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THE FIRST CONFORMAL DIRAC EIGENVALUE ON 2-DIMENSIONAL TORI

BERND AMMANN AND EMMANUEL HUMBERT

Abstract. Let $M$ be a compact manifold with a spin structure $\chi$ and a Riemannian metric $g$. Let $\lambda^2_g$ be the smallest eigenvalue of the square of the Dirac operator with respect to $g$ and $\chi$. The $\tau$-invariant is defined as

$$\tau(M, \chi) := \sup I \sqrt{\lambda^2_g \text{Vol}(M, g)}^{1/n}$$

where the supremum runs over the set of all conformal classes on $M$, and where the infimum runs over all metrics in the given class.

We show that $\tau(T^2, \chi) = 2\sqrt{\pi}$ if $\chi$ is “the” non-trivial spin structure on $T^2$. In order to calculate this invariant, we study the infimum as a function on the spin-conformal moduli space and we show that the infimum converges to $2\sqrt{\pi}$ at one end of the spin-conformal moduli space.

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Contents

1. Introduction 1
2. The spin-conformal moduli space of $T^2$ 3
3. Main results 6
4. Some preliminaries 6
5. Proof of the Main results 8
References 13

1. Introduction

Let $(M, g, \chi)$ be a compact spin manifold of dimension $n \geq 2$. For any metric $\tilde{g}$ in the conformal class $[g]$ of $g$, let $\lambda_1(D^2_{\tilde{g}})$ be the smallest eigenvalue of the square of the Dirac operator. We define

$$\lambda_{\text{min}}(M, g, \chi) = \inf_{\tilde{g} \in [g]} \sqrt{\lambda_1(D^2_{\tilde{g}})} \text{Vol}(M, \tilde{g})^{1/n}.$$ 

Several works have been devoted to the study of this conformal invariant and some variants of it [Hij86, Lott86, Bär92, Am03, Am03a, Am03b]. J. Lott [Lott86, Am03] proved that $\lambda_{\text{min}}(M, g, \chi) = 0$ if and only if $\ker D_g \neq \{0\}$.

From [Hij86, Bär92] we deduce $\lambda_{\text{min}}(S^n) = \frac{n}{2} \omega_n$, where $S^n$ is the sphere with constant sectional curvature 1 and where $\omega_n$ is its volume. Furthermore, in [Am03, AHM03], we have seen that

$$\lambda_{\text{min}}(M, g, \chi) \leq \lambda_{\text{min}}(S^n) = \frac{n}{2} \omega_n$$

(1)

for all Riemannian spin manifolds.

Furthermore we define

$$\tau(M, \chi) := \sup \lambda_{\text{min}}(M, g, \chi)$$

where the supremum runs over all conformal classes on $M$. Obviously, $\tau(M, \chi)$ is an invariant of a differentiable manifold with spin structure.

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We consider it as interesting to determine $\tau$ or at least some bounds for $\tau$ in as many cases as possible. There are several motivations for studying these invariants $\lambda_{\text{min}}(M,g,\chi)$ and $\tau(M,\chi)$.

Our first motivation is the analogy and the relation to Schoen’s $\sigma$-constant, which is defined as

$$\sigma(M) := \sup \inf \frac{\int \text{Scal}_g \, dv_g}{\text{Vol}(M,g^{1/2})}$$

where the infimum runs over all metrics in a conformal class $g \in [g_0]$, and where the supremum runs over all conformal classes.

In the case $\sigma(M) \geq 0$ and $n \geq 3$, there is also an alternative definition of the $\tau$-invariant that is analogous to our definition of the $\tau$-invariant. More exactly, in this case

$$\sigma(M) := \sup \inf \lambda_1(L_g) \text{Vol}(M,g)^{2/n},$$

where $\lambda_1(L_g)$ is the first eigenvalue of the conformal Laplacian $L_g := 4 \frac{n-2}{n} \Delta_g + \text{Scal}_g$. Once again, the infimum runs over all metrics in a conformal class $g \in [g_0]$, and where the supremum runs over all conformal classes.

Many conjectures about the value of the $\sigma$-constant exist, but unfortunately it can be calculated only in very few special cases, e.g. $\sigma(S^n) = n(n-1)\omega_n^{2/n}$, $\sigma(S^{n-1} \times S^1) = n(n-1)\omega_n^{2/n}$, $\sigma(T^n) = 0$ and $\sigma(\mathbb{R}P^3) = n(n-1) (\frac{\omega_n}{4(n-1)})^{2/n}$.

Equality is attained in this inequality if $M = S^n$. Hence, upper bounds for $\tau(M,\chi)$ may help to determine the $\sigma$-constant.

This is one reason for studying the $\tau$-invariant.

Another motivation for studying $\tau(M,\chi)$ and $\lambda_{\text{min}}(M,g,\chi)$ comes from the connection to constant mean curvature surfaces. Let $n = 2$. If $\tilde{g}$ is a minimizer that attains the infimum in the definition of $\lambda_{\text{min}}(M,g,\chi)$, and if $\text{Vol}(M,\tilde{g}) = 1$, then any simply connected open subset $U$ of $M$ can be isometrically embedded into $\mathbb{R}^3$, $(U,\tilde{g}) \hookrightarrow \mathbb{R}^3$, such that the resulting surface has constant mean curvature $\lambda_{\text{min}}(M,g,\chi)$. Vice versa, any constant mean curvature surface gives rise to a stationary point of an associated variational principle. It is shown in [Am03] that minimizers of $\lambda_{\text{min}}(M,g,\chi)$ exist if $\lambda_{\text{min}}(M,g,\chi) < 2\sqrt{\pi}$.

For the third motivation, let again $n \geq 2$ be arbitrary. As indicated above, $\lambda_{\text{min}}(M,g,\chi) > 0$ if and only if $\ker D_g = \{0\}$. Hence, $\tau(M,\chi) > 0$ if and only if $M$ carries a metric with $\ker D = \{0\}$. It follows from the Atiyah-Singer index theorem that any spin manifold $M$ of dimension $4k$, $k \in \mathbb{N}$ with $\tilde{A}(M) \neq 0$ has $\tau = 0$, and the same holds for spin manifolds of dimension $8k + 1$ and $8k + 2$ with non-vanishing $\alpha$-genus. C. Bär conjectures [Bā94, BD02] that in all remaining cases one has $\tau > 0$. Using perturbation methods Maier [Mai97] has verified the conjecture in the case $n \leq 4$. The conjecture also holds if $n \geq 5$ and $\pi_1(M) = \{e\}$. Namely, if $M$ is a compact simply connected spin manifold with vanishing $\alpha$-genus, then building on Gromov-Lawson’s surgery results [GL89] Stolz showed [St92] that $M$ carries a metric $g^+$ of positive scalar curvature. Applying the Schrödinger-Lichnerowicz formula we obtain $\ker D_{g^+} = \{0\}$, and hence $\tau(M,\chi) > \lambda_{\text{min}}(M,g^+,\chi) > 0$ for the unique spin structure $\chi$ on $M$. A good reference for this argument is also [BD02], where the interested reader can also find an analogous statement for the case $\alpha(M,\chi) \neq 0$. The method of Stolz and Bär-Dahl also applies to some other fundamental groups, but the general case still remains open.
In the present article we want to have a closer look at the $\tau$-invariant on surfaces, in particular 2-dimensional tori. The higher dimensional case will be the subject of another publication.

