Optimal Control of Systems of Conservation Laws and Application to Non-Equilibrium Traffic Control

Denis Jacquet, Carlos Canudas de Wit, Damien Koenig

OPTIMAL CONTROL OF SYSTEMS OF CONSERVATION LAWS AND APPLICATION TO NON-EQUILIBRIUM TRAFFIC STEERING

Denis Jacquet Carlos Canudas de Wit
Damien Koenig

Laboratoire d'Automatique de Grenoble
BP. 46, 38 402, St. Martin d'Hères, France

Abstract: This paper proposes an optimization algorithm to solve iteratively optimal control problems involving systems of conservation laws. The irregularity of their solution requires a specific variational analysis that have direct implications on the numerical implementation. This method is applied to the control of non-equilibrium traffic using the Payne-Whitham and the Aw-Rascle-Zhang model.

Keywords: Conservation laws, optimal control, adjoint calculus.

1. INTRODUCTION

Many physical systems are modelled by a system of conservation laws that takes the form of an hyperbolic partial differential equation. When the dynamics is nonlinear as in traffic, aerodynamics, meteorology and elasticity, the distributed state of the system may develop discontinuities that propagate in time even for smooth initial and boundary conditions.

The mathematical theory of conservation laws has undergone tremendous improvements in the last 30 years and an abondant literature is available for the one-dimensional Cauchy problem (Bressan, 2000). Moreover, dedicated integration schemes that handle the lack of regularity are available (Godlewski and Raviart, 1996). Though the behavior of boundary conditions is well understood for scalar problems, their treatment is less clear for systems (Dubois and LeFloch, 1988) and ghost cells are usually used in numerical schemes.

Optimal control of conservation laws has already been treated in the literature. (Messmer and Papageorgiou, 1994) propose an optimal traffic control algorithm by first discretizing the dynamics and then using general purpose nonlinear optimization routines. Nevertheless, the discretization step is questionable as accurate discretization procedures as the Godunov scheme (Godlewski and Raviart, 1996) can not be put in a form suitable for control as a difference equation or an ordinary differential equation. (Sanders and Katopodes, 2000) propose an adjoint based optimal control algorithm to steer shallow-water systems but disregard the possible discontinuities in the linearization and adjoint calculus though such phenomena may happen. If shocks are not common in irrigation channels, they are in traffic as shown on field measurements with strong congestions, making their treatment compulsory. The contribution of this paper is to take into account the possible development of discontinuous waves in computing an optimal (or suboptimal) control for systems of conservation laws.

Distributed traffic models on a bounded domain $x \in [0, L]$ and with aggregated lanes are used in this paper. Freeway traffic fulfils the car conservation principle and can be modelled by conservation laws where the state may be the vehicle density $\rho(x, t)$, the average speed $v(x, t)$, the flow

$\phi(x,t) = \rho(x,t)v(x,t)$ or any combinaison of these variables. In traffic applications, a discontinuity in the state (as in ρ) models a congestion wave that propagates along the traffic stream. There exists two classes of non-equilibrium traffic models. The first one proposed in (Payne, 1971) and (Zhang, 1998) concerns isotropic models where the information can travel forwards and backwards with respect to a vehicle in the steam. Following the criticism of (Daganzo, 1995), the Aw-Rascle-Zhang (ARZ) model have been proposed in (Aw and Rascle, 2000), (Zhang, 2002) and extended in (Greenberg, 2001). To our knowledge, no controller of any kind have been proposed for the ARZ model.

Optimal control is an appealing framework for traffic control problems as the main objective is to maximize the infrastructure usage. A classical approach in optimal control of linear partial differential equations (Lions, 1971) is to use adjoint calculus to characterize the necessary conditions of optimality. For nonlinear problems, the same approach can be used iteratively on the linearized dynamics. Nevertheless, it is not clear for conservation laws if the dynamics can be linearized given the irregularity of the fields they generate. (Bardos and Pironneau, 2003) used distributional calculus to compute a linearization of conservation laws with respect to a parameter in the initial condition. (Godlewski and Raviart, 1999) proposed a variational analysis with respect to a perturbation in the initial condition by differentiating the PDE and the Rankine-Hugoniot jump condition. The authors proposed in (Jacquet *et al.*, 2005) a different approach for scalar conservation laws based on their weak formulation and considering the piecewise-C^1 structure of their solution. This paper is an extension of these results to systems of conservation laws.

2. NON-EQUILIBRIUM TRAFFIC MODELS AND THEIR SOLUTION

We propose to build our design on the two main inviscid non-equilibrium models that have been proposed in the literature. The first one is the isotropic model proposed by Payne (Payne, 1971), which have the form

$$\begin{cases} \partial_t \rho + \partial_x(\rho v) = 0 \\ \partial_t v + v\partial_x v + \dfrac{c^2}{\rho}\partial_x \rho = \dfrac{V_e(\rho) - v}{\tau} \end{cases} \qquad (1)$$

with c the constant traffic sound speed, $V_e(\cdot)$ the equilibrium velocity and τ the relaxation parameter. In the Payne model, information propagates at wave speeds given by $\lambda_1 = v - c < v$ and $\lambda_2 = v + c > v$, which explains its isotropic property. It has two genuinely nonlinear fields and may develop shock waves (discontinuities with different density and speed values on both sides).

