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Hypothesis H and the prime number theorem for automorphic representations

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Dedicated to Jean-Marc Deshouillers on the occasion of his sixtieth birthday

Abstract. For any unitary cuspidal representations $\pi_n$ of $GL_n(\mathbb{Q}_A)$, $n = 2, 3, 4$, respectively, consider two automorphic representations $\Pi$ and $\Pi'$ of $GL_6(\mathbb{Q}_A)$, where $\Pi_p \cong \wedge^2 \pi_{4,p}$ for $p \neq 2, 3$ and $\pi_{4,p}$ not supercuspidal, and $\Pi' = \pi_2 \boxtimes \pi_3$. First, Hypothesis H for $\Pi$ and $\Pi'$ is proved. Then contributions from prime powers are removed from the prime number theorem for cuspidal representations $\pi$ and $\pi'$ of $GL_m(\mathbb{Q}_A)$ and $GL_{m'}(\mathbb{Q}_A)$, respectively. The resulting prime number theorem is unconditional when $m, m' \leq 4$ and is under Hypothesis H otherwise.

Keywords. Hypothesis H, functoriality, prime number theorem

§ 1. Introduction

Recent developments in functoriality by the Langlands-Shahidi method have many profound applications in prime distribution. To name a few, we recall a recent proof of Hypothesis H for any cuspidal representation of $GL_4(\mathbb{Q}_A)$ and for Sym$^4(\pi)$ by Kim [2], where $\pi$ is an automorphic cuspidal representation of $GL_2(\mathbb{Q}_A)$. Here Hypothesis H predicts the convergence of a certain Dirichlet series associated with $(L'/L)'(s, \pi \times \tilde{\pi})$ taken over prime powers.

More precisely, let $\pi = \otimes_p \pi_p$ be a unitary automorphic cuspidal representation of $GL_m(\mathbb{Q}_A)$. Or more generally, let $\pi$ be an automorphic representation irreducibly induced from unitary cuspidal representations, i.e., $\pi = \text{Ind } \sigma_1 \otimes \cdots \otimes \sigma_k$, where $\sigma_j$ is a cuspidal representation of $GL_{m_j}(\mathbb{Q}_A)$, with $m_1 + \cdots + m_k = m$. The local component $\pi_p$ with $p < \infty$ can be parameterized by the Satake parameters $\text{diag}[\alpha_\pi(p, 1), \ldots, \alpha_\pi(p, m)]$. For $\nu \geq 1$ define

$$a_\pi(p^\nu) = \sum_{j=1}^m \alpha_\pi(p, j)^\nu.$$ 

Let $\tilde{\pi}$ be the contragredient representation of $\pi$, and $L(s, \pi \times \tilde{\pi})$ the Rankin-Selberg $L$-function. Then for $\Re s > 1$, we have (see [10], RS 1)

$$\left(\frac{L'}{L}\right)'(s, \pi \times \tilde{\pi}) = \sum_{n=1}^\infty \frac{(\log n)\Lambda(n)|a_\pi(n)|^2}{n^s}.$$ 

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Here $\Lambda(n) = \log p$ if $n = p^r$ and $\Lambda(n) = 0$ otherwise, so that the series in (1.2) is taken over primes and prime powers.

**Hypothesis H.** (Rudnick and Sarnak [10]) For any fixed $\nu \geq 2$, 
\[
\sum_{p} \frac{(\log p)^2 |a_{\pi}(p^r)|^2}{p^{\nu}} < \infty.
\]

Hypothesis H is trivial for $m = 1$. When $m = 2$ it follows from bounds toward the Ramanujan conjecture $|\alpha_{\pi}(p,j)| \leq p^\theta$ with $\theta = 7/64$ (see [9]), another result based on the Langlands-Shahidi method proved by Kim and Sarnak in [1]. For $m = 3$, Hypothesis H follows from the Rankin-Selberg theory [10]. The $GL_4$ case was proved by Kim [2] based on his proof of the (weak) functoriality of the exterior square $\wedge^2 \pi$ from a cuspidal representation $\pi$ of $GL_4(\mathbb{Q}_A)$ (see [1]). Beyond $GL_4$, the only known special case for Hypothesis H is the symmetric fourth power $\text{Sym}^4(\pi)$ of a cuspidal representation $\pi$ of $GL_2(\mathbb{Q}_A)$, which is an automorphic representation of $GL_5(\mathbb{Q}_A)$.

