Cramer-Rao lower bound for the estimation of the degree of polarization in active coherent imagery at low photon level
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Polarization imaging is increasingly used in medical imaging, remote sensing, and industrial control. For example, this technique can reveal contrasts between regions of a scene that have the same intensity reflectivity but different polarimetric properties. These systems often measure the degree of polarization (DOP) of laser light backscattered by a scene. Since illumination is coherent, in the absence of photon noise, the reflected intensity $I$ measured in one pixel of image $X$ is an integer-valued random variable $X$ whose PDF is $P(X)=\exp\left(-\frac{I}{L}\right)\frac{I^X}{X!}$. The PDF of the number of photons measured at pixel $j$ in images $X$ and $Y$, where $n_{X,j},n_{Y,j}$ represent, respectively, the number of photons measured at pixel $j$ in images $X$ and $Y$, one will assume that all the elements of the sample are statistically independent. The $n_{X,j}$ and $n_{Y,j}$ are assumed to have the same average value $I_X$, and the $n_{Y,j}$ the average value $I_Y$. Since the illuminated materials are assumed to be purely depolarizing, these values can be expressed as a function of the total intensity $I_0$ (expressed in number of photons) and of the DOP $P$ as $I_X=I_0(1+P)/2$ and $I_Y=I_0(1-P)/2$. Our goal will be to estimate the two parameters $I_0$ and $P$ from the sample $\chi$.

The probability distribution function (PDF) of the sample values is classically determined by using a semiclassical model of light. Since illumination is coherent, the reflected intensity $I$ measured in one pixel of image $U=X,Y$ is modeled as a Gamma-distributed random variable with mean $I_U$ and order $L$, whose PDF is
\[
P_U(I) = \Gamma[I]^{-1} \frac{I^{L-1}}{\Gamma[L]} \exp\left(-\frac{IL}{I_U}\right)
\] (1)
with $U=X,Y$. For a given realization of the intensity $I$, the number of detected photons is an integer-valued random variable $n$ distributed with a Poisson PDF: $P(n|I)=\exp(-I)I^n/n!$. The PDF of the number of photons averaged over the possible realizations of $I$ can be expressed as
\[
P_U(n)=\int_0^\infty P(n|I)P_U(I)dI
\]
with $U=X,Y$. An explicit expression of this integral can be computed:
\[
P_U(n) = \frac{\Gamma(L+n)}{\Gamma(L)\Gamma(n+1)} \left(1 + \frac{L}{I_U}\right)^{-n} \left(1 + \frac{I_U}{L}\right)^{-L}.
\] (2)
It represents the PDF of the photon number measured for a light of average intensity $I_U$ in the presence of speckle noise of order $L$. The problem we address is the following. One assumes to have observed a sample $\chi$ of size $N$ pixels, defined by $\chi=[n_{X,1},n_{Y,1},n_{X,2},n_{Y,2},\ldots,n_{X,N},n_{Y,N}]$, and order $L$, whose PDF is $P_U(I) = \Gamma[I]^{-1} \frac{I^{L-1}}{\Gamma[L]} \exp\left(-\frac{IL}{I_U}\right)$ (1) with $U=X,Y$. For a given realization of the intensity $I$, the number of detected photons is an integer-valued random variable $n$ distributed with a Poisson PDF: $P(n|I)=\exp(-I)I^n/n!$. The PDF of the number of photons averaged over the possible realizations of $I$ can be expressed as $P_U(n)=\int_0^\infty P(n|I)P_U(I)dI$ with $U=X,Y$. An explicit expression of this integral can be computed:
\[
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To analyze the precision of estimation of parameters $I_0$ and $P$ from this sample, we determine the CRLBs that represent the lowest variance that can be reached by any unbiased estimator. It is an efficient way to characterize the intrinsic difficulty of an estimation task. To determine the CRLBs, one first needs to calculate the Fisher information matrix

$$J = \begin{bmatrix}
-\frac{\partial^2}{\partial P^2} I(\chi) & -\frac{\partial^2}{\partial P \partial I_0} I(\chi) \\
-\frac{\partial^2}{\partial P \partial I_0} I(\chi) & -\frac{\partial^2}{\partial I_0^2} I(\chi)
\end{bmatrix},$$  

(3)

where $I(\chi) = \sum_{i=1}^{N} \log(P_X(n_X)P_Y(n_Y))$ is the loglikelihood of the sample and $\langle \rangle$ corresponds to statistical averaging. Let $\hat{P}(\chi)$ and $\hat{I}_0(\chi)$ be some estimators of, respectively, the polarization $P$ and the intensity $I_0$. They are unbiased if $\langle \hat{P}(\chi) \rangle = P$ and $\langle \hat{I}_0(\chi) \rangle = I_0$, and one can define their covariance matrix $\Gamma$ as

$$\Gamma = \begin{bmatrix}
\langle (\hat{P}(\chi) - P)^2 \rangle & \langle (\hat{P}(\chi) - P)(\hat{I}_0(\chi) - I_0) \rangle \\
\langle (\hat{P}(\chi) - P)(\hat{I}_0(\chi) - I_0) \rangle & \langle (\hat{I}_0(\chi) - I_0)^2 \rangle
\end{bmatrix}.$$  

(4)

The diagonal elements of this matrix are the variances $\sigma_P^2$ of the DOP and $\sigma_{I_0}^2$ of the average intensity. The Cramer–Rao theorem states that for unbiased estimators the covariance matrix $\Gamma$ and the inverse of the Fisher information matrix $J^{-1}$ are related by the following inequality: $\sqrt{\text{trace} \Gamma} \geq \sqrt{\text{det} J^{-1}} \text{v}$, where $\text{v}$ can be any vector. From this inequality, one gets $\sigma_P^2 \geq \kappa_{pp}$ and $\sigma_{I_0}^2 \geq \kappa_{I_0}$, where

$$J^{-1} = \begin{bmatrix}
\kappa_{pp} & \kappa_{P I_0} \\
\kappa_{P I_0} & \kappa_{I_0}
\end{bmatrix}.$$  

(5)

These values are called the Cramer–Rao lower bounds (CRLBs). We first determine the CRLBs when only photon noise or Gamma noise is present. We then address the case where the image is perturbed by both types of noise.

