilon')^2|\psi_\varepsilon|^2 \, dxdt.$$

Adding the nine terms together to form $(M_1\psi_\varepsilon, M_2\psi_\varepsilon)_{L^2(Q')}$ in (2.7) leads to

$$\begin{aligned} &\|M_1\psi_\varepsilon\|^2_{L^2(Q')} + \|M_2\psi_\varepsilon\|^2_{L^2(Q')} \\ &\quad + 6s\lambda^2 \iint_{Q'} \varphi_\varepsilon(\beta_\varepsilon')^2 \, |c_\varepsilon\partial_x\psi_\varepsilon|^2 \, dxdt + 2s^3\lambda^4 \iint_{Q'} \varphi_\varepsilon^3(\beta_\varepsilon')^4|c_\varepsilon\psi_\varepsilon|^2 \, dxdt \\ &\quad + 2s\lambda \sum_{i=0}^{n}\int_0^T \varphi_\varepsilon(t,a_i)\Big([\beta_\varepsilon' \, |c_\varepsilon\partial_x\psi_\varepsilon|^2(t,.)]_{a_i} + [c_\varepsilon^2(\beta_\varepsilon')^3]_{a_i}|s\lambda\varphi_\varepsilon(t,a_i)\psi_\varepsilon(t,a_i)|^2\Big) \, dt \\ &\quad = \|f_s\|^2_{L^2(Q')} - 2(I_{11} + X_{12} + X_{13} + I_{21} + X_{22} + I_{31} + I_{32} + I_{33}). \end{aligned} \tag{2.8}$$

The terms $I_{11}, \ldots, I_{33}$ on the r.h.s. are terms to be 'dominated'. The 'dominating' volume and surface terms are the terms we kept on the l.h.s. of (2.8).

We shall first treat the 'dominated' volume terms and bound them from above uniformly w.r.t. ε.

With β_ε' piecewise constant outside $\mathcal{O}$, the term X_{12} reduces to

$$X_{12} = s\lambda \iint_{(0,T)\times\mathcal{O}} \varphi_\varepsilon(\beta_\varepsilon'') \, |c_\varepsilon\partial_x\psi_\varepsilon|^2 \, dxdt,$$

and we have

$$|X_{12}| \leq s\lambda C \iint_{(0,T)\times\mathcal{O}} |\partial_x\psi_\varepsilon|^2 \, dxdt,$$

with C uniform w.r.t. ε by lemma 2.3. The absolute value of the volume terms in X_{13} can be bounded by [3, 8]

$$C_\delta T^4 s\lambda^4 \iint_Q \varphi_\varepsilon^3|\psi_\varepsilon|^2 \, dxdt + \delta s\lambda^2 \iint_Q \varphi_\varepsilon|\partial_x\psi_\varepsilon|^2 \, dxdt, \quad \delta > 0,$$

with δ arbitrary small, using $\varphi_\varepsilon \leq CT^4\varphi_\varepsilon^3$; the constants C_δ is uniform w.r.t. ε. (recall that c_ε is piecewise constant outside $\mathcal{O}$ and $\|c_\varepsilon - c\|_{\mathscr{C}^1(\overline{\mathcal{O}})} \leq \varepsilon$.) Noting that [8, equations (89)–(91)]

$$|\partial_t\varphi_\varepsilon| \leq T\varphi_\varepsilon^2, \quad |\partial_t\eta_\varepsilon| \leq T\varphi_\varepsilon^2, \quad |\partial_{tt}^2\eta_\varepsilon| \leq 2T^2\varphi_\varepsilon^3,$$

we obtain

$$|I_{21}| \leq s^2\lambda^2 CT \iint_Q \varphi_\varepsilon^3 |\psi_\varepsilon|^2 \, dxdt, \quad |I_{31}| \leq sCT^2 \iint_Q \varphi_\varepsilon^3 |\psi_\varepsilon|^2 \, dxdt,$$
$$|I_{33}| \leq s^2\lambda^2 CT \iint_Q \varphi_\varepsilon^3 |\psi_\varepsilon|^2 dxdt,$$

with the constants uniform w.r.t. ε. Similarly we have

$$|X_{22}| \leq Cs^3\lambda^3 \iint_Q \varphi_\varepsilon^3 \, |\psi_\varepsilon|^2 \, dxdt,$$

with a constant C uniform w.r.t. ε. Finally, the absolute value of the volume terms in I_{32} can be estimated from above by $s^2\lambda^2 CT \iint_Q \varphi_\varepsilon^3 |\psi_\varepsilon|^2 dxdt$ with a constant C uniform w.r.t. ε.

We shall use the properties of β_ε listed in Lemma 2.3 to now estimate from above the ‘dominated’ surface terms.

Lemma 2.5. *Let $\delta > 0$. There exists $C_\delta > 0$ uniform w.r.t. ε such that the absolute value of the surface terms in I_{11}, I_{13} and I_{32} can be bounded by*

$$C_\delta(s\lambda T^3 + s\lambda^3 T^4 + (\lambda + \lambda^3)s^2 T^2) \sum_{i=1}^{n-1} |Y_i - 1| \int_0^T \varphi_\varepsilon^3(t, a_i) |\psi_\varepsilon(t, a_i)|^2 \, dt$$
$$+ s\lambda\delta \sum_{i=1}^{n-1} |Y_i - 1| \int_0^T \varphi_\varepsilon(t, a_i) \, |(c_\varepsilon \partial_x \psi_\varepsilon)(t, a_i^-)|^2 \, dt.$$

Proof. Note first that on the r.h.s. of the open set O ($\beta'_\varepsilon < 0$) we either have $X = Y$ if $Y > 1$ or $X = 2 - Y$, if $Y < 1$. In the first case, $Y - X = 0$ and $Y - X^2 = (1 - Y)Y$; in the second case $X - Y = 2(Y - 1)$ and $Y - X^2 = (Y - 1)(4 - Y)$. On the l.h.s. of O ($\beta'_\varepsilon > 0$) we either have $X = \frac{Y}{2Y-1}$ if $Y > 1$ or $X = Y$ if $Y < 1$. In the first case, $Y - X = \frac{2Y}{2Y-1}(Y - 1)$ and $Y - X^2 = \frac{4Y^2-Y}{(2Y-1)^2}(Y - 1)$; in the second case $Y - X = 0$ and $Y - X^2 = (1 - Y)Y$. Hence, in any case, since

$$0 < \frac{c_{min} - \varepsilon_0}{c_{max} + \varepsilon_0} \leq Y \leq \frac{c_{max} + \varepsilon_0}{c_{min} - \varepsilon_0},$$

we obtain that $|X - Y| \leq C|Y - 1|$ and $|Y - X^2| \leq C|Y - 1|$ with the constant C uniform w.r.t. ε and w.r.t. the considered point of discontinuity of c_ε.

Observing that $[c_\varepsilon \beta'_\varepsilon]_{a_i} = c_\varepsilon(a_i^-)\beta'_\varepsilon(a_i^+)(Y_i - X_i)$ we obtain

$$|I_{11}| \leq s\lambda CT^3 \sum_{i=1}^{n-1} |Y_i - 1| \int_0^T \varphi_\varepsilon^3(t, a_i) |\psi_\varepsilon(t, a_i)|^2 \, dt,$$

with C uniform w.r.t. ε by Lemma 2.3.

To estimate the surface terms in X_{13} we write, with a being one of the a_i, $i = 1, \ldots, n-1$,

$$
\begin{aligned}
2s\lambda^2 &\int_0^T \varphi_\varepsilon(t,a)\psi_\varepsilon(t,a)\,[(\beta_\varepsilon')^2\, c_\varepsilon^2 \partial_x \psi_\varepsilon(t,.)]_a\, dt \\
&= 2s\lambda^2 \int_0^T \varphi_\varepsilon(t,a)\psi_\varepsilon(t,a) c_\varepsilon(a^-)\beta_\varepsilon'(a^+)^2 \left((c_\varepsilon \partial_x \psi_\varepsilon)(a^+)Y - (c_\varepsilon \partial_x \psi_\varepsilon)(a^-)X^2\right) dt \\
&\qquad = 2s\lambda^2 (Y - X^2) c_\varepsilon(a^-)\beta_\varepsilon'(a^+)^2 \int_0^T \varphi_\varepsilon(t,a)\, \psi_\varepsilon(t,a)(c_\varepsilon \partial_x \psi_\varepsilon)(a^-)\, dt \\
&\qquad\qquad + 2s^2\lambda^3 (Y - X) Y c_\varepsilon^2(a^-)\beta_\varepsilon'(a^+)^3 \int_0^T \varphi_\varepsilon^2(t,a)\, |\psi_\varepsilon(t,a)|^2\, dt,
\end{aligned}
$$

