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To cite this version:
Bernd Ammann, Emmanuel Humbert. The first conformal Dirac eigenvalue on 2-dimensional tori. Journal of Geometry and Physics, Elsevier, 2006, 56, pp.623-642. <hal-00101461>

HAL Id: hal-00101461
https://hal.archives-ouvertes.fr/hal-00101461
Submitted on 27 Sep 2006

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THE FIRST CONFORMAL DIRAC EIGENVALUE ON 2-DIMENSIONAL TORI

BERND AMMANN AND EMMANUEL HUMBERT

ABSTRACT. Let $M$ be a compact manifold with a spin structure $\chi$ and a Riemannian metric $g$. Let $\lambda_2^g$ be the smallest eigenvalue of the square of the Dirac operator with respect to $g$ and $\chi$. The $\tau$-invariant is defined as

$$\tau(M, \chi) := \sup \inf \sqrt{\lambda_2^g \text{Vol}(M, g)}^{1/n}$$

where the supremum runs over the set of all conformal classes on $M$, and where the infimum runs over all metrics in the given class.

We show that $\tau(T^2, \chi) = 2\sqrt{\pi}$ if $\chi$ is "the" non-trivial spin structure on $T^2$. In order to calculate this invariant, we study the infimum as a function on the spin-conformal moduli space and we show that the infimum converges to $2\sqrt{\pi}$ at one end of the spin-conformal moduli space.

1 MSC 2000: 53 A 30, 53C27 (Primary) 58 J 50 (Secondary)

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1. Introduction

Let $(M, g, \chi)$ be a compact spin manifold of dimension $n \geq 2$. For any metric $\tilde{g}$ in the conformal class $[g]$ of $g$, let $\lambda_1(D_2^{\tilde{g}})$ be the smallest eigenvalue of the square of the Dirac operator. We define

$$\lambda_{\text{min}}(M, g, \chi) = \inf_{\tilde{g} \in [g]} \sqrt{\lambda_1(D_2^{\tilde{g}}) \text{Vol}(M, \tilde{g})}^{1/n}.$$ 

Several works have been devoted to the study of this conformal invariant and some variants of it [Hij86, Lott86, Bär92, Am03, Am03a]. J. Lott [Lott86, Am03] proved that $\lambda_{\text{min}}(M, g, \chi) = 0$ if and only if $\ker D_g \neq \{0\}$.

From [Hij86, Bär92] we deduce $\lambda_{\text{min}}(S^n) = \frac{n}{2} \omega_n$ where $S^n$ is the sphere with constant sectional curvature 1 and where $\omega_n$ is its volume. Furthermore, in [Am03, AHM03], we have seen that

$$\lambda_{\text{min}}(M, g, \chi) \leq \lambda_{\text{min}}(S^n) = \frac{n}{2} \omega_n$$

for all Riemannian spin manifolds.

Furthermore we define

$$\tau(M, \chi) := \sup \lambda_{\text{min}}(M, g, \chi)$$

where the supremum runs over all conformal classes on $M$. Obviously, $\tau(M, \chi)$ is an invariant of a differentiable manifold with spin structure.

Date: March 2005.

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We consider it as interesting to determine $\tau$ or at least some bounds for $\tau$ in as many cases as possible. There are several motivations for studying these invariants $\lambda_{\min}(M, g, \chi)$ and $\tau(M, \chi)$.

Our first motivation is the analogy and the relation to Schoen’s $\sigma$-constant, which is defined as

$$\sigma(M) := \sup \inf \left( \int \frac{\text{Scal}_g \, dv_g}{\text{Vol}(M, g)} \right),$$

where the infimum runs over all metrics in a conformal class $g \in [g_0]$, and where the supremum runs over all conformal classes.

In the case $\sigma(M) \geq 0$ and $n \geq 3$, there is also an alternative definition of the $\tau$-invariant that is analogous to our definition of the $\tau$-invariant. More exactly, in this case

$$\sigma(M) := \sup \inf \lambda_1(L_g) \text{Vol}(M, g)^{2/n},$$

where $\lambda_1(L_g)$ is the first eigenvalue of the conformal Laplacian $L_g := 4\frac{n-2}{n-1} \Delta_g + \text{Scal}_g$. Once again, the infimum runs over all metrics in a conformal class $g \in [g_0]$, and where the supremum runs over all conformal classes. Many conjectures about the value of the $\sigma$-constant exist, but unfortunately it can be calculated only in very few special cases, e.g., $\sigma(S^n) = n(n-1)\omega_n^{2/n}$, $\sigma(S^{n-1} \times S^1) = n(n-1)\omega_n^{2/n}$, $\sigma(T^n) = 0$ and $\sigma(\mathbb{R}P^3) = n(n-1) \left( \frac{2}{n-1} \right)^{2/n}$.

The reader might consult [BN04] for a very elegant and amazing calculation of $\sigma(\mathbb{R}P^3)$ and for a good overview over further literature.

For other quotients of the sphere $\Gamma \backslash S^n, \Gamma \subset O(n+1)$ it is conjectured that

$$\sigma(\Gamma \backslash S^n) = n(n-1) \left( \frac{\omega_n}{\# \Gamma} \right)^{2/n}.$$

(2)

It is not difficult to show that for any metric conformal to the round metric on $\Gamma \backslash S^n$ one has the inequality $\lambda_1(L_g) \text{Vol}(\Gamma \backslash S^n, g)^{2/n} \geq n(n-1) \left( \frac{\omega_n}{\# \Gamma} \right)^{2/n}$. This immediately implies $\sigma(\Gamma \backslash S^n) \geq n(n-1) \left( \frac{\omega_n}{\# \Gamma} \right)^{2/n}$, i.e. the lower bound on $\sigma$ in (3). However, it is very difficult to obtain the upper bound on $\sigma$.

The $\tau$-invariant is not only a formal analogue to Schoen’s $\sigma$-constant, but it is also tightly related to it via Hijazi’s inequality [Hij86, Hij91, Hij01]. Hijazi’s inequality implies that if $M$ carries a spin structure $\chi$, then

$$\tau(M, \chi)^2 \geq \frac{n^2}{4(n-1)} \sigma(M).$$

(3)

Equality is attained in this inequality if $M = S^n$. Hence, upper bounds for $\tau(M, \chi)$ may help to determine the $\sigma$-constant.

This is one reason for studying the $\tau$-invariant.