The second class concerns the anisotropic models proposed by Aw-Rascle-Zhang (ARZ) in (Aw and Rascle, 2000; Zhang, 2002; Greenberg, 2001). These models correct the isotropy of the Payne model and has not been much studied either by the mathematics or the traffic engineering communities. It has the form

$$\begin{cases} \partial_t \rho + \partial_x(\rho v) = 0 \\ \partial_t\big(v + P(\rho)\big) + v\partial_x\big(v + P(\rho)\big) = \dfrac{V_e(\rho) - v}{\tau} \end{cases} \qquad (2)$$

where we get for different pressure terms $P(\rho)$

(1) the Aw-Rascle model for $P(\rho) = \rho^\gamma$, $\gamma > 0$,
(2) the Zhang model for $P(\rho) = -V(\rho)$.

In such models, one field is genuinely nonlinear with wave speed $\lambda_1 = v - \rho P'(\rho) \leq v$ and the other is linearly degenerate with wave speed $\lambda_2 = v$. Consequently, they are anisotropic as both waves travel at velocities smaller or equal to the traffic velocity and the discontinuities that may appear are either shock waves or contact discontinuities (discontinuity in the density with the same velocity on both sides).

The solution structures of (1) and (2) are rather different as (1) have 2 genuinely nonlinear fields while (2) has a linearly degenerate and a genuinely nonlinear field. However, both models can be put with simple algebraic manipulations in the form of a 1-dimensional couple of conservation laws and then manipulated in this general framework. Moreover, (1) and (2) should be extended to take into account the presence of on and off ramps along the freeway. These inhomogeneities are modelled though Dirac source terms in the density equation. The i^{th} on-ramp has a contribution $u_i\Psi_i(\rho)$ with $u_i(t)$ its metering rate and $\Psi_i(\cdot)$ a smoothed saturation limiting the inflow for large mainlane density. The i^{th} off-ramp has a contribution $-\beta_i\phi$ with $\beta_i(t)$ its known or estimated split ratio. The well-posedness of conservation laws with such irregular source terms is not established at present. Nevertheless, some preliminary results are available (Greenberg *et al.*, 1997) and this approach was successful in controlling equilibrium traffic modelled by a scalar conservation law (Jacquet *et al.*, 2005).

The extended Payne model writes

$$\partial_t \begin{pmatrix} \rho \\ \phi \end{pmatrix} + \partial_x \begin{pmatrix} \phi \\ \dfrac{\phi^2}{\rho} + c^2\rho \end{pmatrix} = \begin{pmatrix} \displaystyle\sum_{i=1}^{N_u} \delta_{\hat{x}_i} u_i \Psi_i(\rho) - \sum_{i=1}^{N_\beta} \delta_{\check{x}_i} \beta_i \phi \\ \dfrac{\Phi_e(\rho) - \phi}{\tau} \end{pmatrix} \qquad (3)$$

with ρ and $\phi = \rho v$ the conserved variables. Similarly, the ARZ model writes

$$\partial_t \begin{pmatrix} \rho \\ \omega \end{pmatrix} + \partial_x \begin{pmatrix} \omega - \rho P(\rho) \\ \dfrac{\omega^2}{\rho} - \omega P(\rho) \end{pmatrix} = \begin{pmatrix} \sum_{i=1}^{N_u} \delta_{\hat{x}_i} u_i \Psi_i(\rho) - \sum_{i=1}^{N_\beta} \delta_{\check{x}_i} \beta_i (\omega - \rho P(\rho)) \\ \dfrac{\Phi_e(\rho) - \omega + \rho P(\rho)}{\tau} \end{pmatrix} \quad (4)$$

with ρ and $\omega = \rho(v+P(\rho))$ its conserved variables. For the ARZ model, the flow variable is noted $\phi = \phi(\rho, \omega) = \omega - \rho P(\rho)$. Both of these models are in conservative form and have an irregular source term due to the on and off ramps.

As the states ρ, ϕ and ω are discontinuous at the ramp locations $x = \hat{x}_i$ and $x = \check{x}_j$, the products $\delta_{\hat{x}_i}\Psi_i(\rho)$, $\delta_{\check{x}_i}\phi$ and $\delta_{\hat{x}_i}(\omega - \rho P(\rho))$ should be defined appropriately. From traffic heuristics, the saturation $\Psi_i(\cdot)$ should apply to the maximal value of the density around $x = \hat{x}_i$ at on-ramps and the flow is considered from its left limit, i.e. as $x \uparrow \check{x}_j$, at off-ramps.