where we have used transmission condition (2.6). We thus obtain that the absolute value of the surface terms in X_{13} can be estimated uniformly w.r.t. ε by

$$
\begin{aligned}
s\lambda^2 C &\sum_{i=1}^{n-1} |Y_i - 1| \int_0^T \varphi_\varepsilon(t,a_i)\, \psi_\varepsilon(t,a_i)(c_\varepsilon \partial_x \psi_\varepsilon)(a_i^-)\, dt \\
&\qquad + s^2\lambda^3 C \sum_{i=1}^{n-1} |Y_i - 1| \int_0^T \varphi_\varepsilon^2(t,a_i)\, |\psi_\varepsilon(t,a_i)|^2\, dt \\
&\le C_\delta (s\lambda^3 T^4 + s^2\lambda^3 T^2) \sum_{i=1}^{n-1} |Y_i - 1| \int_0^T \varphi_\varepsilon^3(t,a_i)\, |\psi_\varepsilon(t,a_i)|^2\, dt \\
&\qquad + \delta s\lambda \sum_{i=1}^{n-1} |Y_i - 1| \int_0^T \varphi_\varepsilon(t,a_i)\, |(c_\varepsilon \partial_x \psi_\varepsilon)(t,a_i^-)|^2\, dt,
\end{aligned}
$$

for $\delta > 0$ arbitrary small, by Young's inequality and using $\varphi_\varepsilon^2 \le C\varphi_\varepsilon^3 T^2$ and $\varphi_\varepsilon \le C\varphi_\varepsilon^3 T^4$.

Finally, we estimate the absolute value of the surface terms in I_{32} uniformly w.r.t. ε by

$$
s^2\lambda C T \sum_{i=1}^{n-1} |Y_i - 1| \int_0^T \varphi_\varepsilon^3(t,a_i)|\psi_\varepsilon(t,a_i)|^2\, dt,
$$

which concludes the proof of Lemma 2.5. ∎

Continuation of the proof of Proposition 2.4. We now pass to the task of estimating from below the volume and surface 'dominating' terms. We first treat the volume terms, restricting the domain of integration to $(\Omega \setminus \omega_0) \times (0,T)$. Since $|\beta_\varepsilon'(x)| \ge \min(\beta_\varepsilon'(0), |\beta_\varepsilon'(1)|) = \min(\beta'(0), |\beta'(1)|) > 0$ on $\Omega \setminus \omega_0$, from the construction we gave above, we obtain

$$
\begin{aligned}
6s\lambda^2 &\int_0^T\!\!\int_{\Omega\setminus\omega_0} \varphi_\varepsilon (\beta_\varepsilon')^2\, |c_\varepsilon \partial_x \psi_\varepsilon|^2\, dxdt + 2s^3\lambda^4 \int_0^T\!\!\int_{\Omega\setminus\omega_0} \varphi_\varepsilon^3 (\beta_\varepsilon')^4 |c_\varepsilon \psi_\varepsilon|^2\, dxdt \\
&\qquad \ge C\left(s\lambda^2 \int_0^T\!\!\int_{\Omega\setminus\omega_0} \varphi_\varepsilon |c_\varepsilon \partial_x \psi_\varepsilon|^2\, dxdt + s^3\lambda^4 \int_0^T\!\!\int_{\Omega\setminus\omega_0} \varphi_\varepsilon^3 |\psi_\varepsilon|^2\, dxdt\right),
\end{aligned}
$$

where the constant C is uniform w.r.t. ε.

As in the proof of the previous lemma, to treat the surface terms, we write a as one of the a_i, $i = 1, \ldots, n-1$. The 'dominating' surface terms in (2.8) are sums of terms of the form

$$\mu := 2s\lambda \int_0^T \varphi_\varepsilon(t,a) \Big([\beta'_\varepsilon \, |c_\varepsilon \partial_x \psi_\varepsilon|^2(t,.)]_a + [c_\varepsilon^2 (\beta'_\varepsilon)^3]_a |s\lambda \varphi_\varepsilon(t,a)\psi_\varepsilon(t,a)|^2 \Big) \, dt.$$

Applying transmission condition (2.6) we obtain

$$\begin{aligned}[]
[\beta'_\varepsilon \, |c_\varepsilon \partial_x \psi_\varepsilon|^2(t,.)]_a = [\beta'_\varepsilon]_a \, |c_\varepsilon(a^-)\partial_x\psi_\varepsilon(t,a^-)|^2 + s^2\lambda^2\varphi_\varepsilon^2(t,a)\beta'_\varepsilon(a^+)[c_\varepsilon\beta'_\varepsilon]_a^2 \, |\psi_\varepsilon(t,a)|^2 \\
+ 2s\lambda\varphi_\varepsilon(t,a)\beta'_\varepsilon(a^+)[c_\varepsilon\beta'_\varepsilon]_a \, (c_\varepsilon\partial_x\psi_\varepsilon)(t,a^-)\psi_\varepsilon(t,a),
\end{aligned}$$

which gives

$$\begin{aligned}
\mu := s\lambda \int_0^T \varphi_\varepsilon(t,a) \Big(& \, [\beta'_\varepsilon]_a \, |(c_\varepsilon\partial_x\psi_\varepsilon)(t,a^-)|^2 \\
& + \Big(\beta'_\varepsilon(a^+)[c_\varepsilon\beta'_\varepsilon]_a^2 + [c_\varepsilon^2(\beta'_\varepsilon)^3]_a\Big) \, |s\lambda\varphi_\varepsilon(t,a)\psi_\varepsilon(t,a)|^2 \\
& + 2\beta'_\varepsilon(a^+)[c_\varepsilon\beta'_\varepsilon]_a \, (c_\varepsilon\partial_x\psi_\varepsilon)(t,a^-)(s\lambda\varphi_\varepsilon(t,a)\psi_\varepsilon(t,a))\Big) \, dt \\
& = s\lambda \int_0^T \varphi_\varepsilon(t,a) \Big(Au(t,a), u(t,a)\Big) \, dt,
\end{aligned}$$

with $u(t,a) = ((c_\varepsilon\partial_x\psi_\varepsilon)(t,a^-), s\lambda\varphi_\varepsilon(t,a)\psi_\varepsilon(t,a))^t$ and the symmetric matrix A given by

$$A = \begin{pmatrix} [\beta'_\varepsilon]_a & \beta'_\varepsilon(a^+)[c_\varepsilon\beta'_\varepsilon]_a \\ \beta'_\varepsilon(a^+)[c_\varepsilon\beta'_\varepsilon]_a & \beta'_\varepsilon(a^+)[c_\varepsilon\beta'_\varepsilon]_a^2 + [c_\varepsilon^2(\beta'_\varepsilon)^3]_a \end{pmatrix}.$$

The matrix A is positive definite by Lemma 2.3 and Lemma 1.1. However, we need to estimate its eigenvalues from below, which is the object of the following lemma.

Lemma 2.6. *The eigenvalues ν_1, ν_2 of the matrix A satisfy $\nu_i \geq C|Y-1|$, $i = 1,2$, with C uniform w.r.t. ε and $i \in \{1,\dots,n\}$.*

Proof. We have several cases to consider. Consider first the r.h.s. of O, that is in the region where $\beta'_\varepsilon < 0$. In the case $Y > 1$, we have made the choice, $X = Y$ and the matrix A then reduces to

$$A = \begin{pmatrix} \beta'_\varepsilon(a^+)(1-Y) & 0 \\ 0 & c_\varepsilon^2(a^-)(\beta'_\varepsilon(a^+))^3 Y^2(1-Y) \end{pmatrix}.$$

and the result follows (recall that $0 < Y_{min} \leq Y \leq Y_{max}$, Y_{min} and Y_{max} uniform w.r.t. ε and $0 < c_{min} - \varepsilon_0 \leq c_\varepsilon \leq c_{max} + \varepsilon_0$ and $|\beta'_\varepsilon(a^+)| \geq |\beta'_\varepsilon(1)| = |\beta'(1)| > 0$).

In the case $Y < 1$ we have $X = 2 - Y$. The matrix A is then equal to

$$A = \beta'_\varepsilon(a^+)(Y-1)\underline{A}, \quad \text{with } \underline{A} = \begin{pmatrix} 1 & 2c_\varepsilon(a^-)\beta'_\varepsilon(a^+) \\ 2c_\varepsilon(a^-)\beta'_\varepsilon(a^+) & c_\varepsilon^2(a^-)(\beta'_\varepsilon(a^+))^2(Y^2+4) \end{pmatrix}.$$

Observe that $\det(\underline{A}) = Y^2 c_\varepsilon^2(a^-)(\beta'_\varepsilon(a^+))^2 = c_\varepsilon^2(a^+)(\beta'_\varepsilon(a^+))^2$ thus $\det(\underline{A}) \geq C_1 > 0$ and $0 < \operatorname{tr}(\underline{A}) \leq C_2$. The constants are uniform w.r.t. ε. We thus obtain that $\nu_i \geq \beta'_\varepsilon(a^+)(Y - 1)\frac{C_1}{C_2}$, $i = 1,2$, since ν_1 and ν_2 are both positive by Lemma 2.3 and Lemma 1.1.