Another motivation for studying $\tau(M, \chi)$ and $\lambda_{\min}(M, g, \chi)$ comes from the connection to constant mean curvature surfaces. Let $n = 2$. If $\tilde{g}$ is a minimizer that attains the infimum in the definition of $\lambda_{\min}(M, g, \chi)$, and if $\text{Vol}(M, \tilde{g}) = 1$, then any simply connected open subset $U$ of $M$ can be isometrically embedded into $\mathbb{R}^3$, $(U, \tilde{g}) \hookrightarrow \mathbb{R}^3$, such that the resulting surface has constant mean curvature $\lambda_{\min}(M, g, \chi)$. Vice versa, any constant mean curvature surface gives rise to a stationary point of an associated variational principle. It is shown in [Am03] that minimalizers of $\lambda_{\min}(M, g, \chi)$ exist if $\lambda_{\min}(M, g, \chi) \leq 2\sqrt{n}$. For the third motivation, let again $n \geq 2$ be arbitrary. As indicated above, $\lambda_{\min}(M, g, \chi) > 0$ if and only if $\ker D_g = \{0\}$. Hence, $\tau(M, \chi) > 0$ if and only if $M$ carries a metric with $\ker D = \{0\}$. It follows from the Atiyah-Singer index theorem that any spin manifold $M$ of dimension $4k$, $k \in \mathbb{N}$ with $\hat{A}(M) = 0$ has $\tau = 0$, and the same holds for spin manifolds of dimension $8k + 1$ and $8k + 2$ with non-vanishing $\alpha$-genus. C. Bär conjectures [Bâ94, BD02] that in all remaining cases one has $\tau > 0$. Using perturbation methods Maier [Mai97] has verified the conjecture in the case $n \leq 4$. The conjecture also holds if $n \geq 5$ and $\pi_1(M) = \{e\}$. Namely, if $M$ is a compact simply connected spin manifold with vanishing $\alpha$-genus, then building on Gromov-Lawson’s surgery results [GLS93] Stolz showed [Sto92] that $M$ carries a metric $g_+ \geq 0$ of positive scalar curvature. Applying the Schrödinger-Lichnerowicz formula we obtain $\ker D_{g_+} = \{0\}$, and hence $\tau(M, \chi) \geq \lambda_{\min}(M, g_+, \chi) > 0$ for the unique spin structure $\chi$ on $M$. A good reference for this argument is also [BD02], where the interested reader can also find an analogous statement for the case $\alpha(M, \chi) \neq 0$. The method of Stolz and Bär-Dahl also applies to some other fundamental groups, but the general case still remains open.
In the present article we want to have a closer look at the \(\tau\)-invariant on surfaces, in particular 2-dimensional tori. The higher dimensional case will be the subject of another publication.

On surfaces the Yamabe operator cannot be defined as above. The Gauss-Bonnet theorem says that the \(\sigma\)-constant of a surface does not depend on the metric:

\[
\sigma(M) = \sup_{g} \inf_{\lambda} \int 2K_g \, dv_g = 4\pi \chi(M).
\]

It was conjectured by Lott \cite{Lott86} and proved by C. Bär \cite{Bär92} that equation (3) also holds in dimension 2. This amounts in showing \(\tau(S^2) = 2\sqrt{\pi}\). If \(M\) is a compact orientable surface of higher genus, then inequality (3) is trivial.

We will calculate the \(\tau\)-invariant for the 2-dimensional torus \(T^2\). The 2-dimensional torus \(T^2\) has 4 different spin structures. The diffeomorphism group \(\text{Diff}(T^2)\) acts on the space of spin structures by pullback, and the action has two orbits: one orbit consisting of only one spin structure, the so-called trivial spin structure \(\chi_0\) and another orbit consisting of three spin structures. The torus \(T^2\) equipped with the trivial spin structure has non-vanishing \(\alpha\)-genus, thus \(\tau = 0\). The main result of this article is the following theorem.

**Theorem 1.1.** Let \(\chi\) be a non-trivial spin structure on the 2-dimensional torus \(T^2\). Then

\[
\tau(T^2, \chi) = 2\sqrt{\pi} = (\lambda_{\min}(S^2)).
\]

More exactly, for a fixed non-trivial spin structure \(\chi\) we will study \(\lambda_{\min}(M, g, \chi)\) as a function on the spin-conformal moduli space \(\mathcal{M}\). We show that it is continuous (Proposition 3.1), and we show that it can be continuously extended to the natural 2-point compactification of \(\mathcal{M}\), i.e. the compactification where both ends are compactified by one point each. It will be easy to show that \(\lambda_{\min}(M, g, \chi) \to 0\) at one of the ends. However, it is much more involved to prove Theorem 3.2 which states that \(\lambda_{\min}(M, g, \chi) \to \lambda_{\min}(S^2) = 2\sqrt{\pi}\) at the other end.

It is evident that Theorem 3.3 implies Theorem 1.1.

For the proof of Theorem 3.2 we have to establish a qualitative lower bound for the eigenvalues. One important ingredient in the proof of Theorem 3.2 is to study a suitable covering of the 2-torus by a cylinder, and to lift a test spinor to this covering. Using a cut-off argument in a way similar to \cite{AB02} we obtain a compactly supported test spinor on the cylinder. After compactifying the cylinder conformally to the sphere \(S^2\), we can use Bär’s 2-dimensional version of (3), to prove \(\lambda_{\min}(M, g, \chi) \to \lambda_{\min}(S^2) = 2\sqrt{\pi}\) at the other end.

Theorem 3.2 and Theorem 1.1 should be seen as a spinorial analogue of \cite{Sch91}. In that article, Schoen studies the Yamabe invariant on the moduli space of \(O(n)\)-invariant conformal structures on \(S^1 \times S^{n-1}\), \(n \geq 3\). He shows that at one end of this moduli space, the Yamabe invariant converges to the Yamabe invariant of \(S^n\), and hence \(\sigma(S^1 \times S^{n-1}) = \sigma(S^n)\). Combining this result with the Hijazi inequality and Theorem 1.1, one obtains

**Corollary 1.2.** Let \(n \geq 2\). Then

\[
\tau(S^{n-1} \times S^1, \chi) = \begin{cases} 
0 & \text{if } n = 2 \text{ and if } \chi \text{ is trivial,} \\
2 \omega_1^{1/n} & \text{otherwise.}
\end{cases}
\]

The structure of the article is as follows.

In section 3 we define the spin-conformal moduli \(\mathcal{M}\) space of 2-tori and recall some well known facts. In section 4 we state and explain our results. In sections 5 we recall some preliminaries which will be useful for the proof of Theorem 3.2. In Section 6 the proof is carried out.

**Acknowledgement.** The authors want to thank the referee for many useful comments.

## 2. The spin-conformal moduli space of \(T^2\)

At the beginning of this section we will recall the definition of a spin structure. We will only give it in the case \(n = 2\). For more information and for the case of general dimension we refer to standard text books \cite{Fr00, LM85, Ro88, GSV91}. More details about the 2-dimensional case can be obtained in \cite{AB02} and \cite{Am98, BS92}.

Let \((M, g)\) be an oriented surface with a Riemannian metric \(g\). Let \(P_{SO}(M, g)\) denote the set of oriented orthonormal frames over \(M\). The base point map \(P_{SO}(M, g) \to M\) is an \(S^1\) principal bundle. Let \(\alpha : S^1 \to S^1\)
be the non-trivial double covering, i.e. $\alpha(z) = z^2$. A spin structure on $(M, g)$ is by definition a pair $(P, \chi)$ where $P$ is an $S^1$ principal bundle over $M$ and where $\chi : P \rightarrow P_{SO}(M, g)$ is a double covering, such that the diagram

$$
\begin{array}{ccc}
P \times S^1 & \rightarrow & P \\
\downarrow \chi \times \alpha & & \downarrow \chi \\
P_{SO}(M) \times S^1 & \rightarrow & P_{SO}(M)
\end{array}
$$

(4)

commutes (in this diagram the horizontal flashes denote the action of $S^1$ on $P$ and $P_{SO}(M)$). By slightly abusing the notation we will sometimes write $\chi$ for the spin structure, assuming that the domain $P$ of $\chi$ is implicitly given. Two spin structures $(P, \chi)$ and $(\tilde{P}, \tilde{\chi})$ are isomorphic if there is an $S^1$-equivariant bijection $b : P \rightarrow \tilde{P}$ such that $\tilde{\chi} = \chi \circ b$.