On surfaces the Yamabe operator cannot be defined as above. The Gauss-Bonnet theorem says that the $\sigma$-constant of a surface does not depend on the metric:

$$\sigma(M) = \sup \inf \int 2K_g \, dv_g = 4\pi \chi(M).$$

It was conjectured by Lott [Lott86] and proved by C. Bär [Bär92] that equation (3) also holds in dimension 2. This amounts in showing $\tau(S^2) = 2\sqrt{\tau}$. If $M$ is a compact orientable surface of higher genus, then inequality (3) is trivial.

We will calculate the $\tau$-invariant for the 2-dimensional torus $T^2$. The 2-dimensional torus $T^2$ has 4 different spin structures. The diffeomorphism group Diff($T^2$) acts on the space of spin structures by pullback, and the action has two orbits: one orbit consisting of only one spin structure, the so-called trivial spin structure $\chi_T$ and another orbit consisting of three spin structures. The torus $T^2$ equipped with the trivial spin structure has non-vanishing $\alpha$-genus, thus $\tau = 0$. The main result of this article is the following theorem.

**Theorem 1.1.** Let $\chi$ be a non-trivial spin structure on the 2-dimensional torus $T^2$. Then

$$\tau(T^2, \chi) = 2\sqrt{\tau} \ (= \lambda_{\min}(S^2)).$$

More exactly, for a fixed non-trivial spin structure $\chi$ we will study $\lambda_{\min}(M, g, \chi)$ as a function on the spin-conformal moduli space $\mathcal{M}$. We show that it is continuous (Proposition 3.1) and we show that it can be continuously extended to the natural 2-point compactification of $\mathcal{M}$, i.e. the compactification where both ends are compactified by one point each. It will be easy to show that $\lambda_{\min}(M, g, \chi) \to 0$ at one of the ends. However, it is much more involved to prove Theorem 3.2 which states that $\lambda_{\min}(M, g, \chi) \to \lambda_{\min}(S^2) = 2\sqrt{\tau}$ at the other end.

It is evident that Theorem 3.3 implies Theorem 1.1.

For the proof of Theorem 3.2 we have to establish a qualitative lower bound for the eigenvalues. One important ingredient in the proof of Theorem 3.2 is to study a suitable covering of the 2-torus by a cylinder, and to lift a test spinor to this covering. Using a cut-off argument in a way similar to [AB02] we obtain a compactly supported test spinor on the cylinder. After compactifying the cylinder conformally to the sphere $S^2$, we can use Bär’s 2-dimensional version of (5), to prove $\lambda_{\min}(M, g, \chi) \to \lambda_{\min}(S^2) = 2\sqrt{\tau}$ at the other end.

Theorem 3.2 and Theorem 1.1 should be seen as a spinorial analogue of [Sch91]. In that article, Schoen studies the Yamabe invariant on the moduli space of $O(n)$-invariant conformal structures on $S^1 \times S^{n-1}$, $n \geq 3$. He shows that at one end of this moduli space, the Yamabe invariant converges to the Yamabe invariant of $S^n$, and hence $\sigma(S^1 \times S^{n-1}) = \sigma(S^n)$. Combining this result with the Hijazi inequality and Theorem 1.1, one obtains

**Corollary 1.2.** Let $n \geq 2$. Then

$$\tau(S^{n-1} \times S^1, \chi) = \begin{cases} 0 & \text{if } n = 2 \text{ and if } \chi \text{ is trivial,} \\ \frac{\sqrt{\tau}}{2} \omega_3^{1/n} & \text{otherwise.} \end{cases}$$

The structure of the article is as follows.

In section 3 we define the spin-conformal moduli $\mathcal{M}$ space of 2-tori and recall some well known facts. In section 4 we state and explain our results. In sections 5 we recall some preliminaries which will be useful for the proof of Theorem 3.2. In Section 6 the proof is carried out.

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2. The spin-conformal moduli space of $T^2$

At the beginning of this section we will recall the definition of a spin structure. We will only give it in the case $n = 2$. For more information and for the case of general dimension we refer to standard text books [Fr00, LM89, Ro88, BGV91]. More details about the 2-dimensional case can be obtained in [AB02] and [Am98, BS92].

Let $(M, g)$ be an oriented surface with a Riemannian metric $g$. Let $P_{SO}(M, g)$ denote the set of oriented orthonormal frames over $M$. The base point map $P_{SO}(M, g) \to M$ is an $S^1$ principal bundle. Let $\alpha : S^n \to S^1$
be the non-trivial double covering, i.e. $\alpha(z) = z^2$. A spin structure on $(M, g)$ is by definition a pair $(P, \chi)$ where $P$ is an $S^1$ principal bundle over $M$ and where $\chi : P \to P_{SO}(M, g)$ is a double covering, such that the diagram

$$
P \times S^1 \quad \to \quad P
\downarrow \chi \times \alpha \quad \downarrow \chi
\quad \to \quad M
$$

commutes (in this diagram the horizontal flashes denote the action of $S^1$ on $P$ and $P_{SO}(M)$). By slightly abusing the notation we will sometimes write $\chi$ for the spin structure, assuming that the domain $P$ of $\chi$ is implicitly given. Two spin structures $(P_1, \chi)$ and $(P_2, \chi)$ are isomorphic if there is an $S^1$-equivariant bijection $b : P \to P$ such that $\bar{\chi} = \chi \circ b$.

If $\bar{g} = f^2 g$ is a metric conformal to $g$. Then $P_{SO}(M, \bar{g}) \to P_{SO}(M, g)$, $(e_1, e_2) \mapsto (fe_1, fe_2)$ defines an isomorphism of $S^1$ principal bundles. The pullback of a spin structure on $(M, g)$ is a spin structure on $(M, \bar{g})$.

In a similar way, if $(M_1, g_1) \to (M_2, g_2)$ is an orientation preserving conformal map, but not necessarily a diffeomorphism, then any spin structure on $(M_2, g_2)$ pull back to a spin structure on $(M_1, g_1)$.