Consider now the l.h.s. of O, that is in the region where $\beta'_\varepsilon > 0$. In the case $Y < 1$ we made the choice $X = Y$ and the result follows as above. In the case $Y > 1$ we have $X = \frac{Y}{2Y-1}$. The matrix A is then equal to $\beta'_\varepsilon(a^+)(Y-1)\underline{A}$ with

$$\underline{A} = \begin{pmatrix} \frac{X}{Y} & 2\alpha X \\ 2\alpha X & \alpha^2(4X^2(Y-1) + \frac{X^3}{Y}(8Y^2 - 4Y + 1)) \end{pmatrix},$$

where $\alpha = c_\varepsilon(a^-)\beta'_\varepsilon(a^+)$. Observe that $\det(\underline{A}) = c_\varepsilon^2(a^+)(\beta'_\varepsilon(a^+))^2\frac{1}{(2Y-1)^4} \geq C_1 > 0$ and $0 < \mathrm{tr}(\underline{A}) \leq C_2$. Thus result thus follows as above. ■

End of the proof of Proposition 2.4. With the estimations provided above we can absorb the 'dominated' terms by the 'dominating' ones, taking the parameters s and λ sufficiently large. More precisely we obtain

$$\|M_1\psi_\varepsilon\|^2_{L^2(Q')} + \|M_2\psi_\varepsilon\|^2_{L^2(Q')} + s\lambda^2 \iint_Q \varphi_\varepsilon e^{-2s\eta_\varepsilon}|\partial_x q|^2\,dxdt + s^3\lambda^4 \iint_Q \varphi_\varepsilon^3 e^{-2s\eta_\varepsilon}|q|^2\,dxdt$$
$$\leq C\|e^{-s\eta_\varepsilon} f\|^2_{L^2(Q')} + Cs\lambda^2 \int_0^T\!\!\int_{\omega_0} \varphi_\varepsilon e^{-2s\eta_\varepsilon}|\partial_x q|^2\,dxdt + Cs^3\lambda^4 \int_0^T\!\!\int_{\omega_0} \varphi_\varepsilon^3 e^{-2s\eta_\varepsilon}|q|^2\,dxdt,$$

for $\lambda \geq \lambda_1 = \lambda_1(\Omega, \mathcal{O}, c)$, $s \geq s_1 = \sigma_1(\Omega, \mathcal{O}, c, \lambda_1)(T + T^2)$, with σ_1, λ_1 and C uniform w.r.t. ε. As in [8, Estimate (100)], we have the following estimate, uniformly w.r.t. ε, because of the properties of β_ε on $\mathcal{O}$ (see Lemma 2.3)

$$s\lambda^2 \int_0^T\!\!\int_{\omega_0} \varphi_\varepsilon e^{-2s\eta_\varepsilon}|\partial_x q|^2\,dxdt \leq C\|e^{-s\eta_\varepsilon} f\|^2_{L^2(Q')} + C\Big(s^3\lambda^4 + s^2\lambda^2(\lambda^2T^2 + T) + s\lambda^2(\lambda T^4 + \lambda T^2 + T^3)\Big)\int_0^T\!\!\int_{\mathcal{O}} \varphi_\varepsilon^3 e^{-2s\eta_\varepsilon}|q|^2\,dxdt. \tag{2.9}$$

For $\lambda \geq \lambda_1$ and $s \geq s_1$, we then obtain

$$\|M_1\psi_\varepsilon\|^2_{L^2(Q')} + \|M_2\psi_\varepsilon\|^2_{L^2(Q')} + s\lambda^2 \iint_Q \varphi_\varepsilon e^{-2s\eta_\varepsilon}|\partial_x q|^2\,dxdt + s^3\lambda^4 \iint_Q \varphi_\varepsilon^3 e^{-2s\eta_\varepsilon}|q|^2\,dxdt$$
$$\leq C\|e^{-s\eta_\varepsilon} f\|^2_{L^2(Q')} + Cs^3\lambda^4 \int_0^T\!\!\int_{\omega_0} \varphi_\varepsilon^3 e^{-2s\eta_\varepsilon}|q|^2\,dxdt,$$

with the constant C uniform w.r.t. ε. To incorporate the higher order terms on the l.h.s. and obtain Carleman estimate (1.3) we follow the classical procedure (see e.g. [10]) which can be done uniformly w.r.t. ε. ■

For c_ε defined as above, converging to c in L^∞, we shall now analyse the convergence of each term in Carleman estimate (1.3), that holds for the operators $\partial_t \pm \partial_x(c_\varepsilon\partial_x)$, as $|c_\varepsilon - c|_\infty$ goes to zero. For this purpose, we define the following weight functions associated to β by

$$\varphi(x,t) = \frac{e^{\lambda\beta(x)}}{t(T-t)}, \qquad \eta(x,t) = \frac{e^{\lambda\overline{\beta}} - e^{\lambda\beta(x)}}{t(T-t)}. \tag{2.10}$$

The constant $\overline{\beta}$ used is the same used in the definition of η_ε in (2.3), since $\overline{\beta}_\varepsilon$ can be chosen uniformly w.r.t. ε as mentioned above.

At first, we consider $f \in \mathscr{C}^1([0,T], L^2(\Omega))$, with $f(0) \in H^1_0(\Omega)$, and q (weak) solution to

$$\begin{cases} \partial_t q \pm \partial_x(c\partial_x q) = f & \text{in } Q, \\ q = 0 & \text{on } \Sigma, \\ q(T,x) = q_0(x) \text{ (resp. } q(0,x) = q_0(x)) & \text{in } \Omega. \end{cases} \tag{2.11}$$

We also define q_ε as the (weak) solution to

$$\begin{cases} \partial_t q_\varepsilon \pm \partial_x(c_\varepsilon \partial_x q_\varepsilon) = f & \text{in } Q, \\ q_\varepsilon = 0 & \text{on } \Sigma, \\ q_\varepsilon(T,x) = q_{0,\varepsilon}(x) \text{ (resp. } q_\varepsilon(0,x) = q_{0,\varepsilon}(x)) & \text{in } \Omega. \end{cases} \tag{2.12}$$

The final (resp. initial) conditions are chosen such that

$$\partial_x(c\partial_x q_0) = \mu, \quad \text{and } \partial_x(c_\varepsilon \partial_x q_{0,\varepsilon}) = \mu,$$

with $\mu \in H_0^1(\Omega)$. Then we find

$$\|q_0 - q_{0,\varepsilon}\|_{H_0^1(\Omega)} \leq C\|c - c_\varepsilon\|_\infty \|\mu\|_{L^2(\Omega)}. \tag{2.13}$$

For the solutions q and q_ε we have the following lemma.

Lemma 2.7. *The solutions to (2.11) and (2.12) satisfy*

$$\|q(t,.) - q_\varepsilon(t,.)\|_{L^2(\Omega)} + \|\partial_x q - \partial_x q_\varepsilon\|_{L^2(Q)} \leq C\|c - c_\varepsilon\|_\infty(\|f\|_{L^2(Q)} + \|\mu\|_{L^2(\Omega)}), \tag{2.14}$$

for $t \in [0,T]$ *and*

$$\begin{aligned} \|\partial_t q(t,.) - \partial_t q_\varepsilon(t,.)\|_{L^2(\Omega)} + \|\partial_x(c\partial_x q)(t,.) - \partial_x(c_\varepsilon \partial_x q_\varepsilon)(t,.)\|_{L^2(\Omega)} \\ \leq C\|c - c_\varepsilon\|_\infty(\|\partial_t f\|_{L^2(Q)} + \|f(0)\|_{L^2(\Omega)} + \|\mu\|_{L^2(\Omega)}), \quad t \in [0,T]. \end{aligned} \tag{2.15}$$