If $\tilde{g} = f^2 g$ is a metric conformal to $g$. Then $P_{SO}(M, \tilde{g}) \rightarrow P_{SO}(M, g)$, $(e_1, e_2) \mapsto (fe_2, fe_2)$ defines an isomorphism of $S^1$ principal bundles. The pullback of a spin structure on $(M, g)$ is a spin structure on $(M, \tilde{g})$.

In a similar way, if $(M_1, g_1) \rightarrow (M_2, g_2)$ is an orientation preserving conformal map, but not necessarily a diffeomorphism, then any spin structure on $(M_2, g_2)$ pull back to a spin structure on $(M_1, g_1)$.

**Examples 2.1.**

1. If $g_0$ is the standard metric on $S^2$. Then $P_{SO}(S^2, g_0) = SO(3)$, and the base point map $SO(3) \rightarrow S^2$ is the map that associates to a matrix in $SO(3)$ the first column. The double cover $SU(2) \rightarrow SO(3)$ defines a spin structure on $(S, g_0)$.

2. Let $\tilde{g}$ be an arbitrary metric on $S^2$. After a possible pullback by a diffeomorphism $S^2 \rightarrow S^2$ we can write $\tilde{g} = f^2 g_0$. The pullback of the spin structure given in (1) under the isomorphism $P_{SO}(S^2, \tilde{g}) \rightarrow P_{SO}(S^2, g_0)$ defines a spin structure on $(S^2, \tilde{g})$.

3. Let $g_1$ be a flat metric on the torus $T^2$. Then a parallel frame gives rise to a (global) section of $P_{SO}(T^2, g_1) \rightarrow T^2$. Hence, this is a trivial $S^1$ principal bundle. The trivial fiberwise double covering $T^2 \times S^1 \rightarrow T^2 \times S^1$, $(p, z) \mapsto (p, z^2)$ defines a spin structure on $(T^2, g_1)$, the so-called trivial spin structure $\chi_{tr}$, on $(T^2, g_1)$.

4. If $\tilde{g}$ is an arbitrary metric on $T^2$. Then we can write $\tilde{g} = f^2 g_1$ where $g_1$ is a flat metric. As above, the trivial spin structure on $(T^2, g_1)$ defines a spin structure on $(T^2, \tilde{g})$. This spin structure is also called the trivial spin structure $\chi_{tr}$.

5. For $(x_0, y_0) \in \mathbb{R}^2 \setminus \{0\}$ we define $Z_{x_0,y_0} = \mathbb{R}^2 / \langle (x_0, y_0) \rangle$ where $\langle (x_0, y_0) \rangle$ is the subgroup of $\mathbb{R}^2$ spanned by $(x_0, y_0)$. We will assume that it carries the metric induced by the euclidean metric $g_{eucl}$ on $\mathbb{R}^2$. Then $P_{SO}(Z_{x_0,y_0})$ is a trivial bundle, and a natural trivialization is obtained by a parallel frame. The map $Z_{x_0,y_0} \times S^1 \rightarrow Z_{x_0,y_0} \times S^1$, $(p, z) \mapsto (p, z^2)$ defines a spin structure on $Z_{x_0, y_0}$, the trivial spin structure on $Z_{x_0, y_0}$.

Assume that $\chi : P \rightarrow P_{SO}(M, g)$ is a spin structure on a surface, and assume that $\beta : \pi_1(M) \rightarrow \{-1, +1\}$ is a group homomorphism. Then there is a $\{-1, +1\}$ principal bundle $B_\beta \rightarrow M$ with holonomy $\beta$. Let $B_\beta$ be the quotient of $P \times B_\beta$ by the diagonal action of $\{-1, +1\}$. Then $P_{SO}(M, g)$ is also a spin structure on $(M, g)$. Conversely, if $(\tilde{P}, \tilde{\chi})$ is another spin structure, then one can show that there is a unique $\beta : \pi_1(M) \rightarrow \{-1, +1\}$ such that $(\tilde{P}, \tilde{\chi})$ and $(P_\beta, \chi_\beta)$ are isomorphic. Thus, we see that the space of spin structures is an affine space over the $\{-1, +1\}$-vector space $\text{Hom}(\pi_1(M), \{-1, +1\}) = H^1(M, \{-1, +1\})$.

**Examples 2.2.**

1. Any compact oriented surface $M$ carries a spin structure. If $k$ denotes the genus of $M$, then there are $4^k$ homomorphisms $\pi_1(M) \rightarrow \{-1, +1\}$, hence there are $4^k$ isomorphism classes of spin structures. In particular, the spin structure on $S^2$ is unique.

2. Because of $\pi_1(Z_{x_0,y_0}) = \mathbb{Z}$, there are exactly two spin structures on $Z_{x_0,y_0}$, the trivial one and another one called the non-trivial spin structure.

From now on, let $M = T^2 = \mathbb{R}^2 / \Gamma$ where $\Gamma$ is a lattice in $\mathbb{R}^2$. The trivial spin structure defined above can be used to identify $\text{Hom}(\pi_1(M), \{-1, +1\})$ with the set of isomorphism classes of spin structures. By slightly
abusing the language we will always write $\chi$ for the spin structure $(P,\chi)$ and also for the homomorphism $\pi_1(M) \to \{-1,+1\}$.

The following lemma summarizes some well-known equivalent characterizations of triviality of $\chi$ (see e.g. [LM89], [Mil63], [Am98], [Fr84]).

**Lemma 2.3.** With the above notations, the following statements are equivalent

1. The spin structure is trivial (in the above sense);
2. $\chi(\gamma) = 1$ for all $\gamma \in \Gamma$;
3. The spin structure is invariant under the natural action of the diffeomorphism group $\text{Diff}(T^2)$;
4. $(T^2, \chi)$ is the non-trivial element in the 2-dimensional spin-cobordism group;
5. The $\alpha$-genus of $(T^2, \chi)$ is the non-trivial element in $\mathbb{Z}/2\mathbb{Z}$;
6. The Dirac operator has a non-trivial kernel;
7. The kernel of the Dirac operator has complex dimension $2$.

In particular, we easily see

$$\tau(T^2, \chi_{tr}) = 0.$$

From now on, in the rest of this article, we assume that $\chi$ is not the trivial spin structure, i.e. $\chi(\gamma) = -1$ for some $\gamma \in \Gamma$.

**Definition 2.4.** Two 2-dimensional tori with Riemannian metrics, orientations and spin structures are said to be spin-conformal if there is a conformal map between them preserving the orientation and the spin structure. “Being spin-conformal” is obviously an equivalence relation. The spin-conformal moduli space $\mathcal{M}$ of $T^2$ with the non-trivial spin structure is defined to be the set of these equivalence classes. Furthermore we define

$$\mathcal{M}_1 := \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \middle| |x| \leq \frac{1}{2}, \quad y^2 + \left( |x| - \frac{1}{2} \right)^2 \geq \frac{1}{4}, \quad y > 0 \right\}$$

(see also Fig. 1).