With appropriate initial and boundary conditions on the domain $(x,t) \in (0,L)\times(0,T)$, these models can be rewritten as a couple of conservation laws with a finite dimensional control variable $\mathbf{u} \in \mathbb{R}^{N_u}$ in the source term

$$\begin{cases} \partial_t \mathbf{y} + \partial_x \mathbf{f}(\mathbf{y}) = \mathbf{s}(\mathbf{y}, \mathbf{u}) \\ \mathbf{y}(x,0) = \mathbf{y}_{\mathrm{I}}(x) \\ \mathbf{y}(0,t) \sim \mathbf{y}_{\mathrm{Up}}(x) \text{ and } \mathbf{y}(L,t) \sim \mathbf{y}_{\mathrm{Do}}(x) \end{cases} \quad (5)$$

or with y_1 and y_2 the conserved quantities

$$\begin{cases} \partial_t y_1 + \partial_x f_1(y_1, y_2) = s_1(y_1, y_2, u_1, ..., u_{N_u}) \\ \partial_t y_2 + \partial_x f_2(y_1, y_2) = s_2(y_1, y_2, u_1, ..., u_{N_u}) \end{cases}$$

In (5), $\mathbf{y} = (\rho \quad \phi)^T$ or $\mathbf{y} = (\rho \quad \omega)^T$ is the state vector depending on the system we consider (Payne or ARZ), $\mathbf{f}(\mathbf{y})$ is the flux function and $\mathbf{s}(\mathbf{y}, \mathbf{u})$ the source term. The symbol $\sim$ underlines the fact that boundary conditions are only proposed and may not apply depending on the trace of $\mathbf{y}$ at the boundaries. The theoretical treatment of boundary conditions for systems of nonlinear conservation laws can be found in (Dubois and LeFloch, 1988) and its numerical counterpart in (Godlewski and Raviart, 1996). Equation (5) should be interpreted in the weak sense (Bressan, 2000) to allow for the development of discontinuous waves, i.e.

$$0 = \int_0^T\!\!\int_0^L \Big\{ \mathbf{y} \cdot \partial_t \theta + \mathbf{f}(\mathbf{y}) \cdot \partial_x \theta + \mathbf{s}(\mathbf{y}, \mathbf{u}) \cdot \theta \Big\} \mathrm{d}x\mathrm{d}t + \int_0^L \mathbf{y}(x,0) \cdot \theta(x,0) \mathrm{d}x \text{ , } \forall\ \theta \in C_0^1(]0,L[\times[0,\infty[) \quad (6)$$

with C_0^1 the space of continuously differentiable functions with compact support. Theoretically, Equation (6) gives a solution in the space BV of functions with bounded variations (Bressan, 2000). Nevertheless, physical problems usually lead to piecewise-C^1 solutions with a finite number of differentiable curves of discontinuity. We use this setting in this paper as it is more appropriate for the problem at hand.

3. OPTIMAL CONTROL OF SYSTEMS OF CONSERVATION LAWS

In this section, we propose to compute the gradient of the abstract optimal control problem

$$\underset{\mathbf{u}}{\mathbf{Min}} \quad \mathcal{J}(\mathbf{y}, \mathbf{u}) \quad \mathbf{Subj.\ to} \ (5) \quad (7)$$

with the cost functional defined by

$$\begin{aligned} \mathcal{J}(\mathbf{y}, \mathbf{u}) &= \mathcal{J}_{\mathrm{obs}}(\mathbf{y}) + \mathcal{J}_{\mathrm{bar}}(\mathbf{u}) \\ &= \int_0^T\!\!\int_0^L g(\mathbf{y}) \,\mathrm{d}x\mathrm{d}t - \gamma \sum_{i=1}^{N_u} \int_0^T \ln(u_i(1-u_i))\mathrm{d}t \end{aligned} \quad (8)$$

where $g(\cdot)$ weights the distributed value of the state. $\mathcal{J}_{\mathrm{obs}}(\mathbf{y})$ defines the traffic management objective while $\mathcal{J}_{\mathrm{bar}}(\mathbf{u})$ ensures with a barrier technique that $\mathbf{u} \in [0,1]^{N_u}$, the metering rates being bounded quantities.

The program to compute the gradient is the following. First, we perform a linearization of (5). Then, we compute the adjoint system, taking into account the piecewise-C^1 structure of the solution. Finally, the adjoint identity is used to evaluate gradients of the cost functional with respect to the decision variable $\mathbf{u}$.

Taking into account the real time constraint and the adaptation requirement of the method, this gradient evaluation routine can be used in the following two ways:

- **Receding horizon**: At time t, $\nabla_{\mathbf{u}}\mathcal{J}$ is used iteratively to find the local minimum of (7) on the time horizon $[t, t+T_1]$. Then the optimal control strategy $\mathbf{u}^*$ is applied in the time window $[t, t+T_2]$ with $T_2 \leq T_1$. At time $t+T_2$, the same procedure is applied.
- **Instantaneous control:** At time t, $\nabla_u \mathcal{J}$ is computed for an horizon T and we apply the updated control $\mathbf{u}_{[t,t+T]} = \mathbf{u}_{[t-T,t]} - \nabla_{\mathbf{u}}\mathcal{J}$.

The linearization of (5) is given by

$$\begin{cases} \partial_t \tilde{\mathbf{y}} + \partial_x (D\mathbf{f}(\bar{\mathbf{y}})\tilde{\mathbf{y}}) = D_{\mathbf{y}}\mathbf{s}(\bar{\mathbf{y}}, \bar{\mathbf{u}})\tilde{\mathbf{y}} + D_{\mathbf{u}}\mathbf{s}(\bar{\mathbf{y}}, \bar{\mathbf{u}})\tilde{\mathbf{u}} \\ \tilde{y}(t=0) = 0 \\ \tilde{y}(x=0) = 0 \text{ and } \tilde{y}(x=L) = 0 \ \text{ when applicable} \end{cases} \quad (9)$$

where the perturbed variables $u = \bar{u} + \tilde{u}$ and $y = \bar{y} + \tilde{y}$ were plugged in the weak formulation (6) and the nonlinear terms removed after some Taylor expansions. We recall that the solution $\bar{\mathbf{y}}$ of (5) is a vector of 2 piecewise-C^1 fields, each field having the same curves of discontinuity Γ_i parameterized in time by the shock locations $s_i(t)$, i.e.