The first goal of the present paper is to prove Hypothesis H for two types of automorphic representations of $GL_6(\mathbb{Q}_A)$.

**Theorem 1.** Let $\pi$ be a cuspidal representation of $GL_4(\mathbb{Q}_A)$. Denote by $T$ the set of places consisting of $p = 2, 3$ and those $p$ at which $\pi_p$ is supercuspidal. Let $\Pi$ be the automorphic representation of $GL_6(\mathbb{Q}_A)$ such that $\Pi_p \cong \wedge \pi_p$ if $p \not\in T$, according to [1]. Then Hypothesis H holds for $\Pi$.

**Theorem 2.** Let $\pi_1$ (resp. $\pi_2$) be a cuspidal representation of $GL_2(\mathbb{Q}_A)$ (resp. $GL_3(\mathbb{Q}_A)$). Let $\Pi'$ be the automorphic representation of $GL_6(\mathbb{Q}_A)$ equal to $\pi \boxtimes \pi_2$ according to [3]. Then Hypothesis H holds for $\Pi'$.

As an application, one can use Hypothesis H to deduce the following Mertens’ theorem for automorphic representations, or the so-called Selberg orthogonality conjecture, from unconditional results on similar sums taken over primes and prime powers:

(1.3) \[
\sum_{p \leq x} \frac{|a_{\pi}(p)|^2}{p} = \log \log x + O(1);
\]

(1.4) \[
\sum_{p \leq x} \frac{a_{\pi}(p)a_{\pi'}(p)}{p} = O(1),
\]

when $\pi \not\cong \pi'$. Here (1.3) was proved by Rudnick and Sarnak [10], while (1.4) was proved by Liu, Wang and Ye ([6], [4]). Results in (1.3) and (1.4) played crucial roles in the $n$-level correlation of nontrivial zeros of automorphic $L$-functions and random matrix theory ([10], [5], [7]).

Another application of Hypothesis H is on the prime number theorem for automorphic representations. For any self-dual cuspidal representation $\pi$ of $GL_m(\mathbb{Q}_A)$, Liu, Wang and Ye [4] showed that there is a constant $c > 0$ such that

(1.5) \[
\sum_{n \leq x} \Lambda(n)|a_{\pi}(n)|^2 = x + O(x \exp(-c\sqrt{\log x})).
\]
In [8], Liu and Ye proved that
\[ \sum_{n \leq x} \Lambda(n) a_\pi(n) \pi'(n) \]
\[ (1.6) \]
\[ = \begin{cases} 
\frac{x^{1+i\tau_0}}{1+i\tau_0} + O(x \exp(-c\sqrt{\log x})) & \text{if } \pi' \equiv \pi \otimes |det|^{\tau_0} \text{ for some } \tau_0 \in \mathbb{R}; \\
O(x \exp(-c\sqrt{\log x})) & \text{if } \pi' \not\equiv \pi \otimes |det|^{\tau} \text{ for any } \tau \in \mathbb{R},
\end{cases} \]
where \( \pi \) and \( \pi' \) are cuspidal representations of \( GL_m(\mathbb{Q}_\ell) \) and \( GL_{m'}(\mathbb{Q}_\ell) \), respectively, such that at least one of them is self-dual.

The second goal of the present paper is to use Hypothesis H to remove terms on prime powers from the left side of (1.6) and deduce a prime number theorem over primes.