**Lemma 2.5.** Let $g$ be a Riemannian metric on $T^2 = \mathbb{R}^2/\mathbb{Z}^2$, and let $\chi : \mathbb{Z}^2 \to \{-1,+1\}$ be a non-trivial spin structure. Then there is a lattice $\Gamma \subset \mathbb{R}^2$, a spin structure $\chi' : \Gamma \to \{-1,+1\}$, such that

1. $\Gamma$ is generated by $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} x \\ y \end{pmatrix}$ with $\begin{pmatrix} x \\ y \end{pmatrix} \in \mathcal{M}_1$
2. $(T^2, g, \chi)$ is spin-conformal to $((\mathbb{R}^2/\Gamma, g_{\text{eucl}}, \chi'))$
3. $\chi'\left( \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right) = +1$ and $\chi'\left( \begin{pmatrix} x \\ y \end{pmatrix} \right) = -1$. 

![Figure 1. The spin conformal moduli space is $\mathcal{M} = \mathcal{M}_1/\sim$, where $\sim$ means identifying $(x,y) \in \partial \mathcal{M}_1$ with $(-x,y)$.

\[x\]
Proof. Because of the uniformization theorem we can assume without loss of generality that \( g \) is a flat metric. The lemma then follows from elementary algebraic arguments. \( \square \)

Note that \( x \) and \( y \) are uniquely determined if \( (x, y) \) is in the interior of \( M_1 \), i.e. if \( |x| < 1/2 \) and \( y^2 + (|x| - 1/2)^2 > 1/4 \). If \( (x, y) \) is on the boundary of \( M_1 \), then \( y \) and \( |x| \) are determined, but not the sign of \( x \). Hence, after gluing \( (x, y) \in \partial M_1 \) with \( (-x, y) \) we obtain the spin-conformal moduli space \( M \).

Notation. Let \((x_0, y_0) \in M_1 \). The lattice generated by \((1, 0)\) and \((x, y)\) is noted as \( \Gamma_{x_0, y_0} \). Furthermore, we write \( T_{x_0, y_0} \) for the 2-dimensional torus \( \mathbb{R}^2 / \Gamma_{x_0, y_0} \), equipped with the euclidean metric. The quantity \( \lambda_{\min}(T^2, g, \sigma) \) is a spin-conformal invariant, hence \( \lambda_{\min} \) can be viewed as a function on \( M \) or on \( M_1 \).

3. Main results

In this article, we study \( \lambda_{\min} \) as a function on the spin-conformal moduli space with the non-trivial spin structure. This function takes values in \([0, \lambda_{\min}(S^2)]\) because of (1). As the spin structure is non-trivial, Lott’s results states that 0 is not attained. As a preliminary result we will prove that this function is continuous.

Proposition 3.1. The function
\[
\lambda_{\min} : M_1 \to [0, \lambda_{\min}(S^2)]
\]
is continuous on \( M_1 \).

The spin-conformal moduli space \( M \) (resp. \( M_1 \)) has two ends. We will compactify each end by adding one point. The point added at the end \( y \to \infty \) will be denoted by \( \infty \) and the point added at the end \( y \to 0 \) is denoted by \((0, 0)\).

Theorem 3.2. The function
\[
\lambda_{\min} : M_1 \to [0, \lambda_{\min}(S^2)]
\]
extends continuously to \( M_1 \cup \{(0, 0), \infty\} \) by setting \( \lambda_{\min}^0 = \lambda_{\min}(S^2) \) and \( \lambda_{\min}^\infty = 0 \).

The continuous extension at \( \infty \) is is easy to see. The first eigenvalue of the Dirac operator on \((T_{x_0, y_0}, g_{\text{eucl}}, \chi_{x_0, y_0})\), is \( \pi/y_0 \), the area is \( y_0 \), hence
\[
\lambda_{\min}^{x_0, y_0} \leq \pi/\sqrt{y_0} \to 0 \quad \text{for} \quad y_0 \to \infty.
\]

However, the limit \((x_0, y_0) \to (0, 0)\) is much more difficult to obtain. Clearly, Theorem 3.2 implies Theorem 1.1.

4. Some preliminaries

Variational characterization of \( \lambda_{\min} \). Let \((M, g, \chi)\) be a compact spin manifold of dimension \( n \geq 2 \) with \( \ker D_g = \{0\} \). For \( \psi \in \Gamma(\Sigma M) \), we define
\[
J_g(\psi) = \left( \int_M |D\psi|^2_d \nu_g \right)^{\frac{1}{n+1}}.
\]
Lott [Lott86] proved that
\[
\lambda_{\min}(M, [g], \chi) = \inf_{\psi} J_g(\psi)
\]
where the infimum is taken over the set of smooth spinor fields for which
\[
\left( \int_M (D\psi, \psi) \, dv_0 \right) \neq 0.
\]
The functional \( J_g \) for the torus \( T_{x_0,y_0} \) is noted as \( J^{x_0,y_0} \).

**Remark 4.1.** The exponents in \( J_g \) are chosen such that \( J_g \) is conformally invariant. More exactly, if \( g \) and \( \tilde{g} \) are conformal, then the spinor bundles of \((M, g, \chi)\) and \((M, \tilde{g}, \chi)\) can be identified in such a way that \( J_g(\psi) = J_{\tilde{g}}(\psi) \).

### Cylinders and doubly pointed spheres.
Let \( Z_{x_0,y_0} \) be defined as in Examples 2.2 (3).

**Lemma 4.2.** (Mercator, around 1569). Let \( N, S \in S^2 \) be respectively the North pole and the South pole of \( S^2 \). Then there is a conformal diffeomorphism \( F_{x_0,y_0} \) from \((Z_{x_0,y_0}, g_{eucl})\) to \((S^2 \setminus \{N, S\})\).

**Proof.** In the case \((x_0, y_0) = (0, 2\pi)\) we see that the application
\[
F_{0,2\pi} : (x, y) \mapsto \left( \frac{\sin y}{\cos x}, \frac{\cos y}{\sin x}, \tanh x \right)
\]
is conformal and defines a conformal bijection \( Z_{0,2\pi} \rightarrow S^2 \setminus \{N, S\} \). The general case follows by composing with a linear conformal map \( Z_{x_0,y_0} \rightarrow Z_{0,2\pi} \).

The map \( F \) induces a map between the frame bundles.
\[
\tilde{F}_{x_0,y_0} : P_{SO}(Z_{x_0,y_0}) \rightarrow P_{SO}(S^2)
\]
\[
\tilde{F}_{x_0,y_0}((p, X, Y)) := \left( F_{x_0,y_0}(p), \frac{dF_{x_0,y_0}(X)}{dF_{x_0,y_0}(Y)} \right)
\]
\[
X, Y \in T_p\{Z_{x_0,y_0} \text{ are orthonormal and oriented}
\]
The unique spin structure on \( S^2 \) pulls back to a spin structure on \( Z_{x_0,y_0} \), that we will denote as \( \chi_{x_0,y_0} \).

**Lemma 4.3.** The spin structure \( \chi_{x_0,y_0} \) is the non-trivial spin structure on \( Z_{x_0,y_0} \).

**Proof.** We will show the lemma for the case \((x_0, y_0) = (0, 2\pi)\). As before, the general case then follows by composing with a linear map \( Z_{x_0,y_0} \rightarrow Z_{0,2\pi} \).