**Examples 2.1.**

1. If $g_0$ is the standard metric on $S^2$. Then $P_{SO}(S^2, g_0) = SO(3)$, and the base point map $SO(3) \to S^2$ is the map that associates to a matrix in $SO(3)$ the first column. The double cover $SU(2) \to SO(3)$ defines a spin structure on $(S, g_0)$.
2. Let $\bar{g}$ be an arbitrary metric on $S^2$. After a possible pullback by a diffeomorphism $S^2 \to S^2$ we can write $\bar{g} = f^2 g$. The pullback of the spin structure given in (1) under the isomorphism $P_{SO}(S^2, \bar{g}) \to P_{SO}(S^2, g)$ defines a spin structure on $(S^2, \bar{g})$.
3. Let $g_1$ be a flat metric on the torus $T^2$. Then a parallel frame gives rise to a (global) section of $P_{SO}(T^2, g_1) \to T^2$. Hence, this is a trivial $S^1$ principal bundle. The trivial fiberwise double covering $T^2 \times S^1 \to T^2 \times S^1$, $(p, z) \mapsto (p, z^2)$ defines a spin structure on $(T^2, g_1)$, the so-called trivial spin structure $\chi_{tr}$ on $(T^2, g_1)$.
4. If $\bar{g}$ is an arbitrary metric on $T^2$. Then we can write $\bar{g} = f^2 g_1$ where $g_1$ is a flat metric. As above, the trivial spin structure on $(T^2, g_1)$ defines a spin structure on $(T^2, \bar{g})$. This spin structure is also called the trivial spin structure $\chi_{tr}$.
5. For $(x_0, y_0) \in \mathbb{R}^2 \setminus \{0\}$ we define

$$Z_{x_0, y_0} = \mathbb{R}^2 / \langle (x_0, y_0) \rangle$$

where $\langle (x_0, y_0) \rangle$ is the subgroup of $\mathbb{R}^2$ spanned by $(x_0, y_0)$. We will assume that it carries the metric induced by the euclidean metric $g_{euc}$ on $\mathbb{R}^2$. Then $P_{SO}(Z_{x_0, y_0})$ is a trivial bundle, and a natural trivialization is obtained by a parallel frame. The map $Z_{x_0, y_0} \times S^1 \to Z_{x_0, y_0} \times S^1$, $(p, z) \mapsto (p, z^2)$ defines a spin structure on $Z_{x_0, y_0}$, the trivial spin structure on $Z_{x_0, y_0}$.

Assume that $\chi : P \to P_{SO}(M, g)$ is a spin structure on a surface, and assume that $\beta : \pi_1(M) \to \{-1, +1\}$ is a group homomorphism. Then there is a $\{-1, +1\}$ principal bundle $B_\beta \to M$ with holonomy $\beta$. Let $B_\beta$ be the quotient of $P \times B_\beta$ by the diagonal action of $\{-1, +1\}$. Then $B_\beta$ together with the induced map $\chi_\beta : B_\beta \to P_{SO}(M, g)$ is also a spin structure on $(M, g)$. Conversely, if $(P, \bar{\chi})$ is another spin structure, then one can show that there is a unique $\beta : \pi_1(M) \to \{-1, +1\}$ such that $(P, \bar{\chi})$ and $(P, \chi_\beta)$ are isomorphic. Thus, we see that the space of spin structures is an affine space over the $\{-1, +1\}$-vector space $\text{Hom}(\pi_1(M), \{-1, +1\}) = H^1(M, \{-1, +1\})$.

**Examples 2.2.**

1. Any compact oriented surface $M$ carries a spin structure. If $k$ denotes the genus of $M$, then there are $4^k$ homomorphisms $\pi_1(M) \to \{-1, +1\}$, hence there are $4^k$ isomorphism classes of spin structures. In particular, the spin structure on $S^2$ is unique.

2. Because of $\pi_1(Z_{x_0, y_0}) = \mathbb{Z}$, there are exactly two spin structures on $Z_{x_0, y_0}$, the trivial one and another one called the non-trivial spin structure.

From now on, let $M = T^2 = \mathbb{R}^2 / \Gamma$ where $\Gamma$ is a lattice in $\mathbb{R}^2$. The trivial spin structure defined above can be used to identify $\text{Hom}(\pi_1(M), \{-1, +1\})$ with the set of isomorphism classes of spin structures. By slightly
abusing the language we will always write $\chi$ for the spin structure $(P, \chi)$ and also for the homomorphism $\pi_1(M) \to \{-1, +1\}$.

The following lemma summarizes some well-known equivalent characterizations of triviality of $\chi$ (see e.g. [LM89], [Mil63], [Am98], [Fr84]).

Lemma 2.3. With the above notations, the following statements are equivalent

1. The spin structure is trivial (in the above sense);
2. $\chi(\gamma) = 1$ for all $\gamma \in \Gamma$;
3. The spin structure is invariant under the natural action of the diffeomorphism group $\text{Diff}(T^2)$;
4. $(T^2, \chi)$ is the non-trivial element in the 2-dimensional spin-cobordism group;
5. The $\alpha$-genus of $(T^2, \chi)$ is the non-trivial element in $\mathbb{Z}/2\mathbb{Z}$;
6. The Dirac operator has a non-trivial kernel;
7. The kernel of the Dirac operator has complex dimension 2.

In particular, we easily see

$$\tau(T^2, \chi_{tr}) = 0.$$ 

From now on, in the rest of this article, we assume that $\chi$ is not the trivial spin structure, i.e. $\chi(\gamma) = -1$ for some $\gamma \in \Gamma$.

Definition 2.4. Two 2-dimensional tori with Riemannian metrics, orientations and spin structures are said to be spin-conformal if there is a conformal map between them preserving the orientation and the spin structure. “Being spin-conformal” is obviously an equivalence relation. The spin-conformal moduli space $\mathcal{M}$ of $T^2$ with the non-trivial spin structure is defined to be the set of these equivalence classes. Furthermore we define

$$\mathcal{M}_1 := \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \middle| |x| \leq \frac{1}{2}, \ y^2 + \left( |x| - \frac{1}{2} \right)^2 \geq \frac{1}{4}, \ y > 0 \right\}$$

(see also Fig. 1).

Lemma 2.5. Let $g$ be a Riemannian metric on $T^2 = \mathbb{R}^2/\mathbb{Z}^2$, and let $\chi : \mathbb{Z}^2 \to \{-1, +1\}$ be a non-trivial spin structure. Then there is a lattice $\Gamma \subset \mathbb{R}^2$, a spin structure $\chi' : \Gamma \to \{-1, +1\}$, such that

1. $\Gamma$ is generated by \( \begin{pmatrix} 1 \\ 0 \end{pmatrix} \) and \( \begin{pmatrix} x \\ y \end{pmatrix} \) with \( \begin{pmatrix} x \\ y \end{pmatrix} \in \mathcal{M}_1 \)
2. $(T^2, g, \chi)$ is spin-conformal to $(\mathbb{R}^2/\Gamma, g_{\text{eucl}}, \chi')$
3. $\chi'(\begin{pmatrix} 1 \\ 0 \end{pmatrix}) = +1$ and $\chi'(\begin{pmatrix} x \\ y \end{pmatrix}) = -1$. 

Proof. Because of the uniformization theorem we can assume without loss of generality that \( g \) is a flat metric. The lemma then follows from elementary algebraic arguments. □

Note that \( x \) and \( y \) are uniquely determined if \( (x, y) \) is in the interior of \( M_1 \), i.e. if \( |x| < 1/2 \) and \( y^2 + (|x| - 1/2)^2 > 1/4 \). If \( (x, y) \) is on the boundary of \( M_1 \), then \( y \) and \( |x| \) are determined, but not the sign of \( x \). Hence, after gluing \( (x, y) \in \partial M_1 \) with \( (-x, y) \) we obtain the spin-conformal moduli space \( M \).

