Remark on the Kato smoothing effect for Schrödinger equation with superquadratic potentials

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Résumé

The aim of this note is to extend recent results of Yajima-Zhang [Y-Z1, Y-Z2] on the $\frac{1}{t^2}$ smoothing effect for Schrödinger equation with potential growing at infinity faster than quadratically.

1 Introduction

The aim of this note is to extend a recent result by Yajima-Zhang [Y-Z1, Y-Z2]. In this paper these authors considered the Hamiltonian $H = -\Delta + V(x)$ where $V$ is a real and $C^\infty$ potential on $\mathbb{R}^n$ satisfying for some $m > 2$ and $\langle x \rangle = (1 + |x|^2)^{1/2}$,

\begin{equation}
|\partial^\alpha V(x)| \leq C_\alpha \langle x \rangle^{m-|\alpha|}, \quad x \in \mathbb{R}^n, \quad \alpha \in \mathbb{N}^n,
\end{equation}

for large $|x|$, $V(x) \geq C_1 |x|^m$, $C_1 > 0$, and they proved the following. For any $T > 0$ and $\chi \in C^\infty_0(\mathbb{R}^n)$ one can find $C > 0$ such that for all $u_0$ in $L^2(\mathbb{R}^n)$,

\begin{equation}
\int_0^T \|\chi(I - \Delta)^{\frac{m}{2}} e^{-itH} u_0\|_{L^2(\mathbb{R}^n)}^2 dt \leq C \|u_0\|_{L^2(\mathbb{R}^n)}^2
\end{equation}

where $\Delta$ is the flat Laplacian. In this note, using the ideas contained in Doi [D3] we shall show that one can handle variable coefficients Laplacian with time dependent potentials, one can remove the condition (1.2), one can replace the cut-off function $\chi$ in (1.3) by $\langle x \rangle^{-\frac{1+\nu}{2}}$ with any $\nu > 0$ and finally that the weight $\langle x \rangle^{-\frac{1}{2}}$ is enough for the tangential derivatives.

When $V = 0$ the estimate (1.3) goes back to Constantin-Saut [C-S], Sjölin [S], Vega [V], Yajima [Y] who extended to the Schrödinger equation a phenomenon discovered by T. Kato [K] on the KdV equation. Later on their results where extended to the variable coefficients operators by Doi in a series of papers [D1, D2, D3, D4] which contained the case $m = 2$ of Theorem 1.1 below.
Let us describe more precisely our result. It will be convenient to introduce the Hörmander’s metric

\[(1.4) \quad g = \frac{dx^2}{\langle x \rangle^2} + \frac{d\xi^2}{\langle \xi \rangle^2} \]

to which we associate the usual class of symbols $S(M, g)$ if $M$ is a weight. Recall that $q \in S(M, g)$ if $q \in C^\infty(\mathbb{R}^{2n})$ and

$$\forall \alpha, \beta \in \mathbb{N}^n \exists C_{\alpha\beta} > 0, \ |\partial^\beta_\xi \partial^\alpha_q q(x, \xi)| \leq C_{\alpha\beta} M(x, \xi) \langle x \rangle^{-|\beta|} \langle \xi \rangle^{-|\alpha|}, \ \forall (x, \xi) \in T^*(\mathbb{R}^n)$$

If $T > 0$ we shall set

\[(1.5) \quad S_T(M, g) = L^\infty([0, T], S(M, g)). \]

We shall consider here an operator $P$ of the form

\[(1.6) \quad P = \sum_{j,k=1}^n (D_j - a_j(t, x)) g^{jk}(x)(D_k - a_k(t, x)) + V(t, x) \]

and we shall denote by $p$ the principal symbol of $P$, namely

\[(1.7) \quad p(x, \xi) = \sum_{j,k=1}^n g^{jk}(x) \xi_j \xi_k. \]

We shall make the following structure and geometrical assumptions.