We define the loop \( \gamma : [0, 2\pi] \rightarrow Z_{0,2\pi}, \gamma(t) := (0, t) \) and the parallel section
\[
\alpha : t \mapsto \left( \frac{\partial}{\partial x} |_{\gamma(t)}, \frac{\partial}{\partial y} |_{\gamma(t)} \right)
\]
of \( P_{SO}(Z_{0,2\pi}) \) along \( \gamma \). The spin structure \( (P, \chi_{0,2\pi}) \) on \( Z_{0,2\pi} \) is trivial if and only if there is a section \( \tilde{\alpha} \) of \( P \) along \( \gamma \) such that \( \chi_{0,2\pi} \circ \tilde{\alpha} = \alpha \) and \( \tilde{\alpha}(0) = \tilde{\alpha}(2\pi) \).

The composition \( \tilde{F}_{0,2\pi} \circ \alpha \) is a section of \( P_{SO}(S^2) = SO(3) \) along \( F_{0,2\pi} \circ \gamma \). One checks that
\[
\tilde{F}_{0,2\pi} \circ \alpha(t) = \left( \frac{\partial F_{0,2\pi}}{\partial x} |_{(0,t)}, \frac{\partial F_{0,2\pi}}{\partial y} |_{(0,t)}, F(0,t) \right) = \begin{pmatrix} 0 & \cos y & \sin y \\ 0 & -\sin y & \cos y \\ 1 & 0 & 0 \end{pmatrix}
\]
We lift this loop to a path \( \tilde{\alpha} \) in SU(2), then one easily sees that \( \tilde{\alpha}(0) = -\tilde{\alpha}(2\pi) \). As \( \chi_{0,2\pi} \) is defined as the pullback of the spin structure on \( S^2 \), we see any lift \( \tilde{\alpha} \) of \( \alpha \) also satisfies \( \tilde{\alpha}(0) \neq \tilde{\alpha}(2\pi) \). Hence, we have proved non-triviality of \( \chi_{0,2\pi} \).

**Corollary 4.4.** Let \( Z_{x_0,y_0} \) carry its non-trivial spin structure. Then,
\[
\frac{\left( \int_{Z_{x_0,y_0}} |D\psi| dx \right)^2}{\int_{Z_{x_0,y_0}} \langle \psi, D\psi \rangle dx} \geq \lambda_{\min}(S^2)
\]
for any compactly supported spinor \( \psi \in \Gamma(\Sigma Z_{x_0,y_0}) \) such that \( \int \langle \psi, D\psi \rangle \neq 0 \).
Let $f : Z_{x_0,y_0} \to [0, +\infty[$ be such that $F_{x_0,y_0} = f^2 g_{cuc}$. It is well known (see for example [Hit74, Hij86]) that $F_{x_0,y_0}$ induces a pointwise isometry

$$\Sigma(T_{x_0,y_0}, g_{cuc}) \to \Sigma(S^2 \setminus \{N, S\}, g_0)$$

such that

$$\bar{D} f^{-\frac{1}{2}} \psi = f^{-\frac{1}{2}} D\psi,$$

where $\bar{D}$ denotes the Dirac operator on $S^2$. Moreover, $\psi$ is smooth on $S^2$ since $\psi \equiv 0$ in a neighborhood of $N$ and $S$. It is well known that the functional $J$ defined at the beginning of section 4 is conformally invariant. This implies that

$$\left(\int_{Z_{x_0,y_0}} |D\psi|^2 dx\right)^{\frac{2}{3}} = \left(\int_{Z_{x_0,y_0}} \langle \psi, D\psi \rangle dx\right)^{\frac{2}{3}} \geq \lambda_{\min}(S^2).$$

5. Proof of the Main results

For the proof we will need the following well known elliptic estimates. These estimates are a consequence of techniques explained for example in [Ta81], see also [Aub98]. However, in our special situation a proof is much easier. Hence, for the convenience of the reader we will include an elementary proof here.

**Lemma 5.1 (Elliptic estimates).** Let $(x_0, y_0) \in M_1$, and note $T^2$ for $T_{x_0,y_0}$. There exists $C > 0$ depending only on $x_0$ and $y_0$ such that

$$\int_{T^2} |D\psi|^2 \, dv_g \geq C \int_{T^2} |\nabla \psi|^2 \, dv_g$$

and

$$\left(\int_{T^2} |\psi|^4 \, dv_g\right)^{\frac{1}{2}} \leq C \int_{T^2} |\nabla \psi|^2 \, dv_g$$

for any smooth spinor $\psi$.

**Proof.** Let $q = \frac{4}{3}$. Assume that (6) is false. Then, for all $\varepsilon > 0$, we can find a smooth spinor $\psi_\varepsilon \in \Gamma(\Sigma(T^2))$ such that

$$\int_{T^2} |D\psi_\varepsilon|^q \, dv_g \leq \varepsilon \quad \text{and} \quad \int_{T^2} |\nabla \psi_\varepsilon|^q \, dv_g = 1.$$  

(8)

Now, assume that

$$\lim_{\varepsilon \to 0} \left(\int_{T} |\psi_\varepsilon|^q \, dv_g\right)^{\frac{1}{q}} = +\infty.$$  

Then, we set

$$\psi'_\varepsilon = \frac{\psi_\varepsilon}{\left(\int_{T} |\psi_\varepsilon|^q \, dv_g\right)^{\frac{1}{q}}}.$$  

The sequence $(\psi'_\varepsilon)$ is bounded in $W^{1,q}(T^2)$ and since $W^{1,q}(T^2)$ is reflexive, we can find $\psi'_0 \in W^{1,q}(T^2)$ such that there is sequence $\varepsilon_i \to 0$, with $\lim_{i \to \infty} \psi'_i = \psi'_0$ weakly in $W^{1,q}(T^2)$. Then, we would have

$$\int_{T^2} |\nabla \psi'_0|^q \, dv_g \leq \liminf_{\varepsilon \to 0} \int_{T^2} |\nabla \psi'_\varepsilon|^q \, dv_g = 0.$$  

We would get that $\psi'_0$ is parallel which cannot occur since the structure on $T^2$ is not trivial. This proves that $(\psi_\varepsilon)$ is bounded in $L^q(T^2)$ and hence, by (8), in $W^{1,q}(T^2)$. Again by reflexivity of $W^{1,q}(T^2)$, we get the existence of a spinor $\psi_0$, weak limit of a subsequence $\psi_\varepsilon$, in $W^{1,q}(T^2)$. By weak convergence of $D\psi_\varepsilon$, to $D\psi_0$ in $L^q(T^2)$, we have

$$\int_{T^2} |D\psi_0|^q \, dv_g \leq \liminf_{\varepsilon \to 0} \int_{T^2} |D\psi_\varepsilon|^q \, dv_g = 0.$$  

This is impossible since the Dirac operator on $T^2$ has a trivial kernel. This proves (6). As one can check, relation (6) can be proved with the same type of arguments. \qed
Proof of Proposition 3.1. The proposition states that $\lambda_{\text{min}}$ is continuous on $M_1$. Let $(x_k, y_k)_k \in M_1$ be a sequence tending to $(x_0, y_0) \in M_1$. We identify $T^2$ with $\mathbb{R}^2/\mathbb{Z}^2$. The conformal structures corresponding to $(x_k, y_k)$ and $(x_0, y_0)$ are represented by flat metrics $g_{x_k, y_k}$ and $g_{x_0, y_0}$ on $\mathbb{R}^2/\mathbb{Z}^2$, that are invariant under translations, and such that $g_{x_k, y_k} \to g_{x_0, y_0}$ in the $C^\infty$-topology.