$\Gamma_i = \{(s_i(t), t) : t \geq t_i^I\}$. Given the linearization procedure we used, (9) should be interpreted in the weak sense. It can be shown as in (Jacquet *et al.*, 2005) that its solution has singular measures at the discontinuity locations in $\bar{\mathbf{y}}$. Homogeneous initial and boundary conditions are provided when applicable for the linearized system as they are provided to the original problem and cannot be changed. Note that the boundary conditions are applicable for the incoming characteristics variables (Godlewski and Raviart, 1996) identified by the eigenvalue decomposition of the matrix $D\mathbf{f}(\bar{\mathbf{y}})$.

The adjoint equation $\text{PDE}^\star(\lambda) = 0$ is computed using the identity $<\lambda, \text{PDE}(y)>=<\text{PDE}^\star(\lambda), y>$ where the duality paring $< \cdot, \cdot >$ is similar to the L^2 scalar product and the adjoint equation is obtained using Green's formula (integration by parts). The technicality here is to use the following generalization of the Green's formula for piecewise-C^1 fields

$$\int_\Omega g.\text{div} f = -\int_{\Omega\setminus\cup_i\Gamma_i} \nabla g \cdot f + \int_{\partial\Omega} g.f \cdot \nu + \sum_{i=1}^{N_s} \int_{t_i^I}^{t_i^F} -[g.f_x]_{|x=s_i(t)} + \dot{s}_i(t)[g.f_t]_{|x=s_i(t)}$$

Following this framework, the adjoint equation is then defined by

$$\begin{aligned} &<\lambda, \partial_t\tilde{\mathbf{y}} + \partial_x(D\mathbf{f}(\bar{\mathbf{y}})\tilde{\mathbf{y}}) - D_\mathbf{y}\mathbf{s}(\bar{\mathbf{y}},\bar{\mathbf{u}})\tilde{\mathbf{y}} - D_\mathbf{u}\mathbf{s}(\bar{\mathbf{y}},\bar{\mathbf{u}})\tilde{\mathbf{u}}> = \\ &< \tilde{\mathbf{y}}, -\partial_t\lambda - D\mathbf{f}(\bar{\mathbf{y}})^T\partial_x\lambda - D_\mathbf{y}\mathbf{s}(\bar{\mathbf{y}},\bar{\mathbf{u}})^T\lambda > \\ &\quad + \int_0^L \left[\lambda^T\tilde{\mathbf{y}}\right]_0^T \mathrm{d}x + \int_0^T \left[\lambda^T D\mathbf{f}(\bar{\mathbf{y}})\tilde{\mathbf{y}}\right]_0^L \mathrm{d}t \\ &\quad + \sum_{i=1}^{N_s}\int_{t_i^I}^T \dot{s}_i\left[\lambda^T(\tilde{\mathbf{y}} - D\mathbf{f}(\bar{\mathbf{y}})\tilde{\mathbf{y}})\right]_{|x=s_i(t)}\mathrm{d}t \\ &\quad - < \tilde{\mathbf{u}}, D_\mathbf{u}\mathbf{s}(\bar{\mathbf{y}},\bar{\mathbf{u}})^\star\lambda >= 0 \end{aligned} \quad (10)$$

with $[\xi]_{|x=s}$ the jump in ξ at $x = s$. The gradient of the cost functional with respect to the control variable $\mathbf{u}$ can then be evaluated using the following theorem.

Theorem (Gradient evaluation)

The gradient $\nabla_\mathbf{u}\mathcal{J}$ of the optimal control problem (7-8) along the trajectory defined by $(\bar{u}, \bar{y})$ is

$$\nabla_\mathbf{u}\mathcal{J}(\bar{\mathbf{u}}, \bar{\mathbf{y}}) = D_\mathbf{u}\mathbf{s}(\bar{\mathbf{y}},\bar{\mathbf{u}})^\star\lambda - \gamma\begin{pmatrix} 1/\bar{u}_1 - 1/(1-\bar{u}_1) \\ \vdots \\ 1/\bar{u}_{N_u} - 1/(1-\bar{u}_{N_u}) \end{pmatrix} \quad (11)$$

with the adjoint variable λ defined by

$$\begin{cases} -\partial_t\lambda - D\mathbf{f}(\bar{\mathbf{y}})^T\partial_x\lambda - D_\mathbf{y}\mathbf{s}(\bar{\mathbf{y}},\bar{\mathbf{u}})^T\lambda = \mathbf{g}'(\bar{\mathbf{y}}) \\ \lambda(t=T) = 0 \\ \lambda(x=0) = \lambda(x=L) = 0 \quad \text{when applicable} \\ \lambda_{|\Gamma_i} = 0 \end{cases} \quad (12)$$

Note that $D_\mathbf{u}\mathbf{s}(\bar{\mathbf{y}},\bar{\mathbf{u}})^\star$ is the transpose of $D_\mathbf{u}\mathbf{s}(\bar{\mathbf{y}},\bar{\mathbf{u}})$ where Dirac distributions are replaced by pointwise evaluations.