Structure assumptions. We shall assume the following,

\[(1.8) \quad \begin{cases} 
(i) \text{ the coefficients } a_j, g^{jk}, V \text{ are real valued for } j, k = 1, \ldots, n, \\
(ii) \quad p \in S(\langle \xi \rangle^2, g) \text{ and } \nabla g^{jk}(x) = o(|x|^{-1}), \ |x| \to +\infty, 1 \leq j, k \leq n \\
(iii) \quad a_j \in S_T(\langle x \rangle^{\frac{m}{2}}, g), 1 \leq j \leq n, \ V \in S_T(\langle x \rangle^{m}, g) \ m \geq 2 
\end{cases} \]

\[(1.9) \quad \exists \delta > 0, \ p(x, \xi) \geq \delta |\xi|^2, \ \forall (x, \xi) \in T^*(\mathbb{R}^n). \]

\[(1.10) \quad \text{For any fixed } t \in [0, T] \text{ the operator } P \text{ is essentially self adjoint on } L^2(\mathbb{R}^n) \]

Geometrical assumptions. Let $\phi_t$ be the bicharacteristic flow of $p$. It is easy to see that under the conditions (1.8), (1.9) it is defined for all $t \in \mathbb{R}$. Let us set $S^*(\mathbb{R}^n) = \{(x, \xi) \in T^*(\mathbb{R}^n) : p(x, \xi) = 1\}$. Then we shall assume that,

\[(1.11) \quad \forall K \text{ compact } \subseteq S^*(\mathbb{R}^n) \ \exists t_K > 0 \text{ such that } \Phi_t(K) \cap K = \emptyset, \ \forall t \geq t_K. \]

This is the so-called ”non trapping condition” which is equivalent to the fact that if $\Phi_t(x; \xi) = (x(t), (\xi(t)))$ then $\lim_{t \to +\infty} |x(t)| = +\infty$.

We shall consider $u \in C^1([0, T], S(\mathbb{R}^n))$ and we set

\[(1.12) \quad f(t) = (D_t + P)u(t) \]

For $s \in \mathbb{R}$ let $e_s(x, \xi) = (1 + |\xi|^2 + |x|^m)^{\frac{s}{2}}$ and $E_s$ be the Weyl quantized pseudo-differential operator with symbol $e_s$.

Our first result is the following.
**Theorem 1.1** Let $T > 0$. Let $P$ be defined by (1.6) which satisfies (1.8), (1.9), (1.10), and (1.11). Then for any $\nu > 0$ one can find $C = C(\nu, T) > 0$ such that for any $u \in C^1([0, T], \mathcal{S}(\mathbb{R}^n))$ and all $t$ in $[0, T]$ we have,

$$\|u(t)\|_{L^2}^2 + \int_0^T \|\langle x \rangle^{-\frac{\nu}{2}} E_{\lambda} u(t)\|_{L^2}^2 dt \leq C (\|u(0)\|_{L^2}^2 + \int_0^T \|\langle x \rangle^{-\frac{\nu}{2}} E_{-\lambda} f(t)\|_{L^2}^2 dt).$$

Here $L^2 = L^2(\mathbb{R}^n)$ and $f(t)$ is defined by (1.6).

Now even when $P$ is the flat Laplacian it is known that the estimate in the above Theorem does not hold with $\nu = 0$. However we have the following result. Let us set

(1.13) $$\ell_{jk} = \frac{x_j \xi_k - x_k \xi_j}{\langle x \rangle \langle \xi \rangle}, \quad 1 \leq j, k \leq n,$$

and let us denote by $\ell_{jk}^w$ its Weyl quantization.

**Theorem 1.2** Let $T > 0$. Let $P$ be defined by (1.6) with real coefficients satisfying (1.9), (1.10), and (1.11) and

(1.14) $$\left\{ \begin{array}{ll}
(i) & g_{jk} = \delta_{jk} + b_{jk}, \quad b_{jk} \in S(\langle x \rangle^{-\sigma_0}, g), \text{ for some } \sigma_0 > 0, \\
(ii) & a_j \in S_T(\mathbb{R}^n, g), \quad V \in S_T(\langle x \rangle^m, g).
\end{array} \right.$$ 

Then for any $\nu > 0$ one can find $C = C(\nu, T)$ such that for any $u \in C^1([0, T], \mathcal{S}(\mathbb{R}^n))$ and $f(t) = (D_t + P)u(t)$ we have

$$\sum_{j,k=1}^n \int_0^T \|\langle x \rangle^{-\frac{\nu}{2}} E_{\lambda} \ell_{jk}^w u(t)\|_{L^2}^2 dt \leq C (\|u(0)\|_{L^2}^2 + \int_0^T \|\langle x \rangle^{-\frac{\nu}{2}} E_{-\lambda} f(t)\|_{L^2}^2 dt).$$

Here are some remarks and examples.