Let $\varepsilon > 0$ be small and let $\psi_0$ and $(\psi_k)_k$ be smooth spinors such that

$$J_{x_0, y_0}(\psi_0) \leq \lambda_{\text{min}}^{x_0, y_0} + \varepsilon \text{ and } J_{x_k, y_k}(\psi_k) \leq \lambda_{\text{min}}^{x_k, y_k} + \varepsilon.$$ 

At first, since $(g_{x_k, y_k})_k$ tends to $g_{x_0, y_0}$, it is easy to see that

$$\lim_k J_{x_k, y_k}(\psi_0) = J_{x_0, y_0}(\psi_0)$$

and hence $\limsup_k \lambda_{\text{min}}^{x_k, y_k} \leq \lambda_{\text{min}}^{x_0, y_0} + \varepsilon$ for the given $\varepsilon > 0$ that we can choose as small as we want. Thus

$$\limsup_k \lambda_{\text{min}}^{x_k, y_k} \leq \lambda_{\text{min}}^{x_0, y_0}.$$

Now, let us prove that

$$\limsup_k J_{x_0, y_0}(\psi_k) \leq \liminf_k J_{x_k, y_k}(\psi_k) \quad (9)$$

We let $(v, w)$ be a orthonormal basis for $g_{x_0, y_0}$ and $(v_k, w_k)_k$, orthonormal basis for $g_{x_k, y_k}$ which tends to $(v, w)$. One can write for all $k$, $v_k = a_k v + b_k w$ and $w_k = c_k v + d_k w$ with $\lim_k a_k = \lim_k d_k = 1$ and $\lim_k b_k = \lim_k c_k = 0$. We have

$$\left( \int_{T^2} |D_{x_k, y_k} \psi_k|^4 dv_{g_{x_k, y_k}} \right)^{\frac{1}{4}} \leq \left( \int_{T^2} |D_{x_0, y_0} \psi_k|^4 dv_{g_{x_0, y_0}} \right)^{\frac{1}{4}} + \alpha_k \left( \int_{T^2} |D_{x_0, y_0} \psi_k|^4 dv_{g_{x_0, y_0}} \right)^{\frac{1}{4}} \quad (10)$$

where $\lim_k \alpha_k' = 0$. Together with Lemma 7.3, we get that

$$\left( \int_{T^2} |D_{x_0, y_0} \psi_k|^4 dv_{g_{x_0, y_0}} \right)^{\frac{1}{4}} \leq \left( \int_{T^2} |D_{x_k, y_k} \psi_k|^4 dv_{g_{x_k, y_k}} \right)^{\frac{1}{4}}$$

where $C$ is a positive constant independent of $k$. Now, in the same way, we can write

$$\int_{T^2} \langle \psi_k, D_{x_0, y_0} \psi_k \rangle dv_{g_{x_0, y_0}} \geq \int_{T^2} \langle \psi_k, D_{x_k, y_k} \psi_k \rangle dv_{g_{x_k, y_k}} - \beta_k \int_{T^2} |\psi_k||\nabla\psi_k| dv_{g_{x_0, y_0}}$$

where $\lim_k \beta_k = 0$. Using Hölder inequality, we have

$$\int_{T^2} |\psi_k||\nabla\psi_k| dv_{g_{x_0, y_0}} \leq \left( \int_{T^2} |\psi_k|^4 dv_{g_{x_0, y_0}} \right)^{\frac{1}{4}} \left( \int_{T^2} |\nabla\psi_k|^2 dv_{g_{x_0, y_0}} \right)^{\frac{1}{2}}.$$

Using (3) and (7), this gives

$$\int_{T^2} |\psi_k||\nabla\psi_k| dv_{g_{x_0, y_0}} \leq C \left( \int_{T^2} |D\psi_k|^4 dv_{g_{x_0, y_0}} \right)^{3/2}.$$

We obtain

$$\int_{T^2} \langle \psi_k, D_{x_0, y_0} \psi_k \rangle dv_{g_{x_0, y_0}} \geq \int_{T^2} \langle \psi_k, D_{x_k, y_k} \psi_k \rangle dv_{g_{x_k, y_k}} - \beta_k \left( \int_{T^2} |D\psi_k|^4 dv_{g_{x_0, y_0}} \right)^{3/2}.$$
Together with (1), we get (3). This immediately implies that

$$\liminf_k \lambda_{\min}^{x_k,y_k} \geq \lambda_{\min}^{x_0,y_0}$$

and ends the proof of the proposition. \qed

**Proof of Theorem 3.2.** Any calculation in this proof will be carried out in Riemannian normal coordinates with respect to a flat metric. In the following, \(e_1, e_2\) will denote the canonical basis of \(\mathbb{R}^2\).

In order to prove \(\lim_{(x_0, y_0) \to (0,0)} \lambda_{\min}^{x_0,y_0} = \lambda_{\min}(S^2)\) we will show that there is no sequence \((x_k, y_k) \to (0,0)\) such that \(\lim_{(x_k, y_k) \to (0,0)} \lambda_{\min}^{x_k,y_k} < \lambda_{\min}(S^2)\). We may assume that \(\lambda_{\min}^{x_k,y_k} < \lambda_{\min}(S^2)\) for all \(k\). Note that the spectrum of \(D\) is symmetric in dimension 2. By [Am03a], we then can find a sequence of spinors \(\psi_k\) of class \(C^1\) such that on \(T_{x_k, y_k}\)

$$D \psi_k = \lambda_{\min}^{x_k,y_k} |\psi_k|^2 \psi_k \quad (11)$$

and such that

$$\int_{T_{x_k, y_k}} |\psi_k|^4 \, dx = 1. \quad (12)$$

Moreover, we have

$$J_{x_k, y_k}(\psi_k) = \lambda_{\min}^{x_k,y_k} \quad (13)$$

Sometimes we will identify \(\psi_k\) with its pullback to \(\mathbb{R}^2\). In this picture \(\psi_k\) is a doubly periodic spinor on \(\mathbb{R}^2\).

**Step 1.** There exists \(C > 0\) such that for all \(k\), we have \(\lambda_{\min}^{x_k,y_k} \geq Cy_k^{1/2}\).

Here and in the sequel, \(C\) will always denote a positive constant which does not depend on \(k\).

For the proof of the first step, we let \(\Omega = \{(x, y) \in M_1 \mid 1/2 \leq y \leq 3/2\}\). Since \(\Omega\) is compact and since \(\lambda_{\min}\) is continuous and positive, there exists \(C > 0\) such that for all \(\lambda_{\min} \geq C\) on \(\Omega\).