**Theorem 3.** Let \( \pi \) and \( \pi' \) be as above. (i) If \( m, m' \leq 4 \), then
\[ \sum_{p \leq x} (\log p) a_\pi(p) \pi'(p) \]
\[ (1.7) \]
\[ = \begin{cases} 
\frac{x^{1+i\tau_0}}{1+i\tau_0} + O(x \exp(-c\sqrt{\log x})) & \text{if } \pi' \equiv \pi \otimes |det|^{\tau_0} \text{ for some } \tau_0 \in \mathbb{R}; \\
O(x \exp(-c\sqrt{\log x})) & \text{if } \pi' \not\equiv \pi \otimes |det|^{\tau} \text{ for any } \tau \in \mathbb{R},
\end{cases} \]
(ii) If \( \max(m, m') \geq 5 \), (1.7) is true under Hypothesis H with error terms replaced by \( O(x/\log x) \).

We remark that (i) is an unconditional result.

### § 2. Proof of Theorems 1 and 2

**Lemma 2.1.** Let \( \pi \) be a unitary cuspidal representation for \( GL_m(\mathbb{Q}_\ell) \), or an automorphic representation irreducibly induced from unitary cuspidal representations. Then for any \( \nu_0 \geq (m^2 + 1)/2 + 1 \), \( \epsilon > 0 \), and integer \( \ell \geq 0 \),
\[ (2.1) \]
\[ \sum_{\nu \geq \nu_0, \nu' \leq x} (\log p) |a_\pi(p')|^2 \ll x^{1-2/(m^2+1)+1/\nu_0} \log x, \]
\[ (2.2) \]
\[ \sum_{p} \frac{(\log p)^\ell |a_\pi(p)|^2}{p^{1+\epsilon}} < \infty. \]

**Proof.** From (1.1) and the bound toward the Ramanujan conjecture ([10])
\[ (2.3) \]
\[ |a_\pi(p, j)| \leq p^{1/2-1/(m^2+1)} \quad (j = 1, \ldots, m), \]
we know that
\[ |a_\pi(p')|^2 \leq m^2 p^{1-2/(m^2+1)} \nu. \]

Then
\[ \sum_{\nu \geq \nu_0, \nu' \leq x} (\log p) |a_\pi(p')|^2 \leq m^2 \sum_{\nu_0 \leq \nu' \leq 2 \log x} p^{1/2-1/(m^2+1)} \nu \]
\[ \ll m x^{1-2/(m^2+1)+1/\nu_0} \log x. \]
Eqn. (2.2) follows from the fact that the \( \ell \)th-derivation of \( \log L(s, \pi \times \pi') \) converges absolutely for \( \Re s > 1 \). \[ \square \]
Lemma 2.2. Let $\pi'$ (resp. $\pi''$) be a unitary cuspidal representation, or an automorphic representation irreducibly induced from unitary cuspidal representations, for $\text{GL}_{m'}(\mathbb{Q}_k)$ (resp. $\text{GL}_{m''}(\mathbb{Q}_k)$). Let $\nu \geq 2$ be an integer and $\mathcal{P}$ a set of prime numbers. If there are fixed constants $\delta' \in (0, 1]$ and $\delta'' \in (0, \frac{1}{2}]$ such that

$$\left| a_{\pi'}(p^\nu) \right|^2 \ll_{\nu, \epsilon} |a_{\pi''}(p)|^2 p^{(1-\delta')(\nu-1) + p^{(1/2-\delta'\nu)}}$$

for all $p \in \mathcal{P}$, then for any $\epsilon > 0$ we have

$$\sum_{p^\nu \leq x, p \in \mathcal{P}} (\log p) |a_{\pi'}(p^\nu)|^2 \ll_{\nu, \epsilon} x^{1-\delta}$$

with $\delta := \min\{\delta'/2 + \delta' - \epsilon, \delta''\}$.