Proof:

The first variation of the cost in (7-8) with respect to $\mathbf{y}$ is given by $\tilde{\mathcal{J}}_\mathbf{y} =<\mathbf{g}'(\bar{\mathbf{y}}), \tilde{\mathbf{y}}>$. By setting

$$\begin{cases} -\partial_t\lambda - D\mathbf{f}(\bar{\mathbf{y}})^T\partial_x\lambda - D_\mathbf{y}\mathbf{s}(\bar{\mathbf{y}},\bar{\mathbf{u}})^T\lambda = \mathbf{g}'(\bar{\mathbf{y}}) \\ \lambda(t=T) = 0 \\ \lambda_{|\Gamma_i} = 0 \end{cases}$$

the remaining terms in the identity (10) are

$$<\mathbf{g}'(\bar{\mathbf{y}}), \tilde{\mathbf{y}}>=<\tilde{\mathbf{u}}, D_\mathbf{u}\mathbf{s}(\bar{\mathbf{y}},\bar{\mathbf{u}})^\star\lambda> - \int_0^T \left[\lambda^T D\mathbf{f}(\bar{\mathbf{y}})\tilde{\mathbf{y}}\right]_0^L \mathrm{d}t$$

To remove the second term, the applicability of the boundary conditions (Godlewski and Raviart, 1996) for the linearized dynamics (9) and the adjoint equation (12) should be studied. In non-conservative form, (9) and (12) can be rewritten $\partial_t\tilde{\mathbf{y}} + D\mathbf{f}(\bar{\mathbf{y}})\partial_x\tilde{\mathbf{y}} = \mathbf{S}_\mathbf{y}$ and $\partial_\tau\lambda - D\mathbf{f}(\bar{\mathbf{y}})^T\partial_x\lambda = \mathbf{S}_\lambda$ with $\mathbf{S}_\mathbf{y}$ and $\mathbf{S}_\lambda$ some source terms and τ the reversed time. Let note $D\mathbf{f}(\bar{\mathbf{y}}) = T\Lambda T^{-1}$ the eigenvalue decomposition of $D\mathbf{f}(\bar{\mathbf{y}})$. The splitting of the operator $\Lambda = \Lambda^- + \Lambda^+$ in its negative and positive eigenvalues tells which characteristic variable can be assigned. We have for the remaining term

$$\begin{aligned} \lambda^T D\mathbf{f}(\bar{\mathbf{y}})\tilde{\mathbf{y}} &= \lambda^T T\Lambda T^{-1}\tilde{\mathbf{y}} \\ &= \lambda^T T\Lambda^- T^{-1}\tilde{\mathbf{y}} + \lambda^T T\Lambda^+ T^{-1}\tilde{\mathbf{y}} \end{aligned}$$

Consider for instance the boundary $x = 0$. As homogeneous boundary conditions apply to the linearized equation, we have $\Lambda^+ T^{-1}\tilde{\mathbf{y}}|_{x=0} = 0$ where Λ^+ selects the appropriate characteristic variables. It remains

$$\begin{aligned} \lambda|_{x=0}^T D\mathbf{f}(\bar{\mathbf{y}})\tilde{\mathbf{y}}|_{x=0} &= \lambda|_{x=0}^T T\Lambda^- T^{-1}\tilde{\mathbf{y}}|_{x=0} \\ &= \tilde{\mathbf{y}}|_{x=0}^T T^{-T}\Lambda^- T^T\lambda|_{x=0} \end{aligned}$$

Let note $-D\mathbf{f}(\bar{\mathbf{y}})^T = P\Pi P^{-1}$. With appropriate eigenvalue ordering and eigenvector normalization, we have $\Pi = -\Lambda$, implying that $\Pi^- = \Lambda^+$ and $\Pi^+ = \Lambda^-$, and $T^T = P^{-1}$. Applying homogeneous boundary conditions to the reversed time adjoint equation implies that

$$\Pi^+ P^{-1}\lambda|_{x=0} = \Lambda^- T^T\lambda|_{x=0} = 0$$

so $\lambda^T D\mathbf{f}(\bar{\mathbf{y}})\tilde{\mathbf{y}} = 0$ and the times of active boundary conditions for (9) and (12) are complementarity. $\nabla_\mathbf{u}\mathcal{J} = D_\mathbf{u}\mathbf{s}(\bar{\mathbf{y}},\bar{\mathbf{u}})^\star\lambda + \nabla_\mathbf{u}\mathcal{J}_{\text{bar}}$ concludes the proof.□

An interesting interpretation of the adjoint based gradient evaluation is the following. $\mathbf{g}'(\mathbf{y})$ is used to trigger the adjoint variables where improvements are possible. Then, the adjoint value is transported backwards using the adjoint equation towards regions where decision variables are available. Note that $\lambda(t = T) = 0$, $\lambda(x = 0) = 0$ and $\lambda(x = L) = 0$ make sense as no improvement come form the boundaries. $\lambda_{|\Gamma_i} = 0$ implies that a virtual boundary is needed at the shock locations due to the entropy condition (Bressan, 2000) that requires a value on both side of Γ_i for one characteristic field (Jacquet *et al.*, 2005). Note that this virtual boundary condition requires a numerical shock detection routine for $\bar{\mathbf{y}}$.