**Remark 1.3** 1) We know that one can find $\psi \in C_0^\infty(|x| < 1)$ and $\phi \in C_0^\infty(\frac{1}{2} \leq |x| \leq 2)$ positive such that $\psi(x) + \sum_{j=0}^{+\infty} \phi(2^{-j}x) = 1$, for all $x$ in $\mathbb{R}^n$. Let $V = |x|^m \sum_{j \text{ even}} \phi(2^{-j}x) - |x|^2 \sum_{j \text{ odd}} \phi(2^{-j}x)$. Then $V \in S(\langle x \rangle^m, g)$ and since $V \geq -|x|^2$ the operator $P = -\Delta + V$ is essentially self adjoint on $C_0^\infty(\mathbb{R}^n)$. It follows that (1.9), (1.10), (1.11) and (1.14) are satisfied, therefore Theorem 1.1 and 1.2 apply. However the lower bound (1.2) assumed in [Y-Z22] is not satisfied.

2) Assume that $p(x, \xi) = |\xi|^2 + \varepsilon \sum_{j,k=1}^n b_{jk}(x) \xi_j \xi_k$ with $b_{jk} \in S(\langle x \rangle^{-\sigma_0}, g)$ for some $\sigma_0 > 0$. Then if $\varepsilon$ is small enough the non trapping condition (1.11) is satisfied.
2 Proofs of the results

Let us consider the symbol \( a_0(x, \xi) = \frac{x \cdot \xi}{\langle \xi \rangle} \). A straightforward computation shows that under condition (1.8) (ii) one can find \( C_0, C_1, R \) positive such that

\[
(2.1) \quad H_p a_0(x, \xi) \geq C_0|\xi| - C_1, \quad if \quad (x, \xi) \in T^*(\mathbb{R}^n) \ and \ \ |x| \geq R.
\]

where \( H_p \) denotes the Hamiltonian field of the symbol \( p \).

Then we have the following result due to Doï [D3].

**Lemma 2.1** Assume moreover that (1.11) is satisfied then there exist \( a \in S(\langle x \rangle, g) \) and positive constants \( C_2, C_3 \) such that

(i) \( H_p a(x, \xi) \geq C_2|\xi| - C_3, \quad \forall (x, \xi) \in T^*(\mathbb{R}^n) \),

(ii) \( a(x, \xi) = a_0(x, \xi), \quad if \quad |x| \text{ is large enough.} \)

The symbol \( a \) is called a global escape function for \( p \). Here is the form of this symbol. Let \( \chi \in C_0^\infty(\mathbb{R}^n) \) be such that \( \chi(x) = 1 \ if \ |x| \leq 1, \ \chi(x) = 0 \ if \ |x| \geq 2 \) and \( 0 \leq \chi \leq 1 \). With \( R \) large enough and \( M \geq 2R \) we have,

\[
a(x, \xi) = a_0(x, \xi) + M^2 \chi(\frac{x}{M}) a_1(x, \frac{\xi}{\sqrt{p(x, \xi)}}) (1 - \theta(\sqrt{p(x, \xi)})
\]

where

\[
a_1(x, \xi) = - \int_0^{+\infty} \chi(\frac{1}{R} \pi(\Phi_t(x, \xi))) \ dt
\]

and \( \pi(\Phi_t(x, \xi)) = x(t; x, \xi) \), \( \theta(t) = 1 \ if \ 0 \leq t \leq 1, \ \theta(t) = 0 \ if \ t \geq 2, \ 0 \leq \theta \leq 1 \). Details can be found in [D3].

**Proof of Theorem 1.1**

Let \( \psi \in C^\infty(\mathbb{R}^n) \) be such that \( \text{supp} \ \psi \in [\varepsilon, +\infty[, \ \psi(t) = 1 \ in \ [2\varepsilon, +\infty[ \) (where \( \varepsilon > 0 \) is a small constant chosen later on) and \( \psi'(t) \geq 0 \) for \( t \in \mathbb{R} \). Following Doï [D3] we set,

\[
(2.2) \quad \begin{cases} 
\psi_0(t) = 1 - \psi(t) - \psi(-t) = 1 - \psi(|t|) \\
\psi_1(t) = \psi(-t) - \psi(t) = -\text{sgn } t \ \psi(|t|)
\end{cases}
\]

Then \( \psi_j \in C^\infty(\mathbb{R}) \), for \( j = 0, 1 \) and we have

\[
(2.3) \quad \psi_0'(t) = -\text{sgn } t \ \psi'(|t|) \quad \text{and} \quad \psi_1'(t) = -\psi'(|t|).
\]