Proof. By (2.4) and the Rankin-Selberg theory, for any $\eta > 0$ we can write

$$\sum_{p^\nu \leq x, p \in \mathcal{P}} (\log p) |a_{\pi'}(p^\nu)|^2 \ll_{\nu, \epsilon} \sum_{p^\nu \leq x, p \in \mathcal{P}} (\log p) |a_{\pi''}(p)|^2 p^{(1-\delta')(\nu-1) + x^{1/2+1/\nu-\delta'\nu}}$$

$$\ll_{\nu, \epsilon} x^\eta \sum_{p^\nu \leq x, p \in \mathcal{P}} (\log p) |a_{\pi''}(p)|^2 + x \sum_{x^{\delta'} < p^\nu \leq x, p \in \mathcal{P}} (\log p) |a_{\pi''}(p)|^2 + x^{1-\delta''\nu}.$$  

By (2.2) with $\pi = \pi''$ and $\ell = 1$, it follows that

$$\sum_{p^\nu \leq x, p \in \mathcal{P}} (\log p) |a_{\pi''}(p)|^2 \ll 1$$

and

$$\sum_{x^{\delta'} < p^\nu \leq x, p \in \mathcal{P}} (\log p) |a_{\pi''}(p)|^2 \leq \frac{1}{(x^{\eta/\nu})^{\delta'(\nu-1)-\epsilon}} \sum_{x^{\delta'} < p^\nu \leq x, p \in \mathcal{P}} (\log p) |a_{\pi''}(p)|^2 \ll_{\nu, \epsilon} x^{1-\delta''(\nu-1)-\epsilon/\nu}.$$  

Inserting these two estimates into the preceding inequality, we find

$$\sum_{p^\nu \leq x, p \in \mathcal{P}} (\log p) |a_{\pi'}(p^\nu)|^2 \ll_{\nu, \epsilon} x^\eta + x^{1-\eta[\delta'(\nu-1)-\epsilon/\nu] + x^{1-\delta''\nu}}.$$  

Taking $\eta = \nu/(1 + \delta'\nu - \delta') + \epsilon$, we obtain

$$\sum_{p^\nu \leq x, p \in \mathcal{P}} (\log p) |a_{\pi'}(p^\nu)|^2 \ll_{\nu, \epsilon} x^{\nu/(1+\delta')(\nu-1) + \epsilon} + x^{1-\delta''\nu} \ll_{\nu, \epsilon} x^{1-\delta'/(2+\delta') + \epsilon} + x^{1-\delta''\nu} \ll_{\nu, \epsilon} x^{1-\delta}.$$  

In the second inequality, we have used the fact that $\nu \geq 2$. \hfill \Box

Remark. In proving Hypothesis H, an inequality of the form of (2.4) plays a crucial role. Lemma 2.2 has more flexibility as $\pi''$ is allowed to be different from $\pi'$. 
Lemma 2.3. Let $\Pi'$ be either $\Pi$ or $\Pi$ as in Theorems 1 and 2. Then for any $\varepsilon > 0$, we have

$$
\sum_{\nu \geq 2, \, p^\nu \leq x} (\log p) |a_{\Pi'}(p^\nu)|^2 \ll_{\varepsilon} x^{1-1/38+\varepsilon}.
$$

Proof. In view of (2.1) with the choice of $m = 6$ and $\nu_0 = [37 \times 38/39] + 1$, it suffices to show that for any fixed $\varepsilon > 0$ and $\nu \geq 2$ we have

$$
\sum_{p^\nu \leq x} (\log p) |a_{\Pi'}(p^\nu)|^2 \ll_{\nu, \varepsilon} x^{1-1/38+\varepsilon},
$$

First let us consider the case of $\Pi$. Let $\pi = \otimes \pi_p$ be a cuspidal automorphic representation for $GL_4(A\mathbb{Q})$. Recall that $\Pi$ is irreducibly induced from unitary cuspidal representations. Let $S_0$ be the set of places where $\Pi_p$ is tempered. Then

$$
\sum_{p \in S_0} (\log p)^2 |a_{\Pi}(p^\nu)|^2 < \infty.
$$

Eqn. (2.8) is also true if we replace $S_0$ by $T$, which is given in Theorem 1, because at most two terms for $p = 2, 3$ will then be added to (2.8).