Notation. Let \((x_0, y_0) \in M_1\). The lattice generated by \((1, 0)\) and \((x, y)\) is noted as \( \Gamma_{x_0, y_0} \). Furthermore, we write \( T_{x_0, y_0} \) for the 2-dimensional torus \( \mathbb{R}^2/\Gamma_{x_0, y_0} \) equipped with the euclidean metric.

The quantity \( \lambda_{\min}(T^2, g, \sigma) \) is a spin-conformal invariant, hence \( \lambda_{\min} \) can be viewed as a function on \( M \) or on \( M_1 \).

3. Main results

In this article, we study \( \lambda_{\min} \) as a function on the spin-conformal moduli space with the non-trivial spin structure. This function takes values in \([0, \lambda_{\min}(S^2)]\) because of (1). As the spin structure is non-trivial, Lott’s results states that 0 is not attained. As a preliminary result we will prove that this function is continuous.

**Proposition 3.1.** The function

\[
\lambda_{\min} : \begin{array}{ccc}
M_1 & \rightarrow & [0, \lambda_{\min}(S^2)] \\
(x_0, y_0) & \mapsto & \lambda_{x_0, y_0}^{\min}
\end{array}
\]

is continuous on \( M_1 \).

The spin-conformal moduli space \( M \) (resp. \( M_1 \)) has two ends. We will compactify each end by adding one point. The point added at the end \( y \to \infty \) will be denoted by \( \infty \) and the point added at the end \( y \to 0 \) is denoted by \((0, 0)\).

**Theorem 3.2.** The function

\[
\lambda_{\min} : \begin{array}{ccc}
M_1 & \rightarrow & [0, \lambda_{\min}(S^2)] \\
(x_0, y_0) & \mapsto & \lambda_{x_0, y_0}^{\min}
\end{array}
\]

extends continuously to \( M_1 \cup \{ (0, 0), \infty \} \) by setting \( \lambda_{x_0, y_0}^{\min} = \lambda_{\min}(S^2) \) and \( \lambda_{\min}^\infty = 0 \).

The continuous extension at \( \infty \) is is easy to see. The first eigenvalue of the Dirac operator on \((T_{x_0, y_0}, g_{\text{eucl}}, \chi_{x_0, y_0})\), is \( \pi/y_0 \), the area is \( y_0 \), hence \( \lambda_{x_0, y_0}^{\min} \leq \pi/\sqrt{y_0} \to 0 \) for \( y_0 \to \infty \).

However, the limit \((x_0, y_0) \to (0, 0)\) is much more difficult to obtain.

Clearly, Theorem 3.2 implies Theorem 1.1.

4. Some preliminaries

**Variational characterization of \( \lambda_{\min} \).** Let \((M, g, \chi)\) be a compact spin manifold of dimension \( n \geq 2 \) with \( \ker D_g = \{0\} \). For \( \psi \in \Gamma(\Sigma M) \), we define

\[
J_g(\psi) = \left( \int_M |D\psi|^2 + dvol \right)^{\frac{n}{n+1}}.
\]

Lott [Lott86] proved that

\[
\lambda_{\min}(M, \{g\}, \chi) = \inf_{\psi} J_g(\psi)
\]

(5)
where the infimum is taken over the set of smooth spinor fields for which
\[
\left( \int_M (D\psi, \psi) \, dv_g \right) \neq 0.
\]
The functional \( J_g \) for the torus \( T_{x_0,y_0} \) is noted as \( J^{x_0,y_0} \).

**Remark 4.1.** The exponents in \( J_g \) are chosen such that \( J_g \) is conformally invariant. More exactly, if \( g \) and \( \tilde{g} \) are conformal, then the spinor bundles of \((M, g, \chi)\) and \((M, \tilde{g}, \chi)\) can be identified in such a way that \( J_g(\psi) = J_{\tilde{g}}(\psi) \).

**Cylinders and doubly pointed spheres.**

Let \( Z_{x_0,y_0} \) be defined as in Examples [2] [3].

**Lemma 4.2.** (Mercator, around 1569). Let \( N, S \in S^2 \) be respectively the North pole and the South pole of \( S^2 \).
Then there is a conformal diffeomorphism \( F_{x_0,y_0} \) from \((Z_{x_0,y_0}, g_{eucl})\) to \((S^2 \setminus \{N,S\})\).

**Proof.** In the case \((x_0,y_0) = (0, 2\pi)\) we see that the application
\[
F_{0,2\pi} : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} \frac{\sin y}{\cos x} \\ \frac{\sin x}{\cos y} \end{pmatrix}
\]
is conformal and defines a conformal bijection \( Z_{0,2\pi} \to S^2 \setminus \{N,S\} \). The general case follows by composing with a linear conformal map \( Z_{x_0,y_0} \to Z_{0,2\pi} \).

The map \( F \) induces a map between the frame bundles.
\[
\tilde{F}_{x_0,y_0} : P_{SO}(Z_{x_0,y_0}) \to P_{SO}(S^2)
\]
\[
\tilde{F}_{x_0,y_0}((p, X, Y)) := \left( F_{x_0,y_0}(p), \frac{dF_{x_0,y_0}(X)}{dF_{x_0,y_0}(Y)} \right)
\]
\(X, Y \in T_p Z_{x_0,y_0}\) are orthonormal and oriented

The unique spin structure on \( S^2 \) pulls back to a spin structure on \( Z_{x_0,y_0} \), that we will denote as \( \chi_{x_0,y_0} \).

**Lemma 4.3.** The spin structure \( \chi_{x_0,y_0} \) is the non-trivial spin structure on \( Z_{x_0,y_0} \).

**Proof.** We will show the lemma for the case \((x_0,y_0) = (0, 2\pi)\). As before, the general case then follows by composing with a linear map \( Z_{x_0,y_0} \to Z_{0,2\pi} \).

We define the loop \( \gamma : [0, 2\pi] \to Z_{0,2\pi}, \gamma(t) := (0, t) \) and the parallel section
\[
\alpha : t \mapsto \left( \frac{\partial}{\partial x} |_{\gamma(t)}, \frac{\partial}{\partial y} |_{\gamma(t)} \right)
\]
of \( P_{SO}(Z_{0,2\pi}) \) along \( \gamma \). The spin structure \((P, \chi_{0,2\pi})\) on \( Z_{0,2\pi} \) is trivial if and only if there is a section \( \hat{\alpha} \) of \( P \) along \( \gamma \) such that \( \chi_{0,2\pi} \circ \hat{\alpha} = \alpha \) and \( \hat{\alpha}(0) = \hat{\alpha}(2\pi) \).