An optimal (or suboptimal) control can be obtained with the following iterative steepest gradient descent algorithm. Iterations on the barrier parameter γ can be added.

Require: $\mathbf{u} := \mathbf{u}_{\text{init}}$, ϵ
  while $||\nabla_{\mathbf{u}}\mathcal{J}|| > \epsilon$ **do**
    Compute $\mathbf{y}$ from (5)
    Compute λ from (12)
    Compute $\nabla_{\mathbf{u}}\mathcal{J}$ from (11)
    Normalize $\nabla_{\mathbf{u}}\mathcal{J}$
    Update $\mathbf{u} := \mathbf{u} - \nabla_{\mathbf{u}}\mathcal{J}$ with crude line search
  end while

---

4. APPLICATION TO OPTIMAL RAMP METERING

Meaningful objectives for traffic applications are

- maximize the Vehicle-Miles-Travelled (VMT)

$$\mathcal{J}_{\text{VMT}}(\mathbf{y}) = -\int_0^T\int_0^L \phi(x,t)\ \mathrm{d}x\mathrm{d}t$$

- minimize the Total-Travel-Time (TTT)

$$\mathcal{J}_{\text{TTT}}(\mathbf{y}) = -\int_0^T\int_0^L \rho(x,t)\ \mathrm{d}x\mathrm{d}t$$

For the Payne model (3), the parameters of the linearized dynamics are

$$\begin{cases} D\mathbf{f}(\bar\rho,\bar\phi) = \begin{pmatrix} 0 & 1 \\ c^2 - \dfrac{\bar\phi^2}{\bar\rho^2} & \dfrac{2\bar\phi}{\bar\rho}\end{pmatrix} \\ D_{\mathbf{y}}\mathbf{s}(\bar\rho,\bar\phi,\bar u) = \begin{pmatrix} \sum \delta_{\hat x_i}\bar u_i \Psi_i'(\bar\rho) & -\sum \delta_{\tilde x_i}\beta_i \\ \dfrac{\Phi_e'(\bar\rho)}{\tau} & -\dfrac{1}{\tau}\end{pmatrix} \\ D_{\mathbf{u}}\mathbf{s}(\bar\rho,\bar\phi,\bar u) = \begin{pmatrix} \delta_{\hat x_1}\Psi_1(\bar\rho) & \cdots & \delta_{\hat x_{N_u}}\Psi_{N_u}(\bar\rho) \\ 0 & \cdots & 0\end{pmatrix}\end{cases}$$

For the ARZ model (4), they are

$$\begin{cases} D\mathbf{f}(\bar\rho,\bar\omega) = \begin{pmatrix} -P(\bar\rho) - \bar\rho P'(\bar\rho) & 1 \\ -\dfrac{\bar\omega^2}{\bar\rho^2} - \bar\omega P'(\bar\rho) & \dfrac{2\bar\omega}{\bar\rho} - P(\bar\rho)\end{pmatrix} \\ D_{\mathbf{y}}\mathbf{s}(\bar\rho,\bar\omega,\bar u) = \\ \begin{pmatrix} \sum\delta_{\hat x_i}\bar u_i\Psi'(\bar\rho) - \sum\delta_{\tilde x_i}\beta_i(-P(\bar\rho)-\rho P'(\bar\rho)) & -\sum\delta_{\tilde x_i}\beta_i \\ \dfrac{\Phi_e'(\bar\rho)+P(\bar\rho)+\bar\rho P'(\bar\rho)}{\tau} & -\dfrac{1}{\tau}\end{pmatrix} \\ D_{\mathbf{u}}\mathbf{s}(\bar\rho,\bar\omega,\bar u) = \begin{pmatrix} \delta_{\hat x_1}\Psi_1(\bar\rho) & \cdots & \delta_{\hat x_{N_u}}\Psi_{N_u}(\bar\rho) \\ 0 & \cdots & 0\end{pmatrix}\end{cases}$$

Note that as soon as $\bar\rho$, $\bar\phi$, $\bar\omega$, $\bar u$ and β are known, the entries of the matrices $D\mathbf{f}$, $D_{\mathbf{y}}\mathbf{f}$ and $D_{\mathbf{u}}\mathbf{f}$ become simple piecewise-C^1 functions that depends on x and t only.