If $p \notin S_0 \cup T$, the Satake parameters of $\pi_p$ are in one of the following forms:

$$
\begin{align*}
S_1 & : \text{diag}[u_1p^a, u_2p^a, u_1p^{-a}, u_2p^{-a}], & \text{where } & 0 < a \leq \frac{1}{2} - \frac{1}{17}, \\
S_2 & : \text{diag}[u_1p^a, u_2, u_3, u_1p^{-a}], & \text{where } & 0 < a \leq \frac{1}{2} - \frac{1}{17}, \\
S_3 & : \text{diag}[u_1p^{a_1}, u_2p^{a_2}, u_1p^{-a_1}, u_2p^{-a_2}], & \text{where } & 0 < a_2 < a_1 \leq \frac{1}{2} - \frac{1}{17},
\end{align*}
$$

where $u_1, u_2, u_3$ are complex numbers of absolute value 1 and we have suppressed their dependence on $p$ for the simplicity of notation. As in [1], the corresponding Satake parameters of $\Pi_p \simeq \Lambda^2 \pi_p$ are as follows:

$$
\begin{align*}
S_1 & : \text{diag}[u_1u_2p^{2a}, u_1u_2, u_1^2, u_2, u_1u_2, u_1u_2p^{-2a}], \\
S_2 & : \text{diag}[u_1u_2p^a, u_1u_3p^a, u_1^2, u_2u_3, u_1u_2p^{-a}, u_1u_3p^{-a}], \\
S_3 & : \text{diag}[u_1u_2p^{a_1+a_2}, u_1u_2p^{a_1-a_2}, u_1^2, u_2, u_1u_2p^{-(a_1-a_2)}, u_1u_2p^{-(a_1+a_2)}].
\end{align*}
$$

Since $\Pi$ is an automorphic representation for $GL_4(A\mathbb{Q})$ which is irreducibly induced from unitary cuspidal, (2.3) gives

$$
\begin{align*}
0 < 2a & \leq \frac{1}{2} - \frac{1}{37} & \text{if } p \in S_1, \\
0 < a & \leq \frac{1}{2} - \frac{1}{37} & \text{if } p \in S_2, \\
0 < a_2 < a_1 & \leq \frac{1}{2} - \frac{1}{37} & \text{if } p \in S_3, \\
\text{and } a_1 + a_2 & \leq \frac{1}{2} - \frac{1}{37} & \text{if } p \in S_3.
\end{align*}
$$

If $p \in S_1$, then

$$
\begin{align*}
|a_{\Pi}(p^\nu)| &= |(u_1u_2)^\nu (p^{2a\nu} + p^{-2a\nu} + 2) + u_1^\nu + u_2^\nu| \leq p^{2a\nu} + 5, \\
|a_{\Pi}(p)| &= |u_1u_2(p^{2a} + p^{-2a} + 2) + u_1^2 + u_2^2| \geq p^{2a}.
\end{align*}
$$

From these and (2.3) with $m = 6$, we deduce that

$$
|a_{\Pi}(p^\nu)|^2 \leq (|a_{\Pi}(p)|^\nu + 5)^2 \ll_{\nu} |a_{\Pi}(p)|^{2\nu} + 1 \ll_{\nu} |a_{\Pi}(p)|^2 p^{(1-2/37)(\nu-1)} + 1,
$$

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where the implied constants are all independent of $p$.