Proof. We treat here the case of the operators $\partial_t - \partial_x(c\partial_x)$ and $\partial_t - \partial_x(c_\varepsilon\partial_x)$. The other case follows similarly. The solution to (2.11) satisfies

$$\iint_{Q_t} (\partial_t q\phi + c\partial_x q\partial_x \phi)\, dxdt = \iint_{Q_t} f\phi\, dxdt, \quad \phi \in L^2(0,T,H_0^1(\Omega)),$$

for $Q_t = (0,t) \times \Omega$, $t \in [0,T]$. We write a similar weak formulation for the solution to (2.12), from which we obtain

$$\begin{aligned} \iint_{Q_t} (\partial_t(q - q_\varepsilon)\phi + c_\varepsilon \partial_x(q - q_\varepsilon)\partial_x\phi)\, dxdt \\ = \iint_{Q_t} (c_\varepsilon - c)\partial_x q \partial_x \phi\, dxdt, \quad \phi \in L^2(0,T,H_0^1(\Omega)), \end{aligned} \tag{2.16}$$

which with $\phi = q - q_\varepsilon$ yields

$$\iint_{Q_t} (\frac{1}{2}\partial_t|q - q_\varepsilon|^2 + c_\varepsilon|\partial_x(q - q_\varepsilon)|^2\, dxdt = \iint_{Q_t} (c_\varepsilon - c)\partial_x q \partial_x(q - q_\varepsilon)\, dxdt.$$

It follows that

$$\begin{aligned} \frac{1}{2}\|q(t) - q_\varepsilon(t)\|^2_{L^2(\Omega)} + (c_{min} - \delta)\|\partial_x(q - q_\varepsilon)\|^2_{L^2(Q)} \\ \leq C_\delta \|c_\varepsilon - c\|^2_\infty \|\partial_x q\|^2_{L^2(Q)} + \frac{1}{2}\|q_0 - q_{0,\varepsilon}\|^2_{L^2(\Omega)}, \end{aligned}$$

which yields (2.14) from a classical energy estimate and (2.13).

From the regularity assumption made on f, q and q_ε are in $\mathscr{C}^1([0,T], L^2(\Omega))$. In fact, for q, we can write, by Duhamel's formula [18, Chapter 4, Section 2]

$$q(t) = S(t)q_0 + \int_0^t S(t-s) f(s)\, ds,$$

where S is the semigroup generated by $A = \partial_x(c\partial_x)$. Since q_0 is in the domain of A, the first term is in $\mathscr{C}^1([0,T], L^2(\Omega))$ (see Theorem 2.4.c in [18, Chapter 1, Section 2]). The second term, $q_2(t)$, is differentiable w.r.t. t on $[0,T]$ with

$$\partial_t q_2(t) = S(t) f(0) + \int_0^t S(s)\partial_t f(t-s)\, ds,$$

which is continuous on $[0,T]$ using the continuity of $S(t)$ and the uniform continuity of $\partial_t f$ in $L^2(\Omega)$ on $[0,T]$.

Consider now $p = \partial_t q$. Then the function p is solution to

$$\begin{cases} \partial_t p - \partial_x(c\partial_x p) = \partial_t f & \text{in } Q, \\ p = 0 & \text{on } \Sigma, \\ p(0,x) = \mu + f(0) & \text{in } \Omega. \end{cases} \tag{2.17}$$

Similarly $p_\varepsilon = \partial_t q_\varepsilon$ is solution to

$$\begin{cases} \partial_t p_\varepsilon - \partial_x(c_\varepsilon\partial_x p_\varepsilon) = \partial_t f & \text{in } Q, \\ p_\varepsilon = 0 & \text{on } \Sigma, \\ p_\varepsilon(0,x) = \mu + f(0) & \text{in } \Omega. \end{cases} \tag{2.18}$$

Thus $p(0,.)$ and $p_\varepsilon(0,.)$ are in $H_0^1(\Omega)$, since $f(0) \in H_0^1(\Omega)$. With the previous procedure we obtain

$$\begin{aligned} &\|p(t,.) - p_\varepsilon(t,.)\|_{L^2(\Omega)} + \|\partial_x p - \partial_x p_\varepsilon\|_{L^2(Q)} \\ &\qquad \leq C\|c - c_\varepsilon\|_\infty (\|\partial_t f\|_{L^2(Q)} + \|f(0)\|_{L^2(\Omega)} + \|\mu\|_{L^2(\Omega)}), \quad t \in [0,T], \end{aligned}$$

which yields (2.15). ∎

To study the convergence of the term $\iint_Q e^{-2s\eta_\varepsilon}\varphi_\varepsilon^3\, |q_\varepsilon|^2\, dxdt$ in the Carleman estimate for the operators $\partial_t \pm \partial_x(c_\varepsilon\partial_x)$, we write

$$\begin{aligned} &\left| \iint_Q e^{-2s\eta}\varphi^3\, |q|^2\, dxdt - \iint_Q e^{-2s\eta_\varepsilon}\varphi_\varepsilon^3\, |q_\varepsilon|^2\, dxdt \right| \\ &\qquad \leq \iint_Q |e^{-2s\eta}\varphi^3 - e^{-2s\eta_\varepsilon}\varphi_\varepsilon^3|\, |q_\varepsilon|^2\, dxdt + \iint_Q e^{-2s\eta_\varepsilon}\varphi_\varepsilon^3\, \left||q|^2 - |q_\varepsilon|^2\right| dxdt \\ &\qquad \leq \iint_Q |e^{-2s\eta}\varphi^3 - e^{-2s\eta_\varepsilon}\varphi_\varepsilon^3|\, |q_\varepsilon|^2\, dxdt + \iint_Q e^{-2s\eta_\varepsilon}\varphi_\varepsilon^3\, |q - q_\varepsilon|\, (|q| + |q_\varepsilon|) dxdt, \end{aligned}$$

which converges to zero by Cauchy-Schwarz inequalities and dominated convergence. Recall that β_ε converges everywhere to β and thus $e^{-2s\eta_\varepsilon}$ and φ_ε converge everywhere to $e^{-2s\eta}$ and φ.

Similar arguments yield the following convergences, using Lemma 2.7,

$$\lim_{\varepsilon\to 0}\iint_Q e^{-2s\eta_\varepsilon}\varphi_\varepsilon\, |\partial_x q_\varepsilon|^2\, dxdt = \iint_Q e^{-2s\eta}\varphi\, |\partial_x q|^2\, dxdt.$$

$$\begin{aligned}\lim_{\varepsilon\to 0}&\iint_Q e^{-2s\eta_\varepsilon}\varphi_\varepsilon^{-1}\, (|\partial_t q_\varepsilon|^2 + |\partial_x(c_\varepsilon\partial_x q_\varepsilon)|^2)\, dxdt\\ &= \iint_Q e^{-2s\eta}\varphi^{-1}\, (|\partial_t q|^2 + |\partial_x(c\partial_x q)|^2)\, dxdt.\end{aligned}$$

In the case $\mu \in H_0^1(\Omega)$ and $f \in \mathscr{C}^1([0,T], L^2(\Omega))$, with $f(0) \in H_0^1(\Omega)$, from the Carleman estimate associated to q_ε and the operators $\partial_t \pm \partial_x(c_\varepsilon\partial_x)$, we thus obtain that (1.3) holds for q and $\partial_t \pm \partial_x(c\partial_x)$ with the same constants C, s_1 and λ_1. With such an estimate at hand, we can now relax the assumptions made on the final (resp. initial) condition and on the function f, by a density argument.

Hence, with the convergence results above, Proposition 2.4, Carleman estimate (1.3) and Remark 1.3, we have proven

Theorem 2.8. *Let $\mathcal{O} \Subset \Omega$ be a non-empty open set and $c \in BV(\Omega)$ with $0 < c_{min} \leq c \leq c_{max}$ and c of class $\mathscr{C}^1$ in $\overline{\mathcal{O}}$. There exists $\lambda_1 = \lambda_1(\Omega,\mathcal{O}) > 0$, $s_1 = s_1(\lambda_1, T) > 0$ and a positive constant $C = C(\Omega,\mathcal{O})$ so that Carleman estimate (1.3) holds for $s \geq s_1$, $\lambda \geq \lambda_1$ and for all q (weak) solution to*

$$\begin{cases}\partial_t q \pm \partial_x(c\partial_x q) = f & \text{in } Q,\\ q = 0 & \text{on } \Sigma,\\ q(T,x) = q_0(x)\ (\text{resp. } q(0,x) = q_0(x)) & \text{in } \Omega,\end{cases}$$

with $q_0 \in L^2(\Omega)$ and $f \in L^2(Q)$. The weight functions used are those defined in (2.10) and Lemma 2.3.