The composition \( \tilde{F}_{0,2\pi} \circ \alpha \) is a section of \( P_{SO}(S^2) = SO(3) \) along \( F_{0,2\pi} \circ \gamma \). One checks that
\[
\tilde{F}_{0,2\pi} \circ \alpha(t) = \left( \frac{\partial F_{0,2\pi}}{\partial x} |_{(0,t)}, \frac{\partial F_{0,2\pi}}{\partial y} |_{(0,t)}, F(0,t) \right) = \begin{pmatrix} 0 & \cos y & \sin y \\ 0 & -\sin y & \cos y \\ 1 & 0 & 0 \end{pmatrix}
\]
We lift this loop to a path \( \hat{\alpha} \) in \( SU(2) \), then one easily sees that \( \hat{\alpha}(0) = -\hat{\alpha}(2\pi) \). As \( \chi_{0,2\pi} \) is defined as the pullback of the spin structure on \( S^2 \), we see that any lift \( \hat{\alpha} \) of \( \alpha \) also satisfies \( \hat{\alpha}(0) \neq \hat{\alpha}(2\pi) \). Hence, we have proved non-triviality of \( \chi_{0,2\pi} \).

**Corollary 4.4.** Let \( Z_{x_0,y_0} \) carry its non-trivial spin structure. Then,
\[
\left( \int_{Z_{x_0,y_0}} |D\psi|^2 \, dx \right) \geq \lambda_{\min}(S^2)
\]
for any compactly supported spinor \( \psi \in \Gamma(\Sigma Z_{x_0,y_0}) \) such that \( \int \langle \psi, D\psi \rangle dx \neq 0 \).
Let $f : Z_{x_0,y_0} \to [0, +\infty[$ be such that $F^\ast_{x_0,y_0} g_0 = f^2 g_{\text{eucl}}$. It is well known (see for example \cite{Hit74, Hij86}) that $F_{x_0,y_0}$ induces a pointwise isometry

$$
\Sigma(T_{x_0,y_0}, g_{\text{eucl}}) \overset{\psi}{\longrightarrow} \Sigma(S^2 \setminus \{N, S\}, g_0)
$$

such that

$$
\bar{D} f^{-\frac{1}{2}} \bar{\psi} = f^{-\frac{1}{2}} \bar{D} \bar{\psi}.
$$

where $\bar{D}$ denotes the Dirac operator on $S^2$. Moreover, $\bar{\psi}$ is smooth on $S^2$ since $\bar{\psi} \equiv 0$ in a neighborhood of $N$ and $S$. It is well known that the functional $J$ defined at the beginning of section \[3\] is conformally invariant. This implies that

$$
\left( \int_{Z_{x_0,y_0}} |D\psi|^{\frac{2}{q}} \, dx \right)^{\frac{q}{2}} \leq \left( \int_{Z_{x_0,y_0}} \langle \psi, \bar{D} \bar{\psi} \rangle \, dx \right)^{\frac{q}{2}} \leq \lambda_{\min}(S^2).
$$

5. Proof of the Main Results

For the proof we will need the following well known elliptic estimates. These estimates are a consequence of techniques explained for example in \cite{Ta81}, see also \cite{Aub98}. However, in our special situation a proof is much easier. Hence, for the convenience of the reader we will include an elementary proof here.

**Lemma 5.1 (Elliptic estimates).** Let $(x_0, y_0) \in \mathcal{M}_1$, and note $T^2$ for $T_{x_0,y_0}$. There exists $C > 0$ depending only on $x_0$ and $y_0$ such that

$$
\int_{T^2} |D\psi|^{\frac{2}{q}} \, dv_g \geq C \int_{T^2} |\nabla \psi|^{\frac{2}{q}} \, dv_g \tag{6}
$$

and

$$
\left( \int_{T^2} |\nabla \psi|^{\frac{2}{q}} \, dv_g \right)^{\frac{q}{4}} \leq C \int_{T^2} |D\psi|^{\frac{2}{q}} \, dv_g \tag{7}
$$

for any smooth spinor $\psi$.

**Proof.** Let $q = \frac{4}{3}$. Assume that \[(6)\] is false. Then, for all $\varepsilon > 0$, we can find a smooth spinor $\psi_\varepsilon \in \Gamma(\Sigma(T^2))$ such that

$$
\int_{T^2} |D\psi_\varepsilon|^q \, dv_g \leq \varepsilon \quad \text{and} \quad \int_{T^2} |\nabla \psi_\varepsilon|^q \, dv_g = 1. \tag{8}
$$

Now, assume that

$$
\lim_{\varepsilon \to 0} \left( \int_{T^2} |\psi_\varepsilon|^q \, dv_g \right)^{\frac{1}{q}} = +\infty.
$$

Then, we set

$$
\psi_\varepsilon' = \frac{\psi_\varepsilon}{\left( \int_{T^2} |\psi_\varepsilon|^q \, dv_g \right)^{\frac{1}{q}}}. \tag{9}
$$

The sequence $(\psi_\varepsilon')$ is bounded in $W^{1,q}(T^2)$ and since $W^{1,q}(T^2)$ is reflexive, we can find $\psi_0' \in W^{1,q}(T^2)$ such that there is sequence $\varepsilon_i \to 0$, with $lim_{i \to \infty} \psi_\varepsilon_i = \psi_0'$ weakly in $W^{1,q}(T^2)$. Then, we would have

$$
\int_{T^2} |\nabla \psi_0'|^q \, dv_g \leq \liminf_{\varepsilon_i} \int_{T^2} |\nabla \psi_\varepsilon_i|^q \, dv_g = 0
$$

We would get that $\psi_0'$ is parallel which cannot occur since the structure on $T^2$ is not trivial. This proves that $(\psi_\varepsilon)$ is bounded in $L^9(T^2)$ and hence, by \[(8)\] in $W^{1,q}(T^2)$. Again by reflexivity of $W^{1,q}(T^2)$, we get the existence of a spinor $\psi_0$, weak limit of a subsequence $\psi_\varepsilon$, in $W^{1,9}(T^2)$. By weak convergence of $D\psi_\varepsilon_i$ to $D\psi_0$ in $L^3(T^2)$, we have

$$
\int_{T^2} |D\psi_0|^q \, dv_g \leq \lim\inf_{\varepsilon_i} \int_{T^2} |D\psi_\varepsilon_i|^q \, dv_g = 0.
$$

This is impossible since the Dirac operator on $T^2$ has a trivial kernel. This proves \[(9)\]. As one can check, relation \[(9)\] can be proved with the same type of arguments. $\square$
Proof of Proposition 3.1. The proposition states that \( \lambda_{\text{min}} \) is continuous on \( M_1 \). Let \((x_k, y_k) \in M_1 \) be a sequence tending to \((x_0, y_0) \in M_1 \). We identify \( T^2 \) with \( \mathbb{R}^2 / \mathbb{Z}^2 \). The conformal structures corresponding to \((x_k, y_k)\) and \((x_0, y_0)\) are represented by flat metrics \( g_{x_k,y_k} \) and \( g_{x_0,y_0} \) on \( \mathbb{R}^2 / \mathbb{Z}^2 \), that are invariant under translations, and such that \( g_{x_k,y_k} \rightarrow g_{x_0,y_0} \) in the \( C^\infty \)-topology.