Using the results stated above, the gradient evaluation of the VMT objective for both models is

$$\nabla_{u_i}\mathcal{J}_{\text{VMT}} = \Psi_i(\bar\rho(\hat x_i,t))\lambda_1(\hat x_i,t) - \gamma\left(\frac{1}{\bar u_i} - \frac{1}{1-\bar u_i}\right) \tag{13}$$

For the Payne model, the adjoint system is

$$\begin{cases} -\partial_t\lambda_1 - \left(c^2 - \dfrac{\bar\phi^2}{\bar\rho^2}\right)\partial_x\lambda_2 = \sum\delta_{\hat x_i}\bar u_i\Psi_i'(\bar\rho)\lambda_1 + \dfrac{\Phi_e'(\bar\rho)}{\tau}\lambda_2 \\ -\partial_t\lambda_2 - \partial_x\lambda_1 - \dfrac{2\bar\phi}{\bar\rho}\partial_x\lambda_2 = -\sum\delta_{\tilde x_i}\beta_i\lambda_1 - \dfrac{1}{\tau}\lambda_2 - 1\end{cases} \tag{14}$$

For the ARZ model, the adjoint is

$$\begin{cases} -\partial_t\lambda_1 + \Big(P(\bar\rho)+\bar\rho P'(\bar\rho)\Big)\partial_x\lambda_1 + \left(\dfrac{\bar\omega^2}{\bar\rho^2} + \bar\omega P'(\bar\rho)\right)\partial_x\lambda_2 = \\ \qquad \sum\delta_{\hat x_i}\bar u_i\Psi'(\bar\rho)\lambda_1 - \sum\delta_{\tilde x_i}\beta_i(-P(\bar\rho)-\rho P'(\bar\rho))\lambda_1 \\ \qquad + \dfrac{\Phi_e'(\bar\rho)+P(\bar\rho)+\bar\rho P'(\bar\rho)}{\tau}\lambda_2 - P(\bar\rho) - \bar\rho P(\bar\rho) \\ -\partial_t\lambda_2 - \partial_x\lambda_1 - \left(\dfrac{2\bar\omega}{\bar\rho} - P(\bar\rho)\right)\partial_x\lambda_2 = -\sum\delta_{\tilde x_i}\beta_i\lambda_1 - \dfrac{1}{\tau}\lambda_2 - 1\end{cases} \tag{15}$$

They are both linear hyperbolic systems in non-conservative form that can be integrated using the schemes given in (Godlewski and Raviart, 1996).

The gradient keep the same form as (13) for the TTT objective and only the source terms of the adjoint equations (14) and (15) are slightly modified as $\mathbf{g}' = (-1\ \ 0)^T$ in this case.

5. NUMERICAL IMPLEMENTATION

Several specific methods have been proposed to integrate conservation laws. We propose here to use the Roe average method with an upwind scheme (Bermudez and Vazquez, 1994) as it capture accurately discontinuities and is devoid of oscillating behavior. The time stepping algorithm is

$$\mathbf{y}_i^{n+1} = \mathbf{y}_i^n - \frac{\Delta t}{\Delta x}\left(\tilde{\mathbf{f}}(\mathbf{y}_i^n,\mathbf{y}_{i+1}^n) - \tilde{\mathbf{f}}(\mathbf{y}_{i-1}^n,\mathbf{y}_i^n)\right) + \Delta t\,\tilde{\mathbf{s}}(\mathbf{y}_{i-1}^n,\mathbf{y}_i^n,\mathbf{y}_{i+1}^n) \tag{16}$$

with $\tilde{\mathbf{f}}(\cdot)$ the numerical flux given by

$$\tilde{\mathbf{f}}(\mathbf{y}_i^n,\mathbf{y}_{i+1}^n) = \frac{1}{2}\Big(\mathbf{f}(\tilde{\mathbf{y}}_{i+1/2}) - |D\mathbf{f}(\tilde{\mathbf{y}}_{i+1/2})|(\mathbf{y}_{i+1}-\mathbf{y}_i)\Big)$$

and $\tilde{\mathbf{s}}(\cdot)$ the numerical source term given by

$$\begin{aligned}\tilde{\mathbf{s}}(\mathbf{y}_{i-1}^n,\mathbf{y}_i^n,\mathbf{y}_{i+1}^n) = \\ \frac{1}{2}\Big(I + D\mathbf{f}(\tilde{\mathbf{y}}_{i-1/2})|D\mathbf{f}(\tilde{\mathbf{y}}_{i-1/2})|\Big)\frac{\mathbf{y}_{i-1}^n + \mathbf{y}_i^n}{2} \\ + \frac{1}{2}\Big(I - D\mathbf{f}(\tilde{\mathbf{y}}_{i+1/2})|D\mathbf{f}(\tilde{\mathbf{y}}_{i+1/2})|\Big)\frac{\mathbf{y}_i^n + \mathbf{y}_{i+1}^n}{2}\end{aligned}$$

where $|A| = T\ \mathrm{diag}(|\lambda_i|)\ T^{-1}$ with $A = T\Lambda T^{-1}$. $\tilde{\mathbf{y}}_{i+1/2}$ is the Roe average at the cell interface $i/i+1$. For the Payne model, Roe averages are

$$\begin{cases}\tilde\rho_{i+1/2} = \sqrt{\rho_i\rho_{i+1}} \\ \tilde v_{i+1/2} = \dfrac{\sqrt{\rho_i}v_i + \sqrt{\rho_{i+1}}v_{i+1}}{\sqrt{\rho_i}+\sqrt{\rho_{i+1}}} \\ \tilde\phi_{i+1/2} = \tilde\rho_{i+1/2}\tilde v_{i+1/2}\end{cases}$$

The linear adjoint equation (12) is simulated backwards in time with the following upwind method.

$$\lambda_i^{n-1} = \lambda_i^n - \frac{\Delta t}{\Delta x}\Big(- D\mathbf{f}(\bar{\mathbf{y}}_i^n)^T\Big)^+ (\lambda_i^n - \lambda_{i-1}^n)$$
$$- \frac{\Delta t}{\Delta x}\Big(- D\mathbf{f}(\bar{\mathbf{y}}_i^n)^T\Big)^- (\lambda_{i+1}^n - \lambda_i^n) + \Delta t\ \mathbf{S}_\lambda$$

where $A^+ = T\Lambda^+ T^{-1}$ and $A^- = T\Lambda^- T^{-1}$. For all partial differential equations, the boundary conditions are implemented through ghost cells forced to the boundary data, their applicability being directly handled by the discretization methods.