Remark 2.9. Similarly, for c in $BV(\Omega)$, we can obtain a Carleman estimate with a side observation, say in $\{0\}$, i.e. an estimate of the form

$$\begin{aligned}(2.19)\quad s^{-1}&\iint_Q e^{-2s\eta}\varphi^{-1}\, (|\partial_t q|^2 + |\partial_x(c\partial_x q)|^2)\, dxdt + s\lambda^2\iint_Q e^{-2s\eta}\varphi\, |\partial_x q|^2\, dxdt\\ &+ s^3\lambda^4\iint_Q e^{-2s\eta}\varphi^3\, |q|^2\, dxdt\\ &\leq C\left[s\lambda\int_0^T \varphi(t,0)e^{-2s\eta(t,0)}|\partial_x q|^2(t,0)\, dt + \iint_Q e^{-2s\eta}\, |f|^2\, dxdt\right],\end{aligned}$$

for $s \geq s_1$, $\lambda \geq \lambda_1$. The proof is similar and makes use of such a Carleman estimate for a piecewise-$\mathscr{C}^1$ coefficient proven in [4, 3]. Note however that to obtain (2.19), we need not assume that c is of class $\mathscr{C}^1$ in some inner region of Ω.

3 A Carleman estimate for the heat equation with a right-hand side in $L^2(0,T,H^{-1}(\Omega))$

Following [14], from Theorem 2.8, we can derive a Carleman estimate for (1) in the case of a r.h.s., f, in H^{-1}. Such a estimate will be used in the next section to obtain controllability results for classes of semilinear parabolic equations.

We set

$$\widetilde{\aleph}_{\pm} = \Big\{q \in \mathscr{C}([0,T], H_0^1(\Omega));\, q(t) \in D(A) \text{ for all } t \in [0,T] \\ \text{and } \partial_t q \pm \partial_x(c\partial_x q) = F_0 + \partial_x F_1 \text{ with } F_0, F_1 \in L^2(Q)\Big\}.$$

In the case of a diffusion coefficient c in BV, yet $\mathscr{C}^1$ in some open region, we have

Theorem 3.1. *Let $\mathcal{O} \Subset \Omega$ be a non-empty open set and $c \in BV(\Omega)$ with $0 < c_{min} \leq c \leq c_{max}$ and c of class $\mathscr{C}^1$ in $\overline{\mathcal{O}}$. There exists $\lambda_2 = \lambda_2(\Omega, \mathcal{O}, c) > 0$, $s_2 = s_2(\Omega, \mathcal{O}, c, \lambda_2, T) > 0$ and a positive constant $C = C(\Omega, \mathcal{O}, c)$ so that the following estimate holds*

$$s\lambda^2 \iint_Q e^{-2s\eta}\varphi\, |\partial_x q|^2 \, dxdt + s^3\lambda^4 \iint_Q e^{-2s\eta}\varphi^3\, |q|^2 \, dxdt \\ \leq C\Big[\, s^3\lambda^4 \iint_{(0,T)\times\mathcal{O}} e^{-2s\eta}\varphi^3\, |q|^2 \, dxdt + \iint_Q e^{-2s\eta}\, |F_0|^2 \, dxdt \\ + s^2\lambda^2 \iint_Q e^{-2s\eta}\, \varphi^2 |F_1|^2 \, dxdt \Big], \tag{3.1}$$

for $s \geq s_2$, $\lambda \geq \lambda_2$ and for all $q \in \widetilde{\aleph}_{\pm}$.

The proof can be adapted from the proof given in [10, Lemma 2.1]. We only highlight the main points in the proof.

Proof. We treat the case of $q \in \widetilde{\aleph}_{+}$ with $\partial_t q + \partial_x(c\partial_x q) = F_0 + \partial_x F_1$. The other case can be treated similarly. With the notations $\mathcal{L} = \partial_t - \partial_x(c\partial_x)$ and $\mathcal{L}^* = -\partial_t - \partial_x(c\partial_x)$, we define the bilinear form

$$\kappa(p, p') = \iint_Q e^{-2s\eta}\mathcal{L}^* p \mathcal{L}^* p' \, dxdt + s^3\lambda^4 \iint_{(0,T)\times\mathcal{O}} e^{-2s\eta}\varphi^3 \, pp' \, dxdt, \tag{3.2}$$

which is a scalar product on $P_0 = \mathscr{C}^2([0,T], D(A))$ from Carleman estimate (1.3). We denote by P the Hilbert space defined as the completion of P_0 for the norm $\|p\|_P = (\kappa(p,p))^{1/2}$. We find, from Riesz Theorem, that there exists a unique $p \in P$ such that

$$\kappa(p, p') = l(p'), \quad \forall p' \in P, \tag{3.3}$$

where l is the continuous form on P defined by $l(p') = -s^3\lambda^4 \iint_Q e^{-2s\eta}\varphi^3 \, qp' \, dxdt$. Observe that the elements of P are functions in Q for which the l.h.s. of (1.3) is finite. In particular observe that $e^{-s\eta}p \in L^2(Q)$ and $e^{-s\eta}\varphi^{-1/2}\mathcal{L}^* p \in L^2(Q)$.

If we now solve the parabolic problem

$$\begin{cases} \mathcal{L}\hat{z} = +s^3\lambda^4 e^{-2s\eta}\varphi^3\, (p 1_{\mathcal{O}} + q) & \text{in } Q, \\ \hat{z} = 0 & \text{on } \Sigma, \\ \hat{z}(0) = 0 & \text{in } \Omega, \end{cases}$$

there is a unique weak solution $\hat{z} \in L^2(0,T, H_0^1(\Omega)) \cap \mathscr{C}([0,T], L^2(\Omega))$ [17]. We now observe that $\hat{z} = -e^{-2s\eta}\mathcal{L}^* p$ from (3.3). Since $e^{-s\eta}\varphi^{-1/2}\mathcal{L}^* p \in L^2(Q)$, we then have

$\hat{z}(T) = 0$, because $\hat{z} \in \mathscr{C}([0,T], L^2(\Omega))$. The function p defined above is thus a weak solution to

$$\begin{cases} \mathcal{L}(e^{-2s\eta}\mathcal{L}^* p) = -s^3\lambda^4 e^{-2s\eta}\varphi^3\,(p1_O + q) & \text{in } Q,\\ p = 0,\ e^{-2s\eta}\mathcal{L}^* p = 0 & \text{on } \Sigma,\\ (e^{-2s\eta}\mathcal{L}^* p)(0) = (e^{-2s\eta}\mathcal{L}^* p)(T) = 0 & \text{in } \Omega. \end{cases}$$

Introducing $\hat{u} = s^3\lambda^4 e^{-2s\eta}\varphi^3 p1_O$, and $G = s^3\lambda^4 e^{-2s\eta}\varphi^3 q + \hat{u}$, we note that

$$\begin{cases} \mathcal{L}\hat{z} = G & \text{in } Q,\\ \hat{z} = 0 & \text{on } \Sigma,\\ \hat{z}(0) = \hat{z}(T) = 0 & \text{in } \Omega. \end{cases}$$

From the equation satisfied by $q \in \widetilde{\aleph}_+$ we obtain

$$\int_0^T \langle G(t), q(t)\rangle\, dt = -\int_0^T \langle F_0(t) + \partial_x F_1(t), \hat{z}(t)\rangle, \tag{3.4}$$

where $\langle .,.\rangle$ denotes the duality brackets for $H_0^1(\Omega)$ and $H^{-1}(\Omega)$. Noting that the function β, and the weight functions φ and η are in $W^{1,\infty}$ w.r.t. the space variable, we can follow the proof of Lemma 2.1 in [10] to prove

$$\begin{aligned} s^{-3}\lambda^{-4}\iint_{(0,T)\times O} e^{2s\eta}\varphi^{-3}|\hat{u}|^2\, dxdt + \iint_Q e^{2s\eta}|\hat{z}|^2\, dxdt\\ + s^{-2}\lambda^{-2}\iint_Q e^{2s\eta}\varphi^{-2}|\partial_x\hat{z}|^2\, dxdt \le Cs^3\lambda^4\iint_Q e^{-2s\eta}\varphi^3|q|^2\, dxdt, \end{aligned} \tag{3.5}$$

for $s \ge s_2'(T + T^2)$ and $\lambda \ge \lambda_2'$ (Inequality (2.20) in [10]).

From (3.5) and (3.4), we first obtain (see [10])

$$\begin{aligned} s^3\lambda^4\iint_Q e^{-2s\eta}\varphi^3\,|q|^2\, dxdt \le C\Bigg[s^3\lambda^4\iint_{(0,T)\times O} e^{-2s\eta}\varphi^3\,|q|^2\, dxdt\\ + \iint_Q e^{-2s\eta}\,|F_0|^2\, dxdt + s^2\lambda^2\iint_Q e^{-2s\eta}\,\varphi^2|F_1|^2\, dxdt\Bigg], \end{aligned} \tag{3.6}$$

for $s \ge s_2''(T + T^2)$ and $\lambda \ge \lambda_2''$.