Let \( \varepsilon > 0 \) be small and let \( \psi_0 \) and \( (\psi_k)_k \) be smooth spinors such that
\[
J_{x_0,y_0}(\psi_0) \leq \lambda_{\text{min}}^{x_0,y_0} + \varepsilon \text{ and } J_{x_k,y_k}(\psi_k) \leq \lambda_{\text{min}}^{x_k,y_k} + \varepsilon. 
\]

At first, since \((g_{x_k}, y_k))_k \) tends to \( g_{x_0,y_0} \), it is easy to see that
\[
\lim_k J_{x_k,y_k}(\psi_0) = J_{x_0,y_0}(\psi_0)
\]
and hence \( \limsup_k \lambda_{\text{min}}^{x_k,y_k} \leq \lambda_{\text{min}}^{x_0,y_0} + \varepsilon \) for the given \( \varepsilon > 0 \) that we can choose as small as we want. Thus
\[
\limsup_k \lambda_{\text{min}}^{x_k,y_k} \leq \lambda_{\text{min}}^{x_0,y_0}.
\]

Now, let us prove that
\[
\limsup_k J_{x_0,y_0}(\psi_k) \leq \liminf_k J_{x_k,y_k}(\psi_k)
\]

We let \((v, w)\) be an orthonormal basis for \( g_{x_0,y_0} \) and \( (v_k, w_k)_k \), orthonormal basis for \( g_{x_k,y_k} \), which tends to \((v, w)\). One can write for all \( k \), \( v_k = a_k v + b_k w \) and \( w_k = c_k v + d_k w \) with \( \lim_k a_k = \lim_k d_k = 1 \) and \( \lim_k b_k = \lim_k c_k = 0 \).

We have
\[
\left( \int_{T^2} |D_{x_k,y_k} \psi_k|^4 dv_{g_{x_k,y_k}} \right)^{\frac{2}{3}} \geq \left( \int_{T^2} |D_{x_0,y_0} \psi_k|^4 dv_{g_{x_0,y_0}} \right)^{\frac{2}{3}} - \alpha'_k \left( \int_{T^2} |\nabla \psi_k|^4 dv_{g_{x_0,y_0}} \right)^{\frac{2}{3}}
\]
where \( \lim_k \alpha'_k = 0 \). Together with Lemma 5.1, we get that
\[
(1 - C \alpha'_k) \left( \int_{T^2} |D_{x_0,y_0} \psi_k|^4 dv_{g_{x_0,y_0}} \right)^{\frac{2}{3}} \leq \left( \int_{T^2} |D_{x_k,y_k} \psi_k|^4 dv_{g_{x_k,y_k}} \right)^{\frac{2}{3}}
\]
where \( C \) is a positive constant independent of \( k \). Now, in the same way, we can write
\[
\int_{T^2} \langle \psi_k, D_{x_0,y_0} \psi_k \rangle dv_{g_{x_0,y_0}} \geq \int_{T^2} \langle \psi_k, D_{x_k,y_k} \psi_k \rangle dv_{g_{x_k,y_k}} - \beta_k \int_{T^2} |\psi_k| |\nabla \psi_k| dv_{g_{x_0,y_0}}
\]
where \( \lim_k \beta_k = 0 \). Using Hölder inequality, we have
\[
\int_{T^2} |\psi_k| |\nabla \psi_k| dv_{g_{x_0,y_0}} \leq \left( \int_{T^2} |\psi_k|^4 dv_{g_{x_0,y_0}} \right)^{\frac{1}{2}} \left( \int_{T^2} |\nabla \psi_k|^2 dv_{g_{x_0,y_0}} \right)^{\frac{1}{2}}.
\]
Using (13) and (14), this gives
\[
\int_{T^2} |\psi_k| |\nabla \psi_k| dv_{g_{x_0,y_0}} \leq C \left( \int_{T^2} |D \psi_k|^4 dv_{g_{x_0,y_0}} \right)^{3/2}.
\]

Using (15) and (16), this gives
\[
\int_{T^2} |\psi_k| |\nabla \psi_k| dv_{g_{x_0,y_0}} \leq C \left( \int_{T^2} |D \psi_k|^4 dv_{g_{x_0,y_0}} \right)^{3/2}.
\]

We obtain
\[
\int_{T^2} \langle \psi_k, D_{x_0,y_0} \psi_k \rangle dv_{g_{x_0,y_0}} \geq \int_{T^2} \langle \psi_k, D_{x_k,y_k} \psi_k \rangle dv_{g_{x_k,y_k}} - \beta_k \left( \int_{T^2} |D \psi_k|^4 dv_{g_{x_0,y_0}} \right)^{3/2}.
\]
Together with (11), we get (12). This immediately implies that
\[ \liminf_k \lambda_{\min}^{x_k,y_k} \geq \lambda_{\min}^{x_0,y_0} \]
and ends the proof of the proposition. \qed

**Proof of Theorem 5.4.** Any calculation in this proof will be carried out in Riemannian normal coordinates with respect to a flat metric. In the following, \((e_1, e_2)\) will denote the canonical basis of \(\mathbb{R}^2\).

In order to prove \(\lim_{(x, y) \to (0, 0)} \lambda_{\min}^{x,y} = \lambda_{\min}(\mathbb{S}^2)\) we will show that there is no sequence \((x_k, y_k) \to (0, 0)\) such that \(\lim_{(x, y) \to (0, 0)} \lambda_{\min}^{x,y} < \lambda_{\min}(\mathbb{S}^2)\). We may assume that \(\lambda_{\min}^{x_k,y_k} < \lambda_{\min}(\mathbb{S}^2)\) for all \(k\). Note that the spectrum of \(D\) is symmetric in dimension 2. By [Am03a], we then can find a sequence of spinors \(\psi_k\) of class \(C^1\) such that on \(T_{x_k,y_k}\)

\[ D\psi_k = \lambda_{\min}^{x_k,y_k} |\psi_k|^2 \psi_k \]

and such that
\[ \int_{T_{x_k,y_k}} |\psi_k|^2 dx = 1. \]  

Moreover, we have
\[ J_{x_k,y_k}(\psi_k) = \lambda_{\min}^{x_k,y_k}. \]

Sometimes we will identify \(\psi_k\) with its pullback to \(\mathbb{R}^2\). In this picture \(\psi_k\) is a doubly periodic spinor on \(\mathbb{R}^2\).

**Step 1.** There exists \(C > 0\) such that for all \(k\), we have \(\lambda_{\min}^{x_k,y_k} \geq C y_k^{1/2}\).

Here and in the sequel, \(C\) will always denote a positive constant which does not depend on \(k\).

For the proof of the first step, we let \(\Omega = \{(x, y) \in M_1 \mid 1/2 \leq y \leq 3/2\}\). Since \(\Omega\) is compact and since \(\lambda_{\min}\) is continuous and positive, there exists \(C > 0\) such that for all \(\lambda_{\min} \geq C\) on \(\Omega\).