In the absence of speckle noise, the number of photons is distributed with a Poisson PDF. A direct application of Eq. (3) leads, after some calculus, to $\kappa_{P I_0} = 0$ and to

$$\kappa_{PP} = \frac{1 - P^2}{N I_0}, \quad \kappa_{I_0} = \frac{I_0}{N},$$  

(6)

where the superscript $\pi$ stands for Poisson. On the other hand, in the absence of photon noise, the measurements are distributed with the Gamma PDF defined in Eq. (1). A direct application of Eq. (3) yields, after some calculus,

$$\kappa_{PP}^S = \frac{(1 - P^2)^2}{2LN}, \quad \kappa_{I_0}^S = \frac{I_0^2 (1 + P^2)}{2LN},$$  

(7)

and $\kappa_{P I_0}^S = 1/(2LN) I_0 P (1 - P^2)$, where the superscript $S$ stands for speckle. In contrast to the Poisson noise case, one can expect correlation in the fluctuations of the estimation of $P$ and $I_0$, since the non diagonal element of $J^{-1}$ is nonzero.

In the presence of both speckle and Poisson noise, the data are distributed with the PDF defined in Eq. (2). By application of Eq. (3), a somewhat involved, yet direct calculus yields

$$\kappa_{PP}^M = \frac{1}{2LN} (1 - P^2)(1 - P^2 + 2L/I_0),$$  

$$\kappa_{I_0}^M = \frac{I_0^2}{2LN} (1 + P^2 + 2L/I_0),$$  

(8)

and $\kappa_{P I_0}^M = I_0 P (1 - P^2)$.

Several remarks can be made about this expression. Let us first consider the case $L/I_0 \gg 1$, in which light intensity is very low and Poisson noise is dominant. In this case, it is seen that Eqs. (8) lead to the CRLBs of the Poisson noise case [see Eqs. (6)]. On the other hand, if $L/I_0 \ll 1$, speckle noise is dominant (the photon flux is high), and it is seen that the CRLBs reduce to that of the speckle-only case. More unexpectedly, one can notice the following property:

$$\kappa_{PP}^M = \kappa_{PP}^S + \kappa_{PP}^F, \quad \kappa_{I_0}^M = \kappa_{I_0}^S + \kappa_{I_0}^F,$$  

(9)

The CRLBs in the presence of mixed speckle and Poisson noise are simply the sum of the CRLBs that result from each source of fluctuations. It can also be noticed that the nondiagonal term of $J^{-1}$ in the mixed case is due only to speckle.

Let us first consider estimation of intensity $I_0$. The signal-to-noise ratio (SNR) can be defined as $\eta = I_0 / \sqrt{\kappa_{I_0}}$. When only photon noise is present, the SNR is proportional to $\sqrt{I_0}$ (a well-known property of the Poisson noise) and is independent of $P$. On the other hand, when only speckle noise is present, the SNR is independent of $I_0$ but decreases as $P$ increases. It is higher for totally depolarized light. Let us define the ratios

$$\rho_\pi = \frac{\kappa_{I_0}^\pi}{\kappa_{I_0}} = \frac{I_0}{2L} (1 + P^2), \quad \rho_p = \frac{\kappa_{PP}}{\kappa_{PP}} = \frac{I_0}{2L} (1 - P^2).$$  

(10)

For intensity estimation, the crossover between the two regimes characterized by a dominant Poisson noise or a dominant speckle noise can be defined as $\rho_\pi = 1$. As expected, this crossover depends on $L$ and $I_0$ only through the ratio $L/I_0$. The value of $I_0$ corresponding to this crossover (in the case of $L = 1$), denoted $I_0^\pi(L)$, has been plotted as a function of $P$ in Fig. 1 (dashed curve). For totally depolarized light, speckle noise overcomes Poisson noise when the
crossover occurs for $I_0$. However, it is seen to decrease as $L$ increases and even to become null when $L=0.9$. For example, estimation precision is better for $I_0=0.7$ and $L=1$ than for $I_0=0.5$ and $L=100$. On the other hand, when speckle noise is dominant ($I_0>2L$), it is more efficient to increase the speckle order. For example, estimation precision is better for $I_0=3$ and $L=10$ than for $I_0=100$ and $L=1$. If the average number of photons is large enough to be in the speckle-dominant regime, increasing the speckle order is thus an efficient way of improving the estimation precision of the DOP.

When we have both photon and speckle noise, it has been shown that photon noise has a greater influence on DOP estimation than on intensity estimation, especially when light is highly polarized. Moreover, at low intensity levels, increasing the speckle order is not efficient for improving precision. It will be interesting to analyze different measurement strategies and to compare their performance with those analyzed in this Letter. Taking into account possible correlation of the two components of the reflected light is also a challenging problem.

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References