Similarly if $p \in S_2$, then

$$|a_\Pi(p^\nu)| = |u_1^*(u_2^a + u_3^b)(p^{\alpha\nu} + p^{-\alpha\nu}) + u_2^2\nu + (u_2u_3)^\nu| \leq 2p^{\alpha\nu} + 4, \quad |a_\pi(p)| = |u_1(p^a + p^{-a}) + u_2 + u_3| \geq p^a - 2.$$ 

These and (2.3) with $m = 4$ imply

$$(2.11) \quad |a_\Pi(p^\nu)|^2 \leq 2(|a_\pi(p)| + 2)^{2\nu} + 4 \ll_{\nu} |a_\pi(p)|^{2\nu} + 1 \ll_{\nu} |a_\pi(p)|^{2\nu}p^{(1-2/17)(\nu-1)} + 1.$$ 

Finally if $p \in S_3$, then

$$|a_\Pi(p^\nu)| \leq 2p^{(a_1+a_2)\nu} + 4, \quad |a_\Pi(p)| \geq p^{a_1+a_2} - 1,$$

from which we deduce, as before,

$$(2.12) \quad |a_\Pi(p^\nu)|^2 \leq 2(|a_\Pi(p)| + 1)^{2\nu} + 4 \ll_{\nu} |a_\Pi(p)|^{2\nu} + 1 \ll_{\nu} |a_\Pi(p)|^{2\nu}p^{(1-2/37)(\nu-1)} + 1.$$ 

Now we apply Lemma 2.2 with the choice of parameters

$$(\pi', \pi'', \delta', \delta'') = \begin{cases} (\Pi, \Pi, \frac{2}{17}, \frac{1}{2}) & \text{if } \mathcal{P} = S_1 \text{ or } S_3, \\ (\Pi, \pi, \frac{2}{17}, \frac{1}{2}) & \text{if } \mathcal{P} = S_2, \end{cases}$$

to write

$$(2.13) \quad \sum_{p^\nu \leq x, p \in S_3} (\log p)|a_\Pi(p^\nu)|^2 \ll_{\nu} \begin{cases} x^{1-1/38+\varepsilon} & \text{if } j = 1, 3, \\ x^{1-1/19+\varepsilon} & \text{if } j = 2, \end{cases}$$

Now the required estimate (2.7) for $\Pi$ follows from (2.10) and (2.13).

Next let us turn to the case of $\Pi'$. Let $\pi_1 = \oplus_p \pi_{1,p}$ (resp. $\pi_2 = \oplus_p \pi_{2,p}$) be a cuspidal representation of $GL_2(\mathbb{Q}_p)$ (resp. $GL_3(\mathbb{Q}_p)$). We may just consider those $p$ such that at least one of $\pi_1,p$ and $\pi_2,p$ is not tempered. Then the Satake parameters of $\pi_1,p$ and $\pi_2,p$ are as follows:

$\pi_1,p : \text{diag}[u_1p^a, u_1p^{-a}]$, where $0 \leq a \leq \frac{2}{37},$

$\pi_2,p : \text{diag}[u_2p^b, u_3, u_2p^{-b}]$, where $0 \leq b \leq \frac{1}{2} - \frac{1}{17},$

where $u_1, u_2, u_3$ are complex numbers of absolute value 1. Thus the Satake parameters of $\Pi'_p = \pi_1,p \otimes \pi_2,p$ are:

$$\text{diag}[u_1u_2p^{a+b}, u_1u_2p^{b-a}, u_1u_3p^a, u_1u_3p^{-a}, u_1u_2p^{-(b-a)}, u_1u_2p^{-(a+b)}]$$

with

$$(2.14) \quad 0 < a + b \leq \frac{1}{2} - \frac{1}{17}.$$ 

Then

$$(2.15) \quad |a_{\Pi'}(p^\nu)| = |(u_1u_2)^\nu(p^{(a+b)\nu} + p^{(a-b)\nu} + p^{(b-a)\nu} + p^{-(a+b)\nu}) + (u_1u_3)^\nu(p^{a\nu} + p^{-a\nu})|. $$
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From (2.15) we can see that

\[(2.16) \quad |a_{\Pi}(p^\nu)| \leq 6p^{a+b}\nu, \quad |a_{\Pi}(p)| \geq p^{a+b} - p^a.\]