Now, assume that
\[ \lim_k \lambda_{\min}^{x_k,y_k} y_k^q = 0. \]

We can find a sequence \((N_k)\) which tends to \(+\infty\) such that \((3^{N_k}x_k, 3^{N_k}y_k) \in \Omega\). Note that the locally isometric covering \(T_{3^{N_k}x_k,3^{N_k}y_k} \to T_{x_k,y_k}\), \(p \in \mathbb{N}\), preserves the spin structures if and only if \(p\) is odd. Let \(\tilde{\psi}_k\) be the pullback of \(\psi_k\) with respect to covering \(T_{3^{N_k}x_k,3^{N_k}y_k} \to T_{x_k,y_k}\). We now have
\[ \int_{T_{3^{N_k}x_k,3^{N_k}y_k}} |D\tilde{\psi}_k|^2 dx = 3^{N_k} \int_{T_{x_k,y_k}} |D\psi_k|^2 dx \]

and
\[ \int_{T_{3^{N_k}x_k,3^{N_k}y_k}} \langle \psi_k, D\psi_k \rangle dx = 3^{N_k} \int_{T_{x_k,y_k}} \langle \psi_k, D\psi_k \rangle dx. \]

We then get by (14) that
\[ C \leq \lambda_{\min}^{x_k,y_k} y_k^{3^{N_k}} \leq J_{x_k,y_k}(\psi_k) \leq 3^{N_k} \lambda_{\min}^{x_k,y_k} \leq C y_k^{-1/2} \lambda_{\min}^{x_k,y_k}. \]

**Step 2.** There exists \(C > 0\) such that for all \(k\), we have \(\lambda_{\min}^{x_k,y_k} \geq C\).

Let \(\eta : \mathbb{R} \to [0, 1]\) be a cut-off function defined on \(\mathbb{R}\) which is equal to 0 on \(\mathbb{R} \setminus [-1, 2]\) and which is equal to 1 on \([0, 1]\). We may assume that \(\eta\) is smooth. Let \(v_k = (x_k, y_k)\). Since \((e_1, v_k)\) is a basis of \(\mathbb{R}^2\), we can define \(\eta_k : \mathbb{R}^2 \to [0, 1]\) by
\[ \eta_k(tv_k + se_1) = \eta(s) \]

Since \(v_k\) is asymptotically orthogonal to \(e_1\), we can find \(C > 0\) independent of \(k\) such that
\[ |\nabla \eta_k| \leq C \]  

(15)
Moreover, by corollary 4.4, we have
\[
\left( \frac{\int_{Z_{k,y_k}} |D\eta_k \psi_k|^\frac{4}{3} dx}{\int_{Z_{k,y_k}} \langle \eta_k \psi_k, D\eta_k \psi_k \rangle dx} \right)^\frac{3}{4} \geq \lambda_{\min}(S^2).
\]  
(16)

Now, we write that
\[
\left( \int_{Z_{k,y_k}} |D\eta_k \psi_k|^\frac{4}{3} dx \right)^\frac{3}{4} = \left( \int_{Z_{k,y_k}} |\nabla \eta_k \cdot \psi_k + \eta_k D\psi_k|^\frac{4}{3} dx \right)^\frac{3}{4} \leq \left( \int_{Z_{k,y_k}} |\nabla \eta_k \cdot \psi_k|^\frac{4}{3} dx \right)^\frac{3}{4} + \left( \int_{Z_{k,y_k}} |\eta_k D\psi_k|^\frac{4}{3} dx \right)^\frac{3}{4}.
\]

By (15) and Hölder inequality, we have
\[
\left( \int_{Z_{k,y_k}} |\nabla \eta_k \cdot \psi_k|^\frac{4}{3} dx \right)^\frac{3}{4} \leq C \left( \int_{Z_{k,y_k} \cap \text{Supp}(\nabla \eta_k)} |\psi_k|^\frac{4}{3} dx \right)^\frac{3}{4} \leq C \left( \int_{Z_{k,y_k} \cap \text{Supp}(\nabla \eta_k)} |\psi_k|^4 dx \right)^\frac{3}{4} \text{Vol}(Z_{k,y_k} \cap \text{Supp}(\nabla \eta_k))^\frac{1}{4}.
\]

We then have
\[
\text{Vol}(Z_{k,y_k} \cap \text{Supp}(\nabla \eta_k)) \leq 3y_k.
\]

By (12) and step 1, this gives that
\[
\left( \int_{Z_{k,y_k}} |\nabla \eta_k \cdot \psi_k|^\frac{4}{3} dx \right)^\frac{3}{4} \leq C y_k^\frac{1}{4} \leq C \lambda_{\min}^{x_k,y_k}.
\]

With the same argument and using relations (11) and (12), it follows that
\[
\left( \int_{Z_{k,y_k}} |\eta_k D\psi_k|^\frac{4}{3} dx \right)^\frac{3}{4} \leq 3^\frac{3}{4} \lambda_{\min}^{x_k,y_k} \left( \int_{Z_{k,y_k}} |\psi_k|^4 dx \right)^\frac{3}{4} \leq C \lambda_{\min}^{x_k,y_k}.
\]

Finally, we get that
\[
\left( \int_{Z_{k,y_k}} |D\eta_k \psi_k|^\frac{4}{3} dx \right)^\frac{3}{4} \leq C (\lambda_{\min}^{x_k,y_k})^2.
\]  
(17)

We now write that
\[
\int_{Z_{k,y_k}} \langle \eta_k \psi_k, D\eta_k \psi_k \rangle dx = \int_{Z_{k,y_k}} \langle \eta_k \psi_k, \nabla \eta_k \cdot \psi_k + \eta_k D\psi_k \rangle dx.
\]

Moreover, the left hand side of this equality is real since \(D\) is an autoadjoint operator. Since
\[
\int_{Z_{k,y_k}} \langle \eta_k \psi_k, \nabla \eta_k \cdot \psi_k \rangle dx \in i\mathbb{R}.
\]

Together with equation (11), this implies that
\[
\int_{Z_{k,y_k}} \langle \eta_k \psi_k, D\eta_k \psi_k \rangle dx = \int_{Z_{k,y_k}} \eta_k^2 \lambda_{\min}^{x_k,y_k} |\psi_k|^4 dx.
\]

Using (12), we obtain that
\[
\int_{Z_{k,y_k}} \langle \eta_k \psi_k, D\eta_k \psi_k \rangle dx \geq \lambda_{\min}^{x_k,y_k} \int_{Z_{k,y_k}} |\psi_k|^4 dx = \lambda_{\min}^{x_k,y_k}.
\]  
(18)

Finally, plugging (17) and (18) in (16), we obtain that \(\lambda_{\min}(S^2) \leq C \lambda_{\min}^{x_k,y_k}\). This proves the step.