To obtain the first term on the l.h.s. of (3.1) we multiply $\partial_t q + \partial_x(c\partial_x q) = F_0 + \partial_x F_1$ by $e^{-2s\eta}\varphi q$ and we integrate over Q. This then yields

$$\begin{aligned} -\frac{1}{2}\iint_Q \partial_t(e^{-2s\eta}\varphi)|q|^2\, dxdt - \iint_Q e^{-2s\eta}\varphi c|\partial_x q|^2\, dxdt\\ - \iint_Q \partial_x(e^{-2s\eta}\varphi)cq\partial_x q\, dxdt = \iint_Q \Big(F_0 e^{-2s\eta}\varphi q\ - F_1\partial_x(e^{-2s\eta}\varphi q)\Big)\, dxdt. \end{aligned} \tag{3.7}$$

As the function β, and the weight functions φ and η are in $W^{1,\infty}$ w.r.t. the space variable, the integration by part w.r.t. the space variable is justified since $q(t,.) \in D(A)$. We observe that

$$|\partial_x(e^{-2s\eta}\varphi)| = |s\lambda(\partial_x\beta)\varphi^2 e^{-2s\eta} + \lambda(\partial_x\beta)\varphi e^{-2s\eta}| \le Cs\lambda\varphi^2 e^{-2s\eta} + \lambda\varphi e^{-2s\eta},\ a.e. \text{ in } \Omega,$$

which yields

$$\left|\iint_Q \partial_x(e^{-2s\eta}\varphi)cq\partial_x q\; dxdt\right| \leq \varepsilon \iint_Q \varphi e^{-2s\eta}|\partial_x q|^2\; dxdt$$
$$+ C_\varepsilon s^2\lambda^2 \iint_Q \varphi^3 e^{-2s\eta}|q|^2\; dxdt + C_\varepsilon \lambda^2 \iint_Q \varphi e^{-2s\eta}|q|^2\; dxdt,$$

for any $\varepsilon > 0$. Next, we estimate the first term on the l.h.s. of (3.7) and the r.h.s. of (3.7), as in [10], to obtain

$$\left|\iint_Q \partial_t(e^{-2s\eta}\varphi)|q|^2\; dxdt\right| \leq Cs^2 \iint_Q \varphi^3 e^{-2s\eta}|q|^2\; dxdt,$$

and

$$\left|\iint_Q \left(F_0 e^{-2s\eta}\varphi q\; - F_1\partial_x(e^{-2s\eta}\varphi q)\right) dxdt\right| \leq Cs^2\lambda^2 \iint_Q \varphi^3 e^{-2s\eta}|q|^2\; dxdt$$
$$+ +\varepsilon \iint_Q \varphi e^{-2s\eta}|\partial_x q|^2\; dxdt + Cs^{-2}\lambda^{-2} \iint_Q \varphi^{-1} e^{-2s\eta}|F_0|^2\; dxdt$$
$$+ (C + C_\varepsilon) \iint_Q \varphi e^{-2s\eta}|F_1|^2\; dxdt,$$

for any $\varepsilon > 0$ and for $s \geq C(T + T^2)$. Using $1 \leq C\varphi T^2$, and taking ε sufficiently small, we obtain

$$\left|\iint_Q \varphi e^{-2s\eta}|\partial_x q|^2\; dxdt\right| \leq C\Bigg[s^2\lambda^2 \iint_Q e^{-2s\eta}\varphi^3\; |q|^2\; dxdt$$
$$+s^{-1}\lambda^{-2} \iint_Q e^{-2s\eta}\; |F_0|^2\; dxdt + s \iint_Q e^{-2s\eta}\; \varphi^2|F_1|^2\; dxdt\Bigg],$$

for $s \geq s_2'''(T + T^2)$ and $\lambda \geq \lambda_2'''$. This last estimate, along with (3.6), gives the desired Carleman estimate. ■

4 Controllability results

The Carleman estimate proven in Section 3 allows to give observability estimates that yield null controllability results for classes of semilinear heat equations. We let $\omega \Subset \Omega$ be a non-empty open set and $c \in BV(\Omega)$ with $0 < c_{min} \leq c \leq c_{max}$ and c of class $\mathscr{C}^1$ on $\overline{\mathcal{O}}$, with $\mathcal{O}$ some open subset of ω.

We first state observability results with L^2 and L^1 observations. We let a and b be in $L^\infty(Q)$ and $q_T \in L^2(\Omega)$. From Carleman estimate (3.1) we obtain

Lemma 4.1. *The solution q to*

$$\begin{cases} -\partial_t q - \partial_x(c\partial_x q) + aq - \partial_x(bq) = 0 & \text{in } Q, \\ q = 0 & \text{on } \Sigma, \\ q(T) = q_T & \text{in } \Omega, \end{cases} \tag{4.1}$$

satisfies

$$\|q(0)\|^2_{L^2(\Omega)} \le e^{CK(T,\|a\|_\infty,\|b\|_\infty)} \iint_{(0,T)\times\omega} |q|^2 \, dxdt, \tag{4.2}$$

where $K(T,\|a\|_\infty,\|b\|_\infty) = 1 + \frac{1}{T} + T\|a\|_\infty + \|a\|_\infty^{2/3} + (1+T)\|b\|^2_\infty$.

The proof of this lemma can be found in [10, 8, 7]. From Lemma 4.1, we can then obtain the following observability results with an L^1 observation, which will yield controls in $L^\infty((0,T)\times\omega)$ below.

Lemma 4.2. *The solution q to system (4.1) satisfies*

$$\|q(0)\|^2_{L^2(\Omega)} \le e^{CH(T,\|a\|_\infty,\|b\|_\infty)} \left(\iint_{(0,T)\times\omega} |q| \, dxdt \right)^2, \tag{4.3}$$

where

$$H(T,\|a\|_\infty,\|b\|_\infty) = 1 + \frac{1}{T} + T + (T + T^{1/2})\|a\|_\infty + \|a\|_\infty^{2/3} + (1+T)\|b\|^2_\infty. \tag{4.4}$$

Since the coefficient c is $\mathscr{C}^1$ on the open set ω, the proof of [7, Theorem 2.5, Lemma 2.5] can be adapted. See also [8, Proposition 4.2, Lemma 4.3].

Consider now the following *linear* system

$$\begin{cases} \partial_t y - \partial_x(c\partial_x y) + ay + b\partial_x y = 1_\omega v & \text{in } Q, \\ y = 0 & \text{on } \Sigma, \\ y(0) = y_0 & \text{in } \Omega, \end{cases} \tag{4.5}$$

with a and b in $L^\infty(Q)$ and $y_0 \in L^2(\Omega)$. If $v \in L^2(Q)$, we consider its unique weak solution in $\mathscr{C}([0,T],L^2(\Omega)) \cap L^2(0,T,H^1_0(\Omega))$ [17, 6]. We have the following null controllability result for (4.5)

Theorem 4.3. *For all $T > 0$ and for all y_0 in $L^2(\Omega)$, there exists $v \in L^\infty((0,T)\times\omega)$, such that the solution y_v to (4.5) satisfies $y_v(T) = 0$. Moreover, the control v can be chosen such that*

$$\|v\|_{L^\infty((0,T)\times\omega)} \le e^{CH(T,\|a\|_\infty,\|b\|_\infty)} \|y_0\|_{L^2(\Omega)}, \tag{4.6}$$

with $H(T,\|a\|_\infty,\|b\|_\infty)$ as given in (4.4).

The proof of Theorem 3.1 in [7] can be adapted to the present case. It is based on the argument developed in [9]. It makes use of the observability result in Lemma 4.2.

For the null controllability of the quasi-linear heat equation we shall need estimates for the solution to the following linear system

$$\begin{cases} \partial_t y - \partial_x(c\partial_x y) + ay + b\partial_x(y) = f & \text{in } Q, \\ y = 0 & \text{on } \Sigma, \\ y(0) = y_0 & \text{in } \Omega, \end{cases} \tag{4.7}$$

with a and b in $L^\infty(Q)$ and $y_0 \in L^2(\Omega)$, $f \in L^2(Q)$. We have the following classical estimates.