Thus in view of (2.14), (2.16) and the fact that \(a \leq \frac{7}{37}\), we can deduce

\[(2.17) \quad |a_{\Pi'}(p^\nu)|^2 \ll (|a_{\Pi'}(p)| + p^b)^{2\nu} \ll \nu \quad |a_{\Pi'}(p)|^{2\nu} + p^{2a}\nu \ll \nu \quad |a_{\Pi'}(p)|^{(1-2/37)(\nu-1)} + p^{(1/2-9/32)\nu}.\]

Applying Lemma 2.2 with \(\pi' = \pi'' = \Pi'\), \(\delta' = \frac{2}{37}\) and \(\delta'' = \frac{9}{32}\), we now conclude that

\[\sum_{p^\nu \leq x} (\log p)|a_{\Pi'}(p^\nu)|^2 \ll x^{1-1/38+\varepsilon}.\]

This completes the proof. \(\square\)

The proof of Theorems 1 and 2.

Let \(\Pi''\) be either \(\Pi\) or \(\Pi'\). We can write

\[\sum_{p^\nu \leq x, \nu \geq 2} \frac{(\log p)^2|a_{\Pi'}(p^\nu)|^2}{p^\nu} = \sum_{j \geq 0} \sum_{2j+1 \leq p^\nu \leq 2j+1, \nu \geq 2} \frac{(\log p)^2|a_{\Pi'}(p^\nu)|^2}{p^\nu} \leq \sum_{j \geq 0} \frac{\log(2j+1)x}{2^j x} \sum_{2j < p^\nu \leq 2j+1, \nu \geq 2} (\log p)|a_{\Pi'}(p^\nu)|^2.\]

Using Lemma 2.3, we have

\[\sum_{p^\nu \leq x, \nu \geq 2} \frac{(\log p)^2|a_{\Pi'}(p^\nu)|^2}{p^\nu} \ll \sum_{j \geq 0} \frac{\log(2j+1)x}{2^j x} (2j+1)x^{1-1/38+\varepsilon} \ll \sum_{j \geq 0} \frac{\log(2j+1)x}{(2j+1)x^{1/38-\varepsilon}} \ll x^{-1/38+2\varepsilon}.\]

This implies the required result. \(\square\)

§ 3. Proof of Theorem 3

Theorem 3 follows immediately from (1.6) and the following lemma.

Lemma 3.1. Let \(\pi\) be a unitary automorphic cuspidal representation for \(GL_m(\mathbb{Q}_A)\).

(i) For each \(m \in \{1, \ldots, 4\}\), there is a constant \(\delta_m > 0\) such that

\[\sum_{p^\nu \leq x, \nu \geq 2} (\log p)|a_{\pi}(p^\nu)|^2 \ll x^{-\delta_m}.\]

(ii) If \(m \geq 5\), under Hypothesis H we have

\[\sum_{p^\nu \leq x, \nu \geq 2} (\log p)|a_{\pi}(p^\nu)|^2 \ll x / \log x.\]
Proof. In view of (2.1) of Lemma 2.1 with a suitable choice of \( \nu_0 \), it suffices to show, for fixed \( \nu \geq 2 \), that (i)

\[
(3.1) \quad \sum_{p^\nu \leq x} (\log p)|a_\pi(p^\nu)|^2 \ll_{\nu} x^{1-\delta_m},
\]

if \( m \leq 4 \), and (ii)

\[
(3.2) \quad \sum_{p^\nu \leq x} (\log p)|\alpha_\pi(p^\nu)|^2 \ll_{\nu} x / \log x
\]

if \( m \geq 5 \) under Hypothesis H.

First we prove (3.2):

\[
\sum_{p^\nu \leq x} (\log p)|\alpha_\pi(p^\nu)|^2 = \sum_{p^\nu \leq x^{1/2}} (\log p)|\alpha_\pi(p^\nu)|^2 + \sum_{x^{1/2} < p^\nu \leq x} (\log p)|\alpha_\pi(p^\nu)|^2 \\
\leq x^{1/2} \sum_{p^\nu \leq x^{1/2}} (\log p)^2|\alpha_\pi(p^\nu)|^2 + \frac{2x}{\log x} \sum_{x^{1/2} < p^\nu \leq x} (\log p)^2|\alpha_\pi(p^\nu)|^2
\]

which is \( \ll x / \log x \) under Hypothesis H.