Let \( \chi \in C^\infty(\mathbb{R}) \) be such that \( \chi(t) = 1 \ if \ t \leq \frac{1}{2}, \ \chi(t) = 0 \ if \ t \geq 1 \) and \( \chi(t) \in [0,1] \). With \( a \) given by Lemma 2.1 we set

\[
(2.4) \quad \begin{cases} 
\theta(x, \xi) = \frac{a(x, \xi)}{\langle x \rangle}, \quad (x, \xi) \in T^*(\mathbb{R}^n), \\
r(x, \xi) = \frac{a(x, \xi)}{\sqrt{p(x, \xi)}} \ 
\end{cases} \quad (x, \xi) \in T^*(\mathbb{R}^n) \ \setminus \ 0.
\]
Finally we set

\[
\lambda = \left( \frac{a}{\langle x \rangle} \psi_0(\theta) - (M_0 - \langle a \rangle) \psi_1(\theta) \right) p^{\frac{1}{m}} - \frac{1}{2} \chi(r),
\]

where \( \nu > 0 \) is an arbitrary small constant and \( M_0 \) a large constant to be chosen.

The main step of the proof is the following Lemma.

**Lemma 2.2** (i) One can find \( M_0 > 0 \) such that for any \( \nu > 0 \) there exist positive constants \( C, C' \) such that

\[
-H_p \lambda(x, \xi) \geq C \langle x \rangle^{-1-\nu} (|\xi|^2 + |x|^m)^{\frac{1}{p}} - C', \quad \forall (x, \xi) \in T^*(\mathbb{R}^n),
\]

(ii) \( \lambda \in S(1, g) \),

(iii) \([P, \lambda^w] - \frac{1}{i} (H_p \lambda)^w \in Op^{w} S_T(1, g)\).

**Proof**

First of all on the support of \( \chi(r) \) we have \( \langle x \rangle^\frac{2}{p} \leq \sqrt{p(x, \xi)} \leq C|\xi| \). It follows that \( |\xi| \sim \langle x \rangle \) and \( |\xi| \leq |\xi| + \langle x \rangle^\frac{2}{p} \leq C'|\xi| \). Now

\[
-H_p \lambda = \sum_{j=1}^{6} A_j
\]

where the \( A_j \)'s are defined below.

1) \( A_1 = (H_p \langle x \rangle^{-1}) p^{\frac{1}{m}} - \frac{1}{2} a \psi_0(\theta) \chi(r) \). Since on the support of \( \psi_0(\theta) \) we have \( |a| \leq 2 \varepsilon \langle x \rangle \), it is easy to see that

\[
|A_1| \leq C_1 \varepsilon \langle x \rangle^{-1} |\xi|^\frac{2}{p} \left( 1 - \psi(|\theta|) \right) \chi(r).
\]

2) \( A_2 = \langle x \rangle^{-1} p^{\frac{1}{m}} - \frac{1}{2} (H_p \psi_0(\theta) \chi(r) \). By Lemma 2.1 (i) we have

\[
A_2 \geq C_2 \langle x \rangle^{-1} (|\xi| + \langle x \rangle^\frac{2}{p})^{\frac{2}{m}} \left( 1 - \psi(|\theta|) \right) \chi(r) - C'_2.
\]

3) \( A_3 = \langle x \rangle^{-1} p^{\frac{1}{m}} - \frac{1}{2} a \psi_0(\theta) \chi(r) \). It follows from (2.3), (2.4) that

\[
A_3 = -p^{\frac{1}{m}} - \frac{1}{2} |\theta| (H_p \psi_0(\theta) \psi(|\theta|) \chi(r)
\]

4) \( A_4 = p^{\frac{1}{m}} - \frac{1}{2} (H_p \langle a \rangle - \nu) \psi_1(\theta) \chi(r) \). Here we have \( H_p \langle a \rangle - \nu = -\nu \langle a \rangle - \varepsilon \langle x \rangle \). It follows from (2.2) that \( A_4 = \nu p^{\frac{1}{m}} - \frac{1}{2} |\theta| (H_p \langle a \rangle) \psi(|\theta|) \chi(r) \). Now on the support of \( \psi(|\theta|) \) we have \( \varepsilon \langle x \rangle \leq |a| \) and since \( a \in S(\langle x \rangle, g) \) we have \( |a| \leq C \langle x \rangle \). It follows from Lemma 2.1 (i) that

\[
A_4 \geq C_3 \langle x \rangle^{-1} (|\xi| + \langle x \rangle^\frac{2}{p})^{\frac{2}{m}} \psi(|\theta|) \chi(r) - C'_3.
\]