Lemma 4.4. *The solution y to system (4.7) satisfies*

$$(4.8)\quad \|y(t)\|^2_{L^2(\Omega)} + \|\partial_x y\|^2_{L^2(Q)} + \|y\|^2_{L^2(Q)} \leq K_1(T, \|a\|_\infty, \|b\|_\infty)(\|f\|^2_{L^2(Q)} + \|y(0)\|^2_{L^2(\Omega)}),$$

for $0 \leq t \leq T$, with $K_1(T, \|a\|_\infty, \|b\|_\infty) = e^{C(1+T+T\|a\|_\infty+T\|b\|^2_\infty)}$. If $y_0 \in H^1_0(\Omega)$ then, $y \in \mathscr{C}([0,T], H^1_0(\Omega))$ and

$$(4.9)\quad \|\partial_x y(t)\|^2_{L^2(\Omega)} + \|\partial_t y\|^2_{L^2(Q)} + \|\partial_x(c\partial_x y)\|^2_{L^2(Q)} \\ \leq K_2(T, \|a\|_\infty, \|b\|_\infty)(\|f\|^2_{L^2(Q)} + \|y(0)\|^2_{H^1_0(\Omega)}), \quad 0 \leq t \leq T,$$

with $K_2(T, \|a\|_\infty, \|b\|_\infty) = e^{C(1+T+(T+T^{1/2})\|a\|_\infty+(T+T^{1/2})\|b\|^2_\infty)}$.

With further regularity on f and y_0 we actually obtain

Lemma 4.5. *Let $f \in L^\infty(0,T,L^2(\Omega))$ and $y_0 \in D(A)$. The solution y to system (4.7) satisfies*

$$(4.10)\qquad \|\partial_x y(t)\|_{L^\infty(\Omega)} \leq K_3(T, \|a\|_\infty, \|b\|_\infty)(\|f\|_{L^\infty(0,T,L^2(\Omega))} + \|y\|_{D(A)}),$$

with

$$(4.11)\qquad K_3(T, \|a\|_\infty, \|b\|_\infty) = e^{C(1+T+(T+l_s(T))\|a\|_\infty+(T+l_s(T)^2)\|b\|^2_\infty)},$$

for l a non-negative increasing function such that $l(0) = 0$. More precisely, $l_S(t) = \int_0^t (\frac{1}{t} + \frac{1}{\sqrt{t}})^s (\frac{1}{\sqrt{t}})^{1-s}\, d\tau$ with $\frac{1}{2} < s < 1$.

The domain of $A = \partial_x(c\partial_x)$, $D(A)$, is furnished with the norm of the graph denoted by $\|.\|_{D(A)}$. Note that in the proof we make use of the fact that Ω is one-dimensional.

Proof. We first recall some properties of the semigroup $S(t)$ generated by $A = \partial_x(c\partial_x)$. Consider the system

$$(4.12)\qquad \begin{cases} \partial_t u - \partial_x(c\partial_x u) = 0 & \text{in } Q, \\ u = 0 & \text{on } \Sigma, \\ u(0) = u_0 & \text{in } \Omega, \end{cases}$$

with $u_0 \in L^2(\Omega)$. The solution is given by $u(t) = S(t)u_0$. Since the semigroup $S(t)$ is analytic, we have [18, 6]

$$\|u(t)\|_{L^2(\Omega)} \leq \|u_0\|_{L^2(\Omega)}, \text{ and } \|Au(t)\|_{L^2(\Omega)} \leq \tfrac{1}{t}\|u_0\|_{L^2(\Omega)}, \quad 0 < t \leq T.$$

We can then write

$$|(Au(t), u(t))_{L^2(\Omega)}| \leq \tfrac{1}{t}\|u_0\|_{L^2(\Omega)}\|u(t)\|_{L^2(\Omega)} \leq \tfrac{1}{t}\|u_0\|^2_{L^2(\Omega)}, \quad 0 < t \leq T,$$

which by integration by parts yields

$$\|c\partial_x u(t)\|_{L^2(\Omega)} \leq \tfrac{1}{\sqrt{t}}\|u_0\|_{L^2(\Omega)}, \quad 0 < t \leq T.$$

As $\|c\partial_x u(t)\|_{H^1(\Omega)} \leq (\frac{1}{t} + \frac{1}{\sqrt{t}})\|u_0\|_{L^2(\Omega)}$, the interpolation inequality [17]

$$\|\phi\|_{H^s(\Omega)} \leq \|\phi\|^s_{H^1(\Omega)}\, \|\phi\|^{1-s}_{L^2(\Omega)},$$

for $0 \leq s \leq 1$, yields

$$\|c\partial_x u(t)\|_{H^s(\Omega)} \leq h_s(t)\|u_0\|_{L^2(\Omega)}. \tag{4.13}$$

with $h_s(t) = (\frac{1}{t} + \frac{1}{\sqrt{t}})^s(\frac{1}{\sqrt{t}})^{1-s} \sim_{t\to 0} t^{-\frac{s+1}{2}}$. We choose $\frac{1}{2} < s < 1$. Then $h_s(t)$ is integrable on $[0, T]$.

The solution to (4.7) can be written by Duhamel's formula [18]

(4.14)
$$y(t) = S(t)y_0 + \int_0^t S(t-\tau)f(\tau)\,d\tau - \int_0^t S(t-\tau)\,(ay)(\tau)\,d\tau - \int_0^t S(t-\tau)\,(b\partial_x y)(\tau)\,d\tau.$$

For the first term in (4.14), $y_1(t) = S(t)y_0$, we have $Ay_1(t) = S(t)Ay_0$ [18], which yields

$$\|A(y_1)(t)\|_{L^2(\Omega)} \leq \|A(y_0)\|_{L^2(\Omega)}.$$

By Lemma 4.4, we have $\|c\partial_x y_1\|_{L^2(\Omega)} \leq e^{C(1+T)}\|y_0\|_{H^1_0(\Omega)}$, which gives

$$\|c\partial_x y_1(t)\|_{H^1(\Omega)} \leq e^{C(1+T)}\|y_0\|_{D(A)}. \tag{4.15}$$

For the second term, y_2, in (4.14) we have

$$\|c\partial_x y_2(t)\|_{H^s(\Omega)} \leq \int_0^t \|c\partial_x(S(t-\tau)f(\tau))\|_{H^s(\Omega)}\,d\tau \leq \int_0^t h_s(t-\tau)\|f(\tau)\|_{L^2(\Omega)}\,d\tau$$

by (4.13). We set $l_s(t) = \int_0^t h_s(t-\tau)d\tau = \int_0^t h_s(\tau)d\tau$, and obtain

$$\|c\partial_x y_2(t)\|_{H^s(\Omega)} \leq \left(\int_0^t h_s(t)\,d\tau\right)\|f\|_{L^\infty(0,T,L^2(\Omega))} = l_s(t)\|f\|_{L^\infty(0,T,L^2(\Omega))}. \tag{4.16}$$

For the third term, y_3, in (4.14) we have

$$\begin{aligned}\|c\partial_x y_3(t)\|_{H^s(\Omega)} &\leq \int_0^t \|c\partial_x(S(t-\tau)\,(ay)(\tau))\|_{H^s(\Omega)}\,d\tau\\ &\leq \int_0^t h_s(t)\|ay(\tau)\|_{L^2(\Omega)}\,d\tau \leq l_s(t)\|a\|_\infty\|y\|_{L^\infty(0,T,L^2(\Omega))}\\ &\leq l_s(t)\|a\|_\infty K_1(T, \|a\|_\infty, \|b\|_\infty)\,(\|f\|_{L^2(Q)} + \|y(0)\|_{L^2(\Omega)}),\end{aligned}$$

by Lemma 4.4. Observe that the function l_s is increasing. This yields

$$\|c\partial_x y_3(t)\|_{H^s(\Omega)} \leq e^{C(1+T+(T+l_s(T))\|a\|_\infty+T\|b\|_\infty)}\,(\|f\|_{L^2(Q)} + \|y(0)\|_{L^2(\Omega)}). \tag{4.17}$$

Finally, for the fourth term, y_4, in (4.14) we have

$$\begin{aligned}\|c\partial_x y_4(t)\|_{H^s(\Omega)} &\leq Cl_s(t)\|b\|_\infty\|\partial_x y\|_{L^\infty(0,T,L^2(\Omega))}\\ &\leq l_s(t)\|b\|_\infty K_2(T, \|a\|_\infty, \|b\|_\infty)(\|f\|_{L^2(Q)} + \|y(0)\|_{H^1_0(\Omega)})\\ &\leq e^{C(1+T+(T+T^{1/2})\|a\|_\infty+(T+l_s(T)^2)\|b\|^2_\infty)}.\end{aligned} \tag{4.18}$$

Collecting estimates (4.15), (4.16), (4.17), and (4.18) we obtain

$$\|c\partial_x y(t)\|_{H^s(\Omega)} \leq e^{C(1+T+(T+l_s(T))\|a\|_\infty+(T+l_s(T)^2)\|b\|^2_\infty)}\,(\|f\|_{L^\infty(0,T,L^2(\Omega))} + \|y_0\|_{D(A)}). \tag{4.19}$$

Since the space $H^s(\Omega)$ can be continuously injected in $\mathscr{C}(\overline{\Omega})$ because Ω is one dimensional (see e.g. [17]), for $s > \frac{1}{2}$, the result follows, since $c \geq c_{min} > 0$. ∎

We are now ready to prove the null controllability result for system (2) which is based on a fixed point argument.