Now, assume that

$$\lim_{k} \lambda_{\min}^{x_k,y_k} = 0.$$

We can find a sequence \((N_k)\) which tends to \(+\infty\) such that \((3^{N_k} x_k, 3^{N_k} y_k) \in \Omega\). Note that the locally isometric covering \(T_{p x_k, p y_k} \to T_{x_k, y_k}, p \in \mathbb{N}\), preserves the spin structures if and only if \(p\) is odd. Let \(\tilde{\psi}_k\) be the pullback of \(\psi_k\) with respect to covering \(T_{3^{N_k} x_k, 3^{N_k} y_k} \to T_{x_k, y_k}\). We now have

$$\int_{T_{3^{N_k} x_k, 3^{N_k} y_k}} |D \tilde{\psi}_k|^4 \, dx = 3^{N_k} \int_{T_{x_k, y_k}} |D \psi_k|^4 \, dx$$

and

$$\int_{T_{3^{N_k} x_k, 3^{N_k} y_k}} \langle \psi_k, D \psi_k \rangle \, dx = 3^{N_k} \int_{T_{x_k, y_k}} \langle \psi_k, D \psi_k \rangle \, dx.$$

We then get by (14) that

$$C \leq 3^{N_k} \lambda_{\min}^{x_k,y_k} \leq J_{3^{N_k}(x_k,y_k)}(\psi_k) = 3^{2N_k} \lambda_{\min}^{3^{N_k} x_k,3^{N_k} y_k} \geq Cy_k^{-1/2} \lambda_{\min}^{x_k,y_k}.$$

**Step 2.** There exists \(C > 0\) such that for all \(k\), we have \(\lambda_{\min}^{x_k,y_k} \geq C\).

Let \(\eta : \mathbb{R} \to [0, 1]\) be a cut-off function defined on \(\mathbb{R}\) which is equal to 0 on \(\mathbb{R} \setminus [-1, 2]\) and which is equal to 1 on \([0, 1]\). We may assume that \(\eta\) is smooth. Let \(v_k = (x_k, y_k)\). Since \((e_1, v_k)\) is a basis of \(\mathbb{R}^2\), we can define \(\eta_k : \mathbb{R}^2 \to [0, 1]\) by

$$\eta_k(tv_k + s e_1) = \eta(s)$$

Since \(v_k\) is asymptotically orthogonal to \(e_1\), we can find \(C > 0\) independent of \(k\) such that

$$|\nabla \eta_k| \leq C \quad (15)$$
Moreover, by corollary \ref{c3}, we have
\[
\left( \frac{\int_{Z_{x_k,y_k}} |D\eta_k \psi_k|^4 dx}{\int_{Z_{x_k,y_k}} \langle \eta_k \psi_k, D\eta_k \psi_k \rangle dx} \right)^{\frac{3}{4}} \geq \lambda_{\text{min}}(S^2).
\] (16)

Now, we write that
\[
\left( \int_{Z_{x_k,y_k}} |D\eta_k \psi_k|^4 dx \right)^{\frac{3}{4}} = \left( \int_{Z_{x_k,y_k}} |\nabla \eta_k \cdot \psi_k + \eta_k D\psi_k|^4 dx \right)^{\frac{3}{4}} \leq \left( \int_{Z_{x_k,y_k}} |\nabla \eta_k \cdot \psi_k|^4 dx \right)^{\frac{3}{4}} + \left( \int_{Z_{x_k,y_k}} |\eta_k D\psi_k|^4 dx \right)^{\frac{3}{4}}.
\]

By (15) and Hölder inequality, we have
\[
\left( \int_{Z_{x_k,y_k}} |\nabla \eta_k \cdot \psi_k|^4 dx \right)^{\frac{3}{4}} \leq C \left( \int_{Z_{x_k,y_k} \cap \text{Supp}(\nabla \eta_k)} |\psi_k|^4 dx \right)^{\frac{3}{4}} \leq C \left( \int_{Z_{x_k,y_k} \cap \text{Supp}(\nabla \eta_k)} |\psi_k|^4 dx \right)^{\frac{3}{4}} \text{Vol}(Z_{x_k,y_k} \cap \text{Supp}(\nabla \eta_k) \right)^{\frac{1}{4}}.
\]

We then have
\[
\text{Vol}(Z_{x_k,y_k} \cap \text{Supp}(\nabla \eta_k)) \leq 3y_k.
\]

By (12) and step 1, this gives that
\[
\left( \int_{Z_{x_k,y_k}} |\nabla \eta_k \cdot \psi_k|^4 dx \right)^{\frac{3}{4}} \leq C y_k^{\frac{3}{4}} \leq C \lambda_{\text{min}}^{x_k,y_k}.
\]

With the same argument and using relations (11) and (12), it follows that
\[
\left( \int_{Z_{x_k,y_k}} |\eta_k D\psi_k|^4 dx \right)^{\frac{3}{4}} \leq 3^\frac{3}{4} \lambda_{\text{min}}^{x_k,y_k} \left( \int_{Z_{x_k,y_k}} |\psi_k|^4 dx \right)^{\frac{3}{4}} \leq C \lambda_{\text{min}}^{x_k,y_k}.
\]

Finally, we get that
\[
\left( \int_{Z_{x_k,y_k}} |D\eta_k \psi_k|^4 dx \right)^{\frac{3}{4}} \leq C (\lambda_{\text{min}}^{x_k,y_k})^2.
\] (17)

We now write that
\[
\int_{Z_{x_k,y_k}} \langle \eta_k \psi_k, D\eta_k \psi_k \rangle dx = \int_{Z_{x_k,y_k}} \langle \eta_k \psi_k, \nabla \eta_k \cdot \psi_k + \eta_k D\psi_k \rangle dx.
\]

Moreover, the left hand side of this equality is real since $D$ is an autoadjoint operator. Since
\[
\int_{Z_{x_k,y_k}} \langle \eta_k \psi_k, \nabla \eta_k \cdot \psi_k \rangle dx \in i\mathbb{R}.
\]

Together with equation (11), this implies that
\[
\int_{Z_{x_k,y_k}} \langle \eta_k \psi_k, D\eta_k \psi_k \rangle dx = \int_{Z_{x_k,y_k}} \eta_k^2 \lambda_{\text{min}}^{x_k,y_k} |\psi_k|^4 dx.
\]

Using (12), we obtain that
\[
\int_{Z_{x_k,y_k}} \langle \eta_k \psi_k, D\eta_k \psi_k \rangle dx \geq \lambda_{\text{min}}^{x_k,y_k} \int_{Z_{x_k,y_k}} |\psi_k|^4 dx = \lambda_{\text{min}}^{x_k,y_k}.
\] (18)

Finally, plugging (17) and (18) in (16), we obtain that $\lambda_{\text{min}}(S^2) \leq C \lambda_{\text{min}}^{x_k,y_k}$. This proves the step.

**Step 3.** The function $\lambda_{\text{min}}$ can be extended continuously to $\mathcal{M}_1 \cup \{(0,0)\}$ by setting $\lambda_{\text{min}}^{0,0} = \lambda_{\text{min}}(S^2)$. 

In other words, we show that \( \lim_{k} \lambda_{\min}^{v_k} = \lambda_{\min}(\mathbb{S}^2) \). The method is quite similar to the one of previous step. Let \( \zeta_k : \mathbb{R} \to [0, 1] \) be a smooth cut-off function defined on \( \mathbb{R} \) which is equal to 0 on \( \mathbb{R} \setminus [-y_k, 1 + y_k] \), which is equal to 1 on \([0, 1]\) and which satisfies \( |\nabla \zeta_k| \leq \frac{2}{y_k} \). As in the last step, we can define \( \gamma_k : \mathbb{R}^2 \to [0, 1] \) by \( \gamma_k (tv_k + se_1) = \zeta_k(s) \).