5) \( A_5 = -p^{\frac{1}{m}} - \frac{1}{2} (M_0 - \langle a \rangle) \psi_1(\theta) \chi(r) \). It follows from (2.3) that

\[
A_5 = p^{\frac{1}{m}} - \frac{1}{2} (M_0 - \langle a \rangle) \psi_0(\theta) \psi(|\theta|) \chi(r)
\]

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We deduce from (2.10) and (2.12) that
\[ A_3 + A_5 = p^{\frac{1}{m} - \frac{1}{2}} (M_0 - \langle a \rangle - \langle |\theta| \rangle (H_p \theta) \psi'(\langle |\theta| \rangle) \chi(r) \]

Now \( H_p \theta = \langle x \rangle^{-1} H_p a + a H_p \langle x \rangle^{-1}. \) Since \( |a| \leq 2\varepsilon|\theta| \) on the support of \( \psi'(\langle |\theta| \rangle) \) we deduce that \( H_p \theta \geq C_4 \langle x \rangle^{-1}|\xi| - C_5 \geq -C_5. \) Taking \( M_0 \geq 2 \) and using the facts that \( \psi' \geq 0, \chi \geq 0 \) and \( \varepsilon \leq |\theta| \leq 2\varepsilon \) on the support of \( \psi'(\langle |\theta| \rangle) \) we obtain
\[ (2.13) \quad A_3 + A_5 \geq -C_6 \]

6) \( A_6 = \langle x \rangle^{-1} a \psi_3(\theta) - (M_0 - \langle a \rangle - \langle \psi_1(\theta) \rangle) p^{\frac{1}{m} - \frac{1}{2}} H_p \chi(r). \) We have \( H_p \chi(r) = \frac{1}{\sqrt{p}} (H_p \langle x \rangle^{\frac{m}{p}}) \chi'(r). \) On the support of \( \chi'(r) \) we have \( \langle x \rangle \sim \langle |\xi| \rangle^{\frac{2}{m}} \); this implies that
\[ p^{\frac{1}{m} - \frac{1}{2}} |H_p \chi(r)| \leq C \langle |\xi| \rangle^{\frac{2}{m} - 1} \frac{|\langle x \rangle^{\frac{m}{p}}|}{|\xi|} |\chi'(r)| \leq C_7. \]

Therefore we obtain
\[ (2.14) \quad |A_6| \leq C_8. \]

Gathering the estimates obtained in (2.8) to (2.14) we obtain
\[ (2.15) \quad -H_p \lambda \geq C_9 \langle x \rangle^{-1 - \nu} (|\xi| + \langle |\xi| \rangle)^{\frac{2}{m}} \chi(r) - C_{10}. \]

Now on the support of \( 1 - \chi(r) \) we have \( \langle |\xi| \rangle \leq C_{11} \langle x \rangle^{\frac{m}{p}} \) so \( \langle x \rangle^{1 - \nu} (|\xi| + \langle |\xi| \rangle)^{\frac{2}{m}} \leq C_{12}. \) Therefore writing \( 1 = 1 - \chi + \chi \) and using (2.15) we obtain (2.6).

(ii) We use the symbolic calculus in the classes \( S(M, g) \). We have \( \langle x \rangle^{-1} \in S(\langle x \rangle^{-1}, g), \ a \in S(\langle x \rangle, g), p \in S(\langle x \rangle^2, g) \) so \( p^{\frac{1}{m} - \frac{1}{2}} \in S(\langle x \rangle^{\frac{m}{p} - 1}, g) \) since \( p \geq C > 0 \) on supp \( \chi(r) \). Moreover \( \chi(r) \in S(1, g) \) and on supp \( \chi(r) \) we have \( \langle x \rangle^{\frac{m}{p}} \leq C \langle |\xi| \rangle \). It follows that \( \lambda \in S(\langle x \rangle^{\frac{m}{p} - 1}, g) \subset S(1, g). \)