Theorem 4.6. *We let $\omega \Subset \Omega$ be a non-empty open set and $c \in BV(\Omega)$ with $0 < c_{min} \leq c \leq c_{max}$ and c of class $\mathscr{C}^1$ on some non-empty open subset of ω. We assume that $\mathscr{G}$ is locally Lipschitz. Let $T > 0$:*

1. *Local null controllability: There exists $\varepsilon > 0$ such that for all y_0 in $L^2(\Omega)$ with $\|y_0\|_{L^2(\Omega)} \leq \varepsilon$, there exists a control $v \in L^\infty((0,T)\times\omega)$ such that the corresponding solution to system (2) satisfies $y(T) = 0$.*
2. *Global null controllability: Let $\mathscr{G}$ satisfy in addition Assumption 1. Then for all y_0 in $L^2(\Omega)$, there exists $v \in L^\infty((0,T)\times\omega)$ such that the solution to system (2) satisfies $y(T) = 0$.*

The proof is classical and is along the same lines as those that in [7, 8] and originates from [2, 11].

Proof. We first assume that g and G are continuous. We let $R > 0$ and set $Z = L^2(0,T,H_0^1(\Omega))$. The truncation function T_R is defined as

$$T_R(s) = \begin{cases} s & \text{if } |s| \leq R, \\ R\,\mathrm{sgn}(s) & \text{otherwise.} \end{cases}$$

For $z \in Z$ we consider the following *linear* system

(4.20)
$$\begin{cases} \partial_t y_{z,v} - \partial_x(c\partial_x y_{z,v}) + g(T_R(z), T_R(\partial_x z))y_{z,v} + G(T_R(z), T_R(\partial_x z))\partial_x y_{z,v} = 1_\omega v & \text{in } Q, \\ y_{z,v} = 0 & \text{on } \Sigma, \\ y_{z,v}(0) = y_0 & \text{in } \Omega, \end{cases}$$

Since g and G are continuous, we see that the functions $a_z := g(T_R(z), T_R(\partial_x z))$ and $b_z := G(T_R(z), T_R(\partial_x z))$ are in $L^\infty(Q)$ and have bounds in L^∞ that only depends on g, G, and R. If $y_0 \in L^2(\Omega)$ and if $v = 0$ for $t \in [0,\delta]$, $\delta > 0$, we obtain $y_{z,v}(\delta) \in D(A)$. Without any loss of generality we may thus assume that $y_0 \in D(A)$. We apply Theorem 4.3 to system (4.20). We set

$$T_z = \min(T, \|a_z\|_\infty^{-2/3}, \|a_z\|_\infty^{-1/3}, l_s^{-1}(\|a_z\|_\infty^{-1/3})),$$

with the function l_s defined in Lemma 4.5. Then we have

$$e^{CH(T_z,\|a_z\|_\infty,\|b_z\|_\infty)} \leq \mathfrak{K}, \quad K_2(T_z, \|a_z\|_\infty, \|b_z\|_\infty) \leq \mathfrak{K}, \quad K_3(T_z, \|a_z\|_\infty, \|b_z\|_\infty) \leq \mathfrak{K},$$

with $\mathfrak{K} = e^{(C(T_z)(1+\|a_z\|_\infty^{2/3}+\|b_z\|_\infty^2))}$, for H, K_2 and K_3 the constants in (4.6), (4.9), and (4.11). According to Theorem 4.3, there exists v_z in $L^\infty(Q)$ such that v_z and the associated solution to (4.20), with $v = v_z$ satisfy $y_{z,v}(T) = 0$ and

(4.21) $$\|v_z\|_{L^\infty((0,T)\times\omega)} \leq \mathfrak{H}\|y_0\|_{L^2(\Omega)},$$

(4.22) $$\|y_{z,v}\|_{L^\infty(0,T,W^{1,\infty}(\Omega))} \leq \mathfrak{H}\|y_0\|_{D(A)},$$

with $\mathfrak{H}$ of the same form as $\mathfrak{K}$, by Lemma 4.4 and Lemma 4.5, making use of the continuous injection $H_0^1(\Omega) \hookrightarrow L^\infty(\Omega)$ in the one-dimensional case. Observe also that we have

$$\|y_{z,v}\|_{L^2(0,T,D(A))} + \|\partial_t y_{z,v}\|_{L^2(Q)} \leq \mathfrak{H}\|y_0\|_{H_0^1(\Omega)}, \tag{4.23}$$

by Lemma 4.4. We now set

$$U(z) = \{v \in L^\infty((0,T)\times\omega);\ y_{z,v}(T) = 0,\ (4.21) \text{ holds}\}$$

$$\text{and } \Lambda(z) = \left\{y_{z,v};\ v \in U(z),\ (4.22) \text{ holds}\right\}.$$

The map $z \mapsto \Lambda(z)$ from Z into $\mathscr{P}(Z)$, the power set of Z, satisfies the following properties

1. for all $z \in Z$, $\Lambda(z)$ is a non-empty bounded closed convex set. Boundedness is however uniform w.r.t. to z (and only depends on R);
2. there exists a compact set $\mathcal{K} \subset Z$, such that $\Lambda(z) \subset \mathcal{K}$: by (4.23) $\Lambda(z)$ is uniformly bounded in $L^2(0,T,D(A)) \cap H^1(0,T,L^2(\Omega))$, which injects compactly in $L^2(Q)$ [16, Theorem 5.1, Chapter 1] since $D(A)$ injects compactly in $H_0^1(\Omega)$;
3. adapting the method of [7, pages 811–812] to the present case, we obtain that the map Λ is upper hemicontinuous; the argument uses the continuity of g and G.

These properties allow us to apply Kakutani's fixed point theorem [1, Theorem 1, Chapter 15, Section 3] to the map Λ.

Result 1 follows by choosing ε sufficiently small such that the (essential) supremum on Q of the obtained fixed point is less than R by (4.22).

Result 2 follows if we prove that R can be chosen greater that the (essential) supremum on Q of the obtained fixed point. This is done exactly as in [7, page 813] and makes use of the form of $\mathfrak{H}$, estimate (4.22) and Assumption 1 on $\mathscr{G}$.

To treat the case where g and G are not continuous, we adapt the argument of [7, Section 3.2.1] to the present cases, for both the local and global controllability results. ∎

Arguing as in [13] or e.g. [7] we can actually prove the following null controllability result with a boundary control from Theorem 4.6 :

Theorem 4.7. *We let $c \in BV(\Omega)$ with $0 < c_{min} \leq c \leq c_{max}$. We assume that $\mathscr{G}$ is locally Lipschitz. Let γ be $\{0\}$ or $\{1\}$. Let $T > 0$.*

1. *Local null controllability: There exists $\varepsilon > 0$ such that for all y_0 in $L^2(\Omega)$ with $\|y_0\|_{L^2(\Omega)} \leq \varepsilon$, there exists a control $v \in \mathscr{C}(0,T)$ such that the solution to system*

$$\begin{cases} \partial_t y - \partial_x(c\partial_x y) + \mathscr{G}(y) = 0 & \text{in } Q, \\ y = 0 & \text{on } \Sigma \setminus \gamma, \\ y = v & \text{on } \gamma, \\ y(0) = y_0 & \text{in } \Omega, \end{cases} \tag{4.24}$$

satisfies $y(T) = 0$.

2. *Global null controllability: Assume the function $\mathscr{G}$ satisfies in addition Assumption 1. Then for all y_0 in $L^2(\Omega)$, there exists $v \in \mathscr{C}(0,T)$ such that the solution to system (4.24) satisfies $y(T) = 0$.*

Remark 4.8. 1. Note that for the distributed control (Theorem 4.6) we require that the coefficient c be of class $\mathscr{C}^1$ on an non-empty open subset of ω. On the other hand, for a boundary control (Theorem 4.7) there is no such restriction on the coefficient c, which can have a very singular behavior as the control boundary is approached.

2. Note that as usual, one can replace $y(T) = 0$ by $y(T) = y^*(T)$ in the previous statements, where y^* is any trajectory defined in $[0,T]$ of system (2) (resp. (4.24)), corresponding to some initial data y_0^* and any v^* in $L^\infty((0,T)\times\omega)$ (resp. $\mathscr{C}(0,T)$). For the local controllability result, one has to assume $\|y_0 - y_0^*\|_{L^2(\Omega)} \leq \varepsilon$, with ε sufficiently small.

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