Since \( v_k \) is asymptotically orthogonal to \( e_1 \), we can find \( C > 0 \) independent of \( k \) such that
\[
|\gamma_k| \leq \frac{C}{y_k}
\] (19)

As in step 3, we have
\[
\left( \int_{Z_{x_k,y_k}} |D\gamma_k \psi_k|^{\frac{4}{3}} dx \right)^{\frac{3}{4}} \geq \lambda_{\min}^{v_k}(\mathbb{S}^2).
\] (20)

We first prove that we can assume that
\[
\int_{Z_{x_k,y_k} \cap \text{Supp}(\nabla \gamma_k)} |\psi_k|^4 dx \leq Cy_k.
\] (21)

We let \( n_k = [(2y_k)^{-1}] \) be the integer part of \( 2y_k^{-1} \). For all \( t \in [0, n_k - 1] \), we define
\[
A_{k,t} = \{ te_1 + sv_k \} \in [0, 1] \text{ and } t \in \left[ \frac{l - \frac{1}{n_k}, l + \frac{1}{n_k}}{n_k} \right].
\]

The family of sets \( \{A_{k,t}\}_{t \in [0, n_k - 1]} \) is a partition of \( T_{x_k,y_k} \) which is the image of \( T_{x_k,y_k} \) by the translation of vector \( -\frac{1}{2n_k} \epsilon_1 \). By periodicity, \( \{A_{k,t}\}_{t \in [0, n_k - 1]} \) can be seen as a partition of \( T_{x_k,y_k} \). Consequently, we can write that
\[
1 = \int_{T_{x_k,y_k}} |\psi_k|^4 dx = \sum_{t=0}^{n_k-1} \int_{A_{k,t}} |\psi_k|^4 dx.
\]

Hence, there exists \( l_0 \in [0, n_k - 1] \) such that
\[
\int_{A_{k,l_0}} |\psi_k|^4 dx = \min_{l \in [0, n_k - 1]} \sum_{l=0}^{n_k-1} \int_{A_{k,l}} |\psi_k|^4 dx \leq \frac{1}{n_k}.
\]

Obviously, without loss of generality, we can replace \( \psi_k \) by \( \psi_k \circ t_0 \) where \( t_0 \) is the translation of vector \( -l_0 \epsilon_1 \). In this way, we can assume that \( t_0 = 0 \). By periodicity, \( \text{Supp}(\nabla \gamma_k) \subset A_{k,0} \). Hence,
\[
\int_{Z_{x_k,y_k} \cap \text{Supp}(\nabla \gamma_k)} |\psi_k|^4 dx \leq \frac{1}{n_k}.
\]

Since \( n_k \sim \frac{2}{y_k} \), equation (21) follows.

Now, we proceed as in step 3. We write that
\[
\left( \int_{Z_{x_k,y_k}} |D\gamma_k \psi_k|^{\frac{4}{3}} dx \right)^{\frac{3}{4}} = \left( \int_{Z_{x_k,y_k}} |\nabla \gamma_k \cdot \psi_k + \gamma_k D\psi_k|^{\frac{4}{3}} dx \right)^{\frac{3}{4}} \leq \left( \int_{Z_{x_k,y_k}} |\nabla \gamma_k \cdot \psi_k|^{\frac{4}{3}} dx \right)^{\frac{3}{4}} + \left( \int_{Z_{x_k,y_k}} |\gamma_k D\psi_k|^{\frac{4}{3}} dx \right)^{\frac{3}{4}}.
\]

It follows from (21) and the Hölder inequality that
\[
\left( \int_{Z_{x_k,y_k}} |\nabla \gamma_k \cdot \psi_k|^{\frac{4}{3}} dx \right)^{\frac{3}{4}} \leq \frac{C}{y_k} \left( \int_{Z_{x_k,y_k} \cap \text{Supp}(\nabla \gamma_k)} |\psi_k|^4 dx \right)^{\frac{3}{4}} \leq \frac{C}{y_k} \left( \int_{Z_{x_k,y_k} \cap \text{Supp}(\nabla \gamma_k)} |\psi_k|^4 dx \right)^{\frac{3}{4}} \left( \text{Vol}(Z_{x_k,y_k} \cap \text{Supp}(\nabla \gamma_k)) \right)^{\frac{1}{4}}.
\]
Clearly, we have
\[ \text{Vol}(Z_{x_k,y_k} \cap \text{Supp}(\nabla \gamma_k)) \leq C y_k^2. \]

By (21), we obtain
\[ \left( \int_{Z_{x_k,y_k}} |\nabla \gamma_k \cdot \psi_k|^\frac{4}{3} dx \right)^\frac{3}{4} \leq C y_k^{-1+\frac{1}{2}} + C y_k^{\frac{1}{2}} = o(1). \]

For the other term, we write, using (11)
\[ \left( \int_{Z_{x_k,y_k}} |\gamma_k D \psi_k|^\frac{4}{3} dx \right)^\frac{3}{4} = \lambda_{x_k,y_k}^{\frac{4}{3}} \left( \int_{T_{x_k,y_k}} |\psi_k|^4 dx + \int_{Z_{x_k,y_k} \cap \{0 < \gamma_k < 1\}} |\psi_k|^4 dx \right)^\frac{3}{4}. \]

Clearly, we can construct \( \gamma_k \) such that \( \{0 < \gamma_k < 1\} \subset \text{Supp}(\nabla \gamma_k) \). It then follows from (21) that
\[ \left( \int_{Z_{x_k,y_k}} |\gamma_k D \psi_k|^\frac{4}{3} dx \right)^\frac{3}{4} \leq \lambda_{x_k,y_k}^{\frac{4}{3}} + o(1). \]

Finally, we obtain
\[ \left( \int_{Z_{x_k,y_k}} |D \gamma_k \psi_k|^\frac{4}{3} dx \right)^\frac{3}{4} \leq (\lambda_{x_k,y_k}^{\frac{4}{3}})^2 + o(1). \] (22)

Now, as in step 2, we write that
\[ \int_{Z_{x_k,y_k}} \langle \gamma_k \psi_k, D \gamma_k \psi_k \rangle dx = \int_{Z_{x_k,y_k}} \gamma_k^2 \lambda_{x_k,y_k}^{\frac{4}{3}} |\psi_k|^4 dx. \]

Using (22), we obtain that
\[ \int_{Z_{x_k,y_k}} \langle \gamma_k \psi_k, D \gamma_k \psi_k \rangle dx \geq \lambda_{x_k,y_k}^{\frac{4}{3}} \int_{T_{x_k,y_k}} |\psi_k|^4 dx = \lambda_{x_k,y_k}^{\frac{4}{3}}. \] (23)

Plugging (22) and (23) in (20), we obtain that
\[ \lambda_{\text{min}}(S^2) \leq (\lambda_{x_k,y_k}^{\frac{4}{3}})^2 + o(1) \]
which implies that either \( \lambda_{x_k,y_k}^{\frac{4}{3}} \to 0 \) or \( \lambda_{x_k,y_k}^{\frac{4}{3}} \to \lambda_{\text{min}}(S^2) \). Hence, step 3 yields the statement of the theorem. \( \square \)

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