(iii) By the symbolic calculus \( \{\lambda, V\} \in S_T(\langle |\xi| \rangle^{\frac{m}{p} - 1} \langle x \rangle^{m-1} \langle |\xi|^{-1} \rangle, g) \). Since we have \( \langle x \rangle^{\frac{m}{p}} \leq C \langle |\xi| \rangle \) on its support we will have \( \langle x \rangle^{m-1} \langle |\xi| \rangle^{\frac{m}{p} - 2} \leq C \langle |\xi|^{\frac{m}{p} - (m-1)} \rangle \langle |\xi| \rangle^{\frac{m}{p} - 2} \leq C'. \) Therefore \( \{\lambda, V\} \in S_T(1, g). \) Now if \( b \in S_T(\langle |\xi| \rangle^{\frac{m}{p}}, g) \) we have \( \{\lambda, b \xi\} \in S(\langle |\xi| \rangle^{\frac{m}{p} - 1} \langle x \rangle^{m-1} \langle |\xi|^{-1} \rangle, g) \) and since \( \langle x \rangle^{\frac{m}{p}} \leq C \langle |\xi| \rangle \) we have \( \langle x \rangle^{m-1} \langle |\xi| \rangle^{\frac{m}{p} - 1} \leq C \langle |\xi|^{\frac{m}{p} - (m-1)} \rangle \langle |\xi|^{-1} \rangle^{\frac{m}{p} - 1} \leq C' \) so \( \{\lambda, b \xi\} \in S_T(1, g) \).

Finally \( \text{Op}^w(p, \lambda)^w \) \( \in S(\langle |\xi| \rangle^2 \langle |\xi| \rangle^{\frac{m}{p} - 1} \langle x \rangle^{-2} \langle |\xi| \rangle^{-2}, g) \subset \text{Op}^w S(1, g). \)

End of the proof of Theorem 1.1.

Since \( \lambda \in S(1, g) \) we can set \( M = 1 + \sup_{\langle x, \xi \rangle \in \mathbb{R}^{2n}} |\lambda(x, \xi)|. \) Let us introduce \( N(t) = ((M + \lambda^w) u(t), u(t))_{L^2(\mathbb{R}^n)}. \) Then there exist absolute constants \( C_1 > 0, C_2 > 0 \) such that \( C_1 \|u(t)\|_{L^2}^2 \leq N(u(t)) \leq C_2 \|u(t)\|_{L^2}^2. \) Now
\[
\frac{d}{dt} N(t) = ((M + \lambda^w) \frac{\partial u}{\partial t}(t), u(t))_{L^2} + ((M + \lambda^w) u(t), \frac{\partial u}{\partial t}(t))_{L^2}
\]
Since \( \frac{\partial u}{\partial t}(t) = -iPu(t) + if(t) \) and \( P^* = P \) we obtain
\[
\frac{d}{dt} N(t) = i(\langle P, \lambda^w \rangle u(t), u(t))_{L^2} - 2 \Im((M + \lambda^w) f(t), u(t))_{L^2}
\]
\[
= -((H_p \lambda)^w u(t), u(t))_{L^2} - 2 \Im((M + \lambda^w) f(t), u(t))_{L^2} + O(\|u(t)\|_{L^2}^2)
\]
By lemma 2.2 (iii).

Now by Lemma 2.2 (i) and the sharp Gårding inequality, we obtain
\[
(2.16) \quad ((H_p \lambda)^w u(t), u(t))_{L^2} \geq C\|\langle x \rangle^{-\frac{i+1}{2}} E_m^+ u(t)\|_{L^2}^2 - C\|u(t)\|_{L^2}^2
\]
On the other hand we have for any \( \varepsilon > 0 \)
\[
(2.17) \quad |((M + \lambda^w) f(t), u(t))_{L^2}| \leq \varepsilon \|\langle x \rangle^{-\frac{i+1}{2}} E_m^+ u(t)\|_{L^2}^2 + C\|\langle x \rangle^{\frac{1}{2}} E_m^- f(t)\|_{L^2}^2
\]
Using (2.16) and (2.17) with \( \varepsilon \) small enough, we obtain
\[
\frac{d}{dt} N(t) \leq -C_1 \|\langle x \rangle^{-\frac{i+1}{2}} E_m^+ u(t)\|_{L^2}^2 + C_2 \|\langle x \rangle^{\frac{1}{2}} E_m^- f(t)\|_{L^2}^2 + C_3 N(t)
\]
Integrating this inequality between 0 and \( t \) (in \([0,T]\)) and using Gronwall’s inequality, we obtain the conclusion of Theorem 1.1. \( \blacksquare \)

**Proof of theorem 1.2.**

Let \( \chi \in C_0^\infty(\mathbb{R}^+) \), \( \chi(t) = 1 \) if \( t \in [0,1] \), \( \chi(t) = 0 \) if \( t \geq 2 \). Recall that according to (1.14) we have
\[ p = |\xi|^2 + q(x, \xi) \text{ where } q(x, \xi) = \sum_{j,k=1}^n b^{jk}(x) \xi_j \xi_k \text{ and } b^{jk} \in S(\langle x \rangle^{-\sigma_0}, g) \].