We provide below a simulation example with the VMT objective for a single on-ramp that creates a congestion with a constant inflow of 400 veh/h during 5 min on a 5 km freeway section. The optimizer gives the flow improvement depicted in Figure 1 with the ramp flow of Figure 2 computed in 20 iterations. The new metering rate releases slowly the vehicle and enables to delay the flow drop upstream of the on-ramp location. The improvement is rather local in space due to the finite speed of propagation.

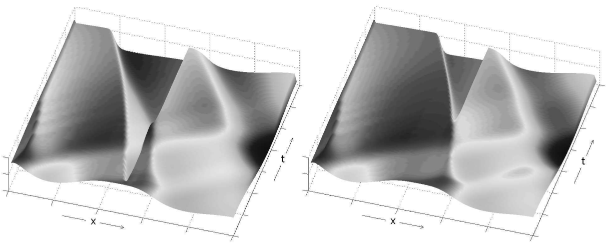

Fig. 1. Initial (left) and optimized (right) flows.

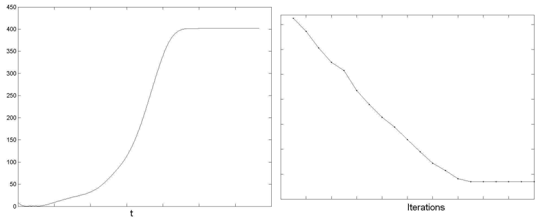

Fig. 2. Optimized control (left) and $\mathcal{J}_{\text{obs}}$ (right).

6. CONCLUSION

This paper proposed an optimal control algorithm for system of conservation laws based on adjoint-based gradient evaluations. The contribution of the paper is to take into account the piecewise-C^1 structure of the flow generated by conservation laws and the specific nature of the boundary conditions that are not applicable for all times. A traffic application was presented that computes the ramp metering rates given the initial and boundary conditions only. Improvement in the numerical algorithm are currently under study.

REFERENCES

Aw, A. and M. Rascle (2000). Resurrection of 'second order' models of traffic flow. *SIAM Journal on Applied Mathematics*.

Bardos, C. and O. Pironneau (2003). Derivatives and control in the presence of shocks. *Computational Fluid Dynamics Journal* **11**(4), 383–392.

Bermudez, A and M.A. Vazquez (1994). Upwind methods for hyperbolic conservation laws with source terms. *Computers Fluids*.

Bressan, A. (2000). *Hyperbolic Systems of Conservation Laws - The One-dimensional Cauchy Problem*. Oxford University Press.

Daganzo, C.F. (1995). Requiem for second-order fluid approximations of traffic flow. *Transportation Research Part B: Methodological*.

Dubois, F. and P.G. LeFloch (1988). Boundary conditions for nonlinear hyperbolic systems of conservation laws. *Journal of Differential Equations*.

Godlewski, E. and P.A. Raviart (1996). *Numerical Approximation of Hyperbolic Systems of Conservation Laws*. Applied Mathematical Sciences. Springer.

Godlewski, E. and P.A. Raviart (1999). The linearized stability of solutions of nonlinear hyperbolic systems of conservation laws. a general numerical approach.. *Mathematics and Computers in Simulation*.

Greenberg, J. M. (2001). Extensions and amplifications of a traffic model of aw and rascle. *SIAM Journal on Applied Mathematics*.

Greenberg, J. M., A. Y. LeRoux, R. Baraille and A. Noussair (1997). Analysis and approximation of conservation laws with source terms. *SIAM Journal on Numerical Analysis*.

Jacquet, D., C. Canudas de Wit and D. Koenig (2005). Traffic control and monitoring with a macroscopic model in the presence of strong congestion waves. In: *Proceedings of the 2005 Conference on Decision and Control*.

Lions, J.L. (1971). *Optimal control of systems governed by partial differential equations*. Springer.

Messmer, A. and M. Papageorgiou (1994). Automatic control methods applied to freeway traffic networks. *Automatica*.

Payne, H.J. (1971). Models of freeway traffic and control. In: *Mathematical Models of Public Systems* (G.A. Bekey, Ed.). Vol. 1 of *Simulation Council Proceedings*.

Sanders, B.F. and N.D. Katopodes (2000). Sensitivity analysis of shallow water flow by adjoint method. *Journal of Engineering Mechanics* **126**(9), 909–919.

Zhang, H. M. (1998). A theory of nonequilibrium traffic flow. *Transportation Research Part B: Methodological*.

Zhang, H. M. (2002). A non-equilibrium traffic model devoid of gas-like behavior. *Transportation Research Part B: Methodological*.