**Step 3.** The function \(\lambda_{\min}\) can be extended continuously to \(M_1 \cup \{(0,0)\}\) by setting \(\lambda_{\min}^{0,0} = \lambda_{\min}(S^2)\).
In other words, we show that \( \lim_k \lambda_{\text{min}}^{y_k} = \lambda_{\text{min}}(S^2) \). The method is quite similar to the one of previous step. Let \( \zeta_k : \mathbb{R} \to [0, 1] \) be a smooth cut-off function defined on \( \mathbb{R} \) which is equal to 0 on \( \mathbb{R} \setminus [-y_k, 1 + y_k] \), which is equal to 1 on \([0, 1]\) and which satisfies \( |\nabla \zeta_k| \leq \frac{2}{y_k} \). As in the last step, we can define \( \gamma_k : \mathbb{R}^2 \to [0, 1] \) by 
\[
\gamma_k(tv_k + se_1) = \zeta_k(s).
\]
Since \( v_k \) is asymptotically orthogonal to \( e_1 \), we can find \( C > 0 \) independent of \( k \) such that 
\[
|\gamma_k| \leq \frac{C}{y_k}.
\]  
(19)
As in step 3, we have 
\[
\left( \int_{Z_{x_k,y_k}} |D\gamma_k \psi_k|^2 \, dx \right)^{\frac{1}{2}} \geq \lambda_{\text{min}}(S^2).
\]  
(20)
We first prove that we can assume that 
\[
\int_{Z_{x_k,y_k} \cap \text{Supp}(\nabla \gamma_k)} |\psi_k|^4 \, dx \leq Cy_k.
\]  
(21)
We let \( n_k = \lfloor (2y_k)^{-1} \rfloor \) be the integer part of \( 2y_k^{-1} \). For all \( t \in [0, n_k - 1] \), we define 
\[
A_{k,t} = \left\{ te_1 + sv_k \mid s \in [0, 1] \text{ and } t \in \left[ \frac{l - \frac{1}{2}, \frac{l + \frac{1}{2}}{n_k} \right] \right\}.
\]
The family of sets \( (A_{k,t})_{t \in [0, n_k - 1]} \) is a partition of \( T_{x_k,y_k} \) which is the image of \( T_{x_k,y_k} \) by the translation of vector \( -\frac{1}{2n_k}e_1 \). By periodicity, \( (A_{k,t})_{t \in [0, n_k - 1]} \) can be seen as a partition of \( T_{x_k,y_k} \). Consequently, we can write that 
\[
1 = \int_{T_{x_k,y_k}} |\psi_k|^4 \, dx = \sum_{l=0}^{n_k-1} \int_{A_{k,l}} |\psi_k|^4 \, dx.
\]
Hence, there exists \( l_0 \in [0, n_k - 1] \) such that 
\[
\int_{A_{k,l_0}} |\psi_k|^4 \, dx = \min_{l \in [0, n_k - 1]} \sum_{l=0}^{n_k-1} \int_{A_{k,l}} |\psi_k|^4 \, dx \leq \frac{1}{n_k}.
\]
Obviously, without loss of generality, we can replace \( \psi_k \) by \( \psi_k \circ t_0 \) where \( t_0 \) is the translation of vector \( -l_0e_1 \). In this way, we can assume that \( t_0 = 0 \). By periodicity, \( \text{Supp}(\nabla \gamma_k) \subset A_{k,0} \). Hence, 
\[
\int_{Z_{x_k,y_k} \cap \text{Supp}(\nabla \gamma_k)} |\psi_k|^4 \, dx \leq \frac{1}{n_k}.
\]
Since \( n_k \sim \frac{2}{y_k} \), equation (21) follows.

Now, we proceed as in step 4. We write that 
\[
\left( \int_{Z_{x_k,y_k}} |D\gamma_k \psi_k|^2 \, dx \right)^{\frac{1}{2}} = \left( \int_{Z_{x_k,y_k}} |\nabla \gamma_k \cdot \psi_k + \gamma_k D\psi_k|^2 \, dx \right)^{\frac{1}{2}} \leq \left( \int_{Z_{x_k,y_k}} |\nabla \gamma_k \cdot \psi_k|^2 \, dx \right)^{\frac{1}{2}} + \left( \int_{Z_{x_k,y_k}} |\gamma_k D\psi_k|^2 \, dx \right)^{\frac{1}{2}}.
\]
It follows from (17) and the Hölder inequality that 
\[
\left( \int_{Z_{x_k,y_k}} |\nabla \gamma_k \cdot \psi_k|^2 \, dx \right)^{\frac{1}{2}} \leq C \frac{1}{y_k} \left( \int_{Z_{x_k,y_k} \cap \text{Supp}(\nabla \gamma_k)} |\psi_k|^4 \, dx \right)^{\frac{1}{4}} \leq C \frac{1}{y_k} \left( \int_{Z_{x_k,y_k} \cap \text{Supp}(\nabla \gamma_k)} |\psi_k|^4 \, dx \right)^{\frac{1}{4}} \left( \text{Vol}(Z_{x_k,y_k} \cap \text{Supp}(\nabla \gamma_k)) \right)^{\frac{1}{4}}.
\]
Clearly, we have
\[ \text{Vol}(Z_{x_k,y_k} \cap \text{Supp}(\nabla \gamma_k)) \leq C y_k^2. \]

By (21), we obtain
\[ \left( \int_{Z_{x_k,y_k}} |\nabla \gamma_k \cdot \psi_k|^{\frac{4}{3}} dx \right)^{\frac{3}{4}} \leq C y_k^{-1 + \frac{2}{3}} \leq C y_k^{\frac{2}{3}} = o(1). \]

For the other term, we write, using (11)
\[ \left( \int_{Z_{x_k,y_k}} |\gamma_k D\psi_k|^{\frac{4}{3}} dx \right)^{\frac{3}{4}} = \lambda_{x_k,y_k}^{x_k,y_k} \left( \int_{Z_{x_k,y_k}} |\psi_k|^{4} dx + \int_{Z_{x_k,y_k} \cap \{0 < \gamma_k < 1\}} |\psi_k|^{4} dx \right)^{\frac{3}{4}}. \]

Clearly, we can construct \( \gamma_k \) such that \( \{0 < \gamma_k < 1\} \subset \text{Supp}(\nabla \gamma_k) \). It then follows from (21) that
\[ \left( \int_{Z_{x_k,y_k}} |\gamma_k D\psi_k|^{\frac{4}{3}} dx \right)^{\frac{3}{4}} \leq \lambda_{\min}^{x_k,y_k} + o(1). \]

Finally, we obtain
\[ \left( \int_{Z_{x_k,y_k}} |D\gamma_k \psi_k|^{\frac{4}{3}} dx \right)^{\frac{3}{4}} \leq (\lambda_{\min}^{x_k,y_k})^{2} + o(1). \]

Now, as in step 2, we write that
\[ \int_{Z_{x_k,y_k}} (\gamma_k \psi_k, D\gamma_k \psi_k) dx = \int_{Z_{x_k,y_k}} \gamma_k \lambda_{\min}^{x_k,y_k} |\psi_k|^{4} dx. \]

Using (12), we obtain that
\[ \int_{Z_{x_k,y_k}} (\gamma_k \psi_k, D\gamma_k \psi_k) dx \geq \lambda_{\min}^{x_k,y_k} \int_{Z_{x_k,y_k}} |\psi_k|^{4} dx = \lambda_{\min}^{x_k,y_k}. \]

Plugging (22) and (23) in (20), we obtain that
\[ \lambda_{\min}(S^2) \leq \frac{(\lambda_{\min}^{x_k,y_k})^{2} + o(1)}{\lambda_{\min}^{x_k,y_k}} \]

which implies that either \( \lambda_{\min}^{x_k,y_k} \to 0 \) or \( \lambda_{\min}^{x_k,y_k} \to \lambda_{\min}(S^2) \). Hence, step 3 yields the statement of the theorem. \( \square \)

**References**


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