Let us set
\[
A_{jk} = \frac{x_j \xi_k - x_k \xi_j}{\langle \xi \rangle}, \quad 1 \leq j, k \leq n
\]
Then we have the following result.

**Lemma 2.3** Let \( a \) be defined in Lemma 2.1. One can find positive constants \( C_0, C_1 \) and \( C_2 \) such that if we set
\[
(2.19) \quad -\lambda = \frac{a}{(1 + a^2 + \sum_{j,k=1}^n A_{jk}^2)^{\frac{i}{2}}} \chi \left( \frac{\langle x \rangle^{\frac{m}{2}}}{\sqrt{p(x, \xi)}} \right)
\]
then
\[
(i) \quad -H_p \lambda \geq C_0\langle x \rangle^{-3}(|\xi| + \langle x \rangle^{\frac{m}{2}})^{\frac{i}{m}} \sum_{j,k=1}^n A_{jk}^2 = C_1 \langle x \rangle^{-1-\sigma_0}(|\xi| + \langle x \rangle^{\frac{m}{2}})^{\frac{i}{m}} - C_2,
\]
\[
(ii) \quad \lambda \in S(\langle \xi \rangle^{\frac{m}{2}-1}, g),
\]
\[
(iii) \quad [P, \lambda^w] - \frac{1}{t}(H_p \lambda)^w \in \text{Op}^w S_T(1, g).
\]
Proof

First of all we have

\[(2.20) \quad |H_p A_{jk}(x, \xi)| \leq C_1 \frac{||\xi||}{\langle x \rangle_{\sigma_0}}, \quad 1 \leq j, k \leq n, \quad (x, \xi) \in T^*(\mathbb{R}^n).\]

Indeed we have \{|\xi|^2, A_{jk}\} = 0 and \{|q, A_{jk}\| \leq C_2 \frac{||\xi||}{\langle x \rangle_{\sigma_0}}.\]

Let us set

\[(2.21) \quad D = 1 + a^2 + \sum_{j,k=1}^n A_{jk}^2.\]

We claim that on the support of \(\chi(\langle x \rangle_{\frac{m}{2}} p^{-\frac{1}{2}})\) we have

\[(2.22) \quad C_3 \langle x \rangle^2 \leq D \leq C_4 \langle x \rangle^2\]

for some positive constants \(C_3\) and \(C_4\).

Indeed a straightforward computation shows that

\[(x, \xi)^2 + \sum_{j,k=1}^n (x_j \xi_k - x_k \xi_j)^2 \geq |x|^2 |\xi|^2.\]

Since by Lemma 2.1 we have \(a(x, \xi) = \frac{x \cdot \xi}{\langle \xi \rangle}\) for \(|x| \geq R_0 \gg 1\) and \(|\xi| \geq C_5 > 0\) on the support of \(\chi\) we deduce that \(D \geq C_6 \langle x \rangle^2\) when \(|x| \geq R_0\). When \(|x| \leq R_0\) we have \(D \geq 1 \geq \frac{1}{1 + R_0^2} \langle x \rangle^2\).

Now we can write with \(r(x, \xi) = \langle x \rangle_{\frac{m}{2}} p^{-\frac{1}{2}},\)

\[(2.23) \quad \left\{ \begin{array}{l} -H_p \lambda = I_1 + I_2 \\
I_1 = D^{-\frac{1}{2}}(D(H_p a) - \frac{1}{2} a(H_p D))p^{\frac{m}{2} - \frac{1}{4}} \chi(r) \\
I_2 = p^{\frac{m}{2} - \frac{1}{4}} aD^{-\frac{1}{2}} H_p (\chi(r)) \end{array} \right.\]

We have

\[DH_p a - \frac{1}{2} a(H_p D) = \left(1 + \sum_{j,k=1}^n A_{jk}^2 \right) H_p a + a^2 H_p a - \frac{1}{2} a(2aH_p a + 2 \sum_{j,k=1}^n A_{jk} H_p A_{jk}) \]

\[= \left(1 + \sum_{j,k=1}^n A_{jk}^2 \right) H_p a - a \sum_{j,k=1}^n A_{jk} H_p A_{jk}.\]
Using (2.18) and (2.20) we see that,

\[(2.24) \quad |a| \sum_{j,k=1}^n |A_{jk}| |H_p A_{jk}| \leq C_7 |x|^2 \frac{|\xi|}{\langle x \rangle^{\alpha_0}}.\]