Next we prove (3.1) for \( m = 4 \), since other cases are easier. As before it suffices to consider the sum on the left side of (3.1) taken over \( p \neq 2, 3 \) with \( \pi_p \) being not tempered. Then for such a \( p \), \( \Pi_p \equiv \lambda^2 \pi_p \). There are then three possibilities.

If \( p \in S_1 \) as in (2.9), using \( \Pi_p \) we get \( 0 < 2a \leq \frac{1}{2} - \frac{1}{17} \) as in (2.10). Then

\[
|\alpha_\pi(p^\nu)|^2 = |(u_1^\nu + u_2^\nu)(p^{a\nu} + p^{-a\nu})|^2 \leq 16p^{(1/2 - 1/37)\nu}.
\]

From this, we deduce that

\[
(3.3) \quad \sum_{p^\nu \leq x, p \in S_1} (\log p)|\alpha_\pi(p^\nu)|^2 \ll \sum_{p^\nu \leq x, p \in S_1} (\log p)p^{(1/2 - 1/37)\nu} \ll x^{1-1/37}.
\]

If \( p \in S_2 \), we have

\[
|\alpha_\pi(p^\nu)| = |u_1^\nu(p^{a\nu} + p^{-a\nu}) + u_2^\nu + u_3^\nu| \leq p^{a\nu} + 3,
\]

\[
|\alpha_\pi(p)| = |u_1(p^{a} + p^{-a}) + u_2 + u_3| \geq p^a - 2
\]

with \( 0 < a \leq 1/2 - 1/17 \). Then

\[
|\alpha_\pi(p^\nu)|^2 \leq \{(|\alpha_\pi(p)| + 2)^\nu + 3\}^2 \ll_{\nu} |\alpha_\pi(p)|^{2\nu} + 1 \\
\ll_{\nu} |\alpha_\pi(p)|^{2\nu}p^{(1-2/17)(\nu-1) + 1}.
\]

Similarly if \( p \in S_3 \), then

\[
|\alpha_\pi(p^\nu)| = |u_1^\nu(p^{a_1\nu} + p^{-a_1\nu}) + u_2^\nu(p^{a_2\nu} + p^{-a_2\nu})| \leq 2p^{a_1\nu} + 2,
\]

\[
|\alpha_\pi(p)| = |u_1(p^{a_1} + p^{-a_1}) + u_2(p^{a_2} + p^{-a_2})| \geq p^{a_1} - 2p^{a_2}.
\]
From this, (2.3) with $m = 4$ and the last inequality of (2.10), we deduce that

\begin{equation}
|\pi(p')^2| \leq \{2(|\pi(p)| + 2p^2)^\nu + 2\}^2 \ll \nu|\pi(p)|^{2\nu} + p^{2\nu}
\ll \nu|\pi(p)|^2p^{(1-2/17)(\nu-1)} + p^{(1/2-1/37)\nu}.
\end{equation}

As before, we can apply Lemma 2.2 with the choice of parameters

\[ (\pi', \pi'', \delta', \delta'') = \begin{cases} 
(\pi, \pi, \frac{2}{17}, \frac{1}{2}) & \text{if } \mathcal{P} = S_2 \\
(\pi, \pi, \frac{3}{17}, \frac{1}{37}) & \text{if } \mathcal{P} = S_3 
\end{cases} \]

to write

\begin{equation}
\sum_{p^e \leq x, p \in S_j} (\log p)|\pi(p')^2| \ll \nu x^{1-j/37} \quad (j = 2, 3).
\end{equation}

Now the required result follows from (3.3) and (3.6). \qed

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