Moreover by Lemma 2.1 we have on the support of \(\chi(r)\),

\[(2.25) \quad p^{\frac{1}{m} - \frac{1}{2}} (1 + \sum_{j,k=1}^n A_{jk}^2) H_p a \geq (1 + \sum_{j,k=1}^n A_{jk}^2) (C_8 (|\xi| + \langle x \rangle^{\frac{m}{2}}) \frac{2}{m} - C_9).\]

Therefore (2.21), (2.23), (2.24), (2.25) show that,

\[I_1 \geq [C_{10} \langle x \rangle^{-3} (|\xi| + \langle x \rangle^{\frac{m}{2}}) \frac{2}{m} \sum_{j,k=1}^n A_{jk}^2 - C_{11} \langle x \rangle^{1+\sigma_0}] \chi(r).\]

On the support of \(1 - \chi(r)\) we have \(|\xi| \leq \langle x \rangle^{\frac{m}{2}}\) so we obtain,

\[(2.26) \quad I_1 \geq C_{12} \langle x \rangle^{-3} (|\xi| + \langle x \rangle^{\frac{m}{2}}) \frac{2}{m} \sum_{j,k=1}^n A_{jk}^2 - C_{13} \langle x \rangle^{1+\sigma_0} \chi(r).\]

On the other hand we have,

\[|H_p(\chi(r))| = |p^{-\frac{1}{2}} \chi'(r) H_p \langle x \rangle^{\frac{m}{2}}| \leq \frac{C_{15}}{|\xi|} |\chi'(r)||\xi||\langle x \rangle^{\frac{m}{2}}|^{-1}.\]

It follows from (2.22) and the estimate \(|a| \leq C_6 \langle x \rangle\) that,

\[(2.27) \quad |I_2| \leq C_{17},\]

since \(\langle x \rangle^{\frac{m}{2}}|\xi|^{-1} \leq C_{18} \).

Then (i) in lemma 2.3 follows from (2.23), (2.26) and (2.27). The proofs of (ii) and (iii) are the same as those in the proof of lemma 2.2. \(\blacksquare\)

End of the proof of Theorem 1.2.

We introduce as before, for \(t\) in \((0,T)\).

\[N(t) = ((M_0 + \lambda^w) u(t), u(t))_{L^2}\]

Where \(M_0\) is a large constant. Then \(N(t) \sim \|u(t)\|^2_{L^2}\).

Now using the equation and Lemma 2.3 (iii) we can write,

\[
\frac{d}{dt} N(t) = -((-H_p \lambda)^w u(t), u(t))_{L^2} - 2 \text{Im}((M_0 + \lambda^w) f(t), u(t))_{L^2} + O(\|u(t)\|^2_{L^2})
\]
Since by (1.13) and (2.19) we have $\langle x \rangle^{-2} A_{jk}^2 = \ell_{jk}^2$, Lemma 2.3 (i) and the sharp Gårding inequality ensure that

$$
\frac{d}{dt} N(t) \leq - C_1 \sum_{j,k=1}^n \| \langle x \rangle^{-\frac{1}{2}} E_m \ell_{jk}^w u(t) \|_{L^2}^2 + C_2 \| \langle x \rangle^{-\frac{1}{2}} E_m^- u(t) \|_{L^2}^2 + \| \langle x \rangle^{-\frac{1}{2}} E_m^- f(t) \|_{L^2}^2 + C_3 N(t).
$$

It follows that for $0 < t < T$,

$$
N(t) + C_1 \int_0^t \sum_{j,k=1}^n \| \langle x \rangle^{-\frac{1}{2}} E_m \ell_{jk}^w u(s) \|_{L^2}^2 ds \leq N(0) + C_2 \int_0^T \| \langle x \rangle^{-\frac{1}{2}} E_m^- u(s) \|_{L^2}^2 ds + \int_0^T \| \langle x \rangle^{-\frac{1}{2}} E_m^- f(s) \|_{L^2}^2 ds + C_3 \int_0^t N(s) ds.
$$

Using Theorem 1.1 to bound the second term in the right hand side and then using the Gronwall inequality we obtain

$$
N(t) \leq C(T)(\| u(0) \|_{L^2}^2 + \int_0^T \| \langle x \rangle^{-\frac{1}{2}} E_m^- f(t) \|_{L^2}^2 dt).
$$

Using again the inequality (2.28) we obtain the conclusion of Theorem 1.2. The proof is complete.

Références


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